Renormalised Field Theory for Ideal Molecular-Beam Epitaxy

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Edward Sherman

Imperial College London

Department of Mathematics

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I, Edward Sherman, declare that this thesis is my own original work, unless otherwise stated and referenced.

In this thesis an overview is given of the renormalisation group as it is applied to equilibrium systems; the methods of field theory are extended to non-equilibrium systems, described by a Langevin equation in the stead of a Hamiltonian; this analysis is applied to a well known model of surface growth driven by molecular-beam epitaxy.

The renormalisation group is a celebrated technique in both hard and soft condensed matter physics for probing the asymptotic behaviour of a model, though in this thesis no examination is made of quantum effects. Several distinct methods exist under the banner of renormalisation, most famously the approaches of Wilson and field theory. The renormalisation group is explored through a comparison of these approaches.

The approach of field theory, with its methods being applied to an equilibrium system where a model is defined by a Hamiltonian, can be extended to analyse non-equilibrium systems, where a model is described by a Langevin equation. One class of the non-equilibrium condensed matter systems which have received extensive attention is that of surface growth. For the last two decades the Villain-Lai-Das Sarma equation has been used to understand conserved surface growth processes such as molecular-beam epitaxy. However, the theory has some aspects that seem incomplete. The mound formation observed experimentally and numerically lacks a complete theoretical narrative for its mechanism. Also, no clear picture has emerged over a disagreement in the literature about the alleged exactness of scaling relations. Using field theory to analyse the original derivation of the Villain-Lai-Das Sarma equation reveals that terms responsible for mound formation are generated under renormalisation, further these terms should have been included initially on symmetry grounds. It is possible to recover several widely studied Langevin equations at the trivial fixed point of the full theory, allowing a more complete theoretical picture to be presented for conserved epitaxial surface growth.

This is for my mother

I would like to thank Gunnar Pruessner for the immensely enjoyable work we did together, for his unfailing friendship, and even for the occasional attempts made at supervision. I would like to thank Andy Parry, in a way this is all your fault as you inspired me to do physics, and my time at Imperial would have been unimaginable without you.

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Contents

Chapter 1

An Introduction to the Renormalisation Group

The renormalisation group (RG) is used in hard and soft condensed matter physics, for both statistical and quantum systems. However, my understanding of it is informed almost entirely by statistical physics. The work I have done is in statistical field theory and uses scalar fields, so no mention of particle physics will be made. Though I am sure there are insights to be offered from that approach, I cannot offer them.

In keeping with tradition, the RG is initially motivated with a view to understanding critical phenomena and phase transitions. When looking at critical phenomena one is not so much interested in modelling the fine details; the experiments are too good anyway compared to the theory (Lipa *et al.*, 1996), so it is best to avoid hubris and embarrassment. The goal of the models becomes to an extent qualitative, aiming to capture the long range behaviour; so tools like the Ising and Ginzburg-Landau models are used.

Conceptually the RG is straight forward, or rather its motivation, purpose and goals are. It also comes with a wealth of mathematical details, while these are enormously fun they can obscure the underlying simplicity. Suppose there is a system, $\mathcal{A}$, described by some Hamiltonian and partition function, in which a calculation is intractable. To deal with this a way could be found of transforming to another system, $\mathcal{A}'$, where the calculation can be done, getting rid of some degrees of freedom to simplify the calculation, but keeping the same long range behaviour. One might say it is the physics of metaphor and simile. This is the core idea of the RG.

The philosophy that informs this desire is the one that is present in the mental juggling done when going between models at different length scales, it is bound up with the role a minimum length has in models. Consider an electron in orbit around a heavy nucleus, say that of lead. The length scale which is relevant in the equations of motion for this atomic electron is the atomic size, a few hundred picometres. The macroscopic world is effectively an infinite distance away. All the physics going on in the nucleus, at a length scale of a few femtometres, is far below the length scale 'visible' to the electronic interactions. It is captured in a handful of parameters, such as the nuclear charge, that appear in the equations of motion of the electron. The details of the nuclear interactions are not relevant on the atomic scale, they take place below the minimum length scale of interest in the problem.

If there were now several thousand of these heavy atoms, enough to make a dilute gas, other effects would emerge. For example sound could now propagated through the system. Using the off the shelf models that are available for this, the minimum length scale of interest is now a few microns, the mean free path of the atoms. The details of the atomic physics have become as irrelevant as the nuclear physic was before, finding expression only in properties like compressibility and viscosity. As the minimum length of interest is changed, in this rather odd box containing a dilute lead gas, the equations of motion change drastically.

While in principle it should be possible to answer questions about sound waves using atomic physics, it is effectively a calculation that cannot be done. However if all that was available were the atomic equations of motion it might be hoped that there was some well posed way to adjust the minimum length of the problem, morphing the equations till they gave something useable, progressively wrapping up the underlying physics into a few parameters until the calculation could be done. Put in this way, it is probably daft to expect any sense to come out of such a process. Sometimes though it does make sense, and this is what the RG is for.

Why this process is expected to make some sense for critical phenomena is the divergence of the correlation length as the system approaches the critical point, suggesting that the microscopic details of a model become unimportant. This is tied to the observation of identical behaviour in different systems with phase transitions; of different substances undergoing a liquid-gas transition, and their similarity to the ferromagnetic transition etc.

Another popular physical example is of screened ions in a fluid. The ion polarises and attracts nearby molecules, so that away from the immediate atomic proximity of the ion its charge is screened by the surrounding polarised molecules, replacing the bare charge with a new effective charge as it is felt at a distance. This cloud of polarised molecules surrounded the ion also gives it a new effective mass. This acquisition of new effective parameters as the length scale is increased is in the same spirit as the example above.

In the following section I introduce the classical approaches of statistical mechanics to the RG, the real-space and momentum-space formulations. In the following chapter the approach of field theory is explored, demonstrating some of the famous, signature features of renormalisation. The mathematical detail is used sparingly in order to quickly outline the salient points and offer a comparison of the different approaches. Some passing familiarity is assumed at first with the Ising model, though the treatment here is only superficial. Chapter 2 deals with field theory as applied to a ϕ^4 model in more depth. Of the many excellent books in the field I would point to a discussion of the Ising model by Huang (Huang, 1987), Christensen and Moloney (Christensen and Moloney, 2005) applying a real-space RG to the Ising model, and Amit (Amit, 1984) for field theory. Though my chief recommendation for both would be a book by Le Bellac (Le Bellac, 1991). A special mention goes to a paper by Shang-keng Ma (Ma, 1973), which I find to be wonderfully clear and insightful. It contains a worked example of the RG applied to the spherical model of Stanley fame (Stanley, 1968), the beauty of which is that it requires no obfuscatory approximations to demonstrate the process of renormalisation. Also the description of the role of a minimum length in models, as given above, is based on the discussion given by Ma in that paper. Indeed my understanding of renormalisation and the philosophy that underpins it, as presented here, owes much to the insights offered in that paper.

1.1 The Classic Approach

1.1.1 Real-space

The process of the real-space RG, of applying Kadanoff's (Kadanoff, 1966a) block spin to a two-dimensional Ising model, is a good illustration of a systematic scale transformation that does make sense.

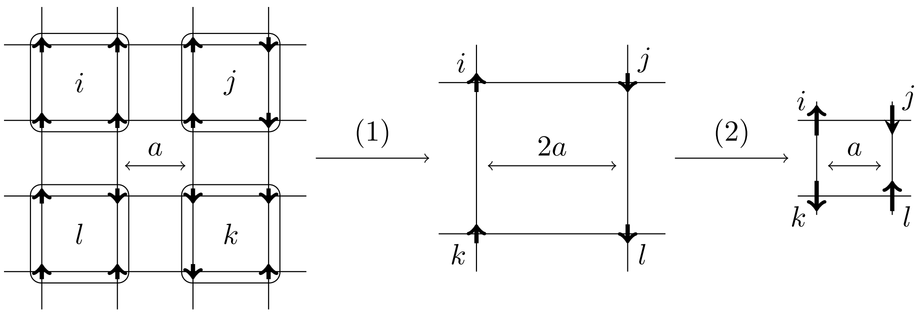
For simplicity take a square lattice in the absence of an external field. A spin is assigned to each lattice site, site i on the lattice has a spin S_i, and each spin takes one of the two values ± 1. Between adjacent lattice sites there is an interaction, strength K, that favours alignment and penalises difference by lowering or raising the energy of a configuration by K. The system is described by the traditional Hamiltonian

$$\mathbf{H}[S_i] = -K \sum_{\langle ij \rangle} S_i S_j, \tag{1.1}$$

formed by summing over the interaction between all nearest neighbour pairs $\langle ij \rangle$. The partition function, $\mathcal{Z}$, is given by summing the Boltzman weight over all possible realisations of the spins in the system, that is

$$\mathcal{Z} = \sum_{[S_i]} \exp\left(-\mathbf{H}[S_i]\right). \tag{1.2}$$

Rather than deal directly with this partition function, a coarse graining procedure can be carried out in order to simplify the situation. The recipe is to first group the spins into blocks, and for each block assign a new overall spin. The spin, S'_α, of the block α is assigned using some arbitrary rule based on spins inside the block, $S'_\alpha = f(S_i)|_{i\in\alpha}$. This coarse graining removes degrees of freedom from the system, simplifying the situation while maintaining the same overall behaviour. This new system of spins is then re-scaled onto the old lattice, allowing direct comparison of the two systems on the same lattice. Performing this on a four by four grid of spins on a lattice, and forming the spins into blocks of two by two would look like



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Intuitively the original systems of spins and the system of block spins will show the same long-range behaviour. More strongly than that it is possible to show that the partition function of the transformed system, $\mathcal{Z}'$, is equal to the old partition function. A particular configuration of block spins, S'_α, will have a Boltzman weight with a new Hamiltonian,

$\mathbf{H}'$. This must be equal to the sum of the Boltzman weights belonging to the original unblocked spin arrangements leading to that particular configuration of blocked spins,

$$\exp\left(-\mathbf{H}'[S'_\alpha]\right) = \sum_{[S_i]} \prod_\alpha \delta\left(S'_\alpha - f(S_i)|_{i\in\alpha}\right) \exp\left(-\mathbf{H}[S_i]\right), \tag{1.3}$$

where the Kronecher delta constrains the free sum to the salient configurations. Given that for a given configuration, $[S_i]$, there is one block configuration,

$$\sum_{[S'_\alpha]} \prod_\alpha \delta\left(S'_\alpha - f(S_i)|_{i\in\alpha}\right) = 1, \tag{1.4}$$

the partition function of the transformed system is

$$\begin{aligned} \mathcal{Z}' &= \sum_{[S'_\alpha]} \exp\left(-\mathbf{H}'[S'_\alpha]\right) \\ &= \sum_{[S'_\alpha]} \sum_{[S_i]} \prod_\alpha \delta\left(S'_\alpha - f(S_i)|_{i\in\alpha}\right) \exp\left(-\mathbf{H}[S_i]\right) \\ &= \sum_{[S_i]} \exp\left(-\mathbf{H}[S_i]\right) = \mathcal{Z}. \end{aligned} \tag{1.5}$$

While the partition functions for the original and transformed systems are equal, the new Hamiltonian will not have the same structure. Generalising from the original Hamiltonian the transformed Hamiltonian may include new interaction terms such as next nearest neighbours, $\langle\langle ij \rangle\rangle$, and plaquettes, $\langle ijkl \rangle$,

$$\mathbf{H}' = -K_1 \sum_{\langle ij \rangle} S_i S_j - K_2 \sum_{\langle\langle ij \rangle\rangle} S_i S_j - K_3 \sum_{\langle ijkl \rangle} S_i S_j S_k S_l + \cdots . \tag{1.6}$$

In fact all possible new interaction terms are generated. The adjustments required in the Boltzman weight to take account of the summed over spin configurations cannot come only from the energy cost of nearest neighbour spin-spin interaction in the new Hamiltonian. The coefficients of the various interaction terms, $K_1, K_2, \ldots, K_n, \ldots$, are also called coupling constants or just couplings and these are used interchangeably. Together they define a parameter space, and any system then corresponds to a point in this parameter space. The original Ising system in (1.1) would be given by

$$\mu = \{K_1, 0, 0, \ldots, 0, \ldots\} \tag{1.7a}$$

and the transformed system by

$$\mu' = \{K'_1, K'_2, \ldots, K'_n, \ldots\}. \tag{1.7b}$$

Repeating the coarse graining procedure the system would be transformed to another point in parameter space

$$\mu'' = \{K_1'', K_2'', \ldots, K_n'', \ldots\}. \tag{1.7c}$$

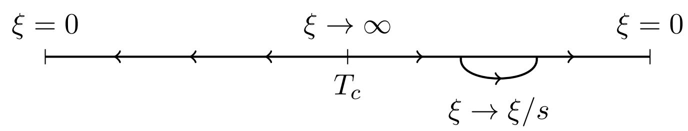
This describes a way of going from one system to another, wrapping up degrees of freedom into new parameters and changing the values of existing parameters. The process of going from one point to another in the parameter space in this way is called a renormalisation group transformation (RGT). These transformations can be represented as

$$\mu' = R_s \mu, \tag{1.8}$$

where s indicates the size of the block formation and re-scaling length. If the spin of a block is defined appropriately, say using a linear transformation that sums over the spins in a block α, so that

$$S'_\alpha = \frac{\lambda(s)}{s^d} \sum_{i\in\alpha} S_i, \tag{1.9}$$

then the remarkably useful feature that the correlation lengths of the two systems are related by $\xi' = \xi/s$ is easy to access. Straight away it can be seen that from this perspective there are two fixed points of an RGT: an unstable one at $\xi \to \infty$ and a stable one at $\xi = 0$. To be at the unstable fixed point requires the couplings to have specific values; the system is then said to be at the critical point, characterised by the critical temperature T_c. The stable fixed point then corresponds to either a high or low temperature limit, $T = \infty$ or $T = 0$, in which the system is effectively non-interacting:



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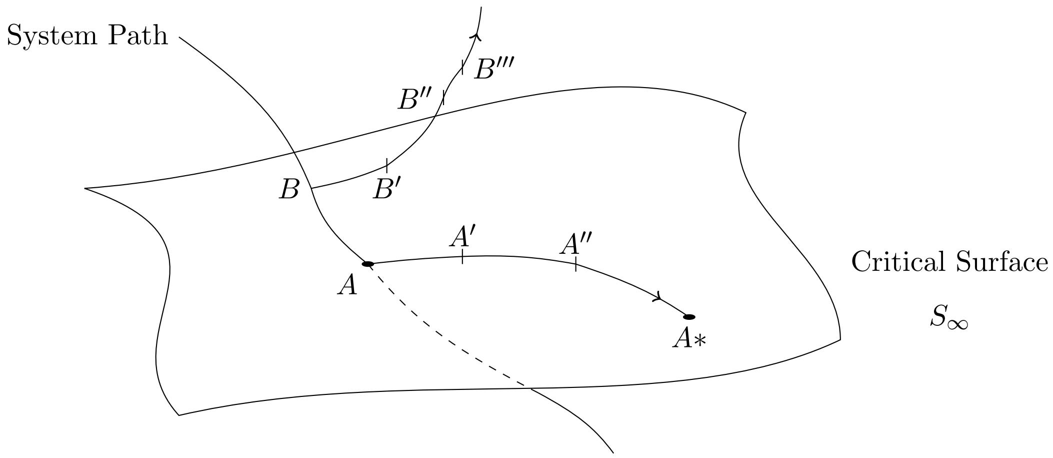
In the parameter space this gives the existence of a critical surface. If the system is at the critical point and an RGT is applied the correlation length will stay infinite and the transformed system remains critical. The set of points in the parameter space for which this is true form a surface, unsurprisingly called the critical surface, and RGTs can only move a point on this surface to another point on the surface. Applying RGTs iteratively to a point just off the critical surface will take that system progressively further away from the critical surface, eventually off to a decoupled system at a high or low temperature limit. The visually analogy of the parameter flow resulting from repeated application of

RGTs, to which ever fixed point the system is attracted to, is that of zooming out from a picture until the picture stops changing its qualitative appearance. It is not clear a priori what the trajectory of points on the critical surface under RGT flow would be, it could be completely random, there could be limit cycles etc. The physically relevant cases are fortunately where the trajectory converges to a fixed point such that $R_s\mu* = \mu*$. Visualising the parameter space; there is the system path, a locus of points formed by varying the physical fields and variables such as temperature in the original system. If RGTs were applied at the critical point of the system, there would be parameter flow on the critical surface to a fixed point. Or if RGTs were applied to the system away from the critical point, the parameter flow would take it further away:



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From which it is possible to define a relevant coupling, one that if perturbed will take the system off the critical surface, and an irrelevant coupling, one whose initial value is unimportant as it will transform along the critical surface to the fixed point. The starting point on the critical surface is also unimportant, any set of parameters, i.e.: any system, in the basin of attraction for the fixed point will end up there. This is where the concept of universality classes comes from, how seemingly different systems have the same behaviour.

This is what the man on the street thinks of when asked about the RG. That it involves flow on the critical surface, then doing some work around the fixed point to determine critical exponents. The take home point that I wished to convey was that it is possible in some cases to define a scale transformation between systems which preserves the long range behaviour and simplifies the situation. This generates new couplings in the Hamiltonian, leading to the study of the parameter/coupling space; where live such creatures as coupling flow under renormalisation group transformations, a critical surface and fixed points.

As well as the introduction to the Ising model by Huang (Huang, 1987) mentioned in the introduction, and the application of a real-space RGT to it discussed by Christensen and Moloney (Christensen and Moloney, 2005), there is a discussion of real-space renormalisation in the book by Le Bellac (Le Bellac, 1991). The calculation presented by Le Bellac is based on work by Niemeijer and van Leeuwen (Domb *et al.*, 1976), in which they carry out a real-space renormalisation on a triangular lattice by performing a cumulant expansion.

1.1.2 Momentum-space

The trouble with real space RGTs is that they are a dark art. Whilst intuitively appealing and an easy way to showcase some aspects of the RG, they are practically very messy and not at all systematic. In particular it is very difficult to get a handle on the effects of the approximations that have to be made, or to assess the accuracy at any point. I think this stems from not being able to form a proper perturbation theory, the expansion is effectively being made about a gradient term.

It turns out to be much easier to work in momentum space, this is usually thought of as the approach of Wilson. The shift to working in momentum space asserts that defining an RGT by forming blocks of spin is equivalent to integrating over fluctuations smaller that the block size, by transforming to the conjugate Fourier space these fluctuations can be directly integrate over. If forming a block of size sa is equivalent to integrating out fluctuations of length a to sa, this translates as an instruction to integrate over the wavenumbers between $\frac{1}{a}$ and $\frac{1}{sa}$. This is physically appealing, as when interest is focused on the long range behaviour, i.e.: long wavelength behaviour, getting rid of the short range fluctuations should leave the system with the same behaviour overall. Removing these fluctuations must surely have simplified the system, and so have achieved the core idea as described in the introduction.

Having moved into momentum-space, the pliable features in the system are no longer discrete spins. Instead continuous fields, $\phi(x)$, now appear in the Hamiltonian, $\mathbf{H}[\phi]$. The partition function involves a path integral over these fields in place of a summation. The RGT in momentum space can straightforwardly be written in three steps:

1. Integration over a shell of the wavevector k:

   $\Lambda/s \leq k \leq \Lambda$

2. Dilation of the unit of length:

   $x \to x' = x/s$

   $k \to k' = sk$

3. Renormalisation of the field:

   $\phi(x) \to \phi'(x') = s^{d_\phi}\phi(x)$

   $\phi(k) \to \phi'(k') = s^{d_\phi - d/2}\phi(k)$

Giving the relation for the Boltzman weight of the transformed Hamiltonian as

$$\exp\left(-\mathbf{H}'[\phi']\right) = \int\limits_{\Lambda/s \le k \le \Lambda} \mathscr{D}\phi(k) \exp\left(-\mathbf{H}[\phi]\right)\Bigg|_{\phi(k) \to s^{d/2-d_\phi}\phi'(sk)}. \tag{1.10}$$

The Gaussian and spherical models both provide very nice, clear examples of RGTs in momentum space. For simplicity sake I will quickly go through the case of the Gaussian model. Starting with the ϕ^4 Hamiltonian (a.k.a.: the Ginzburg-Landau Hamiltonian),

$$\mathbf{H}[\phi] = \int d^d x \left[\frac{c}{2}\big(\nabla\phi(x)\big)^2 + \frac{r_0}{2}\phi^2(x) + \frac{u_0}{4!}\phi^4(x)\right], \tag{1.11}$$

which by introducing the possibility of a double-well in the potential describes a second order phase transition equivalent to the Ising model, then setting $u_0 = 0$ gives the simpler Gaussian model

$$\begin{aligned}\mathbf{H}[\phi] &= \int d^d x \left[\frac{c}{2}\big(\nabla\phi(x)\big)^2 + \frac{r_0}{2}\phi^2(x)\right] \\ &= \frac{1}{2}\int \frac{d^d k}{(2\pi)^d}(r_0 + ck^2)|\phi(k)|^2. \end{aligned} \tag{1.12}$$

This has a two-dimensional parameter space $\mu = \{c, r_0\}$. The integration over the small wavelengths is trivially a product of decoupled Gaussian integrals, giving a constant. Looking only at the transformed Hamiltonian

$$\begin{aligned}\mathbf{H}'[\phi] &= \frac{1}{2}\int\limits_{k \le \Lambda/s} \bar{d}^d k \; (r_0 + ck^2)|\phi(k)|^2 \\ &= \frac{1}{2}\int\limits_{k' \le \Lambda} \bar{d}^d k' \; s^{d-2d_\phi}(r_0 + cs^{-2}k'^2)|\phi'(k')|^2, \end{aligned} \tag{1.13}$$

where the Jacobian from the shift in measure is absorbed as part of the renormalisation of the field. A more compact notation has been introduced for the integral measure,

$$\bar{d}^d k = \frac{d^d k}{(2\pi)^d}. \tag{1.14}$$

The parameters can be read off to be transforming as

$$c' = s^{d-2-2d_\phi} c,$$
$$r_0' = s^{d-2d_\phi} r_0. \tag{1.15}$$

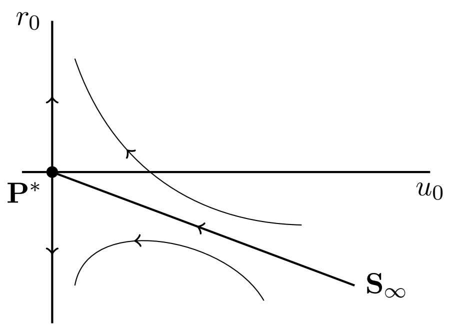
Giving two fixed points:

- $d - 2d_\phi = 0$; r_0 arbitrary; $c = 0$
- $d - 2 - 2d_\phi = 0$; c arbitrary; $r_0 = 0$

The first choice is fairly boring, no interaction would mean a system of decoupled sites. The second choice is more interesting, still containing interaction and having a relevant field $r_0' = s^2 r_0$.[1] By convention the constant c is set equal to one, and promptly forgotten about. These fixed points provide clear demonstration of what it means for a coupling to be relevant or irrelevant under an RGT; if a coupling is rescaled by $s^{n>0}$ then it is relevant, if by $s^{n<0}$ it is irrelevant.

The field renormalisation is shown to be crucial; it provides a handle on which fixed point to approach, it enables the recasting of the transformed Hamiltonian into the same form as the original. The fixed point and the form of the Hamiltonian that is physically interesting is one that keeps the gradient term non-zero. This imposes that interactions are still important, or at least still present, at the critical point. The process of field renormalisation seems to me very like choosing the lattice upon which to compare the transformed system with the original system when conducting a real-space RGT.

Including the ϕ^4 term in the Hamiltonian, by setting $c = 1$ and $u_0 \neq 0$ in (1.11), reveals some further iconic behaviour in parameter space. Assuming for the moment that an appropriate approximation has been used to carry out the RGT, another two-dimensional parameter space is defined: $\mu = \{r_0, u_0\}$. For $d > 4$ there is a trivial fixed point at the origin, with r_0 being relevant and u_0 irrelevant:



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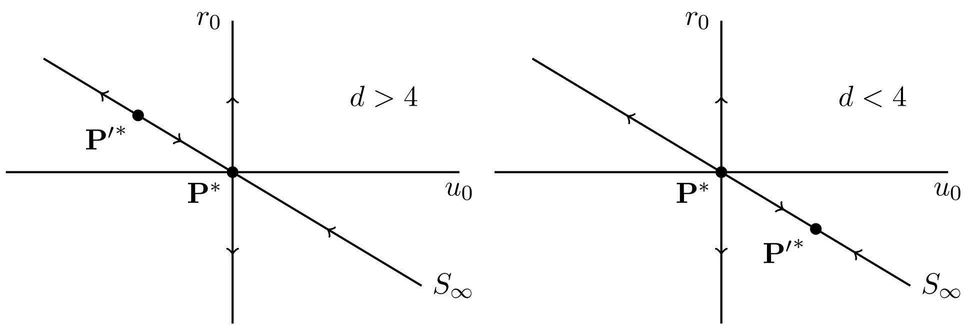
[1] Leading to the mean-field critical exponents $\eta = 0$ and $\nu = 1/2$. See Section 2.1-2

Any other couplings generated by the RGT are irrelevant, indeed more irrelevant than u_0. If the dimension of the system is lowered below $d = 4$, the fixed point at the origin becomes unstable and u_0 becomes relevant. Some further analysis would reveal the presence of a second fixed point that was there all along and whose character is dimensionally dependent. Above $d = 4$ it is unstable and located in the upper-left quadrant. Below $d = 4$ it is stable and located in the lower-right quadrant. As the dimension of the system is altered, the fixed points cross over each other and change character:



This showcases not only the dimensional dependence of the couplings, but also the existence of the upper critical dimension; above which mean field theory is valid, below which non-trivial fixed points and new critical exponents appear.

What I wished to quickly show about the classical/Wilsonian approach to the RG was that it is possible to have a well defined transformation between systems with same long range behaviour. That the generation of couplings occurs under an RGT, and the dynamics in coupling space that result from iterating RGTs lead to a critical surface and to fixed points. This in turn allows relevant and irrelevant parameters to be characterised, and the appearance of dimensional dependence of the couplings. Field renormalisation is seen to be key to building a workable RGT, both enabling it and effectively choosing[2] the fixed point or theory the RGT is geared towards, where the fixed point of interest is unstable in this approach.

[2]Through the value of the exponent η.

Chapter 2

Field Theory in the ϕ^4 Sandpit

While it is possible to introduce and discuss the Wilsonian RG very intuitively, using relatively little mathematics to illustrate the concepts, and without recourse to perturbation theory and Feynman diagrams, it is not possible to do so with field theory. Some further preliminaries will be needed in order to explore the approach of field theory. The Gaussian model is again used as the starting point with which to understand the complications of the ϕ^4 model.

The essential advantage of field theory is that it provides a more systematic, I hesitate to use the word 'rigorous', method for RG calculations and makes more complicated calculations possible. The chief cost is the direct, natural seeming physical connection to the system in question.

2.1 Some Preliminaries

The partition function for a d-dimensional system with real field variable $\phi(x)$ can be written as a functional integral

$$\mathcal{Z}[H] = \int \mathscr{D}\phi(x) \, \exp\left(-\mathcal{H}[\phi] + \int d^d x \, H(x) \, \phi(x) \right), \tag{2.1}$$

where $\mathcal{H}[\phi]$ is the 'microscopic' Hamiltonian of the system. $H(x)$ is a real field analogous to an external magnetic field. The functional integral is defined as the continuum limit of a product of integrals at different lattice points,

$$
\begin{aligned}
\int \mathscr{D}\phi(x) &:= \prod_{i=1}^{N\to\infty} \int d\phi_i \\
&= \prod_x \int d\phi(x),
\end{aligned}
\tag{2.2}
$$

where each lattice point corresponds to a term/point in the sum/integral of the Hamiltonian. The equivalent object to the classical Helmholtz free energy is

$$
\mathcal{W}[H] = \ln \mathcal{Z}[H]. \tag{2.3}
$$

The average of the field $\phi(x)$ at vanishing external field is

$$
\langle \phi(x) \rangle = \frac{1}{\mathcal{Z}[0]} \int \mathscr{D}\phi(x)\; \phi(x) \exp\left(-\mathcal{H}[\phi] + \int d^d x\, H(x)\, \phi(x) \right)\bigg|_{H=0}. \tag{2.4}
$$

That is to say the partition function is a moment generating function, the 'external' field $H(x)$ is the source field, and the Boltzman weight from the Hamiltonian is a probability measure $\mathscr{P}(\phi)$. The moments of the field $\phi(x)$ at vanishing external field can be expressed as

$$
\begin{aligned}
\langle \phi^n \rangle &= \frac{\int \mathscr{D}\phi\; \phi^n\, \mathscr{P}(\phi)}{\int \mathscr{D}\phi\;\; \mathscr{P}(\phi)} \\
&= \frac{1}{\mathcal{Z}[0]} \left. \frac{\delta^n \mathcal{Z}[H]}{\delta H^n} \right|_{H=0}.
\end{aligned}
\tag{2.5}
$$

To generate the cumulants requires a logarithm,

$$
\begin{aligned}
\langle \phi^n \rangle_c &= \left. \frac{\delta^n}{\delta H^n} \right|_{H=0} \ln \frac{\mathcal{Z}[H]}{\mathcal{Z}[0]} \\
&= \left. \frac{\delta^n}{\delta H^n} \right|_{H=0} \mathcal{W}[H].
\end{aligned}
\tag{2.6}
$$

Functional derivatives of the 'Helmholtz' free energy generate the cumulants. The normalising factor of $\mathcal{Z}[0]$ drops out as soon as the first derivative is taken, and so is usually ignored from the start. Functional differentiation can be applied in a fairly intuitive fashion, analogous to its better known cousins. The necessary identities are

$$
\frac{\delta \phi(x)}{\delta \phi(y)} = \delta(x - y), \tag{2.7a}
$$

leading to the sensible looking

$$
I = \int dy\; V\big(\phi(y)\big) \quad ; \quad \frac{\delta I}{\delta \phi(x)} = V'\big(\phi(x)\big), \tag{2.7b}
$$

a straightforward genus of chain-rule

$$\frac{\delta I}{\delta \psi(x)} = \int dy \, \frac{\delta I}{\delta \phi(y)} \frac{\delta \phi(y)}{\delta \psi(x)}, \tag{2.7c}$$

and finally the not immediately obvious

$$I = \int d^d y \, \big(\nabla \phi(y)\big)^2 \quad ; \quad \frac{\delta I}{\delta \phi(x)} = -2\nabla^2 \phi(x). \tag{2.7d}$$

The cumulants are also known as the connected correlation functions, $\mathbf{G}^{(n)}$, where

$$\mathbf{G}^{(n)}(x_1, x_2, \ldots, x_n) = \left. \frac{\delta^n \mathcal{W}[H]}{\delta H(x_n) \ldots \delta H(x_2) \delta H(x_1)} \right|_{H=0}. \tag{2.8}$$

Note that the first cumulant, $\mathbf{G}^{(1)}(x)$, is equal to the first moment:

$$\begin{aligned} \langle \phi(x) \rangle &= \frac{1}{\mathcal{Z}[0]} \left. \frac{\delta \mathcal{Z}[H]}{\delta H(x)} \right|_{H=0} = \left. \frac{\delta \ln \mathcal{Z}[H]}{\delta H(x)} \right|_{H=0} \\ &= \frac{\delta \mathcal{W}[H]}{\delta H(x)} = \mathbf{G}^{(1)}(x). \end{aligned} \tag{2.9}$$

The system is translationally invariant, so the moments and cumulants must be independent of any special position. Starting with

$$\langle \phi(x+a) \rangle = \frac{1}{\mathcal{Z}[0]} \int \mathscr{D}\phi(x) \; \phi(x+a) \exp\left(-\mathcal{H}[\phi] + \int d^d x \, H(x) \, \phi(x) \right) \Bigg|_{H=0}, \tag{2.10a}$$

then introducing the transformation, with Jacobian $\mathcal{J}$,

$$\phi(x+a) = \phi'(x); \quad |\mathcal{J}| = 1; \quad \mathscr{D}\phi \to \mathscr{D}\phi', \tag{2.10b}$$

leads to

$$\langle \phi(x+a) \rangle = \frac{1}{\mathcal{Z}[0]} \int \mathscr{D}\phi'(x) \; \phi'(x) \exp\left(-\mathcal{H}[\phi'] + \int d^d x \, H(x-a) \, \phi'(x) \right) \Bigg|_{H=0}. \tag{2.10c}$$

The re-labelling in the Hamiltonian from ϕ to ϕ' is allowed given the integral over all space, infinite sums are invariant under a shift of the origin. However the term with the external field causes a problem, translational invariance requires that $H(x-a) = H(x)$, only then does $\langle \phi(x+a) \rangle = \langle \phi(x) \rangle$. The external field being uniform is a sufficient condition for the translational invariance of the system. Assuming the external field to be homogeneous at the point we evaluate our expressions, to do so before taking the derivatives would make no sense(!), all the moments and cumulants are translationally invariant,

$$\begin{aligned} \frac{\delta^2 \mathcal{W}[H]}{\delta H(y) \delta H(x)} &= \mathbf{G}^{(2)}(x, y) \\ &= \hat{\mathbf{G}}^{(2)}(x - y). \end{aligned} \tag{2.11}$$

The hat, $\hat{\cdot}$, indicates translational invariance has been used to eliminate an argument. Fourier transforming, translational variance becomes conservation of momentum:

$$\begin{aligned}\int dx \int dy\, e^{i(k_1 x+k_2 y)}\, \mathbf{G}^{(2)}(x,y) &= \int dx \int dy\, e^{i(k_1 x+k_2 y)}\, \hat{\mathbf{G}}^{(2)}(x-y) \\ \mathbf{G}^{(2)}(k_1,k_2) &= \int dx' \int dy\, e^{ik_1 x'}\, e^{i(k_1+k_2)y}\, \hat{\mathbf{G}}^{(2)}(x') \\ &= \int dx'\, \hat{\mathbf{G}}^{(2)}(x')\, e^{ik_1 x'} \cdot (2\pi)^d \delta(k_1+k_2) \\ \mathbf{G}^{(2)}(k_1,k_2) &= \hat{\mathbf{G}}^{(2)}(k_1) \cdot (2\pi)^d \delta(k_1+k_2).\end{aligned} \tag{2.12}$$

Another important quantity is $\Gamma[\theta]$, the equivalent of the Gibbs free energy, formed by taking the Legendre transform of the free energy $\mathcal{W}[H]$. Defining a new variable,

$$\theta(x) = \frac{\delta \mathcal{W}[H]}{\delta H(x)}, \tag{2.13}$$

and transforming to eliminate H as a variable gives

$$\Gamma[\theta] = -\mathcal{W}[H] + \int d^d x\, H(x)\, \theta(x). \tag{2.14}$$

The new potential is also commonly known as the effective action. A similar compact notation is used for its derivatives, referred to as the vertex functions,

$$\Gamma^{(n)}(x_1,x_2,\ldots,x_n) = \frac{\delta^n \Gamma[\theta]}{\delta\theta(x_n)\ldots\delta\theta(x_2)\delta\theta(x_1)}. \tag{2.15}$$

Noting a useful relation for the first derivative,

$$\begin{aligned}\frac{\delta\Gamma[\theta]}{\delta\theta(x)} &= -\int d^d x' \frac{\delta\mathcal{W}[H]}{\delta H(x')}\frac{\delta H(x')}{\delta\theta(x)} + H(x) + \int d^d x' \frac{\delta H(x')}{\delta\theta(x)} \cdot \theta(x') \\ &= H(x),\end{aligned} \tag{2.16}$$

which emphasises that after the Legendre transformation the field $H(x)$ is no longer a free variable, being implicitly defined by the value selected for $\theta(x)$ to evaluate the derivative at. Translational invariance is inherited from $\mathcal{W}[H]$, for example the second derivative can be written

$$\frac{\delta^2\Gamma[\phi]}{\delta\phi(y)\delta\phi(x)} = \hat{\Gamma}^{(2)}(x-y). \tag{2.17}$$

Together these can be used to build useful relations in momentum space between the vertex functions and the connected correlation functions, iconically $\hat{\Gamma}^{(2)}(k)$ and $\hat{G}^{(2)}(k)$:

$$
\begin{aligned}
\frac{\delta\theta(k_1)}{\delta\theta(k_2)} &= \int d^d q \frac{\delta\theta(k_1)}{\delta H^\dagger(q)} \frac{\delta H^\dagger(q)}{\delta\theta(k_2)} \\
&= \int d^d q \frac{\delta}{\delta H^\dagger(q)} \left((2\pi)^d \frac{\delta \mathcal{W}[H]}{\delta H^\dagger(k_1)} \right) \cdot \frac{\delta}{\delta\theta(k_2)} \left((2\pi)^d \frac{\delta\Gamma[\theta]}{\delta\theta(q)} \right) \\
&= \int d^d q \; (2\pi)^{-d} \mathbf{G}^{(2)}(k_1, q) \cdot (2\pi)^{-d} \Gamma^{(2)}(-q, -k_2) \\
&= \int d^d q \; (2\pi)^{-d} \cdot \left((2\pi)^d \hat{\mathbf{G}}^{(2)}(k_1) \delta(k_1 + q) \right) \\
&\qquad \cdot (2\pi)^{-d} \left((2\pi)^d \hat{\Gamma}^{(2)}(-q) \delta(q + k_2) \right) \\
\delta(k_1 - k_2) &= \hat{\mathbf{G}}^{(2)}(k_1) \hat{\Gamma}^{(2)}(k_1) \delta(k_1 - k_2).
\end{aligned}
\tag{2.18}
$$

Which leads to the identity

$$
\hat{\Gamma}^{(2)}(k) = \frac{1}{\hat{\mathbf{G}}^{(2)}(k)}. \tag{2.19}
$$

Relations between higher order connected correlation functions and vertex function can be found by taking further functional derivatives. The reason for such acrobatics is that although the connected correlation functions are the objects of interest, the vertex functions are usually far easier to compute and work with.

If a system is close to its critical point, or to the critical surface, it is known that the connected correlation function has the behaviour

$$
\hat{\mathbf{G}}^2(k) = \frac{1}{k^{2-\eta}} f(k\xi), \tag{2.20}
$$

where the scaling function $f(k\xi)$ tends to a constant at large argument, and the correlation length ξ diverges as an inverse power law in the approach to the critical point,

$$
\xi \sim |t|^{-\nu}, \tag{2.21}
$$

with t characterising the proximity to the critical point. In a magnetic system say this would be the distance to the critical temperature, $t = \dfrac{T - T_c}{T_c}$. The critical exponents η and ν, which are related through scaling laws to the other traditional exponents describing the divergence of different physical properties, can also be accessed from the vertex functions via the relations outlined above. It is also possible to define other exponents for the scaling of whichever fields and couplings appear in the Hamiltonian or 'thermodynamic' potentials for the system.

2.2 The Free Theory

Starting with the eminently tractable Gaussian model, or free theory,

$$
\begin{aligned}
\mathcal{H}[\phi] &= \int d^d x \left[\frac{1}{2}\big(\nabla\phi(x)\big)^2 + \frac{r_0}{2}\phi^2(x)\right], \\
\mathcal{Z}[H] = \int \mathscr{D}\phi(x) \exp&\left(-\int d^d x \;\left[\frac{1}{2}\big(\nabla\phi(x)\big)^2 + \frac{r_0}{2}\phi^2(x) - H(x)\phi(x)\right]\right),
\end{aligned}
\tag{2.22}
$$

the gradient term motivates a Fourier transform of the Hamiltonian to make the path integral tractable,

$$
\mathcal{H}[\phi] = \int đ^d k \;\left[\frac{1}{2}(k^2 + r_0)\phi^\dagger(k)\phi(k)\right]. \tag{2.23}
$$

Naïvely the transformed partition function would be written straight away as

$$
\mathcal{Z}[H] = \int \mathscr{D}\phi(k) \exp\left(-\int đ^d k \;\left[\frac{1}{2}(k^2 + r_0)\phi^\dagger(k)\phi(k) - H(k)^\dagger\phi(k)\right]\right), \tag{2.24}
$$

which could then be evaluated as a product of decoupled Gaussian integrals. However, care must be taken with the path integral as not all the $\phi(k)$ fields are independent. Returning to the definition of the path integral as the limit of a product of integrals at different lattice points,

$$
\begin{aligned}
\int \mathscr{D}\phi(x) &:= \prod_{i=1}^{N\to\infty} \int d\phi_i \\
&= \prod_x \int d\phi(x),
\end{aligned}
\tag{2.25}
$$

the same can be written for momentum space. However the $\phi(k)$ are complex fields, so it would appear that the number of degrees of freedom in the system has doubled from the original real fields $\phi(x)$. This is brought back under control by the condition on the conjugate field that arises from transforming a real field: $\phi^\dagger(k) = \phi(-k)$. The same applies to the source field $H(k)$. In the path integral this leads to a pairing of integrals. Examining the pair of points k_1 and $-k_1$,

$$
\begin{aligned}
\int d\phi(-k_1) \int d\phi(k_1) \exp&\left(-\frac{1}{2}\phi^\dagger(k_1)(r_0 + k^2)\phi(k_1) + H^\dagger(k_1)\phi(k_1)\right) \\
&\cdot \exp\left(-\frac{1}{2}\phi^\dagger(-k_1)(r_0 + k^2)\phi(-k_1) + H^\dagger(-k_1)\phi(-k_1)\right),
\end{aligned}
\tag{2.26}
$$

such pairs of integrals cannot be factorised. When the condition relating a field and its complex conjugate is imposed, this enslaves one integral to the other. With the consequence that the ϕ^2 terms are identical, and the external field terms are conjugates. The second integral is not independent and must track the first. The pair can be re-written as a single integral

$$\int d\phi(k_1)\exp\Big(-\phi^\dagger(k_1)a(k)\phi(k_1)+H^\dagger(k_1)\phi(k_1)+\phi^\dagger(k_1)H(k_1)\Big). \tag{2.27}$$

This is equivalent to condensing the functional integral to the half space

$$\int \mathscr{D}\phi(k)=\prod_{k/2}\int d\phi(k). \tag{2.28}$$

The purpose of functional integral is to vary all of the fields found in the Boltzman factor, giving the partition function in momentum space correctly as

$$\mathcal{Z}[H]=\int \mathscr{D}\phi(k)\exp\left(-\int\limits_{\mathbb{R}^d/2} đ^dk\ \Big[\phi^\dagger(k)(k^2+r_0)\phi(k)-H(k)^\dagger\phi(k)-\phi(k)^\dagger H(k)\Big]\right). \tag{2.29}$$

The number of source terms compared to naïve expression has doubled, as has the factor in front of the quadratic. Completing the square to get the integral into Gaussian form,

$$\mathcal{Z}[H]=\int \mathscr{D}\phi(k)\exp\left(-\int\limits_{\mathbb{R}^d/2} đ^dk\ \left[\phi^\dagger(k)-\frac{H(k)^\dagger}{(k^2+r_0)}\right](k^2+r_0)\left[\phi(k)-\frac{H(k)}{(k^2+r_0)}\right]\right)$$
$$\cdot\exp\left(+\int\limits_{\mathbb{R}^d/2} đ^dk\ \left[\frac{H^\dagger(k)H(k)}{(k^2+r_0)}\right]\right), \tag{2.30}$$

the Jacobian for the transformation $\psi(k)=\phi(k)-\frac{H(k)}{(k^2+r_0)}$ is unity, so the expression can be re-written as

$$\mathcal{Z}[H]=\int \mathscr{D}\psi(k)\exp\left(-\int\limits_{\mathbb{R}^d/2} đ^dk\ \psi^\dagger(k)(k^2+r_0)\psi(k)\right)$$
$$\cdot\exp\left(\int\limits_{\mathbb{R}^d/2} đ^dk\ \left[\frac{H^\dagger(k)H(k)}{(k^2+r_0)}\right]\right)$$
$$=\exp\left(\int\limits_{\mathbb{R}^d} đ^dk\ \left[\frac{H^\dagger(k)H(k)}{2(k^2+r_0)}\right]\right)\cdot\mathcal{Z}[0]. \tag{2.31}$$

$\mathcal{Z}[0]$ can be evaluated as the limit of a product of Gaussian integrals, making use of $detA=\exp\left(Tr\{\ln A\}\right)$,

$$\mathcal{Z}[0]=\exp\left(-\int\limits_{\mathbb{R}^d} d^dk\frac{1}{2}\ln\left(k^2+r_0\right)\right), \tag{2.32}$$

up to a(n infinite) constant, into which the factor of $(2\pi)^d$ has been swept as the logarithm was taken. Giving the free energy for the free theory as

$$\mathcal{W}_0[H] = \text{Const.} - \int_{\mathbb{R}^d} d^d k \frac{1}{2} \ln\left(k^2 + r_0\right) + \int_{\mathbb{R}^d} \bar{d}^d k \left[\frac{H^\dagger(k) H(k)}{2(k^2 + r_0)}\right]. \tag{2.33}$$

The first cumulant is found by taking the derivative

$$\begin{aligned}(2\pi)^d \frac{\delta \mathcal{W}_0[H]}{\delta H^\dagger(k_1)} &= \int d^d k \frac{H(k)}{2(k^2 + r_0)} \delta(k - k_1) + \int d^d k \frac{H^\dagger(k)}{2(k^2 + r_0)} \delta(k + k_1) \\ &= \frac{H(k_1)}{(k_1^2 + r_0)},\end{aligned} \tag{2.34}$$

which goes to zero with the external field. Taking the second derivative,

$$(2\pi)^{2d} \frac{\delta^2 \mathcal{W}_0}{\delta H^\dagger(k_2) \delta H^\dagger(k_1)} = (2\pi)^d \cdot \frac{1}{(k_1^2 + r_0)} \cdot \delta(k_1 + k_2), \tag{2.35}$$

identifies the (two-point) connected correlation function for the Gaussian model,

$$\hat{\mathbf{G}}_0^{(2)}(k) = \frac{1}{(k^2 + r_0)}. \tag{2.36}$$

The partition function can then be re-written as

$$\mathcal{Z}[H] = \exp\left(\frac{1}{2} \int_{\mathbb{R}^d} \bar{d}^d k \; H^\dagger(k) \hat{\mathbf{G}}_0^{(2)}(k) H(k)\right) \cdot \mathcal{Z}[0], \tag{2.37}$$

and through identifying $r_0 = \xi^{-2}$, the exponents $\eta = 0$ and $\nu = \frac{1}{2}$ can be read off. The fixed point of the Gaussian model in the last chapter produces the same values, given the definitions presented in (2.20) and (2.21).

2.3 Gaussian Correlation Functions

It is relatively straight forward to explicitly calculate the Gaussian connected correlation functions in real space from (2.36). Replacing r_0 by ξ^{-2}, the notation in this section for the Gaussian connected correlation function will be simplified to

$$\mathbf{G}(k) = \frac{1}{\mathbf{k}^2 + \xi^{-2}}. \tag{2.38}$$

As well as the translational and rotational invariance of the model there are further expectation for the behaviour from (2.20). In real space this becomes $\mathbf{G}(x) \propto \frac{g(x/\xi)}{x^{d+\eta-2}}$, with the scaling function $g(x/\xi)$ dying exponentially at large x, $g(x/\xi) \to \exp(-x/\xi)$, and tending to a constant at vanishing argument.

2.3.1 d=1

In real-space, in one dimension, (2.38) Fourier transforms to become

$$\mathbf{G}(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dq\, \frac{1}{\mathbf{q}^2+\xi^{-2}} \exp\left(-i\mathbf{q}\cdot\mathbf{x}\right)$$

$$\mathbf{G}(x) = \frac{\xi}{2}\exp\left(-\frac{|x|}{\xi}\right) \quad = x\, g\left(\frac{x}{\xi}\right), \tag{2.39a}$$

from a straight forward contour integral. Given that $|x|$ and ξ are both always real and positive there is no divergence or troubling behaviour. The scaling function is

$$g(y) = \frac{1}{2|y|}\exp\left(-|y|\right). \tag{2.39b}$$

2.3.2 d=2

In two dimensions the real-space connected correlation function becomes

$$\mathbf{G}(x) = \int_{-\infty}^{+\infty} d\!\!{}^-{}^2 q\, \frac{1}{\mathbf{q}^2+\xi^{-2}} \exp\left(-i\mathbf{q}\cdot\mathbf{x}\right)$$

$$= \frac{1}{(2\pi)^2}\int_0^{\infty} dr\, \frac{r}{r^2+\xi^{-2}} \int_{-\pi}^{+\pi} d\theta \exp\left(-ir|\mathbf{x}|\cos\theta\right)$$

$$= \frac{1}{2\pi}\int_0^{\infty} dr\, \frac{r}{r^2+\xi^{-2}}\, \mathbf{J}_0\left(r|\mathbf{x}|\right)$$

$$\mathbf{G}(x) = \frac{1}{2\pi}\mathbf{K}_0\left(\frac{|\mathbf{x}|}{\xi}\right), \tag{2.40}$$

where $\mathbf{J}_\nu(z)$ is a Bessel function of the first kind, and $\mathbf{K}_0(z)$ is a modified Bessel function. The scaling function is simply $\mathbf{K}_0(z)$. Use has been made of the identities (Gradshteĭn *et al.*, 2007):

$$\mathbf{J}_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \int_0^{+\pi} d\phi\, \exp\left(\pm iz\cos\phi\right)\, \sin^{2\nu}\phi, \tag{2.41a}$$

$$\int_0^{\infty} dx\, \frac{x}{x^2+k^2}\, \mathbf{J}_0(ax) = \mathbf{K}_0(ak) \tag{2.41b}$$

for $a > 0$ and $Re[k] > 0$.

Although the presence of a modified Bessel function is slightly unexpected, it has the correct behaviour for the correlation function; it is smooth, decays to zero at large argument and diverges at the origin.

2.3.3 d=3

In three dimensions contour integration is again required,

$$
\begin{aligned}
\mathbf{G}(x) &= \int_{-\infty}^{+\infty} d^3q \, \frac{1}{q^2+\xi^{-2}} \exp\left(-i\mathbf{q}\cdot\mathbf{x}\right) \\
&= \frac{1}{(2\pi)^3} \int_0^{\infty} dr\, r^2 \int_0^{+2\pi} d\phi \int_0^{+\pi} d\theta \, \sin\theta \, \frac{\exp\left(-ir|\mathbf{x}|\cos\theta\right)}{r^2+\xi^{-2}} \\
&= \frac{1}{(2\pi)^2} \int_0^{\infty} dr \, \frac{r^2}{r^2+\xi^{-2}} \int_{-1}^{+1} d(\cos\theta) \exp\left(-ir|\mathbf{x}|\cos\theta\right) \\
&= \frac{1}{(2\pi)^2} \int_0^{\infty} dr \, \frac{r}{r^2+\xi^{-2}} \, \frac{2}{|\mathbf{x}|} \, \sin\left(r|\mathbf{x}|\right) \\
&= \frac{1}{(2\pi)^2} \frac{1}{|\mathbf{x}|} \quad Im \left[\int_{-\infty}^{+\infty} dr \, \frac{r\exp\left(ir|\mathbf{x}|\right)}{r^2+\xi^{-2}} \right] \\
&= \frac{1}{(2\pi)^2} \cdot \frac{1}{|\mathbf{x}|^2} \cdot Re \left[- \int_{-\infty}^{+\infty} dr \left(\frac{\exp\left(ir|\mathbf{x}|\right)}{r^2+\xi^{-2}} - \frac{2}{\xi^2} \cdot \frac{\exp\left(ir|\mathbf{x}|\right)}{(r^2+\xi^{-2})^2} \right) \right] \\
&= \frac{1}{(2\pi)^2} \cdot \frac{1}{|\mathbf{x}|^2} \cdot \left[-\pi\xi \exp\left(-\frac{|\mathbf{x}|}{\xi}\right) + \frac{2}{\xi^2}\, \frac{\pi\xi^3}{2} (|\mathbf{x}|\xi^{-1}+1) \exp\left(-\frac{|\mathbf{x}|}{\xi}\right) \right] \\
\mathbf{G}(x) &= \frac{1}{4\pi} \, \frac{1}{|\mathbf{x}|} \exp\left(-\frac{|\mathbf{x}|}{\xi}\right),
\end{aligned}
\tag{2.42}
$$

with the scaling function being the exponential. Had the only information given been that the real-space correlation function behaved as $e^{-|x|/\xi}$, this would lead to the conclusion that in k-space the behaviour went as

$$
\begin{aligned}
\mathbf{G}(k) &= \int_{-\infty}^{+\infty} d^3x \, \exp\left(i\mathbf{x}\cdot\mathbf{k} - \frac{|\mathbf{x}|}{\xi} \right) \\
&= \frac{8\pi}{\xi} \cdot \frac{1}{(k^2+\xi^{-2})^2},
\end{aligned}
\tag{2.43}
$$

which is wildly different to the (2.38).

2.3.4 Arbitrary dimension

The connected correlation function (2.38) implies that in real-space the connected correlation function is the solution to

$$-\nabla^2 \mathbf{G}(x) + \xi^{-2}\mathbf{G}(x) = \delta(\mathbf{x}), \tag{2.44}$$

which describes the Green function for a modified Helmholtz equation (Ockendon *et al.*, 2003). At the critical point, $\xi \to \infty$, this becomes

$$-\nabla^2 \mathbf{G}(x) = \delta(\mathbf{x}), \tag{2.45}$$

the solution to which is the Green function of the Laplacian. Given that these correlation function depends only on the magnitude of the argument, $x = |\mathbf{x}|$, the left-hand side can be expressed as

$$\begin{aligned}
\nabla^2 \mathbf{G}(x) &= \nabla \left(\nabla x \cdot \frac{\partial}{\partial x} \mathbf{G}(x) \right) \\
&= \nabla \left(\frac{\vec{x}}{x} \cdot \frac{\partial}{\partial x} \mathbf{G}(x) \right) \\
&= \frac{d}{x} \partial_x \mathbf{G}(x) - \frac{1}{x} \partial_x \mathbf{G}(x) + \partial_x^2 \mathbf{G}(x) \\
&= \frac{d-1}{x} \partial_x \mathbf{G}(x) + \partial_x^2 \mathbf{G}(x) \\
\nabla^2 \mathbf{G}(x) &= x^{1-d} \partial_x \left(x^{d-1} \partial_x \mathbf{G}(x) \right).
\end{aligned} \tag{2.46}$$

Substituting this form and solving gives:

$$\begin{aligned}
x^{1-d} \partial_x \left(x^{d-1} \partial_x \mathbf{G}(x) \right) &= -\delta(\mathbf{x}) \\
&= -\frac{\delta(x)}{x^{d-1} S_d} \\
\partial_x \left(x^{d-1} \partial_x \mathbf{G}(x) \right) &= -\frac{\delta(x)}{S_d} \\
\partial_x \mathbf{G}(x) &= \frac{-x^{1-d}}{S_d} \\
\mathbf{G}(x) &= \frac{x^{-d+2}}{(d-2) S_d} + C,
\end{aligned} \tag{2.47}$$

for some arbitrary constant C, and the surface area of a d-dimensional sphere

$$S_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}. \tag{2.48}$$

A moment of care is required in two dimensions. Going back to $\mathbf{G}_x = -\frac{1}{2\pi x}$ leads to the correct

$$\mathbf{G}(x) = \frac{1}{2\pi} \ln\left(\frac{C}{x}\right). \tag{2.49a}$$

In one dimension, at the critical point

$$\mathbf{G}(x) = x, \tag{2.49b}$$

and in three dimensions

$$\mathbf{G}(x) = \frac{1}{4\pi x}. \tag{2.49c}$$

These are the Green functions expected for the Laplacian. Away from the critical point things are a little trickier. For an integral involving a dot product in arbitrary dimension, of the form $\int d^d x f(\mathbf{x}\cdot\mathbf{k})$, it is always possible to arrange the coordinate system such that the free variable points along one axis. Calling this axis x_1, and the angle to that axis θ_1; the dot product is $|x||q|\cos\theta_1$, with the other part of the projection of the position vector from x_1 on to the hyper-plane of the other axes. Such an integral becomes

$$\begin{aligned}\int d^d x f(\mathbf{x}\cdot\mathbf{k}) &= \int r^{d-1} dr\ \sin^{d-2}\theta_1 d\theta_1\ \sin^{d-3}\theta_2 d\theta_2 \ldots d\theta_{d-1} f(r|k|\cos\theta_1) \\ &= S_{d-1}\int dr\ r^{d-1} \int d\theta_1\ \sin^{d-2}\theta_1\ f(r|k|\cos\theta_1).\end{aligned} \tag{2.50}$$

Rather than finding the Green functions for a modified Helmholtz function, (2.38) can be Fourier transformed as

$$\begin{aligned}\mathbf{G}(x) &= \int d^d k\ \frac{1}{k^2+\xi^{-2}} \exp^{(-ik\cdot x)} \\ &= \frac{S_{d-1}}{(2\pi)^d}\int dq\ \frac{q^{d-1}}{q^2+\xi^{-2}} \int d\theta\ \sin^{d-2}\theta\ \exp\left(-iq|\mathbf{x}|\cos\theta\right).\end{aligned} \tag{2.51}$$

Substituting the identity (2.41a) this becomes

$$\begin{aligned}\mathbf{G}(x) &= \frac{S_{d-1}}{(2\pi)^d}\int dq \frac{q^{d-1}}{q^2+\xi^{-2}} \frac{\Gamma\left(\frac{d-1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\left(\frac{qx}{2}\right)^{\frac{d}{2}-1}} \mathbf{J}_{\frac{d}{2}-1}(qx) \\ &= \frac{2^{-\frac{d}{2}}\pi^{-\frac{d+1}{2}}}{(2\pi)^d} \frac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)} \frac{\pi^{\frac{1}{2}}}{x^{\frac{d}{2}-1}} \int dq \frac{q^{\frac{d}{2}}}{q^2+\xi^{-2}} \mathbf{J}_{\frac{d}{2}-1}(qx) \\ &= (2\pi)^{-\frac{d}{2}} x^{-\frac{d}{2}+1} \int_0^\infty dq \frac{q^{\frac{d}{2}}}{q^2+\xi^{-2}} \mathbf{J}_{\frac{d}{2}-1}(qx),\end{aligned} \tag{2.52}$$

which can be solved analytically for $d < 5$ by using the identities (Gradshteĭn *et al.*, 2007)

$$\int_0^\infty t^\mu \mathbf{J}_\nu(t) dt = \frac{2^\mu \Gamma\left(\frac{\nu+\mu+1}{2}\right)}{\Gamma\left(\frac{\nu-\mu+1}{2}\right)} \tag{2.53a}$$

and

$$\int_0^\infty \frac{t^{\nu+1}\mathbf{J}_\nu(at)}{(t^2+z^2)^{\mu+1}} dt = \frac{a^\mu z^{\nu-\mu}}{2^\mu \Gamma(\mu+1)} \mathbf{K}_{\nu-\mu}(az). \tag{2.53b}$$

So that at the critical point

$$\begin{aligned}\mathbf{G}(x) &= \frac{1}{4\pi^{\frac{d}{2}}}\, x^{2-d}\, \Gamma\left(\frac{d-2}{2}\right) \\ &= \frac{\Gamma\left(\frac{d}{2}\right)}{(2-d)2\pi^{\frac{d}{2}}}\, x^{-d+2},\end{aligned} \tag{2.54a}$$

in agreement with (2.47). Away from the critical point

$$\mathbf{G}(x) = (2\pi)^{-\frac{d}{2}} (x\xi)^{-\frac{d}{2}+1}\ \mathbf{K}_{\frac{d}{2}-1}\left(\frac{x}{\xi}\right). \tag{2.54b}$$

2.4 The ϕ^4 Hamiltonian

Building on the free theory, the simplest adjustment that gives non-trivial behaviour is to include a ϕ^4 term in the Hamiltonian,

$$\begin{aligned}\mathcal{H}[\phi] &= \int d^d x\ \left[\frac{1}{2}\big(\nabla\phi(x)\big)^2 + \frac{r_0}{2}\phi^2(x) + \frac{u_0}{4!}\phi^4(x)\right] \\ &= \frac{1}{2}\int đ^d k\ (k^2+r_0)|\phi(k)|^2 \\ &\quad + \frac{u_0}{4!}\int đ^d k_1 \int đ^d k_2 \int đ^d k_3\ \phi(k_1)\phi(k_2)\phi(k_3)\phi\big(-(k_1+k_2+k_3)\big).\end{aligned} \tag{2.55}$$

The field theory nomenclature usually has m_0^2 and g_0 instead of r_0 and u_0, though for the following discussion there is no reason to make this cosmetic adjustment. The partition function is not as tractable as for the free theory, as with the ϕ^4 interaction the modes no longer decouple. To proceed, an expansion of the exponential containing the interaction can be made:

$$\mathcal{H}_0 = \int d^d x \, \left[\frac{1}{2}\big(\nabla\phi(x)\big)^2 + \frac{r_0}{2}\phi^2(x)\right] \qquad \mathcal{H}_{int} = \int d^d x \frac{u_0}{4!}\phi^4(x), \tag{2.56a}$$

$$\begin{aligned} \mathcal{Z}[H] &= \int \mathscr{D}\phi(x)\, \exp\left(-\mathcal{H}[\phi] + \int d^d x \, H(x)\phi(x)\right) \\ &= \int \mathscr{D}\phi(x)\, \exp\left(-\mathcal{H}_0[\phi] - \mathcal{H}_{\rm int}[\phi] + \int d^d x \, H(x)\phi(x)\right), \end{aligned} \tag{2.56b}$$

$$\begin{aligned} \frac{\mathcal{Z}[H]}{\mathcal{Z}[0]} &= \left\langle \exp\left(-\mathcal{H}_{int}[\phi]\right) + \exp\left(\int d^d x \, H(x)\phi(x)\right)\right\rangle_0 \\ &= \left\langle \left[1 - \int d^d x \frac{u_0}{4!}\phi^4(x) + \dots\right] \exp\left(\int d^d x \, H(x)\phi(x)\right)\right\rangle_0 . \end{aligned} \tag{2.56c}$$

Statistical averages in the full theory can then be built perturbatively from the free theory, with the subscript '0' indicating the average has been taken with respect to the distribution given by the Gaussian Hamiltonian,

$$\begin{aligned} \langle\phi(k_1)\phi(k_2)\rangle &= \langle\phi(k_1)\phi(k_2)\rangle_0 \\ &\qquad - \frac{u_0}{4!}\int d\!\!{}^-{}^d q_{123} \, \left\langle \phi(k_1)\phi(k_2)\, \phi_1\phi_2\phi_3\, \phi^\dagger_{1+2+3}\right\rangle_0 + O(u_0^2)\dots \end{aligned} \tag{2.57}$$

In order to clear up and condense the notation subscripts have been introduced; a single measure dx_{12} indicates a double integral over x_1 and x_2, in momentum-space the measure is understood to come with a factor $(2\pi)^d$ as necessary. A function f_{1+2} in the integral would indicate $f(x_1+x_2)$ and so on. A very useful feature of expanding about a Gaussian distribution is Wick's theorem; the averages of the product of multiple fields can be written as the product of averages over pairings of the fields, summed over all unique pairings:

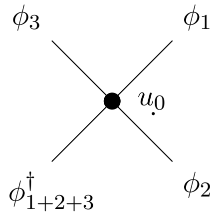
$$\begin{aligned} \langle\phi_1\dots\phi_{2n}\rangle &= \langle\phi_1\phi_2\rangle\langle\phi_3\phi_4\rangle\dots\langle\phi_{2n-1}\phi_{2n}\rangle + \text{permutations} \\ \langle\phi_1\phi_2\phi_3\phi_4\rangle &= \langle\phi_1\phi_2\rangle\langle\phi_3\phi_4\rangle + \langle\phi_1\phi_3\rangle\langle\phi_2\phi_4\rangle + \langle\phi_1\phi_4\rangle\langle\phi_2\phi_3\rangle \\ \langle\phi^{2n}\rangle &= (2n-1)!!\langle\phi^2\rangle^n. \end{aligned} \tag{2.58}$$

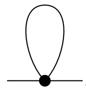
This can be seen from taking differentials of (2.37), it is also why the two-point connected correlation function of the free theory is referred to as the bare propagator for the full theory. In order to keep track of all the integrals a graphical representation is invaluable. Each propagator is given by a line, understood to be connecting the fields:

$$\langle\phi(k_1)\phi(k_2)\rangle_0 = \ \phi(k_1) \ \text{———} \ \phi(k_2) \tag{2.59}$$

Fields connecting at the interaction vertex are given by:

$$\frac{u_0}{4!}\int \text{đ}^d q_{123} = \tag{2.60}$$



The second term in (2.57) would be represented by a diagram of the form . The delta functions that come with the correlation functions quickly reduce the number of integrals, ultimately the number of independent integrals that need to be done can be counted visually as the number of loops in a diagram. Hence the perturbative expansion is sometimes referred to as a loop expansion, and the order of the expansion is counted in loops. In someways the most challenging part of working with the diagrams is counting the symmetry factor for each one, the numerical factor that comes with a diagram counting the number of different pairings of the fields that produce the same integral. When writing out all the diagrams, order by order, for some moment of interest, there will appear numerous sub-diagrams with no connection to the external fields of interest in the moment. These sub-diagrams composed entirely from internal fields are called vacuum fluctuations. The vacuum fluctuations do not need to be calculated as they are eliminated during the normalisation when dividing by $\mathcal{Z}[0]$. What remains after the vacuum fluctuations have been removed are the products of disconnected diagrams, so only connected diagrams need to be calculated and multiplied together as required. It is not the moments but rather the cumulants, the derivatives of the free energy $\mathcal{W}[H]$, that are then of interest. The cumulants consist entirely of connected diagrams; $\mathcal{W}[H]$ is the sum of connected diagrams only (see, e.g.: page 204 (Binney *et al.*, 1992), page 173 (Le Bellac, 1991)), hence 'connected' correlation functions.

The structure of these connected diagrams is of separate sections linked by single propagators, of linked but factorised integrals. It suffices then to know how to calculate these self-contained sections. Such sections, which cannot be split apart by the removal of one propagator, are called one-particle irreducible (1PI). When a 1PI diagram has its 'external' propagators divided out, or amputated, it is called a proper vertex. The effective action, $\Gamma[\phi]$, is the generating functional of amputated 1PI diagrams, of the proper vertices. This means that all the information required can be found via the relations between the connected correlation functions and the vertex functions, requiring only a small number of diagrams to be calculated.

2.5 Renormalising

So far so good it would appear. The partition function is a moment generating functional; the 'Helmholtz' free energy generates the cumulants, which can be calculated from the vertex functions generated by the 'Gibbs' free energy or effective action. The ϕ^4 interaction can be expanded perturbatively and Wick's theorem used, with only the amputated 1PI diagrams that constitute the vertex functions needing to be calculated. In principle the tools are in place to ascertain the behaviour of the model. Unfortunately the vertex functions diverge. The calculation cannot be done. Some cunning transformation to a system with equivalent behaviour is in order. The theory needs to be renormalised.

Wilson advised that: "one cannot write a renormalization cookbook" (Wilson, 1975b), famously used as a quote cited by Niemeijer and van Leeuwen (Domb *et al.*, 1976). The heedlessness of youth prompts an attempted to do so, or rather to continue with the storyboarding project of this introduction.

A theory is renormalisable in some dimension if the divergence of any particular vertex function is the same all orders in the perturbation theory; the coupling constant of the interaction becomes dimensionless and there are a finite number of divergent vertex functions, meaning to fix the divergences a finite number of parameters are needed. Working around the upper critical dimension, $d_c = 4$, the ϕ^4 theory is renormalisable, both Γ^2 and Γ^4 diverge. The divergences of the integrals that constitute Γ^2 and Γ^4 can be controlled at intermediate stages using dimensional regularisation, or imposing some artificial cutoff, while a way is found to fix the infinities. To do so the terms in the expansion are changed around, reshuffling and reordering them; transforming $r_0 \to r$, $u_0 \to u$, and throttling the fields with Z_3, such that the vertices are finite. All the infinities have been swept into the new 'renormalised' couplings, which are related to the old couplings through Z-factors that contain the information about how they diverge. For ϕ^4:

$u = \Gamma^4_R(k=0) \; = Z_3^2\Gamma^4(k=0) \; = u_0 Z_3^2 Z_1^{-1}$	Identifies fixed point
$r = \Gamma^2_R(k=0) \; = Z_3\Gamma^2(k=0) = \; \bar{Z}r_0 + r_{0c}$	Gives exponent ν
$1 = \frac{d}{dk^2}\Big\rvert_{k=0}\Gamma^2_R \; = Z_3 \, \frac{d}{dk^2}\Big\rvert_{k=0}\Gamma^2$	Gives exponent η

Where the subscript 'R' indicates that the vertex function is written in terms of the renormalised couplings, r and u. The renormalised coupling u ensures a sensible perturbation expansion can be written. Holding r fixed keeps the theory intact as the continuum limit taken. The field renormalisation, through Z_3, keeps the propagator in same form. This is the same spirit as when in the classical approach the lattice is re-scaled, or field renormalisation allows the fixed point of interest to be found.

So armed, it is now possible to write down Callan-Symanzik equations to identify the scaling behaviour for the system. The vertex functions of bare and renormalised parameters are related by

$$\Gamma^N(k, r_0, u_0, \Lambda) = Z_3^{-\frac{N}{2}}\left(\frac{\Lambda}{r}, u\right)\Gamma_R^N(k, r, u), \tag{2.61}$$

where Λ is an intermediate cutoff used to regularise the integrals. For $r = 0$, at the critical point, it is necessary to pick an arbitrary momentum scale μ to make sense of the expressions, giving

$$\Gamma^N(k, u_0, \Lambda) = Z_3^{-\frac{N}{2}}\left(\frac{\Lambda}{\mu}, u\right)\Gamma_R^N(k, \mu, u). \tag{2.62}$$

It is possible to set up a number of differential equations that give insight into how the vertex functions scale and how the couplings behave, for example

$$\left[\frac{\partial}{\partial \ln \mu} + \beta(u)\frac{\partial}{\partial u} - \frac{N}{2}\gamma(u)\right]\Gamma_R^N(k, \mu, u) = 0, \tag{2.63}$$

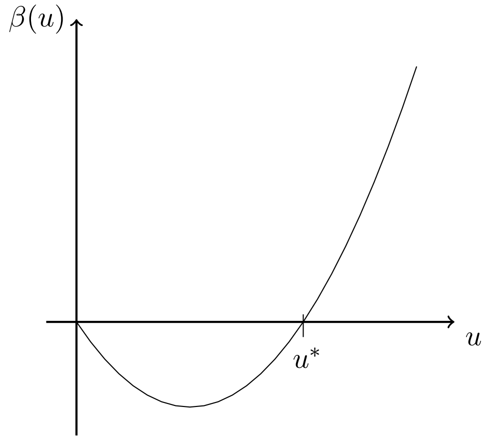
where

$$\beta(u) = \frac{\partial u}{\partial \ln \mu} \quad , \quad \gamma(u) = \frac{\partial \ln Z_3}{\partial \ln \mu} \tag{2.64}$$

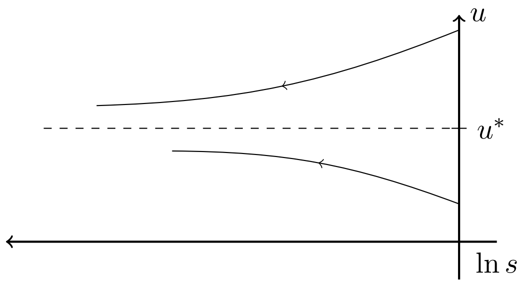
are evaluated at constant bare coupling. The question that needs answering is how the vertex functions behave under changes of scale. Or, in terms of the Wilsonian approach, how they behave during the coupling flow. In the massless case, μ is the convenient tool for such probing. After picking an initial value, s is used to re-scale such that $\mu \to \mu s$, or μ becomes $\mu(s) = \mu s$. This re-parameterisation means that the coupling u now becomes a function of s, this give rise to the cosmetic replacement of $\ln \mu$ with $\ln s$ in the derivatives above. As an example of the kind of scaling behaviour one is looking for, the solution to the example Callan-Symanzik equation would be (see, e.g.: page 248 (Le Bellac, 1991))

$$\Gamma_R^N(sk,u,\mu) = s^{d-N\left(\frac{d}{2}-1\right)} \exp\left(-\frac{N}{2}\int_u^{u(s)} \frac{\gamma(u')du'}{\beta(u')}\right) \cdot \Gamma_R^N\left(k,u(s),\mu\right), \tag{2.65}$$

from which information on the critical exponents can be extracted. Generally it is the γ-functions, the derivatives of the Z-factors, which give the exponents. However there is still some good meat to be had in the β-function. For critical phenomena the expected behaviour, below $d=4$, is typified by that of ϕ^4 to one loop order:



There are two fixed points of the coupling flow, a trivial one at the origin and a more interesting one at $u*$. The stability of this fixed point needs to be ascertained. In critical phenomena, and for ϕ^4 theory, as $s \to 0$ or $\ln s \to -\infty$ the coupling $u(s)$ is driven to the fixed point $u*$, this is infrared stability:



If the non-trivial zero of the β-function had different properties it would not be possible to probe the long range behaviour of the system. It might not be possible to answer any questions, or there might be ultra-violet stability $\big(u(s) \to u* \text{ as } s \to \infty\big)$ allowing the short range behaviour of the system to be understood.

The presence of a non-trivial zero for the β-function is very similar to the behaviour seen in parameter space at the end of the previous chapter, where u_0 became relevant after fixed point crossing.

The writing of a renormalised field theory assumes there is a critical point, for the behaviour of the renormalised r to correspond to a critical correlation length, the system has to already have an appropriate relation between the various couplings, r_0, u_0 and Λ. In field theory the cutoff Λ plays a very different role than in Wilsonian RG, where it is really there and used to integrate out shells in momentum space. In field theory it features as an intermediate step to regularise expressions before the continuum limit is taken. Also the nature of the fixed point of interest has seemingly changed; classically the fixed point of interest on the critical surface is unstable, the slightest move off the surface is expelled out to infinity. The tools of field theory though drive straight to the fixed point of interest, it appears stable under re-scaling. Both approaches however involve transforming a system in search of long range behaviour, their spirit is the same.

Chapter 3

From a Langevin Equation to a Field Theory

Moving out of equilibrium and engaging with time dependent, dynamical behaviour presents a new set of challenges. Over the last forty years several approaches have emerged as canonical techniques in non-equilibrium statistical mechanics. One family deals in equations of motion for the probability density of a process, where a Fokker-Planck equation is sought for the probabilities. The genus originates from a Chapman-Kolmogorov equation for the evolution of the probability density function for a process, leading to a master equation for the transition probabilities (van Kampen, 1992). Then there are Langevin equations, stochastic partial differential equations that form equations of motion dealing directly with the observables in the system. It is quite natural to introduce non-equilibrium analysis with relaxation to equilibrium, where the dynamics arise from the minimisation of the Hamiltonian for a system after some perturbation, extended by implementing noise in some way. The celebrated review by Hohenberg and Halperin (Hohenberg and Halperin, 1977) established the classification of these relaxation models.

As the title of this chapter suggests, there is also a field theoretic approach. A neat analogy for the relationship between these methods can be found in quantum mechanics; where the view of system from the perspective of a Fokker-Planck equation in dealing with the evolution of the probability density is the Schrödinger view, the Langevin equations of time dependent variables is then the Heisenberg picture. Field theory is the Feynman path integral formulation (Zinn-Justin, 2002).

3.1 Response Field Formalism

Given the success of the renormalisation group in analysing critical behaviour in equilibrium systems, it would be extremely useful to extend the same techniques and re-use them for analysing non-equilibrium systems; where dynamics appear in a system, and the observable is described by a Langevin equation rather than the conservative formulation of a Hamiltonian. One way of doing this would be to turn the Langevin equation in question into a field theory. This entails forming an action constrained by the Langevin equation in order to produce a moment generating functional, after which one is home and dry and ready to swing the sledgehammers of quantum field theory. This approach was developed independently by Janssen (Janssen, 1976) and De Dominicis (De Dominicis and Peliti, 1978), below I have given what is the standard presentation to be found in the literature (Janssen, 1979; Folk and Moser, 2006; Täuber, 2005). Starting with a generic Langevin equation for a scalar field $\phi(x,t)$ in a $(d+1)$-dimensional space,

$$\partial_t \phi(x,t) = F[\phi(x,t)] + \eta(x,t), \tag{3.1}$$

where $\eta(x,t)$ is the noise term, whose mean is taken to be zero and has covariance

$$\big\langle \eta(x,t)\eta(x',t')\big\rangle = 2L\,\delta^d(x-x')\delta(t-t'), \tag{3.2}$$

with the noise amplitude, L, being in general an operator. The distribution that produces the noise correlator is

$$\mathcal{K}[\eta] \propto \exp\left\{-\frac{1}{4}\int d^d x \int dt\; \eta(x,t)\left[L^{-1}\eta(x,t)\right]\right\}. \tag{3.3}$$

As L could be an operator, the inverse L^{-1} in this expression should then be interpreted as a Green function. Using the Langevin equation, the noise can be re-written to be viewed as a functional of the fields,

$$\eta[\phi] = \partial_t\phi(x,t) - F[\phi]. \tag{3.4}$$

A probability distribution for the field, written as kernel and measure, would then be

$$\mathcal{K}[\eta]\mathscr{D}\eta = \mathcal{P}[\phi]\mathscr{D}\phi \propto \exp\{-\mathcal{G}[\phi]\}\mathscr{D}\phi, \tag{3.5a}$$

with the statistical weight of a state of the system defined by the Onsager-Machlup functional (Onsager and Machlup, 1953)

$$\mathcal{G}[\phi] = \frac{1}{4}\int d^d x \int dt \big(\partial_t\phi(x,t) - F[\phi]\big)\Big[L^{-1}\big(\partial_t\phi(x,t) - F[\phi]\big)\Big]. \tag{3.5b}$$

This looks like plainly inserting (3.4) into (3.3), rather than a transformation. However the Jacobian coming from the change of variables can be ignored, as will be commented on below. Equations (3.5a) and (3.5b) give, in principle, a field-theoretic representation. However the presence of $F[\phi]L^{-1}F[\phi]$ in the functional makes it more than a little inconvenient to use, the perturbative expansion is a mess and the L^{-1} itself may produce a singularity in the infra-red limit $q \to 0$. To get around these difficulties some sort of transformation is in order. The goal is to calculate averages of some quantity $A[\phi]$ over noise histories,

$$\langle A[\phi] \rangle_\eta \propto \int \mathscr{D}\eta \, A\big[\phi[\eta]\big] \cdot \mathcal{K}[\eta], \tag{3.6}$$

where, to paraphrase Uwe Täuber (Täuber, 2005), a rather labyrinthine unity can be inserted at each space-time point to constrain the fields to the Langevin dynamics:

$$\begin{aligned} 1 &= \int \mathscr{D}\phi \prod_{(x,t)} \delta\left(\partial_t \phi(x,t) - F[\phi](x,t) - \eta(x,t)\right) \\ &= \int \mathscr{D}(i\tilde{\phi}) \int \mathscr{D}\phi \, \exp\left\{ - \int d^d x \int dt \, \tilde{\phi}(x,t) \left(\partial_t \phi(x,t) - F[\phi] - \eta(x,t)\right) \right\}. \end{aligned} \tag{3.7}$$

The functional integral representation of the delta function has in effect introduced a Martin-Siggia-Rose auxiliary field (Martin *et al.*, 1973), $\tilde{\phi}(x,t)$, referred to here as a response field. Using the definition for the noise distribution (3.3), this leads to

$$\begin{aligned} \langle A[\phi] \rangle_\eta = &\int \mathscr{D}(i\tilde{\phi}) \int \mathscr{D}\phi \, \exp\left\{ - \int d^d x \int dt \, \tilde{\phi}(x,t) \left(\partial_t \phi(x,t) - F[\phi]\right) \right\} \\ &\times A[\phi] \int \mathscr{D}\eta \, \exp\left\{ - \int d^d x \int dt \, \frac{1}{4} \eta(x,t) L^{-1} \eta(x,t) - \tilde{\phi}(x,t)\eta(x,t) \right\}. \end{aligned} \tag{3.8}$$

Completing the square, performing the Gaussian integral for the noise, and absorbing any constants along the way into the functional measure, gives a probability distribution for the field $\phi(x,t)$:

$$\mathcal{P}[\phi] = \int \mathscr{D}(i\tilde{\phi}) \exp\left\{ -\mathcal{A}\big[\phi, \tilde{\phi}\big] \right\}, \tag{3.9a}$$

with the statistical weight given by the Janssen-De Dominicis response functional

$$\mathcal{A}\big[\phi, \tilde{\phi}\big] = \int d^d x \int dt \left[\tilde{\phi}(x,t) \left(\partial_t \phi(x,t) - F[\phi]\right) - \tilde{\phi}(x,t) L \tilde{\phi}(x,t) \right]. \tag{3.9b}$$

The quick answer as to why no Jacobian terms appear, either from transforming to get the distribution of the field in (3.5a) or from the insertion of the delta function in (3.8), is that the functional determinate is a constant and can be absorbed into the measure. This can be done provided Itō forward discretisation is assumed to have been used for the noise. More honestly, the missing terms are responsible for causality in the theory. If properly taken care of using a Stratanovitch interpretation of the noise, they exactly cancel the contributions from closed response loops, so one is free to assert Itō discretisation after the fact, drop the Jacobian terms and forbid all vacuum fluctuations as well as terms consisting purely of response fields. This is the approach that will be taken through out. For a good discussion see (Janssen, 1992), also (De Dominicis and Peliti, 1978; Vasil'ev, 2004; Janssen, 1979).

3.2 The i Issue

Aside from the apparently missing Jacobian terms, there was some other fast talking going on above. There is an 'i' suddenly appearing in the measure of the path integral in (3.8). Really this looks like total nonsense, though it is more of a symptom prompting a question about the convergence of the Gaussian integral in (3.9a). Finding no discussion in the literature, I went through the derivation with more care, to clarify to myself what the original authors meant. The objects of interest are averages of some quantity $\mathbf{A}[\psi]$, a functional of the observable $\psi(x,t)$ in a Langevin equation.[1] In place of the ensemble averages of equilibrium statistical mechanics, it is the behaviour of correlations taken over realisations of the noise in the system that need to be calculated. Using the noise $\zeta(x,t)$ with the same distribution, $\mathcal{P}_\zeta[\zeta] = \exp\left\{-\int d^dx \int dt\ \frac{\zeta^2(x,t)}{4\Gamma^2}\right\}$, as in (3.3), but where the noise strength is taken to be a positive number Γ^2, this is:

$$\begin{aligned} \langle \mathbf{A}[\psi] \rangle_\zeta &= \int \mathscr{D}\zeta\ \mathcal{P}_\zeta[\zeta]\ \mathbf{A}\left[\psi[\zeta]\right] \\ &= \int \mathscr{D}\zeta\ \mathcal{P}_\zeta[\zeta] \int \mathscr{D}\psi\ \delta\left(\dot{\psi}(x,t) - \mathbf{F}[\psi] - \zeta(x,t)\right) \mathbf{A}[\psi]. \end{aligned} \tag{3.10}$$

The delta function can be re-written using

$$\begin{aligned} &\delta\left(\dot{\psi}(x,t) - \mathbf{F}[\psi] - \zeta(x,t)\right) = \\ &\qquad \int \mathscr{D}\tilde{\psi}\ \exp\left\{-i \int d^dx \int dt\ \tilde{\psi}(x,t)\left(\dot{\psi}(x,t) - \mathbf{F}[\psi] - \zeta(x,t)\right)\right\}, \end{aligned} \tag{3.11}$$

[1]The change of variable notation in this section was done only to emphasis the difference in derivation taking place.

where the response field $\tilde{\psi}(x,t)$ can be seen to be the conjugate variable to the Langevin equation. Substituting back in, and performing the Gaussian integral over the noise,

$$\begin{aligned}\langle \mathbf{A}[\psi]\rangle_\zeta &= \int \mathscr{D}\zeta\, \mathcal{P}_\zeta[\zeta] \int \mathscr{D}\psi \int \mathscr{D}\tilde{\psi} \\ &\qquad \cdot \exp\left\{-i\int d^dx \int dt\; \tilde{\psi}(x,t)\left(\dot{\psi}(x,t) - \mathbf{F}[\psi] - \zeta(x,t)\right)\right\} \mathbf{A}[\psi] \\ &= \int \mathscr{D}\psi \int \mathscr{D}\tilde{\psi}\; \exp\left\{-i\int d^dx \int dt\; \tilde{\psi}(x,t)\left(\dot{\psi}(x,t) - \mathbf{F}[\psi]\right)\right\} \\ &\qquad \cdot \exp\left\{-\Gamma^2 \int d^dx \int dt\; \tilde{\psi}^2(x,t)\right\} \mathbf{A}[\psi].\end{aligned} \tag{3.12}$$

This equation reproduces the content of equations (3.9a) and (3.9b), except for the rouge 'i' in the first exponential. Splitting $\mathbf{F}[\psi]$ into a bilinear/harmonic part and an interaction/anharmonic part takes a step toward forming a perturbation theory,

$$\mathbf{F}[\psi] = \mathbf{F}_0[\psi] + \mathbf{F}_{\text{int}}[\psi]. \tag{3.13}$$

Leading to, with an implicit Fourier transform of the fields,

$$\begin{aligned}\langle \mathbf{A}[\psi]\rangle &= \int \mathscr{D}\psi \int \mathscr{D}\tilde{\psi}\; \exp\left\{-i\int đ^dq \int đ\omega\; \tilde{\psi}^\dagger(q,\omega)\left(-i\omega + \mathbf{f}(q)\right)\psi(q,\omega)\right\} \\ &\cdot \exp\left\{-\Gamma^2 \int đ^dq \int đ\omega\; \tilde{\psi}^2(q,\omega)\right\} \mathbf{A}[\psi]\; \exp\left\{i\int đ^dq \int đ\omega\; \tilde{\psi}^\dagger(q,\omega)\mathbf{F}_{\text{int}}[\psi]\right\},\end{aligned} \tag{3.14}$$

where $\mathbf{f}(q)$ is some polynomial in q^2. A popular example would be $\mathbf{f}(q) = r_0 + \nu_2 q^2 + \nu_4 q^4$. So far so good, the theory is in the form needed for a perturbation expansion. Generalising slightly, so that the quantity being averaged can depend on the response field too, and writing in a suggestive fashion:

$$\begin{aligned}\left\langle \mathbf{A}\left[\psi,\tilde{\psi}\right]\right\rangle &= \int \mathscr{D}\psi \int \mathscr{D}\tilde{\psi}\; \exp\left\{-\mathcal{A}_0\right\}\mathbf{A}\left[\psi,\tilde{\psi}\right]\exp\left\{-\mathcal{A}_{\text{int}}\right\} \\ &= \left\langle \mathbf{A}[\psi,\tilde{\psi}]\exp\left\{-\mathcal{A}_{\text{int}}\right\}\right\rangle_0, \\ \mathcal{A}_0 &= \int đ^dq \int đ\omega\; \tilde{\psi}^\dagger(q,\omega)\big(i\left[-i\omega + \mathbf{f}(q)\right]\big)\psi(q,\omega) + \Gamma^2\tilde{\psi}^\dagger(q,\omega)\tilde{\psi}(q,\omega), \\ \mathcal{A}_{\text{int}} &= \int đ^dq \int đ\omega\; \tilde{\psi}^\dagger(q,\omega)\left(-i\mathbf{F}_{\text{int}}[\psi]\right).\end{aligned} \tag{3.15}$$

As seen in Section 2.4, the subscript '0' indicates that the averaging has been taken with respect to the harmonic (or 'Gaussian') weight. Proceeding in the traditional fashion, the interaction should be set to zero and the propagators found for the free theory before turning the interaction back on and constructing a perturbative expansion. There is a choice about where to place the noise term, for time being I will follow the usual presentation and keep it with the bilinear term. Note there really is still an 'i' to take account of.

3.2.1 Propagators and the effective theory

From the field theory constructed in (3.15), the harmonic weight $\mathcal{A}_0$ can be written more compactly as

$$
\begin{aligned}
\mathcal{A}_0 &= \frac{1}{2}\int đ^d q \int đ\omega\ \bar{\Psi}^\dagger M \bar{\Psi}, \\
\bar{\Psi} &= \begin{pmatrix} \tilde{\psi}(q,\omega) \\ \psi(q,\omega) \end{pmatrix} \qquad \bar{\Psi}^\dagger = \begin{pmatrix} \tilde{\psi}^\dagger(q,\omega) & \psi^\dagger(q,\omega) \end{pmatrix}, \\
M &= \begin{pmatrix} 2\Gamma^2 & i\big(-i\omega + \mathbf{f}(q)\big) \\ i\big(i\omega + \mathbf{f}(q)\big) & 0 \end{pmatrix}.
\end{aligned} \tag{3.16}
$$

Ignoring $\mathcal{A}_{\rm int}$ for the time being, a bare moment generating functional, $\mathcal{Z}_0\left[j,\tilde{j}\right]$, with source fields $j(q,\omega)$ and $\tilde{j}(q,\omega)$, can be defined as

$$
\begin{aligned}
\mathcal{Z}_0\left[j,\tilde{j}\right] &= \left\langle \exp\left\{ \int đ^d q \int đ\omega\ \tilde{j}^\dagger(q,\omega)\tilde{\psi}(q,\omega) + j^\dagger(q,\omega)\psi(q,\omega) \right\} \right\rangle_0 \\
&= \exp\left\{ \frac{1}{2}\int đ^d q \int đ\omega\ \bar{j}^\dagger M^{-1} \bar{j} \right\}, \\
\bar{j} &= \begin{pmatrix} \tilde{j}(q,\omega) \\ j(q,\omega) \end{pmatrix} \qquad \bar{j}^\dagger = \begin{pmatrix} \tilde{j}^\dagger(q,\omega) & j^\dagger(q,\omega) \end{pmatrix}, \\
M^{-1} &= \frac{1}{\omega^2 + \mathbf{f}^2(q)} \begin{pmatrix} 0 & -i\big(-i\omega + \mathbf{f}(q)\big) \\ -i\big(i\omega + \mathbf{f}(q)\big) & 2\Gamma^2 \end{pmatrix}.
\end{aligned} \tag{3.17}
$$

As discussed earlier vacuum fluctuations are assumed not to make any contribution, so the usual procedure of normalising with a division by $\mathcal{Z}_0[0,0]$ that also gives proper averages is a somewhat moot point. This also brings into sharp focus the question about the convergence of the Gaussian integral, here there is no concern as the matrix M in the weight has eigenvalues with positive real parts. However, if the weight was taken as originally introduced in (3.9b), there would instead be a destabilising factor $-2\Gamma^2$; the weight matrix M would have negative eigenvalues and the Gaussian integral would not converge. Having defined the (bare) moment generating functional and performed the Gaussian integral it is the matter of a couple of derivative to get the bare correlators,

$$
\begin{aligned}
\left\langle \psi(q_1,\omega_1)\tilde{\psi}(q_2,\omega_2) \right\rangle_0 &= \frac{-i}{-i\omega_1 + \mathbf{f}(q_1)} đ^d(q_1+q_2)đ(\omega_1+\omega_2) \\
\left\langle \tilde{\psi}(q_1,\omega_1)\tilde{\psi}(q_2,\omega_2) \right\rangle_0 &= 0 \\
\left\langle \psi(q_1,\omega_1)\psi(q_2,\omega_2) \right\rangle_0 &= \frac{2\Gamma^2}{\omega_1^2 + \mathbf{f}^2(q_1)} đ^d(q_1+q_2)đ(\omega_1+\omega_2),
\end{aligned} \tag{3.18}
$$

where the bar on the delta function indicates a multiplication by the relevant power of 2π, arising from the Fourier normalisation. In the field/response-field propagator, $\langle\psi(q_1,\omega_1)\tilde{\psi}(q_2,\omega_2)\rangle$, there is a factor '$-i$' present in the numerator that would not have been seen if (3.9b) had been used. However the perturbative expansion of the interaction always pulls down another 'i' given its form: $\tilde{\psi}^\dagger(q,\omega)\,(-i\mathbf{F}_{\text{int}}[\psi])$. This means in a diagram all the internal field-response field correlation functions are of the form

$$\left\langle i\psi(q_1,\omega_1)\tilde{\psi}^\dagger(q_2,\omega_2)\right\rangle_0 = \frac{1}{-i\omega_1+\mathbf{f}(q_1)}\bar{\delta}^d(q_1-q_2)\bar{\delta}(\omega_1-\omega_2). \tag{3.19}$$

So for internal propagators it is just as well to drop the factor of 'i'. This still leaves the propagators for the external legs of a diagram, and a factor of 'i' for each of the non-linearities on a field edge of a diagram. However, the external propagators are amputated when calculating 1PI's, and anyway the vertex functions will renormalise identically and display the same scaling behaviour regardless; such things not be affected by a constant like $\pm i$. Which is all to say that the original formulation, with factors of 'i' dropped from the numerators, is fine. The same physics will be found. So it legitimate to build the field theory using the simpler looking action

$$\mathcal{A}[\psi,\tilde{\psi}] = \int d^dx \int dt \left[\tilde{\psi}(x,t)\left(\partial_t\psi(x,t) - F[\psi]\right) - \tilde{\psi}(x,t)\Gamma^2\tilde{\psi}(x,t)\right], \tag{3.20}$$

to pretend that there is no issue with convergence, and calculate using

$$\begin{aligned} M' &= \begin{pmatrix} -2\Gamma^2 & -i\omega+\mathbf{f}(q) \\ i\omega+\mathbf{f}(q) & 0 \end{pmatrix}, \\ \mathcal{Z}_0\left[j,\tilde{j}\right] &= \exp\left\{\frac{1}{2}\int \bar{d}^dq \int \bar{d}\omega\ \bar{j}^\dagger M'^{-1}\bar{j}\right\}, \\ M'^{-1} &= \frac{1}{\omega^2+\mathbf{f}^2(q)}\begin{pmatrix} 0 & -i\omega+\mathbf{f}(q) \\ i\omega+\mathbf{f}(q) & 2\Gamma^2 \end{pmatrix}. \end{aligned} \tag{3.21}$$

The bare connected correlation functions are generated by $\ln\mathcal{Z}_0\left[j,\tilde{j}\right]$, identifying the bare response propagator $\hat{\mathbf{G}}_0^{11}(q,\omega)$ with

$$\begin{aligned} \left((2\pi)^{d+1}\right)^2 \left.\frac{\delta^2\ln\mathcal{Z}_0\left[j,\tilde{j}\right]}{\delta j^\dagger(q_1,\omega_1)\delta\tilde{j}(q_2,\omega_2)}\right|_{j,\tilde{j}=0} &= \frac{1}{-i\omega_1+\mathbf{f}(q_1)}\bar{\delta}^d(q_1-q_2)\bar{\delta}(\omega_1-\omega_2), \\ \hat{\mathbf{G}}_0^{11}(q,\omega) &= \frac{1}{-i\omega+\mathbf{f}(q)}. \end{aligned} \tag{3.22}$$

There is then a choice as to how to account for the response field. Having started as a real field, the factors of 'i' can be explicitly dropped from the field theory and the response field kept as a real field. Alternatively the initially real response field is now to be associated with a factor of 'i' when constructing correlation/diagrams using the above formalism, making it a pure imaginary field. This second interpretation is the one I believe was intended by the original formulation. In either case the same physics will emerge, though care must be taken with the notation, especially regarding conjugates.

3.3 Vertex Functions

The objects that are of practical interest in a field theory calculation ought to be the vertex functions. To this end a moment generating functional, $\mathcal{Z}\left[j,\tilde{j}\right]$, is formed on the basis of (3.20), or indeed (3.9a) and (3.9b),

$$\mathcal{Z}\left[j,\tilde{j}\right] = \int \mathscr{D}\phi\mathscr{D}\tilde{\phi}\ \exp\left\{-\mathcal{A}[\phi,\tilde{\phi}]\right\}\exp\left\{\int d^dx\int dt\ \left(\tilde{\phi}(x,t)\tilde{j}(x,t)+\phi(x,t)j(x,t)\right)\right\}. \tag{3.23}$$

Its logarithm generates the connected correlation functions, and when Legendre transformed produces the effective action, leading to a Dyson-like equation and on to the other relations between the connected correlation functions and the vertex functions. To perform the Legendre transformation two fields are defined,

$$\theta(x,t) = \frac{\delta \ln \mathcal{Z}\left[j,\tilde{j}\right]}{\delta j(x,t)} \qquad \tilde{\theta}(x,t) = \frac{\delta \ln \mathcal{Z}\left[j,\tilde{j}\right]}{\delta \tilde{j}(x,t)}. \tag{3.24}$$

These are the field averages of the Langevin equation, $\langle\phi(x,t)\rangle$ and $\langle\tilde{\phi}(x,t)\rangle$, when evaluated at $j(x,t)$, $\tilde{j}(x,t)=0$. The effective action is

$$\Gamma[\tilde{\theta},\theta] = -\ln\mathcal{Z}\left[j,\tilde{j}\right] + \int d^dx\int dt\left(\tilde{j}(x,t)\tilde{\theta}(x,t)+j(x,t)\theta(x,t)\right). \tag{3.25}$$

Taking a derivative gives

$$\begin{aligned}
\frac{\delta\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(x,t)} &= \tilde{j}(x,t) + \int d^dx\int dt\ \tilde{\theta}(x,t)\frac{\delta\tilde{j}(x,t)}{\delta\tilde{\theta}(x,t)} \\
&\quad + \int d^dx\int dt\ \theta(x,t)\frac{\delta j(x,t)}{\delta\tilde{\theta}(x,t)} - \frac{\delta\ln\mathcal{Z}\left[j,\tilde{j}\right]}{\delta\tilde{\theta}(x,t)} \\
&= \tilde{j}(x,t) + \int d^dx\int dt\left(\tilde{\theta}(x,t)\frac{\delta\tilde{j}(x,t)}{\delta\tilde{\theta}(x,t)} + \theta(x,t)\frac{\delta j(x,t)}{\delta\tilde{\theta}(x,t)}\right) \\
&\quad - \int d^dx\int dt\left(\underbrace{\frac{\delta\ln\mathcal{Z}\left[j,\tilde{j}\right]}{\delta\tilde{j}(x,t)}}_{\tilde{\theta}(x,t)}\frac{\delta\tilde{j}(x,t)}{\delta\tilde{\theta}(x,t)} + \underbrace{\frac{\delta\ln\mathcal{Z}\left[j,\tilde{j}\right]}{\delta j(x,t)}}_{\theta(x,t)}\frac{\delta j(x,t)}{\delta\tilde{\theta}(x,t)}\right) \\
&= \tilde{j}(x,t).
\end{aligned} \tag{3.26}$$

Similarly for a derivative with respect to $\theta(x,t)$, giving the rather pleasing pair

$$\frac{\delta\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)} = \tilde{j}(y) \qquad \frac{\delta\Gamma[\tilde{\theta},\theta]}{\delta\theta(y)} = j(y), \tag{3.27}$$

where a space-time argument (x,t) has been abbreviated to (y). Taking a further derivative gives the perhaps trivial seeming

$$\begin{aligned}\frac{\delta^2\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\tilde{j}(\bar{y})} &= \frac{\delta}{\delta\tilde{j}(\bar{y})}\left(\frac{\delta\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)}\right)\\ &= \delta^{(d+1)}(y-\bar{y}).\end{aligned} \tag{3.28}$$

Exploring the left-hand side a little further,

$$\begin{aligned}\frac{\delta^2\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\tilde{j}(\bar{y})} &= \delta^{(d+1)}(y-\bar{y})\\ &= \int_{y'} \frac{\delta^2\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\tilde{\theta}(y')}\frac{\delta\tilde{\theta}(y')}{\delta\tilde{j}(\bar{y})} + \frac{\delta^2\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\theta(y')}\frac{\delta\theta(y')}{\delta\tilde{j}(\bar{y})}\\ &= \int_{y'} \frac{\delta^2\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\tilde{\theta}(y')}\frac{\delta^2\ln\mathcal{Z}[j,\tilde{j}]}{\delta\tilde{j}(y')\delta\tilde{j}(\bar{y})} + \frac{\delta^2\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\theta(y')}\frac{\delta^2\ln\mathcal{Z}[j,\tilde{j}]}{\delta j(y')\delta\tilde{j}(\bar{y})}\\ \left.\frac{\delta^2\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\tilde{j}(\bar{y})}\right|_{j,\tilde{j}=0} &= \int_{y'} \frac{\delta^2\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\theta(y')}\left\langle\phi(y')\tilde{\phi}(\bar{y})\right\rangle\\ \delta^{(d+1)}(y-\bar{y}) &= \int_{y'} \Gamma^{11}(y,y')\mathbf{G}^{11}(y',\bar{y}).\end{aligned} \tag{3.29}$$

The notation for the integrals above has been condensed in similar fashion to the arguments, so that $\int_y$ indicates is an integral over a space-time y; with an integral over the d-dimensional spatial measure, d^dx, and the one-dimensional time measure dt. The first term in the third line was dropped when evaluated in the limit of vanishing source fields, as it contains a correlation function of response fields only, which is zero by definition. The last line gives a relationship between a vertex function and a connected correlation function, defining the notation for these functions as

$$\mathbf{G}^{nm}(y'_1,\ldots,y'_n,\bar{y}_1,\ldots,\bar{y}_m) = \left.\frac{\delta^{n+m}\ln\mathcal{Z}[j,\tilde{j}]}{\delta j(y'_n)\ldots\delta j(y'_1)\delta\tilde{j}(\bar{y}_m)\ldots\delta\tilde{j}(\bar{y}_1)}\right|_{j,\tilde{j}=0}, \tag{3.30a}$$

$$\Gamma^{nm}(\bar{y}_1,\ldots,\bar{y}_n,y'_1,\ldots,y'_m) = \frac{\delta^{n+m}\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(\bar{y}_n)\ldots\delta\tilde{\theta}(\bar{y}_1)\delta\theta(y'_m)\ldots\delta\theta(y'_1)}. \tag{3.30b}$$

Taking a Fourier transform is the next step. The translational invariance of the correlator, which comes from $\mathcal{Z}[j,\tilde{j}]$, is conferred onto the vertex function via the integral. It could also be noted from (3.24) and (3.25) that translational invariance is passed on. The conjugate variable to y is k, where k represents the momentum-frequency point (q, ω). The notation for an k-integral, $\int_k$, is the same as for y, except there is an implicit division of the measures by the correct normalising power of 2π.

Fourier transforming a translationally invariant object provides the opportunity to drop one of the conjugate arguments and pick up a delta function. For the propagator $\mathbf{G}^{11}$ it might seem natural to keep the field argument, as $\phi(x,t)$ is surely the physical object of interest. Keeping track of the propagator by its business end, ϕ, is also of most practical utility when calculating diagrams; as had started to be seen in the build up to (3.22), with daggered and undaggered response fields in the mix. In keeping with this, the equivalent term, $\tilde{\theta}$, of the corresponding vertex function, Γ^{11}, is kept too:

$$\begin{aligned}
(2\pi)^{d+1}\delta^{(d+1)}(k+\bar{k}) &= \int_{k'} \Gamma^{11}(k,k')\mathbf{G}^{11}(-k',\bar{k}) \\
&= \hat{\Gamma}^{11}(k)\hat{\mathbf{G}}^{11}(-\bar{k})(2\pi)^{d+1}\delta^{(d+1)}(k+\bar{k}) \\
\hat{\Gamma}^{11}(k) &= \frac{1}{\hat{\mathbf{G}}^{11}(k)}, \qquad (3.31a)
\end{aligned}$$

giving a Dyson equation. The hat on $\mathbf{G}$ and Γ indicating that translational invariance has been used to drop an argument and spawn a delta-function. This correct derivation should be held in contrast to the quick and dirty results of (2.18) and (2.19), which were advanced for illustrative value, though lacking in technical clarity. The algebraic details of deriving the needed relations between the vertex functions and the connected correlation functions can be found in Appendix A. Suffice it to say here that other derivative combinations at this order lead to the noise identity,

$$\hat{\Gamma}^{20}(k) = \frac{-\hat{\mathbf{G}}^{20}(k)}{\hat{\mathbf{G}}^{11}(k)\hat{\mathbf{G}}^{11}(-k)}, \qquad (3.31b)$$

and the vertex associated with the pure response field correlator is necessarily zero,

$$\hat{\Gamma}^{02}(k) = 0. \qquad (3.31c)$$

Higher orders relations are derived by taking further source field derivatives, say of (3.28). For example leading to this pair, which may or may not prove to be useful later on:

$$\hat{\Gamma}^{12}(k_1,k_2) = \frac{-\hat{\mathbf{G}}^{12}(k_1,k_2)}{\hat{\mathbf{G}}^{11}(-k_1)\hat{\mathbf{G}}^{11}(-k_2)\hat{\mathbf{G}}^{11}(-k_1-k_2)}, \qquad (3.31d)$$

$$\hat{\Gamma}^{13}(k_1,k_2,k_3) = \frac{-\hat{\mathbf{G}}^{13}(k_1,k_2,k_3)}{\hat{\mathbf{G}}^{11}(-k_1)\hat{\mathbf{G}}^{11}(-k_2)\hat{\mathbf{G}}^{11}(-k_3)\hat{\mathbf{G}}^{11}(-k_1-k_2-k_3)}. \tag{3.31e}$$

Equation (3.31e) has been foreshortened on the understanding that only 1PI terms are to be included, the missing terms acting to subtract out the non-1PI part. The convention used for which arguments to drop from the higher order vertex and connected correlation functions, as a result of translational invariance, is for the single $\tilde{\theta}/\phi$ term to be dropped. This is slightly at odds with the initial inclination of how to define the propagator, but for terms of the form Γ^{1m} and $\mathbf{G}^{1m}$ it seems the most sensible to drop the single term rather than one of the m terms. This judgement will only lead to aesthetic differences, with at most an apparent sign change in arguments from definitional differences.

Chapter 4

Conserved Surface Growth - Homoepitaxy

Homoepitaxial growth is the process of forming a crystal layer by layer, in which a surface is grown on a substrate of the same material as that being deposited. The substrate provides the deposit with a structure, fixing the lattice that forms. A typical example that is given of an application would be to semi-conductor wafers, though there are several different techniques and applications for growing a crystalline surface. In a pair of articles in 2007, Haselwandter and Vvedensky (Haselwandter and Vvedensky, 2007a,b) presented a detailed RG analysis of the following Langevin equation, purporting to describe homoepitaxial growth on a d-dimensional substrate:

$$\begin{aligned}\partial_t\phi(x,t) = \nu_2\nabla^2\phi(x,t) - \nu_4\nabla^4\phi(x,t) + \lambda_{13}\nabla\left(\nabla\phi(x,t)\right)^3 \\ + \lambda_{22}\nabla^2\left(\nabla\phi(x,t)\right)^2 + \eta(x,t).\end{aligned} \tag{4.1}$$

The real field $\phi(x,t)$ is the displacement at a point from the average surface height. The surface growth is driven by the noise term $\eta(x,t)$, a Gaussian white noise with zero mean and the correlator

$$\langle\eta(x,t)\eta(x',t')\rangle = 2L\,\delta^d(x-x')\delta(t-t'). \tag{4.2}$$

A Langevin equation of this form, with the purpose of describing homoepitaxial growth, first appeared in 1991, it was proposed independently by Villain (Villain, 1991) and by Lai and Das Sarma (Lai and Das Sarma, 1991). Though a more limited form, with conserved noise, had been introduced two years earlier (Sun *et al.*, 1989). Haselwandter and Vvedensky derived it in a different manner to its debut, based on considering the transition rules of a lattice model for the surface, and then working towards a continuum equation. After dropping terms considered to be irrelevant for the scaling behaviour, seeking to capture the "qualitative features of the surface morphology" (Haselwandter and Vvedensky, 2007b), they arrive at equation (4.1). Deferring further commentary on the literature till Chapter 6, suffice it to say that the analysis of Haselwandter and Vvedensky was performed using the dynamic RG (DRG), a technique that is an extension of Wilson's approach to RG calculations; in that a cutoff is used, momentum shells are integrated out and as such the differential flow equations are found. It is a popular technique in the field, see for example Appendix B in (Barabasi and Stanley, 1995), though it was original formulated by (Forster *et al.*, 1977).

The situation would seem propitious for analysing the Langevin equation using the field theoretic method as set up in the previous chapter.

4.1 The Villain-Lai-Das Sarma equation as a Field Theory

Following the recipe from Chapter 3, a general Langevin equation

$$\partial_t \phi(x,t) = F[\phi](x,t) + \eta(x,t) \tag{4.3}$$

can be turned into an action

$$\mathcal{A}[\phi,\tilde{\phi}] = \int d^d x \int dt \left[\tilde{\phi}\left(\partial_t \phi - F[\phi]\right) - \tilde{\phi} L \tilde{\phi}\right], \tag{4.4}$$

where L is the noise strength as seen in the correlator (4.2). This action provides a moment generating functional to work with and can be split up into a harmonic/bilinear part, spawning a bare propagator, and an anharmonic/interaction part, to be expanded perturbatively. It was commented on in passing that there is a choice as to where to assign the noise term. If included with the bilinear part there will be two kinds of propagator; a correlation propagator that comes from the field-field correlation $\langle \phi(x,t)\phi(x,t)\rangle_0$, in addition to the response propagator $\hat{\mathbf{G}}_0^{11}$ that comes from the field-response field correlation $\langle \phi(x,t)\tilde{\phi}(x,t)\rangle_0$, as seen in (3.18) and (3.22). However if the noise is included as an interaction there will be only one propagator, the response propagator. The same diagrams and field loops will be formed, but as a result of perturbative expansion with a new vertex.

Having a correlation propagator as well as a response propagator is typical for equilibrium critical dynamics (as discussed by Uwe Täuber in Chapter 4 of (Täuber, 2005)), in relaxation models where there is just one interaction parameter to be expanded. However having a single response propagator is more general; it is easier to explicitly construct the diagrams and follow the momentum flows, and easier to extend to other situations beyond Langevin equations. The diagrams are not as attractive, but I found that it felt more natural when constructing them, especially as the correlation propagator is equivalently constructed with, and in any case renormalised by, the response propagator.

For these reasons the field theory will be assembled using one propagator, with the noise forming part of the interaction. Using the Langevin equation (4.1), with a non-conserved Gaussian white noise, $\eta(x,t)$, given by

$$\langle \eta(x,t)\eta(x',t')\rangle = 2\Gamma^2 \delta^d(x-x')\delta(t-t'), \tag{4.5}$$

with the noise strength, Γ, a real number, not an operator.[1] The bilinear/harmonic part of the action is

$$\mathcal{A}_0[\phi,\tilde{\phi}] = \int d^dx \int dt\ \tilde{\phi}(x,t)\left[\partial_t - \nu_2\nabla^2 + \nu_4\nabla^4\right]\phi(x,t). \tag{4.6a}$$

Moving the noise to the anharmonic part of the action, this is

$$\mathcal{A}_{\text{int}}\left[\phi,\tilde{\phi}\right] = \int d^dx \int dt\ \tilde{\phi}(x,t)\left[-\lambda_{13}\nabla\left(\nabla\phi(x,t)\right)^3 - \lambda_{22}\nabla^2\left(\nabla\phi(x,t)\right)^2 - \Gamma^2\tilde{\phi}(x,t)\right]. \tag{4.6b}$$

Fourier transforming, the harmonic part becomes[2]

$$\mathcal{A}_0[\phi,\tilde{\phi}] = \int đ^dq \int đ\omega\ \tilde{\phi}^\dagger(q,\omega)\left(-i\omega + \nu_2 q^2 + \nu_4 q^4\right)\phi(q,\omega). \tag{4.7}$$

Splitting up the anharmonic part into the different interaction terms, and using the condensed notation introduced previously in Section 2.4, gives

$$\mathcal{A}_{13}[\phi,\tilde{\phi}] = -\lambda_{13}\int đ^dq_{123}\int đ\omega_{123}\ \tilde{\phi}^\dagger_{1+2+3}\ \left(q_3\cdot(q_1+q_2+q_3)\right)(q_1\cdot q_2)\ \phi_1\phi_2\phi_3, \tag{4.8a}$$

[1]See Section 5.1.5 for an additional comment on this.

[2]See comment at the end of Section 3.2. If $\tilde{\phi}(x,t)$ is interpreted as a real field, the dagger is a legitimate notation. If the factors of 'i' are properly tracked and $\tilde{\phi}(x,t)$ is taken to be a purely imaginary field, then a complex conjugate cannot be understood to have been taken as it would give an extra minus. Fourier transforming and integrating away the delta function gives the argument of the response field $\tilde{\phi}(q',\omega')$ as minus the sum of the arguments of the fields: $q' = -\sum q$, $\omega' = -\sum\omega$. This is the proper interpretation of these expressions, the dagger has been used cosmetically for space and should not affect the calculations.

$$\mathcal{A}_{22}[\phi,\tilde{\phi}] = -\lambda_{22} \int đ^d q_{12} \int đ\omega_{12}\ \tilde{\phi}^{\dagger}_{1+2}\ (q_1 \cdot q_2)(q_1+q_2)^2\ \phi_1\phi_2, \tag{4.8b}$$

$$\mathcal{A}_{\eta}[\phi,\tilde{\phi}] = -\Gamma^2 \int đ^d q \int đ\omega\ \tilde{\phi}^{\dagger}(q,\omega)\tilde{\phi}(q,\omega). \tag{4.8c}$$

So the moment generating functional is

$$\begin{aligned}
\mathcal{Z}[j,\tilde{j}] &= \int \mathscr{D}\phi\mathscr{D}\tilde{\phi}\ \exp\left\{-\mathcal{A}[\phi,\tilde{\phi}]\right\} \\
&\quad \cdot \exp\left\{\int d^d x \int dt\ \tilde{j}(x,t)\tilde{\phi}(x,t) + j(x,t)\phi(x,t)\right\} \\
&= \int \mathscr{D}\phi\mathscr{D}\tilde{\phi}\ \exp\left\{-\int đ^d q \int đ\omega\ \tilde{\phi}^{\dagger}(q,\omega)\left(-i\omega+\nu_2 q^2+\nu_4 q^4\right)\phi(q,\omega)\right\} \\
&\quad \cdot \exp\left\{\lambda_{13}\int đ^d q_{123}\int đ\omega_{123}\ \tilde{\phi}^{\dagger}_{1+2+3}\left(q_3\cdot(q_1+q_2+q_3)\right)(q_1\cdot q_2)\phi_1\phi_2\phi_3\right\} \\
&\quad \cdot \exp\left\{\lambda_{22}\int đ^d q_{12}\int đ\omega_{12}\ \tilde{\phi}^{\dagger}_{1+2}\ (q_1\cdot q_2)(q_1+q_2)^2\ \phi_1\phi_2\right\} \\
&\quad \cdot \exp\left\{\Gamma^2\int đ^d q\int đ\omega\ \tilde{\phi}^{\dagger}(q,\omega)\tilde{\phi}(q,\omega)\right\} \\
&\quad \cdot \exp\left\{\int đ^d q\int đ\omega\ \tilde{j}^{\dagger}(q,\omega)\tilde{\phi}(q,\omega) + j^{\dagger}(q,\omega)\phi(q,\omega)\right\}.
\end{aligned} \tag{4.9}$$

Note the formation of the field theory tacitly assumes that the growth process has been running long enough for this to make sense, that the field $\phi(x,t)$ can be considered to run along $\pm\infty$.

The equivalence of having one propagator and an extra vertex to having two propagators can be seen by examining the tree level contribution to the field-field correlator, found by expanding the noise vertex to first order and comparing to the result for the bare correlator in (3.18):

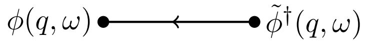
$$\begin{aligned}
\left\langle \phi(k_1,\omega_1)\phi(k_2,\omega_2)\right\rangle_{\text{tree}} &= 2\Gamma^2 \int đ^d q \int đ\omega \left\langle \phi(k_1,\omega_1)\tilde{\phi}(q,\omega)\tilde{\phi}^{\dagger}(q,\omega)\phi(k_2,\omega_2)\right\rangle_0 \\
&= 2\Gamma^2 \int đ^d q \int đ\omega \left\langle \phi(k_1,\omega_1)\tilde{\phi}(q,\omega)\right\rangle_0 \left\langle \tilde{\phi}^{\dagger}(q,\omega)\phi(k_2,\omega_2)\right\rangle_0 \\
&= 2\Gamma^2 \int đ^d q \int đ\omega\ \hat{\mathbf{G}}_0^{11}(k_1,\omega_1)\delta^d(k_1+q)\delta(\omega_1+\omega)\cdot\hat{\mathbf{G}}_0^{11}(k_2,\omega_2)\delta^d(k_2-q)\delta(\omega_2-\omega) \\
&= 2\Gamma^2\left(\hat{\mathbf{G}}_0^{11}(k_1,\omega_1)\right)^2 \delta^d(k_1+k_2)\delta(\omega_1+\omega_2) \\
&= \frac{2\Gamma^2}{\omega_1^2+\mathbf{f}(k_1)}\delta^d(k_1+k_2)\delta(\omega_1+\omega_2).
\end{aligned} \tag{4.10}$$

4.2 Perturbation Components

It is possible to read off the bare propagator from the bilinear part of (4.9) as

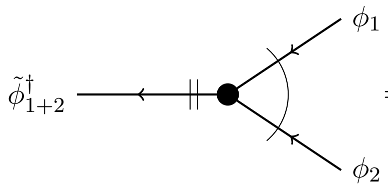
$$\hat{\mathbf{G}}_0^{11}(q,\omega) = \frac{1}{-i\omega + \nu_2 q^2 + \nu_4 q^4}, \tag{4.11}$$

which is given the diagrammatic representation:

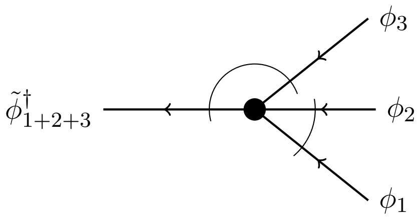


$$= \hat{\mathbf{G}}_0^{11}(q,\omega). \tag{4.12}$$

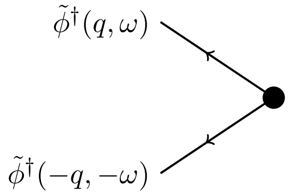
The amputated, perturbative components from the expansion of the interaction exponentials in (4.9) are given the diagrammatic representation:



$$= \lambda_{22}(q_1 \cdot q_2)(q_1 + q_2)^2, \tag{4.13a}$$



$$= \lambda_{13}\left(q_3 \cdot (q_1 + q_2 + q_3)\right)(q_1 \cdot q_2), \tag{4.13b}$$



$$= 2\Gamma^2. \tag{4.13c}$$

The field and response field labels on the legs of the interaction vertices have been included schematically, to indicate the required inputs or attachment protocols for use with the propagators. The bars on, and arcs between, the legs denote multiplication by the dot product of the momentum carried by the fields at those points. The factor two with noise vertex has been included as a definition, though it follows in fact from the symmetry factor, that from reversing the noise vertex, rather than the exponential. Including the factor two with the vertex rather than the symmetry factor eases the comparison with the two propagator version and seemed more natural to deal with. The delta functions have been dropped explicitly, and are accounted for by the proscribed momentum flow seen in the vertices. The diagrams can be built rather like plumbing, the momentum flows causally from right to left. From response fields on the right hand side, through to the fields on the left hand edge. The noise vertex can then be interpreted as the source of the internal momentum and frequency, as it joins the branches of a tree structure, closing them off to form the loops that are be integrated over. The causal flow is indicated by the arrow on the propagator, which is also useful to keep track of its sense when integrating.

4.3 Dimensional Analysis

As a check that all the relevant couplings have been included in the Langevin equation (4.1) a quick dimensional analysis can be performed. Indeed this is required to find out which, if any, of the terms in (4.1) are relevant, and under what conditions it is a renormalisable theory. The noise correlator (4.5) provides the handle to analyse the other terms, giving the dimension of the noise as

$$[\eta(x,t)] = [\Gamma] L^{-d/2} T^{-1/2}. \tag{4.14a}$$

Assigning a dimension to the noise strength, $[\Gamma] = B$, and proceeding by inspection to evaluate the dimension of the fields and the various couplings:

$$\begin{aligned} [\phi(x,t)] &= B L^{-d/2} T^{1/2} & [\phi(q,\omega)] &= B L^{d/2} T^{3/2} \\ [\tilde{\phi}(x,t)] &= B^{-1} L^{-d/2} T^{-1/2} & [\tilde{\phi}(q,\omega)] &= B^{-1} L^{d/2} T^{1/2} \\ [\nu_2] &= L^2 T^{-1} & [\nu_4] &= L^4 T^{-1} \\ [\lambda_{22}] &= B^{-1} L^{4+d/2} T^{-3/2} & [\lambda_{13}] &= B^{-2} L^{4+d} T^{-2}. \end{aligned} \tag{4.14b}$$

The dimension of the vertex functions can then be analysed by proceeding from a schematic version of the identity for the connected correlation functions in momentum-frequency space:

$$\begin{aligned} \hat{\mathbf{G}}^{nm} \bar{\delta}^{d+1} &= \langle \phi_1 \cdots \phi_n \tilde{\phi}_1 \cdots \tilde{\phi}_m \rangle, \\ [\hat{\Gamma}^{nm}] &= \frac{[\hat{\mathbf{G}}^{nm}]}{[\hat{\mathbf{G}}^{11}]^{n+m}} \\ &= \frac{[\phi]^n [\tilde{\phi}]^m}{[\phi\tilde{\phi}]^{n+m}} [\bar{\delta}^{d+1}]^{n+m-1} \\ &= \frac{B^{n-m} L^{d/2(n+m)} T^{3n/2+m/2}}{(L^d T^2)^{n+m}} L^{d(n+m)-d} T^{n+m-1}, \\ [\hat{\Gamma}^{nm}] &= B^{n-m} L^{d/2(n+m)-d} T^{n/2-m/2-1}. \end{aligned} \tag{4.14c}$$

It is possible to see how the vertex functions behave and scale, as introduced in Section 2.5, when time is eliminated by assigning an independent dimension to one of the couplings in the propagator. The choice of which coupling is made independent, and substituted to eliminate time, is in the same spirit as choosing which fixed point of the system to probe, or tuning the critical exponent η to select the desired one to renormalise to.

4.3.1 Independent ν_2

Assigning ν_2 an independent dimension, $A = [\nu_2]$, the dimension of time is re-expressed as: $T = L^2[A]^{-1}$. The dimensions of the couplings in the Langevin equation are:

coupling	ν_2	ν_4	λ_{22}	λ_{13}
coupling dimension	A	AL^2	$A^{3/2}B^{-1}L^{d/2+1}$	$A^2B^{-2}L^d$

For a free ν_2 coupling, all the other couplings must be scaled to zero in the large wavelength/small momentum limit. That is to avoid divergence they must be set to zero initially, or if formed into dimensionless couplings the momentum scale used would scale the dimensionless couplings to zero in the small momentum limit. The system is ultimately governed by a very simple Langevin equation, containing only the ν_2 coupling as a relevant parameter. All the other couplings in (4.1) are driven to zero. Looking at the dimension of different vertex functions in this scenario:

$$[\hat{\Gamma}^{nm}] = A^{1-(n-m)/2}B^{n-m}L^{n(d/2+1)+m(d/2-1)-(d+2)}, \tag{4.15a}$$

$$\begin{aligned} &[\hat{\Gamma}^{20}] \sim L^0 \\ &[\hat{\Gamma}^{11}] \sim L^{-2} \\ &[\hat{\Gamma}^{12}] \sim L^{d/2-3} \to L^{d/2} \\ &[\hat{\Gamma}^{13}] \sim L^{d-4} \to L^d. \end{aligned} \tag{4.15b}$$

The dimension in the last two lines has been adjusted to take account of the diagrammatic form of this theory. As all constructions for Γ^{12} and Γ^{13} have at least one momentum bar on each of the external legs, the dimension must be adjusted accordingly to see the underlying behaviour of the coupling/vertex. The same will be true of any higher order vertex, as it will be built from these components. It is plain that

$$\begin{aligned} \hat{\Gamma}^{1\,m>1} &\sim L^{+ve}, \\ \hat{\Gamma}^{n\geq 2\,m>0} &\sim L^{+ve}. \end{aligned} \tag{4.15c}$$

Giving only two vertices that need to be taken care of, the propagator and the marginal noise. Indeed there is no need to take care of these divergences, as such a linear theory is exactly solvable, and so renormalisation is perhaps not a germane point.

4.3.2 Independent ν_4

The independent dimension is reassigned to ν_4, $A = [\nu_4]$, repeating the exercise above with the time dimension re-expressed using $T = L^4[A]^{-1}$:

coupling	ν_2	ν_4	λ_{22}	λ_{13}
coupling dimension	AL^{-2}	A	$A^{3/2}B^{-1}L^{d/2-2}$	$A^2B^{-2}L^{d-4}$

Showing that $\nu_2 = 0$ needs to be imposed at the infra-red fixed point, or the coupling will diverge in the large wavelength/small momentum limit. As well as a critical point for ν_2, there is also a critical dimension, $d_c = 4$, where the couplings become independent of length scale. Looking at the dimension of the vertex functions:

$$[\hat{\Gamma}^{nm}] = A^{1-(n-m)/2}B^{n-m}L^{n(d/2+2)+m(d/2-2)-(d+4)}, \tag{4.16a}$$

$$\begin{aligned} &[\hat{\Gamma}^{20}] \sim L^0 \\ &[\hat{\Gamma}^{11}] \sim L^{-4} \rightarrow L^{-2} \\ &[\hat{\Gamma}^{12}] \sim L^{d/2-6} \rightarrow L^{d/2-2} \\ &[\hat{\Gamma}^{13}] \sim L^{d-8} \rightarrow L^{d-4}, \end{aligned} \tag{4.16b}$$

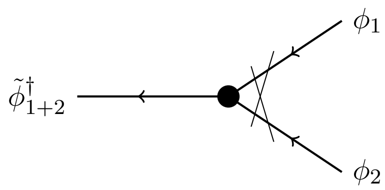
$$\begin{aligned} &[\hat{\Gamma}^{1\,m>3}] \rightarrow L^{+ve\ \ d>3.4}, \\ &[\hat{\Gamma}^{2\,m>0}] \rightarrow L^{+ve\ \ d>2}. \end{aligned} \tag{4.16c}$$

Firstly, there is serious trouble as the dimension is lowered. In the limit $d \rightarrow 2$ an infinite number of couplings become relevant, the theory becomes non-renormalisable. However, about $d = 4$ two marginally relevant vertices appear. They are of corresponding order to the two interaction couplings in equation (4.1). As before the dimensions have been adjusted to discount the momentum carried on the external legs. Note that the dimension of $\hat{\Gamma}^{12}$ has been adjusted by an extra power for symmetry reasons, this should also have been done in the previous section but was not necessary before the term became irrelevant. Not only does the λ_{22} vertex carry four powers of momentum in total on its legs, but the extra adjustment arises in principle as terms that have the order of an odd power of the momentum cannot contribute, as unlike the Langevin equation they are not invariant under the sign exchange of momentum, $k \to -k$.

From these simple power counting arguments it is seen that the theory is general very simple, being dominated by the coupling ν_2. However there is a critical point for this coupling, $\nu_2^c = 0$, where the theory is characterised by the coupling ν_4 and is renormalisable about the dimension $d_c = 4$. There is also a lower critical dimension $d_{lc} = 2$. It would seem at this point that the Langevin equation, as it appears in (4.1), has the correct order of form at this critical point.

4.4 A New Term: κ

Before continuing there is an issue best dealt with at this stage. There is not a problem with the set up of the field theory so far as such, but the dimensional analysis above is not the whole story. A new term is generated under renormalisation of the Langevin equation (4.1). This means that a new coupling had better be included in the Langevin equation from the start, the action must be adjusted accordingly, and a new perturbation component is required. This new coupling is κ, it has the same dimension and symmetry as λ_{22} and forms part of the vertex $\hat{\Gamma}^{12}$, so none of the previous considerations about power counting and so on are invalid. The extra perturbation component is:

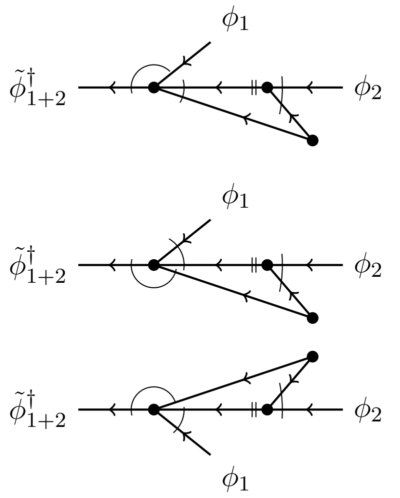


$$= \kappa(q_1 \wedge q_2)^2. \qquad (4.17)$$

Where the crossed lines indicate a cross product squared of the momentum carried by the fields. The adjusted, complete action for the theory is

$$
\begin{aligned}
\mathcal{A}_0[\phi,\tilde{\phi}] &= \int \bar{d}^d q \int \bar{d}\omega \; \tilde{\phi}^\dagger(q,\omega)\left(-i\omega + \nu_2 q^2 + \nu_4 q^4\right)\phi(q,\omega), \\
\mathcal{A}_{\text{int}}[\phi,\tilde{\phi}] &= -\lambda_{13}\int \bar{d}^d q_{123}\int \bar{d}\omega_{123}\; \tilde{\phi}^\dagger_{1+2+3}\left(q_3\cdot(q_1+q_2+q_3)\right)(q_2\cdot q_3)\phi_1\phi_2\phi_3 \\
&\quad -\lambda_{22}\int \bar{d}^d q_{12}\int \bar{d}\omega_{12}\; \tilde{\phi}^\dagger_{1+2}(q_1\cdot q_2)(q_1+q_2)^2\phi_1\phi_2 \\
&\quad -\kappa\int \bar{d}^d q_{12}\int \bar{d}\omega_{12}\; \tilde{\phi}^\dagger_{1+2}\left(q_1\wedge q_2\right)^2\phi_1\phi_2 \\
&\quad -\Gamma^2\int \bar{d}^d q\int \bar{d}\omega\; \tilde{\phi}^\dagger(q,\omega)\tilde{\phi}(q,\omega).
\end{aligned}
\tag{4.18}
$$

The generation of κ under renormalisation, were it not included from the start, arises from a diagram containing an interaction between the λ_{22} and λ_{13} couplings. The diagram in question is:



(4.19)

This diagram will be fully calculated later, for now let it suffice to say these three arrangements produces a term in which the external momentum goes as

$$
q_1^2\left(q_2\cdot(q_1+q_2)\right) + 2\left(q_1\cdot q_2\right)\left(q_1\cdot(q_1+q_2)\right). \tag{4.20}
$$

After symmetrising the arguments, by adding the same term with the external field inputs reversed and dividing by two, this becomes

$$
\begin{aligned}
& q_1^2 q_2^2 + \frac{1}{2}(q_1^2+q_2^2)(q_1\cdot q_2) + (q_1^2+q_2^2)(q_1\cdot q_2) + 2(q_1\cdot q_2)^2 \\
&= q_1^2 q_2^2 - (q_1\cdot q_2)^2 + \frac{3}{2}(q_1^2+q_2^2)(q_1\cdot q_2) + 3(q_1\cdot q_2)^2 \\
&= q_1^2 q_2^2 - (q_1\cdot q_2)^2 + \frac{3}{2}(q_1+q_2)^2(q_1\cdot q_2) \\
&= (q_1\wedge q_2)^2 + \frac{3}{2}(q_1+q_2)^2(q_1\cdot q_2),
\end{aligned}
\tag{4.21}
$$

containing a contribution to λ_{22} and some more terms that cannot be recapitulated into the form of a pre-existing coupling. Re-arranging to the most compact and easy to use form gives rise to the term that then needs to be included in the action as κ. Examining the terms that initially appear after symmetrising that do not immediately form part of λ_{22} more closely,

$$q_1^2 q_2^2 + \frac{1}{2}(q_1^2 + q_2^2)(q_1 \cdot q_2) = \frac{1}{2}(q_1 + q_2)\left[q_1 q_2^2 + q_2 q_1^2\right]. \tag{4.22}$$

This can be interpreted as the momentum signature from the Fourier transform of the real space terms

$$\kappa_{22}\left(\nabla^2\phi(x,t)\right)^2 + \kappa_{13}\nabla\phi(x,t)\cdot\nabla^3\phi(x,t) = \bar{\kappa}\nabla\left(\nabla\phi(x,t)\nabla^2\phi(x,t)\right), \tag{4.23}$$

where $\bar{\kappa} = \kappa_{22} = \kappa_{13}$. The κ designation has been chosen to be consistent with the notation in an unrelated surface growth model presented by Lazerides (Lazarides, 2006), where κ_{13} and κ_{22} appear in a one-dimensional form. Importantly the terms κ_{22} and κ_{13} do not need to be tracked individually. The model as formulated in (4.1) is conservative, as is the new term $\bar{\kappa}$. The hallmark of conservation is that it is possible to pull out a $(q_1 + q_2)\cdot$ or $\nabla\cdot$ in front of the term as a whole. The full Langevin equation to be analysed would then be

$$\begin{aligned}\partial_t\phi(x,t) = \nu_2\nabla^2\phi(x,t) - \nu_4\nabla^4\phi(x,t) + \lambda_{13}\nabla\left(\nabla\phi(x,t)\right)^3 \\ + \bar{\lambda}_{22}\nabla^2\left(\nabla\phi(x,t)\right)^2 + \bar{\kappa}\nabla\left(\nabla\phi(x,t)\nabla^2\phi(x,t)\right) + \eta(x,t).\end{aligned} \tag{4.24}$$

However, this real-space form form is awkward for calculations. Re-writing two of the couplings in a computationally convenient form using $\bar{\lambda}_{22} = \lambda_{22} - \kappa/2$ and $\bar{\kappa} = \kappa$,

$$\begin{aligned}\bar{\lambda}_{22}\nabla^2\left(\nabla\phi(x,t)\right)^2 &+ \bar{\kappa}\nabla\left(\nabla\phi(x,t)\nabla^2\phi(x,t)\right) \\ &= (\lambda_{22} - \frac{\kappa}{2})\nabla^2\Big(\nabla\phi\Big)^2 + \kappa\nabla\left(\nabla\phi\nabla^2\phi\right) \\ &= \lambda_{22}\nabla^2\Big(\nabla\phi\Big)^2 + \kappa\left[\nabla\left(\nabla\phi\nabla^2\phi\right) - \frac{1}{2}\nabla^2\Big(\nabla\phi\Big)^2\right].\end{aligned} \tag{4.25}$$

Giving the full theory for conserved surface growth as

$$\begin{aligned}\partial_t\phi(x,t) = \nu_2\nabla^2\phi(x,t) - \nu_4\nabla^4\phi(x,t) + \lambda_{13}\nabla\left(\nabla\phi(x,t)\right)^3 \\ + \lambda_{22}\nabla^2\Big(\nabla\phi\Big)^2 + \kappa\left[\nabla\left(\nabla\phi\nabla^2\phi\right) - \frac{1}{2}\nabla^2\Big(\nabla\phi\Big)^2\right] + \eta(x,t),\end{aligned} \tag{4.26a}$$

with uncorrelated Gaussian white noise

$$\langle\eta(x,t)\eta(x',t')\rangle = 2\Gamma^2\delta^d(x-x')\delta(t-t'). \tag{4.26b}$$

Which gives the action for the full theory as written in (4.18). As can be seen by the various rearrangements made above, the distinction between the terms that are part of the vertex $\hat{\Gamma}^{12}$ are much more fluid than it initially appears in real-space. In any case it is much simpler to have one diagram for $(q_1 \wedge q_2)^2$, to use $q_1^2 q_2^2 - (q_1 \cdot q_2)^2$ when performing the integrals, and call this term κ.

Chapter 5

One Loop Analysis

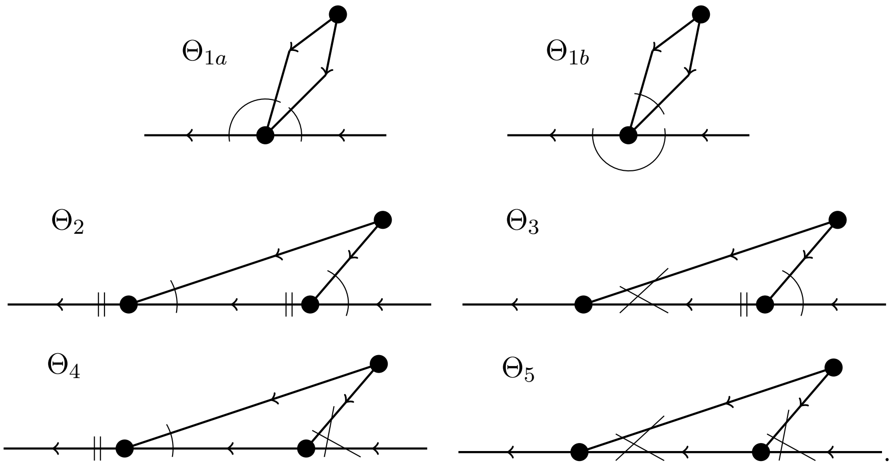
As seen from power counting, the behaviour of the full theory, that of the Langevin equation (4.26), is generically dominated by the ν_2 coupling. There is a critical point, ν_2^c, where the behaviour of the theory is driven by ν_4 for dimensions above the upper critical dimension, $d_c = 4$. For dimensions less than the upper critical one, $d = d_c - \epsilon < d_c$, the non-linearities λ_{13}, λ_{22} and κ are relevant and produce non-trivial scaling behaviour, requiring renormalisation. The renormalisation of vertex functions Γ^{11}, Γ^{12} and Γ^{13} is carried out here using a minimal subtraction scheme, at zero external frequency, to capture their logarithmic divergences. The ultraviolet is dimensionally regularised in a perturbation theory in $\epsilon = d_c - d > 0$. The infrared is regularised in effect by using ν_2 as a mass, imposing $\nu_2 \neq 0$, then using the renormalisation point $\bar{\nu}_2 = 1$ for the dimensionless, renormalised ν_2, and introducing an arbitrary inverse length μ.

In this chapter all the relevant, logarithmically divergent diagrams are listed, along with their contributions and symmetry factors. The symmetry factors do not include the factor two from reversal of the noise, or any factor from permutations of external fields. A comparison is made, in as far as it is possible, with a previous one loop calculation in the literature that was undertaken using a different method. The details of the fixed points of the renormalisation scheme with the associated exponents are given, and the different possible regimes and limits of the full theory (4.26) are explored. The algebraic details of the calculations have been deferred to the appendices, the diagram integrals are to be found in Appendix B, and the β-functions, γ-functions, and fixed points in Appendix C.

5.1 The Diagrams

5.1.1 The $\hat{\Gamma}^{11}(q, 0)$ vertex

There are five distinct diagrams contributing to the renormalisation of $\hat{\Gamma}^{11}(q, 0)$ at one loop, labeled Θ_n. However, because of the different ways the propagators can be connected to the vertices, Θ_1 has two phenotypes, $1a$ and $1b$. This notation will continue to be used to denote the phenotypes of a particular arrangement of vertices. These diagrams can be legitimately referred to as the one loop self-energy contributions, $\Sigma(q, 0)$:



(5.1)

Diagram	Symmetry Factor (S F)	Contribution
Θ_1	$\{3\}$	$\frac{3}{2}\lambda_{13}\mathcal{C}\nu_2 q^2$
Θ_{1a}	2	$\frac{1}{2}\lambda_{13}\mathcal{C}\nu_2 q^2$
Θ_{1b}	1	$2\,\Theta_{1a}$
Θ_2	4	$-\frac{1}{2}\frac{\lambda_{22}^2\mathcal{C}}{\nu_4}\nu_4 q^4$
Θ_3	4	$-\frac{3}{2}\frac{\lambda_{22}\kappa\mathcal{C}}{\nu_4}\nu_4 q^4$
Θ_4	4	0
Θ_5	4	$\frac{5}{4}\frac{\kappa^2\mathcal{C}}{\nu_4}\nu_4 q^4$

The total contribution includes multiplication by the symmetry factor, with

$$\mathcal{C} = \frac{\Gamma^2}{(4\pi)^2\nu_4^2}\left(\frac{\nu_4}{\nu_2}\right)^{\frac{\epsilon}{2}}\Gamma\left(\frac{\epsilon}{2}\right). \tag{5.2}$$

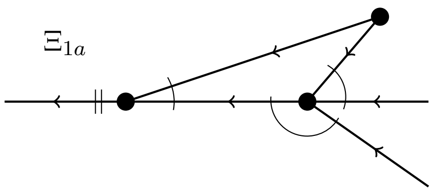
Some sleight of hand has taken place in the contribution for Θ_{1a} and Θ_{1b}, where the gamma function has been expanded using $\Gamma(n+1) = n\Gamma(n)$, in order to express it in a form that uses $\mathcal{C}$. This is symptomatic of the additive renormalisation that this diagram is responsible for that has been swept under the carpet. The Θ_4 diagram has no divergent contribution, but has been included in the list as it would initially be expected to contribute, and needs to be calculated before it can be discarded. The vertex function, with tree level and one loop contribution, can then be written as

$$\begin{aligned}
\hat{\Gamma}^{11}(q,0) &= \frac{1}{\hat{\mathbf{G}}^{11}(q,0)} \\
&= \nu_2 q^2 + \nu_4 q^4 - \Sigma(q,0) \\
&= \nu_2 q^2 + \nu_4 q^4 + \frac{3}{2}\frac{\lambda_{13}}{\frac{\epsilon}{2}-1}\frac{\Gamma^2\nu_2}{(4\pi)^2\nu_4^2}\left(\frac{\nu_4}{\nu_2}\right)^{\frac{\epsilon}{2}}\Gamma\left(\frac{\epsilon}{2}\right)q^2 \\
&\qquad + \left[\frac{\lambda_{22}^2}{2\nu_4} + \frac{3\lambda_{22}\kappa}{2\nu_4} - \frac{5\kappa_\wedge^2}{4\nu_4}\right]\frac{\Gamma^2}{(4\pi)^2\nu_4^2}\left(\frac{\nu_4}{\nu_2}\right)^{\frac{\epsilon}{2}}\Gamma\left(\frac{\epsilon}{2}\right)q^4 \\
&= \nu_2 q^2 + \nu_4 q^4 - \frac{3}{2}\lambda_{13}\mathcal{C}\nu_2 q^2 + \left[\frac{\lambda_{22}^2}{2\nu_4} + \frac{3\lambda_{22}\kappa}{2\nu_4} - \frac{5\kappa^2}{4\nu_4}\right]\mathcal{C}\nu_4 q^4, \\
&= \nu_2^R q^2 + \nu_4^R q^4.
\end{aligned} \tag{5.3}$$

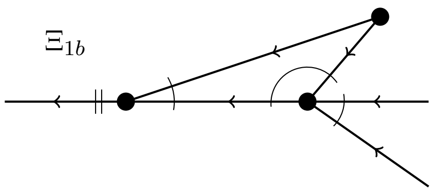
5.1.2 The $\hat{\Gamma}^{12}(q_1, q_2)$ vertex

For $\hat{\Gamma}^{12}(q_1, q_2)$, with inputs from $\tilde{\phi}(q_1, 0)$ and $\tilde{\phi}(q_2, 0)$, there are eight relevant one loop diagrams labelled Ξ_n, albeit with the different pairings of fields associated with the λ_{13} coupling leading to a number of phenotypes:

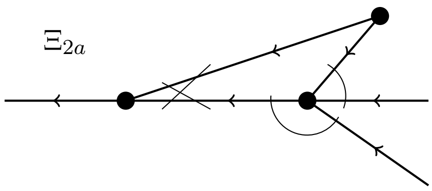
Ξ_{1a}



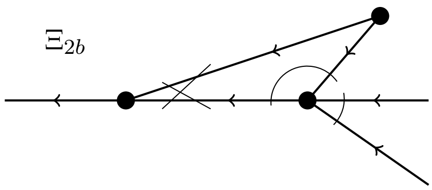
Ξ_{1b}

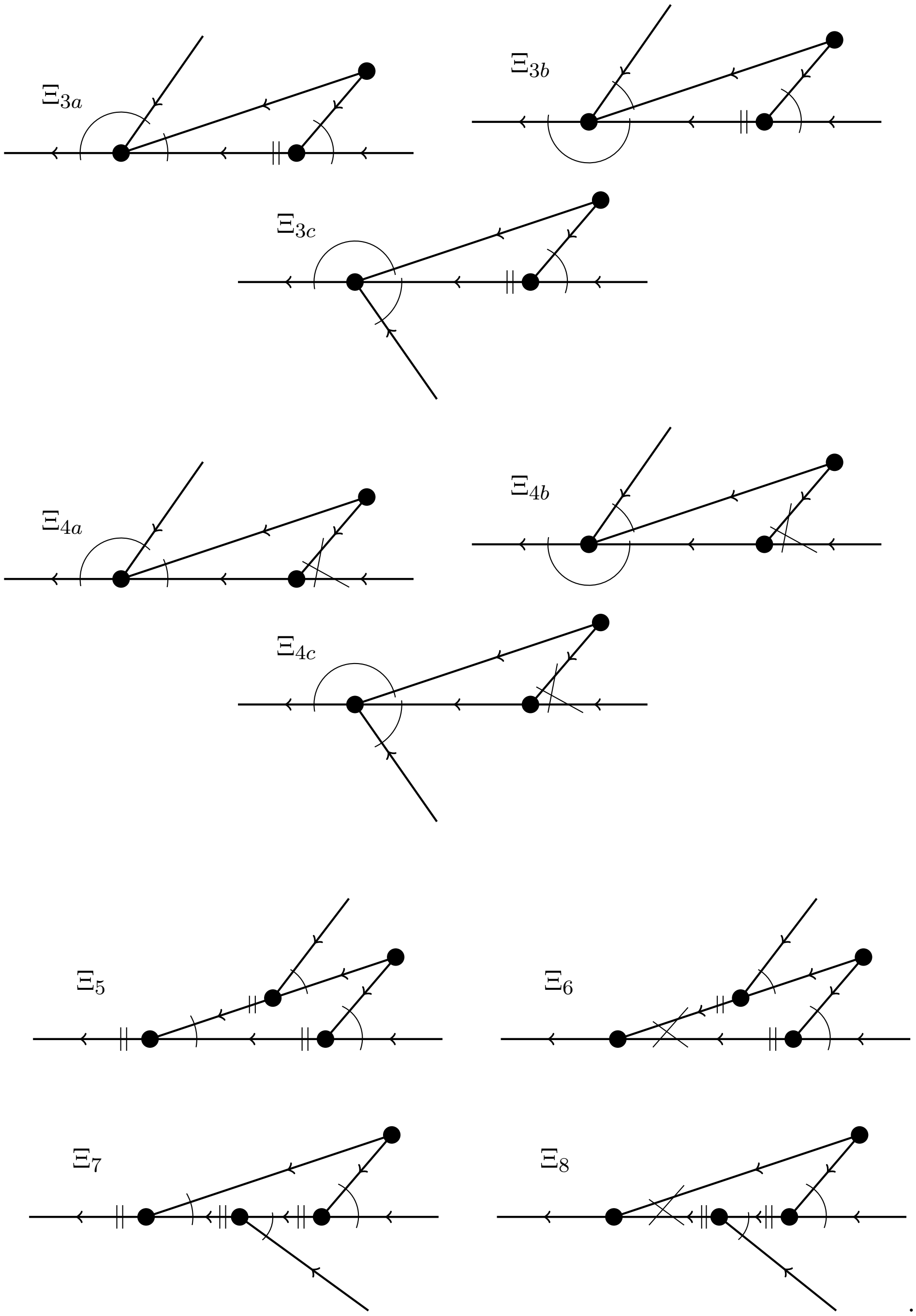


Ξ_{2a}



Ξ_{2b}





(5.4)

Diagram	S F	Contribution
Ξ_1	$\{6\}$	$-\frac{3}{2}\lambda_{13}\mathcal{C}\lambda_{22}(q_1\cdot q_2)(q_1+q_2)^2$
Ξ_{1a}	4	$-\frac{1}{2}\lambda_{13}\mathcal{C}\lambda_{22}(q_1\cdot q_2)(q_1+q_2)^2$
Ξ_{1b}	2	$2\,\Xi_{1a}$
Ξ_2	$\{6\}$	$\frac{\kappa\lambda_{13}\mathcal{C}}{\lambda_{22}}\lambda_{22}(q_1\cdot q_2)(q_1+q_2)^2-\frac{1}{6}\lambda_{13}\mathcal{C}\kappa(q_1\wedge q_2)^2$
Ξ_{2a}	4	$\frac{1}{4}\frac{\kappa\lambda_{13}\mathcal{C}}{\lambda_{22}}\lambda_{22}(q_1\cdot q_2)(q_1+q_2)^2-\frac{1}{6}\lambda_{13}\mathcal{C}\kappa(q_1\wedge q_2)^2$
Ξ_{2b}	2	$\frac{3}{4}\frac{\lambda_{13}\kappa\mathcal{C}}{\lambda_{22}}\lambda_{22}(q_1\cdot q_2)(q_1+q_2)^2$
Ξ_3	$\{12\}$	$-\frac{3}{4}\lambda_{13}\mathcal{C}\lambda_{22}(q_1\cdot q_2)(q_1+q_2)^2-\frac{1}{2}\frac{\lambda_{22}\lambda_{13}\mathcal{C}}{\kappa}\kappa(q_1\wedge q_2)^2$
Ξ_{3a}	4	$-\frac{1}{4}\lambda_{13}\mathcal{C}\lambda_{22}(q_1\cdot q_2)(q_1+q_2)^2-\frac{1}{2}\frac{\lambda_{22}\lambda_{13}\mathcal{C}}{\kappa}\kappa(q_1\wedge q_2)^2$
Ξ_{3b}	4	$-\frac{1}{4}\lambda_{13}\mathcal{C}\lambda_{22}(q_1\cdot q_2)(q_1+q_2)^2$
Ξ_{3c}	4	Ξ_{3b}
Ξ_4	$\{12\}$	$-\frac{\kappa\lambda_{13}\mathcal{C}}{\lambda_{22}}\lambda_{22}(q_1\cdot q_2)(q_1+q_2)^2-\frac{7}{3}\lambda_{13}\mathcal{C}\kappa(q_1\wedge q_2)^2$
Ξ_{4a}	4	$-\frac{3}{4}\frac{\kappa\lambda_{13}\mathcal{C}}{\lambda_{22}}\lambda_{22}(q_1\cdot q_2)(q_1+q_2)^2-\frac{3}{2}\lambda_{13}\mathcal{C}\kappa(q_1\wedge q_2)^2$
Ξ_{4b}	4	$-\frac{1}{8}\frac{\kappa\lambda_{13}\mathcal{C}}{\lambda_{22}}\lambda_{22}(q_1\cdot q_2)(q_1+q_2)^2-\frac{5}{12}\lambda_{13}\mathcal{C}\kappa(q_1\wedge q_2)^2$
Ξ_{4c}	4	Ξ_{4b}
Ξ_5	4	$\frac{1}{2}\frac{\lambda_{22}^2\mathcal{C}}{\nu_4}\lambda_{22}(q_1\cdot q_2)(q_1+q_2)^2$
Ξ_6	8	$-\Xi_5$
Ξ_7	4	$-\frac{1}{4}\frac{\lambda_{22}\kappa\mathcal{C}}{\nu_4}\lambda_{22}(q_1\cdot q_2)(q_1+q_2)^2+\frac{1}{6}\frac{\lambda_{22}^2\mathcal{C}}{\nu_4}\kappa(q_1\wedge q_2)^2$
Ξ_8	8	$-\Xi_7$

Half of the diagrams cancel each other, $\Xi_5+\Xi_6=0$ and $\Xi_7+\Xi_8=0$, so there are in effect only four contributing diagrams. This large scale cancellation can be seen to occur when it is possible to rearrange a diagram between two forms, where the noise either feeds symmetrically into the two vertices receiving in response field inputs, or asymmetrically where the noise connects directly to the output vertex. This leads to a sign change between the two integrals, with otherwise identical contributions. The equivalence between the b and c phenotypes in the Ξ_3 and Ξ_4 diagrams is only established after isolating their divergent contributions, so their identities are kept distinct. The Ξ_3 diagram is the interaction responsible for generating the coupling κ, were it not to be included initially.

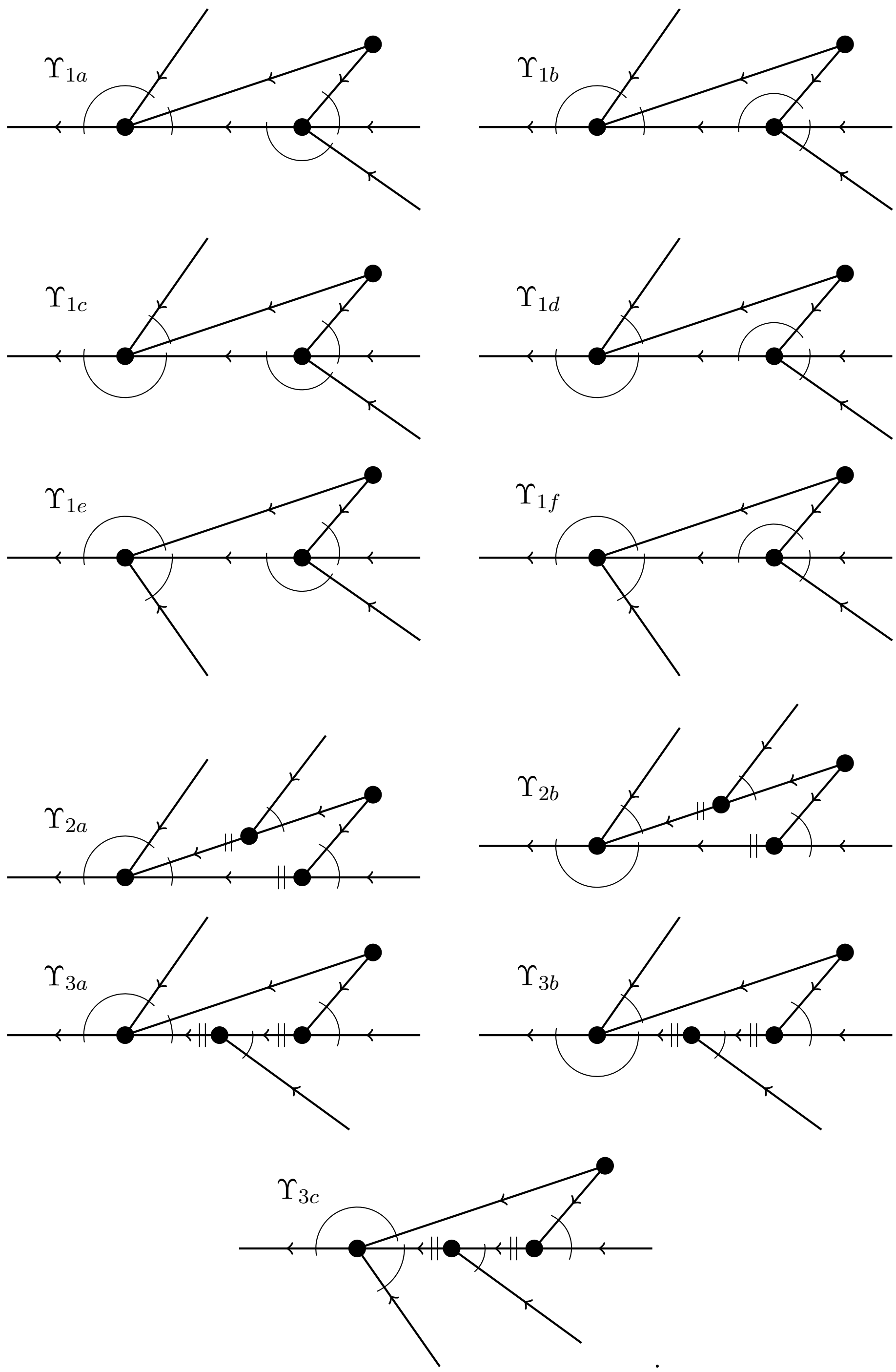
To one loop the vertex function is

$$
\begin{aligned}
\frac{\hat{\Gamma}^{12}(q_1,q_2)}{2} &= \frac{-\hat{\mathbf{G}}^{12}(q_1,q_2)}{\hat{\mathbf{G}}^{11}(-q_1)\hat{\mathbf{G}}^{11}(-q_2)\hat{\mathbf{G}}^{11}(-q_1-q_2)} \\
&= -\lambda_{22}(q_1\cdot q_2)(q_1+q_2)^2 - \kappa(q_1\wedge q_2)^2 \\
&\qquad + \frac{9}{4}\lambda_{13}\mathcal{C}\lambda_{22}(q_1\cdot q_2)(q_1+q_2)^2 + \frac{1}{2}\left(5+\frac{\lambda_{22}}{\kappa}\right)\lambda_{13}\mathcal{C}\kappa(q_1\wedge q_2)^2 \\
&= -\lambda_{22}^R(q_1\cdot q_2)(q_1+q_2)^2 - \kappa_\wedge^R(q_1\wedge q_2)^2,
\end{aligned} \tag{5.5}
$$

where the permutation of the response field inputs has been divided out, as can be seen by considering the tree level contributions.

5.1.3 The $\hat{\Gamma}^{13}(q_1, q_2, q_3)$ vertex

For $\hat{\Gamma}^{13}(q_1, q_2, q_3)$ there are three relevant diagrams, Υ_1, Υ_2 and Υ_3, of which the last two cancel via the same noise symmetric/asymmetric rearrangement mechanism seen in $\hat{\Gamma}^{12}$. This is compensated by a plethora of phenotypes:



(5.6)

Diagram	S F	Contribution
Υ_1	$\{18\}$	$-\frac{5}{2}\mathcal{C}\lambda_{13}^2\left\{\begin{array}{r}(q_1\cdot q_2)(q_3\cdot(q_1+q_2+q_3))\\+(q_1\cdot q_3)(q_2\cdot(q_1+q_2+q_3))\\+(q_2\cdot q_3)(q_1\cdot(q_1+q_2+q_3))\end{array}\right\}$
Υ_{1a}	4	$-\frac{1}{2}\mathcal{C}\lambda_{13}^2\left\{\begin{array}{r}(q_1\cdot q_2)(q_3\cdot(q_1+q_2+q_3))\\+(q_1\cdot q_3)(q_2\cdot(q_1+q_2+q_3))\\+(q_2\cdot q_3)(q_1\cdot(q_1+q_2+q_3))\end{array}\right\}$
Υ_{1b}	2	$2\,\Upsilon_{1a}$
Υ_{1c}	4	$\frac{1}{2}\Upsilon_{1a}$
Υ_{1d}	2	$\frac{1}{2}\Upsilon_{1a}$
Υ_{1e}	4	$\frac{1}{2}\Upsilon_{1a}$
Υ_{1f}	2	$\frac{1}{2}\Upsilon_{1a}$
Υ_2	$\{12\}$	$\frac{\lambda_{22}^2\mathcal{C}}{\nu_4}\lambda_{13}\left\{\begin{array}{r}(q_1\cdot q_2)(q_3\cdot(q_1+q_2+q_3))\\+(q_1\cdot q_3)(q_2\cdot(q_1+q_2+q_3))\\+(q_2\cdot q_3)(q_1\cdot(q_1+q_2+q_3))\end{array}\right\}$
Υ_{2a}	4	$\frac{1}{2}\frac{\lambda_{22}^2\mathcal{C}}{\nu_4}\lambda_{13}\left\{\begin{array}{r}(q_1\cdot q_2)(q_3\cdot(q_1+q_2+q_3))\\+(q_1\cdot q_3)(q_2\cdot(q_1+q_2+q_3))\\+(q_2\cdot q_3)(q_1\cdot(q_1+q_2+q_3))\end{array}\right\}$
Υ_{2b}	8	Υ_{2a}
Υ_3	$\{24\}$	$-\Upsilon_2$
Υ_{3a}	8	$-\Upsilon_{2a}$
Υ_{3b}	8	$-\frac{1}{2}\Upsilon_{2a}$
Υ_{3c}	8	$-\frac{1}{2}\Upsilon_{2a}$

With $\Upsilon_2 + \Upsilon_3 = 0$. The vertex function to one loop is

$$\begin{aligned}
\frac{\hat{\Gamma}^{13}(q_1,q_2,q_3)}{2} &= \frac{-\hat{\mathbf{G}}^{13}(q_1,q_2,q_3)}{\hat{\mathbf{G}}^{11}(-q_1)\hat{\mathbf{G}}^{11}(-q_2)\hat{\mathbf{G}}^{11}(-q_3)\hat{\mathbf{G}}^{11}(-q_1-q_2-q_3)} \\
&= \left(-\lambda_{13} + \frac{5}{2}\lambda_{13}^2\mathcal{C}\right)\left\{\begin{aligned} &(q_1\cdot q_2)(q_3\cdot(q_1+q_2+q_3)) \\ &+(q_1\cdot q_3)(q_2\cdot(q_1+q_2+q_3)) \\ &+(q_2\cdot q_3)(q_1\cdot(q_1+q_2+q_3)) \end{aligned}\right\} \\
&= -\lambda_{13}^R\left\{\begin{aligned} &(q_1\cdot q_2)(q_3\cdot(q_1+q_2+q_3)) \\ &+(q_1\cdot q_3)(q_2\cdot(q_1+q_2+q_3)) \\ &+(q_2\cdot q_3)(q_1\cdot(q_1+q_2+q_3)) \end{aligned}\right\}.
\end{aligned} \tag{5.7}$$

Which takes into account that any of the three response fields, $\tilde{\phi}(q_1,0)$, $\tilde{\phi}(q_2,0)$ or $\tilde{\phi}(q_3,0)$, could be the fore-most one, interacting with the field. In contrast, the permutation of the other two response fields is divided out.

5.1.4 One Loop Couplings

The one loop renormalised couplings, as implicitly defined above, are summarised:

$$\begin{aligned}
\nu_2^R &= \left(1 - \frac{3\lambda_{13}}{2}\mathcal{C}\right)\nu_2 \\
\nu_4^R &= \left(1 + \left[\frac{\lambda_{22}^2}{2\nu_4} + \frac{3\lambda_{22}\kappa}{2\nu_4} - \frac{5\kappa_\wedge^2}{4\nu_4}\right]\mathcal{C}\right)\nu_4 \\
\lambda_{22}^R &= \left(1 - \frac{9\lambda_{13}}{4}\mathcal{C}\right)\lambda_{22} \\
\kappa^R &= \left(1 - \left[5 + \frac{\lambda_{22}}{\kappa}\right]\frac{\lambda_{13}}{2}\mathcal{C}\right)\kappa \\
\lambda_{13}^R &= \left(1 - \frac{5\lambda_{13}}{2}\mathcal{C}\right)\lambda_{13}.
\end{aligned} \tag{5.8}$$

Note there is no renormalisation of the noise strength.

5.1.5 The DRG scheme of Vvedensky and Haselwandter

Presented in one of Haselwandter and Vvedensky's 2007 papers (Haselwandter and Vvedensky, 2007b), as well as in Haselwandter's thesis (Haselwandter, 2007), there is a detailed description of the one loop DRG analysis of equation (4.1). This should allow for some comparison between the two methods. A diagram by diagram comparison is not totally straight forward, as technical differences with the DRG, such making use of a cut off in momentum-space, cloud the issue. The 900lb gorilla in the room is of course the absence of κ from their analysis. However there is still some value and insight in conducting a comparison.

Haselwandter and Vvedensky appear to keep the noise symmetry factor separate, however it is not clear how permutations of the external momenta/fields are accounted for. The contributions of the diagrams include a power of the cut off Λ, an integral element dl rather than a gamma function, and terms

$$\tilde{\mathbf{C}} = \left.\frac{S_d}{(2\pi)^d}\right|_{d=4} D_F, \\ \nu = \nu_2 + \nu_4\Lambda^2, \qquad D_F = D_0 + D_2\Lambda^2 + D_4\Lambda^4. \tag{5.9}$$

Where $\tilde{\mathbf{C}}$ can be interpreted as being equivalent to $\frac{\Gamma^2}{(4\pi)^2}$, and powers of ν as ν_4. The term D_F appears in the same role as the noise strength, and arises from using the noise strength $L = D_0 + D_2\nabla^2 + D_4\nabla^4$. In one case an adjusted expression, $\tilde{\mathbf{C}}'$, uses:

$$D_S = \big((d-4)\nu - 2\nu_4\Lambda^2\big)D_0 + \big((d-2)\nu - 2\nu_4\Lambda^2\big)D_2\Lambda^2 + \big(d\nu - 2\nu_4\Lambda^2\big)D_4\Lambda^4. \tag{5.10}$$

The diagram notation used by Haselwandter and Vvedensky for $\hat{\Gamma}^{11}(q)$ is Φ_i, for the $\hat{\Gamma}^{12}(q_1, q_2)$ diagrams Γ_i, and for the $\hat{\Gamma}^{13}(q_1, q_2, q_3)$ diagrams Υ_i. The diagrams are listed here with their symmetry factor and contributions, with the external momenta divided out, along with the nearest equivalent from my analysis:

Diagram	S F	Contribution	Equivalent	S F
Φ_1	4	$\frac{1}{4}\frac{\lambda_{22}\tilde{\mathbf{C}}'}{\nu^3}\Lambda^d dl$	$\Theta_2 = -\frac{1}{2}\frac{\lambda_{22}^2\mathcal{C}}{\nu_4}\nu_4\ \left(q^4\right)$	4
Φ_2	2	$-\frac{1}{2}\frac{\lambda_{13}\tilde{\mathbf{C}}}{\nu}\Lambda^d dl$	$\Theta_{1b} = \frac{1}{2}\lambda_{13}\mathcal{C}\nu_2\ \left(q^2\right)$	2
Φ_3	1	$2\Phi_2$	$\Theta_{1a} = 2\Theta_{1b}$	1

Diagram	S F	Contribution	Equivalent	S F
Γ_1	4	$\frac{1}{2}\frac{\lambda_{22}^3\tilde{\mathbf{C}}}{\nu^3}\Lambda^{d+2}dl$	$\Xi_5 = \frac{1}{2}\frac{\lambda_{22}^2\mathcal{C}}{\nu_4}\lambda_{22}\cdot(q\text{'s})$	4
Γ_2	8	$-\Gamma_1$	$\Xi_6 = -\Xi_5$	8
Γ_3	8	$-\frac{\lambda_{13}\lambda_{22}\tilde{\mathbf{C}}}{\nu^2}\Lambda^d dl$	$\Xi_{1b} = -\frac{1}{2}\lambda_{13}\mathcal{C}\lambda_{22}\cdot(q\text{'s})$	4
Γ_4	4	$2\Gamma_3$	$\Xi_{1a} = 2\Xi_{1b}$	2

Diagram	S F	Contribution	Equivalent	S F
Ξ_1	8	$-\frac{1}{2}\frac{\lambda_{13}^2\tilde{\mathbf{C}}}{\nu^2}\Lambda^d dl$	$\Upsilon_{1e}+\Upsilon_{1f}=-\frac{1}{2}\lambda_{13}\mathcal{C}\lambda_{13}\cdot(q\text{'s})$	4+4
Ξ_2	4	Ξ_1	$\Upsilon_{1c}+\Upsilon_{1d}=\Upsilon_{1e}+\Upsilon_{1f}$	2+2
Ξ_3	4	Ξ_1	$\Upsilon_{1b}=\Upsilon_{1e}+\Upsilon_{1f}$	4
Ξ_4	2	$2\,\Xi_1$	$\Upsilon_{1a}=2[\Upsilon_{1e}+\Upsilon_{1f}]$	2
Ξ_5	16	$\frac{\lambda_{22}^2\lambda_{13}\tilde{\mathbf{C}}}{\nu^3}\Lambda^{d+2}dl$	$\Upsilon_{2b}=\frac{1}{2}\frac{\lambda_{22}^2\mathcal{C}}{\nu_4}\lambda_{13}\cdot(q\text{'s})$	8
Ξ_6	8	Ξ_5	$\Upsilon_{2a}=\Upsilon_{2b}$	4
Ξ_7	32	$-\Xi_5$	$\Upsilon_{3b}+\Upsilon_{3c}=-\Upsilon_{2b}$	8+8
Ξ_8	16	$-\Xi_5$	$\Upsilon_{3a}=-\Upsilon_{2b}$	8

The one loop renormalised couplings offer a second point of comparison:

$$\begin{aligned}\tilde{\nu_2}&=\nu_2\left(1+\frac{3}{2}\lambda_{13}\tilde{\mathbf{C}}\frac{\Lambda^d dl}{\nu_2\nu}\right)\\ \tilde{\nu_4}&=\nu_4\left(1-\frac{1}{4}\lambda_{22}^2\tilde{\mathbf{C}}'\frac{\Lambda^d dl}{\nu_4\nu^3}\right)\\ \tilde{\lambda}_{13}&=\lambda_{13}\left(1-\frac{5}{2}\lambda_{13}\tilde{\mathbf{C}}\frac{\Lambda^d dl}{\nu^2}\right)\\ \tilde{\lambda}_{22}&=\lambda_{22}\left(1-3\lambda_{13}\tilde{\mathbf{C}}\frac{\Lambda^d dl}{\nu^2}\right).\end{aligned} \tag{5.11}$$

Calculated from $(\nu_2+\nu_4 q^2)(1+\mathbf{G}_0(q,0)\sum\Phi_n)^{-1}$, $\lambda_{13}+\sum\Xi_n$, and $\lambda_{22}+\sum\Gamma_n$.

There is good agreement with the one loop λ_{13} renormalisation in (5.8), that is about it though. Even given the exclusion of κ, and the missed diagram, I would have $-3/2$, rather than -3, in the λ_{22} renormalisation in (5.11). There is a factor two difference for ν_4, and the signs of the renormalisation of ν_2 and ν_4 are reversed. The disagreement over symmetry factors is particularly stark. It is troubling that for the disagreements in Γ^{12} and Γ^{13}, the sets of symmetry factors that differ by a factor of two are for different composition classes of diagram, making a systematic difference less likely.

There is a difference relating to diagram identity; whereas I have a distinction between phenotypes with identical contributions, they have taken this equivalence as fundamental and have a single diagram. However, there is some diagram by diagram agreement. In particular it is encouraging that the same cancellations of diagrams, via the noise symmetric/asymmetric rearrangement mechanism, occurs in both schemes. Also the difference in noise strength is not significant, the extra terms do not alter the scaling behaviour. They are included by Haselwandter and Vvedensky to track the coupling flow more accurately.

5.2 The Fixed Points

5.2.1 The Full Theory

The full theory given by

$$\begin{aligned}\partial_t\phi(x,t) = \nu_2\nabla^2\phi(x,t) - \nu_4\nabla^4\phi(x,t) + \lambda_{13}\nabla\left(\nabla\phi(x,t)\right)^3 \\ + \lambda_{22}\nabla^2\Big(\nabla\phi\Big)^2 + \kappa\left[\nabla\left(\nabla\phi\nabla^2\phi\right) - \frac{1}{2}\nabla^2\Big(\nabla\phi\Big)^2\right] + \eta(x,t),\end{aligned} \tag{5.12a}$$

with spatially-uncorrelated Gaussian white noise

$$\langle\eta(x,t)\eta(x',t')\rangle = 2\Gamma^2\delta^d(x-x')\delta(t-t'), \tag{5.12b}$$

has the vertex functions, in full, to one loop:

$$\begin{aligned}\hat{\Gamma}^{11}(q) =& \nu_2 q^2 - \frac{3}{2}\lambda_{13}\left[\frac{\Gamma^2}{(4\pi)^2\nu_4^2}\left(\frac{\nu_4}{\nu_2}\right)^{\epsilon/2}\Gamma\left(\frac{\epsilon}{2}\right)\right]\nu_2 q^2 \\ &+ \nu_4 q^4 + \left[\frac{\lambda_{22}^2}{2\nu_4} + \frac{3\lambda_{22}\kappa}{2\nu_4} - \frac{5\kappa^2}{4\nu_4}\right]\left[\frac{\Gamma^2}{(4\pi)^2\nu_4^2}\left(\frac{\nu_4}{\nu_2}\right)^{\epsilon/2}\Gamma\left(\frac{\epsilon}{2}\right)\right]\nu_4 q^4\end{aligned} \tag{5.13a}$$

$$\begin{aligned}\frac{\hat{\Gamma}^{12}(q_1,q_2)}{2} =& -\lambda_{22}(q_1\cdot q_2)(q_1+q_2)^2 \\ &+ \frac{9}{4}\lambda_{22}\lambda_{13}\left[\frac{\Gamma^2}{(4\pi)^2\nu_4^2}\left(\frac{\nu_4}{\nu_2}\right)^{\epsilon/2}\Gamma\left(\frac{\epsilon}{2}\right)\right](q_1\cdot q_2)(q_1+q_2)^2 \\ &- \kappa(q_1\wedge q_2)^2 \\ &+ \frac{1}{2}\left(\lambda_{22}\lambda_{13} + 5\kappa\lambda_{13}\right)\left[\frac{\Gamma^2}{(4\pi)^2\nu_4^2}\left(\frac{\nu_4}{\nu_2}\right)^{\epsilon/2}\Gamma\left(\frac{\epsilon}{2}\right)\right](q_1\wedge q_2)^2\end{aligned} \tag{5.13b}$$

$$\begin{aligned}\frac{\hat{\Gamma}^{13}(q_1,q_2,q_3)}{2} =& -\lambda_{13}\left\{\begin{matrix}(q_1\cdot q_2)(q_3\cdot(q_1+q_2+q_3)) \\ +(q_1\cdot q_3)(q_2\cdot(q_1+q_2+q_3)) \\ +(q_2\cdot q_3)(q_1\cdot(q_1+q_2+q_3))\end{matrix}\right\} \\ &+ \frac{5}{2}\lambda_{13}^2\left[\frac{\Gamma^2}{(4\pi)^2\nu_4^2}\left(\frac{\nu_4}{\nu_2}\right)^{\epsilon/2}\Gamma\left(\frac{\epsilon}{2}\right)\right]\left\{\begin{matrix}(q_1\cdot q_2)(q_3\cdot(q_1+q_2+q_3)) \\ +(q_1\cdot q_3)(q_2\cdot(q_1+q_2+q_3)) \\ +(q_2\cdot q_3)(q_1\cdot(q_1+q_2+q_3))\end{matrix}\right\}.\end{aligned} \tag{5.13c}$$

For Γ^{12} and Γ^{13} the factor two from the permutation of two of the incoming response fields has be divided out. The renormalisation procedure is carried out using the reparametrised, dimensionless couplings

$$g = \frac{\Gamma^2}{(4\pi)^2}\frac{\lambda_{13}}{\nu_4^{2-\epsilon/2}}, \tag{5.14a}$$

$$\lambda = \frac{\lambda_{22}^2}{\nu_4\lambda_{13}}, \tag{5.14b}$$

$$\chi = \frac{\kappa}{\lambda_{22}}. \tag{5.14c}$$

Where as the couplings λ and χ are dimensionless by construction, g is dimensionless at the critical dimension and is the loop/expansion parameter. Using an arbitrary inverse length μ, a dimension B for the noise as before, and assigning an independent dimension A to ν_4, the dimensional analysis of (4.14) can be restated as:

$$\begin{aligned}
&[\phi(q,\omega)] = A^{\frac{-3}{2}}B\mu^{-(8-\frac{\epsilon}{2})} \quad [\tilde{\phi}(q,\omega)] = A^{\frac{-1}{2}}B^{-1}\mu^{-(4-\frac{\epsilon}{2})}\\
&[\nu_2] = A\mu^2 \qquad [\nu_4] = A\\
&[\lambda_{22}] = [\kappa] = A^{\frac{3}{2}}B^{-1}\mu^{\frac{\epsilon}{2}} \qquad [\lambda_{13}] = A^2B^{-2}\mu^{\epsilon}\\
&[g] = A^{\frac{\epsilon}{2}}\mu^{\epsilon}\\
&[\hat{\Gamma}^{nm}] = A^{1+\frac{m-n}{2}}B^{n-m}\mu^{\frac{m\epsilon}{2}-n(4-\frac{\epsilon}{2})+8-\epsilon}.
\end{aligned} \tag{5.15}$$

The gamma functions in (5.13) are expanded as $\Gamma\left(\frac{\epsilon}{2}\right) = \frac{2}{\epsilon}+\gamma+O(\epsilon)$, with Euler's constant γ, and only the leading order in ϵ is kept. The dimensionless, renormalised couplings, $\bar{a}$, and Z-factors, Z_a, are:

$$\begin{aligned}
\bar{\nu}_2 &= Z_2\nu_2A^{-1}\mu^{-2} & Z_2 &= 1-\frac{3}{\epsilon}g\nu_2^{-\epsilon/2}\\
\bar{\nu}_4 &= Z_4\nu_4A^{-1} & Z_4 &= 1-\frac{\lambda}{\epsilon}\left(\frac{5}{2}\chi^2-3\chi-1\right)g\nu_2^{-\epsilon/2}\\
\bar{\lambda}_{22} &= Z_{22}\lambda_{22}A^{-3/2}\mu^{-\epsilon/2}B & Z_{22} &= 1-\frac{9}{2\epsilon}g\nu_2^{-\epsilon/2}\\
\bar{\kappa} &= Z_\kappa\kappa A^{-3/2}\mu^{-\epsilon/2}B & Z_\kappa &= 1-\frac{1}{\epsilon}\left(5+\frac{1}{\chi}\right)g\nu_2^{-\epsilon/2}\\
\bar{\lambda}_{13} &= Z_{13}\lambda_{13}A^{-2}\mu^{-\epsilon}B^2 & Z_{13} &= 1-\frac{5}{\epsilon}g\nu_2^{-\epsilon/2}\\
\bar{g} &= Z_g gA^{-\epsilon/2}\mu^{-\epsilon} & Z_g &= 1-\frac{1}{\epsilon}\left(5-2\lambda\left[\frac{5}{2}\chi^2-3\chi-1\right]\right)g\nu_2^{-\epsilon/2}\\
\bar{\lambda} &= Z_\lambda\lambda & Z_\lambda &= 1-\frac{1}{\epsilon}\left(4-\lambda\left[\frac{5}{2}\chi^2-3\chi-1\right]\right)g\nu_2^{-\epsilon/2}\\
\bar{\chi} &= Z_\chi\chi & Z_\chi &= 1-\frac{1}{\epsilon}\left(\frac{1}{2}+\frac{1}{\chi}\right)g\nu_2^{-\epsilon/2}.
\end{aligned} \tag{5.16}$$

There is no field renormalisation as usually seen in equilibrium, instead there is a parameter in front of every term in the Langevin equation. The fixed points are found as the roots of the β-functions, with corresponding γ-functions:

$$\beta_{\bar{a}} = \frac{d\bar{a}}{d\ln\mu}, \qquad \gamma_{\bar{a}} = \frac{d\ \ln Z_{\bar{a}}}{d\ln\mu}, \tag{5.17}$$

where the derivatives are taken with respect to constant bare parameters. It is to be supposed that a scaling solution exists for the vertex functions, of the form

$$\begin{aligned}
&\hat{\Gamma}^{nm}\left(\{q\}\{\omega\};\nu_2,\nu_4,\Gamma^2,g,(\lambda,\chi)\right)\\
&\quad= A^{1+\frac{m-n}{2}}\mu^{\frac{m\epsilon}{2}-n(4-\frac{\epsilon}{2})+8-\epsilon} \qquad (5.18a)\\
&\qquad\cdot\bar{\hat{\Gamma}}^{nm}\left(\left\{\frac{q}{\mu}\right\}\left\{\frac{\omega}{A\mu^4}\right\};\frac{\nu_2}{A\mu^2},\frac{\nu_4}{A},\Gamma^2,\frac{g}{A^{\epsilon/2}\mu^{\epsilon}},(\lambda,\chi)\right)\\
&\quad= l^{\gamma_4(1+\frac{m-n}{2})+\frac{m\epsilon}{2}-n(4-\frac{\epsilon}{2})+8-\epsilon} \qquad (5.18b)\\
&\qquad\cdot\hat{\Gamma}^{nm}\left(\left\{\frac{q}{l}\right\}\left\{\frac{\omega}{l^{\gamma_4+4}}\right\};\nu_2 l^{\gamma_2-\gamma_4-2},\nu_4,\Gamma^2,g l^{\gamma_g-\frac{\epsilon\gamma_4}{2}-\epsilon},\lambda l^{\gamma_\lambda},\chi l^{\gamma_\chi}\right),
\end{aligned}$$

with the original non-linear couplings scaling as

$$\lambda_{13} l^{2\gamma_\Gamma-2\gamma_4-\epsilon+\gamma_{13}}, \quad \lambda_{22} l^{\gamma_{22}+\gamma_\Gamma-\frac{3\gamma_4}{2}-\frac{\epsilon}{2}}, \quad \kappa l^{\gamma_\kappa+\gamma_\Gamma-\frac{3\gamma_4}{2}-\frac{\epsilon}{2}}. \tag{5.18c}$$

The arbitrary inverse length μ is used as the basis for scaling via $\mu(l) = l\mu$, then with A scaling as $A(l) = l^{\gamma_4}A$. The noise does not renormalise, it has a Z-factor of one, its β-function and γ-function are identically zero, it does not scale, and so its independent dimension can be set to $B = 1$. It is straightforward to define a set of scaling exponents in this field theory,

$$\begin{aligned}
&\hat{\Gamma}^{(n,m)}\left(k,\omega;\nu_2,\nu_4,\Gamma^2,[g,\lambda,\chi/\lambda_{13},\lambda_{22},\kappa]\right)\\
&\quad= l^{-\frac{n}{2}(2\eta+8-\epsilon)-\frac{m}{2}(2\tilde{\eta}-\epsilon)+8-\epsilon+\delta}\\
&\qquad\hat{\Gamma}^{(n,m)}\left(\frac{k}{l},\frac{\omega}{l^z};\frac{\nu_2}{l^{1/\nu}},\nu_4,\Gamma^2,\left[\frac{g}{l^{\pi_g}},\frac{\lambda}{l^{\pi_\lambda}},\frac{\chi}{l^{\pi_\chi}}\Big/\frac{\lambda_{13}}{l^{\pi_{13}}},\frac{\lambda_{22}}{l^{\pi_{22}}},\frac{\kappa}{l^{\pi_\kappa}}\right]\right),
\end{aligned} \tag{5.19}$$

which can be read off as:

$$\begin{aligned}
&z = 4+\gamma_4 \qquad \nu = \frac{1}{2+\gamma_4-\gamma_2} \qquad \delta = \gamma_4\\
&\pi_g = \epsilon\left(1+\frac{\gamma_4}{2}\right)-\gamma_g \qquad \pi_\lambda = -\gamma_\lambda \qquad \pi_\chi = -\gamma_\chi\\
&\pi_{13} = \epsilon+2\gamma_4-\gamma_{13} \qquad \pi_{22} = \frac{\epsilon}{2}+\frac{3\gamma_4}{2}-\gamma_{22} \qquad \pi_\kappa = \frac{\epsilon}{2}+\frac{3\gamma_4}{2}-\gamma_\kappa\\
&\eta = \frac{\gamma_4}{2} \qquad \tilde{\eta} = \frac{-\gamma_4}{2}.
\end{aligned} \tag{5.20}$$

The fields still acquire anomalous dimension with η and $\tilde{\eta}$, even without explicit renormalisation. There are three β-functions to be calculated,

$$\beta_g = -\epsilon + \left(5 - 2\lambda\left[\frac{5}{2}\chi^2 - 3\chi - 1\right]\right) g^2, \tag{5.21a}$$

$$\beta_\lambda = \left(4 - \lambda\left[\frac{5}{2}\chi^2 - 3\chi - 1\right]\right) g\lambda, \tag{5.21b}$$

$$\beta_\chi = \left(\frac{\chi}{2} + 1\right) g, \tag{5.21c}$$

with the γ-functions

$$\begin{aligned}
\gamma_2 &= 3g \qquad \gamma_4 = \lambda\left(\frac{5}{2}\chi^2 - 3\chi - 1\right) g \\
\gamma_g &= \left(5 - 2\lambda\left[\frac{5}{2}\chi^2 - 3\chi - 1\right]\right) g \\
\gamma_\lambda &= \left(4 - \lambda\left[\frac{5}{2}\chi^2 - 3\chi - 1\right]\right) g \\
\gamma_\chi &= \left(\frac{1}{2} + \frac{1}{\chi}\right) g \\
\gamma_{22} &= \frac{9}{2} g \quad \gamma_\kappa = \left(5 + \frac{1}{\chi}\right) g \quad \gamma_{13} = 5g.
\end{aligned} \tag{5.22}$$

The simultaneous roots of the β-functions give three fixed points for the theory. One is infrared stable:

$$\chi = -2 \qquad \lambda = 0 \qquad g = \epsilon/5. \tag{5.23}$$

Giving the γ-functions

$$\begin{aligned}
&\gamma_2 = \frac{3\epsilon}{5} && \gamma_4 = 0 \\
&\gamma_g = \epsilon \quad \gamma_\lambda = \frac{4\epsilon}{5} && \gamma_\chi = 0 \\
&\gamma_{13} = \epsilon \quad \gamma_{22} = \frac{9\epsilon}{10} && \gamma_\kappa = \frac{9\epsilon}{10},
\end{aligned} \tag{5.24}$$

and the exponents

$$\begin{aligned}
&z = 4 \quad \nu = \frac{1}{2} + \frac{3\epsilon}{20} && \delta = 0 \\
&\pi_g = 0 \quad \pi_\lambda = -\frac{4\epsilon}{5} && \pi_\chi = 0 \\
&\pi_{13} = 0 \quad \pi_{22} = -\frac{2\epsilon}{5} && \pi_\kappa = -\frac{2\epsilon}{5} \\
&\eta = 0 && \tilde{\eta} = 0.
\end{aligned} \tag{5.25}$$

Having the coupling $\lambda = 0$ at the fixed point would imply that both κ and λ_{22} are driven to zero, while maintaining a fixed ratio given by χ, with the exponents π_κ and π_{22} giving their transient length scale. The renormalisation of ν_2 means there will be a non-trivial critical point, as $\nu_2^c \neq 0$.

There is an infrared unstable fixed point:

$$\chi = -2 \qquad \lambda = \frac{4}{15} \qquad g = \frac{\epsilon}{2\epsilon - 3}. \tag{5.26}$$

The last fixed point is the apparently trivial one:

$$g = 0. \tag{5.27}$$

However, with $g = 0$ all is not well. The coupling λ is the source of this trouble, rather like dangerously irrelevant variable. Interpreting $g = 0$ as the coupling λ_{13} being driven to zero, λ will diverge as can be seen in (5.14). All is not lost, as λ appears will g in the Z-factors, and there is a finite product

$$\psi := g\lambda = \frac{\Gamma^2}{(4\pi)^2} \frac{\lambda_{22}^2}{\nu_4^{3-\epsilon/2}} \qquad [\psi] = A^{\epsilon/2}\mu^{\epsilon}. \tag{5.28}$$

However, it is not possible to recycle any of the results obtained so far, some further calculation is required.

5.2.2 The 'Trivial' Theory

Interpreted as $\lambda_{13} \to 0$, the 'trivial' fixed point is the theory of the Langevin equation

$$\begin{aligned} \partial_t \phi(x,t) =& \nu_2 \nabla^2 \phi(x,t) - \nu_4 \nabla^4 \phi(x,t) + \lambda_{22} \nabla^2 \Big(\nabla\phi\Big)^2 \\ &+ \kappa \left[\nabla\left(\nabla\phi\nabla^2\phi\right) - \frac{1}{2}\nabla^2\Big(\nabla\phi\Big)^2\right] + \eta(x,t). \end{aligned} \tag{5.29}$$

The renormalisation of which requires the new coupling ψ, defined in (5.28), to replace the couplings g and λ used in the full theory. The coupling χ is as before. The Z-factors face a cull, either from vanishing g or from ignoring diagrams containing λ_{13}. After reworking in terms of ψ, the only survivors are Z_4 and the new Z_ψ:

$$\begin{aligned} \bar{\nu}_2 =& \nu_2 A^{-1}\mu^{-2} & Z_2 =& 1 \\ \bar{\nu}_4 =& Z_4 \nu_4 A^{-1} & Z_4 =& 1 - \frac{1}{\epsilon}\left(\frac{5}{2}\chi^2 - 3\chi - 1\right)\psi\nu_2^{-\epsilon/2} \\ \bar{\lambda}_{22} =& \lambda_{22} A^{-3/2}\mu^{-\epsilon/2} B & Z_{22} =& 1 \\ \bar{\kappa} =& \kappa A^{-3/2}\mu^{-\epsilon/2} B & Z_\kappa =& 1 \\ \bar{\psi} =& Z_\psi \psi A^{-\epsilon/2}\mu^{-\epsilon} & Z_\psi =& 1 + \frac{3}{\epsilon}\left(\frac{5}{2}\chi^2 - 3\chi - 1\right)\psi\nu_2^{-\epsilon/2} \\ \chi =& \bar{\chi} & Z_\chi =& 1. \end{aligned} \tag{5.30}$$

There is one β-function, and two γ-functions,

$$\begin{aligned}\beta_\psi &= -\epsilon\psi - \frac{3}{\epsilon}\left(\frac{5}{2}\chi^2 - 3\chi - 1\right)\psi^2 \\ \gamma_\psi = -3\left(\frac{5}{2}\chi^2 - 3\chi - 1\right)\psi &\qquad \gamma_4 = \left(\frac{5}{2}\chi^2 - 3\chi - 1\right)\psi.\end{aligned} \tag{5.31}$$

The fixed point of the β-function gives

$$\psi = \frac{-\epsilon}{3\left(\frac{5}{2}\chi^2 - 3\chi - 1\right)}, \tag{5.32a}$$

with the γ-functions

$$\gamma_\psi = \epsilon \qquad \gamma_4 = -\frac{\epsilon}{3}. \tag{5.32b}$$

By inspection, the scaling form of the new coupling is $\psi l^{\gamma_\psi - \frac{\epsilon\gamma_4}{2} - \epsilon}$, so the scaling exponent is $\pi_\psi = \frac{\epsilon\gamma_4}{2} + \epsilon - \gamma_\psi$. The exponents at the 'trivial' fixed point, that is the exponents for the stable fixed point of the Langevin equation (5.29), are

$$\begin{gathered} z = 4 - \frac{\epsilon}{3} \qquad \nu = \frac{1}{2} + \frac{\epsilon}{12} \qquad \delta = -\frac{\epsilon}{3} \\ \pi_\psi = 0 \qquad \pi_\chi = 0 \qquad \pi_{22} = 0 \qquad \pi_\kappa = 0 \\ \eta = -\frac{\epsilon}{6} \qquad \tilde{\eta} = \frac{\epsilon}{6}. \end{gathered} \tag{5.33}$$

Renormalisation does however impose bounds on coupling χ, or the fixed point of ψ would oblige an unphysical value of ν_4. To one loop, in order to get sensible results, $(5/2)\chi^2 - 3\chi - 1$ needs to be negative. This means $\chi \in [\frac{3-\sqrt{19}}{5}, \frac{3+\sqrt{19}}{5}]$, and sufficiently large κ violates this. The same exponents emerge if implemented without consideration of this, though the conclusion would be invalid.

5.2.3 The Limited Theory

It is not possible to adjust the full theory by setting $\kappa = 0$ at the start, as a theory with the λ_{13} and λ_{22} couplings generates κ. The full theory can be adjusted to have $\lambda_{22} = 0$ at the start, this limited case with the λ_{13} and κ couplings is consistent, at least to one loop order. The Langevin equation for this limited theory is

$$\begin{aligned}\partial_t\phi(x,t) =& \nu_2\nabla^2\phi(x,t) - \nu_4\nabla^4\phi(x,t) + \lambda_{13}\nabla\left(\nabla\phi(x,t)\right)^3 \\ &+ \kappa\left[\nabla\left(\nabla\phi\nabla^2\phi\right) - \frac{1}{2}\nabla^2\Big(\nabla\phi\Big)^2\right] + \eta(x,t).\end{aligned} \tag{5.34}$$

With λ_{13} still present the renormalisation can be carried out using the coupling g as defined in (5.14), but in the absence of λ_{22} a new dimensionless coupling X is needed:

$$X = \frac{\kappa^2}{\nu_4 \lambda_{13}}, \tag{5.35}$$

which is the equivalent of λ in (5.14). The Z-factors are:

$$\begin{aligned}
\bar{\nu}_2 =& Z_2 \nu_2 A^{-1} \mu^{-2} & Z_2 =& 1 - \frac{3}{\epsilon} g \nu_2^{-\epsilon/2} \\
\bar{\nu}_4 =& Z_4 \nu_4 A^{-1} & Z_4 =& 1 - \frac{5X}{2\epsilon} g \nu_2^{-\epsilon/2} \\
\bar{\kappa} =& Z_\kappa \kappa A^{-3/2} \mu^{-\epsilon/2} B & Z_\kappa =& 1 - \frac{5}{\epsilon} g \nu_2^{-\epsilon/2} \\
\bar{\lambda}_{13} =& Z_{13} \lambda_{13} A^{-2} \mu^{-\epsilon} B^2 & Z_{13} =& 1 - \frac{5}{\epsilon} g \nu_2^{-\epsilon/2} \\
\bar{g} =& Z_g g A^{-\epsilon/2} \mu^{-\epsilon} & Z_g =& 1 - \frac{5}{\epsilon} (1 - X) g \nu_2^{-\epsilon/2} \\
\bar{X} =& Z_X X & Z_X =& 1 - \frac{5}{\epsilon} \left(1 - \frac{X}{2}\right) g \nu_2^{-\epsilon/2}
\end{aligned} \tag{5.36}$$

With the β-function and γ-functions,

$$\begin{gathered}
\beta_g = -\epsilon g + 5(1 - X) g^2 \qquad \beta_X = 5X \left(1 - \frac{X}{2}\right) g \\
\gamma_g = 5(1 - X) g \qquad \gamma_X = 5 \left(1 - \frac{X}{2}\right) g \\
\gamma_2 = 3g \quad \gamma_4 = \frac{5X}{2} g \quad \gamma_{13} = 5g \quad \gamma_\kappa = 5g,
\end{gathered} \tag{5.37}$$

leading to the stable the fixed point

$$X = 0 \qquad g = \frac{\epsilon}{5}, \tag{5.38}$$

and the γ-functions

$$\begin{gathered}
\gamma_g = \epsilon \quad \gamma_X = \epsilon \\
\gamma_2 = \frac{3\epsilon}{5} \quad \gamma_4 = 0 \quad \gamma_{13} = \epsilon \quad \gamma_\kappa = \epsilon.
\end{gathered} \tag{5.39}$$

So the exponents at the stable fixed point of the limited theory are

$$\begin{gathered}
z = 4 \qquad \nu = \frac{1}{2} + \frac{3\epsilon}{20} \qquad \delta = 0 \\
\pi_g = 0 \qquad \pi_{13} = 0 \qquad \pi_X = -\epsilon \qquad \pi_\kappa = -\frac{\epsilon}{2} \\
\eta = 0 \qquad \tilde{\eta} = 0,
\end{gathered} \tag{5.40}$$

which are near identical to the exponents for stable fixed point of the full theory in (5.25), the coupling κ again driven to zero. The key difference brought about by the absence of λ_{22} from the start is an adjustment of π_κ, and thus the length scale on which κ can be transiently observed.

5.3 Higher Orders

A brief word on extending the renormalisation to higher orders. To one loop order there is a neat cancellation of the diagrams constructed exclusively from the λ_{22} and κ couplings; they do not renormalise themselves at one loop order, if present renormalisation is driven by the λ_{13} coupling. This does not hold at two loop order, as has been shown in the literature already (Janssen, 1997). At higher orders the λ_{22} and κ couplings do renormalise themselves, which will significantly change the behaviour of the renormalised theory, in particular the 'trivial' case, but also the dynamics of the β-functions generally.

Regarding the two loop calculation itself, it may be advisable to use the momentum-time representation, as the frequency integrals become particularly messy.

Chapter 6

In the Context of Ideal Molecular-Beam Epitaxy

The classic Langevin equation formulation in crystal growth, as introduced at the beginning of Chapter 4, was proposed in 1991. This was done so independently by Villain (Villain, 1991), and by Lai and Das Sarma (Lai and Das Sarma, 1991), with the purpose of describing a growing crystalline interface, evolving with volume conserving surface dynamics, in which no overhangs or bulk defects are present. The Langevin equation proposed for such growth on a d-dimensional substrate was

$$\begin{aligned}\partial_t\phi(x,t) =&\nu_2\nabla^2\phi(x,t) - \nu_4\nabla^4\phi(x,t) + \lambda_{13}\nabla\left(\nabla\phi(x,t)\right)^3 \\ &+ \lambda_{22}\nabla^2\left(\nabla\phi(x,t)\right)^2 + \eta(x,t),\end{aligned} \tag{6.1a}$$

where the scalar field $\phi(x,t)$ is the displacement from the average surface height, and $\eta(x,t)$ is a Gaussian white noise with zero mean and the correlator

$$\langle\eta(x,t)\eta(x',t')\rangle = 2L\delta^d(x-x')\delta(t-t'). \tag{6.1b}$$

This Langevin equation was derived by considering the most general fourth order equation consistent with the symmetries of the problem. These symmetries, explicitly discussed in a non-technical talk by Das Sarma (Das Sarma, 1996), are chiefly given as translation in the growth direction, prohibiting terms that are functions of the height, and rotation in the plane, meaning no odd gradients of the height are allowed.

There are several limits of the general Langevin equation (6.1) that are traditionally of interest in the literature. The linear limit with only the ν_2 coupling, where all the other couplings are set to zero, is the Edwards-Wilkinson (EW) (Edwards and Wilkinson, 1982) equation,

$$\partial_t \phi(x,t) = \nu_2 \nabla^2 \phi(x,t) + \eta(x,t). \tag{6.2}$$

The other linear limit, with only the ν_4 coupling remaining, is commonly known as the Mullins-Herring (MH) (Mullins, 1957; Herring, 1951) equation,

$$\partial_t \phi(x,t) = -\nu_4 \nabla^4 \phi(x,t) + \eta(x,t). \tag{6.3}$$

The limit with the ν_4 and λ_{22} couplings,

$$\partial_t \phi(x,t) = -\nu_4 \nabla^4 \phi(x,t) + \lambda_{22} \nabla^2 \left(\nabla \phi(x,t)\right)^2 + \eta(x,t), \tag{6.4}$$

as settled on in the original papers (Lai and Das Sarma, 1991; Villain, 1991), this limit is demotically titled the Villain-Lai-Das Sarma (VLDS) equation. It is used quite generally to describe surface growth by molecular-beam epitaxy (MBE).

In this chapter, these popular Langevin equations for surface growth will be compared to the results of the analysis presented in Chapter 5. First, an overview will be given of the derivation and accepted meaning of the surface growth Langevin equations, along with a summary of how the literature has developed over the last two decades. Second, the three stable fixed points that emerged from the study of the full theory will be compared to results for the surface growth Langevin equations. Finally, the impact of the new κ interaction is discussed.

6.1 The State of the Literature

Crystal surface growth in general has been extensively studied; with books, such as the unstoppably popular one by Barabasi and Stanley, amongst others (Barabasi and Stanley, 1995; Pimpinelli and Villain, 1998), and numerous review articles, with Krug being particularly prolific (Krug, 1997; Michely and Krug, 2004; Krug, 2005). The generic behaviour of surface growth, in the absence of any restrictions on the growth conditions, belongs to the Kardar-Parisi-Zhang (KPZ) (Kardar *et al.*, 1986) universality class. However, the growth conditions of MBE allow the imposition of conservation laws that prohibit KPZ behaviour.

6.1.1 The set-up

In experimental MBE, the temperature is high enough that relaxation of the incident atoms plays the dominant role in producing smooth growth, but low enough that desorption and overhangs are negligible, and that the system is below the roughening transition, above which thermal fluctuations dominate the structure.[1] The sample size is small enough that the influence of gravity can be ignored, becoming relevant only on scales much larger than the sample size (Marsili *et al.*, 1996), if at all.

As such, ideal MBE (Lai and Das Sarma, 1991) was proposed as "atomistic stochastic growth without any bulk defects or surface overhangs driven by atomic deposition in a chemical-bonding environment where surface relaxation can occur only through the breaking of bonds."

This has a number of implications. The surface dynamics are constrained to obey mass conservation. The deterministic part of the Langevin equation can be written as a continuity equation, as the divergence of some surface current $j(x,t)$:

$$\frac{\partial\phi(x,t)}{\partial t} = -\nabla\cdot j(x,t) + \eta(x,t). \tag{6.5}$$

It is this condition, of volume preserving surface dynamics, that prevents a KPZ term from arising, as $\big(\nabla\phi(x,t)\big)^2$ cannot be written in this form. The no-overhang approximation is also referred as the Monge representation of the surface.

The condition placed on the surface relaxation means that a surface current arises in general from differences in the local chemical potential $\mu(x,t)$, with $j(x,t) = -\nabla\mu(x,t)$ describing atoms/surface deposits drifting to the minima of the chemical potential. If gravity were the source of this chemical potential, $\mu(x,t) \propto \phi(x,t)$, this would lead to the Langevin equation (6.2) (Edwards and Wilkinson, 1982). If the chemical potential derived from the local curvature, $\mu(x,t) \propto \nabla^2\phi(x,t)$, this surface diffusion would produce the Langevin equation (6.3).

Noise enters the system through fluctuations in the beam intensity as new material is deposited, and though thermal fluctuations during surface relaxation. The beam fluctuations are non-conservative, whereas the motion on the surface is necessarily constrained by mass conservation. This gives a noise strength in (6.1b) of $L = D_0 + D_2\nabla^2 + D_4\nabla^4$, as seen before in Section 5.1.5. The beam fluctuations give rise to the D_0 term, and thermal

[1] At the roughening temperature, the line tension of steps in the surface height vanishes.

fluctuations are the origin of D_2 and D_4. However, the conserved part is irrelevant in the RG sense and is therefore often immediately dropped or disregarded from the start, even if thermal fluctuations might initially be considered for inclusion in the model on physical grounds.

A more pressing issue is that there is no obvious chemical potential gradient in such a non-thermodynamic problem. The question arises of what physical process could lead to the terms in the Langevin equation (6.1). In one of the original papers, Lai and Das Sarma (Lai and Das Sarma, 1991) investigated this by examining what each of the proposed terms in the Langevin equation (6.1) did when applied individually to a one-dimensional kink in the surface. They attempted to describe and assign a surface diffusion mechanism to each coupling by applying the associated derivatives in turn on the function $\tanh(x)$, to mimic their diffusive effect on a step in the surface height. From this they interpreted:

ν_2 : Transports particles straightforwardly downhill, so the valley receives more particles than the plateau. Attributed to smoothing by gravity, which is rejected from the model as irrelevant to MBE. It is an equilibrium term which could come from relaxation of a Hamiltonian term $H = \int dx \ (\nabla\phi(x,t))^2$.

ν_4 : Has the effect of gently lowering the peak and raising the valley floor, but also sharpens the slope. It is suggested that it describes the process of particles sticking to small kinks near the top of the slope on the way down, rather than rolling all the way down a slope. Could originate from a Hamiltonian term $H = \int dx \ \left(\nabla^2\phi(x,t)\right)^2$.

λ_{22} : Acts to raises the valley floor and to sharpen the slope. Attributed to a process similar to ν_4, with particles sticking to small kinks; though as a higher temperature correction, that allows particles to hop to lower parts of the slope. No known connection to a Hamiltonian term.

λ_{13} : Has the same effect as ν_2, suggested as a higher order contribution from a Hamiltonian term $H = \int dx \ (\nabla\phi(x,t))^4$

Though this is all limited to one-dimensional speculation.

Tang and Nattermann (Tang and Nattermann, 1991) examined the model that Villain (Villain, 1991) proposed independently of Lai and Das Sarma, concerning itself with "non-equilibrium adatom population" (a rather pleasing phrase I feel). From the general growth equation for the surface height $\mathcal{Z}(x,t)$ with beam intensity $\mathbf{F}$,

$$\frac{\partial \mathcal{Z}}{\partial t} + \Omega\nabla\cdot j = \mathbf{F}, \tag{6.6}$$

a conserved current, $\nabla\cdot j$, that is consistent with in-plane isotropy and continuous translational symmetry in the growth direction was sought. Supposing that $j = j_{eq} + j_{neq}$, where j_{eq} is the quasi-equilibrium part driven by the gradient of the chemical potential,

$$\begin{aligned} j_{eq} &= -\left(\frac{\rho_s D_s}{kT}\right)\nabla\mu + j_0 \\ \mu &= \Omega\frac{\delta\mathcal{F}}{\delta\mathcal{Z}} \\ \mathcal{F} &= \int d^d x\,\frac{\Gamma}{2}(\nabla\mathcal{Z})^2 + \mathbf{V}[\mathcal{Z}] \\ \langle j_0(x,t)j_0(x',t')\rangle &= 2\rho_s D_s\delta^d(x-x')\delta(t-t'), \end{aligned} \tag{6.7}$$

with surface stiffness Γ, atomic volume Ω, surface diffusion coefficient D_s, and "surface density of the diffusing species" ρ_s. In the postulated free energy $\mathcal{F}$, $\mathbf{V}[\mathcal{Z}]$ is the lattice pinning potential, while the other term, $\frac{\Gamma}{2}(\nabla\mathcal{Z})^2$, produces the equivalent of the ν_4 coupling term. The non-equilibrium current, j_{neq}, is proposed as

$$j_{neq} = -\frac{\nu}{\Omega}\nabla\mathcal{Z} - \frac{\sigma}{2\Omega}\nabla(\nabla\mathcal{Z})^2, \tag{6.8}$$

with the first term translating as the ν_2 coupling, and the second as λ_{22}. They comment that, although in equilibrium ν_2 should be absent as gravity is negligible, such a current is possible under growth conditions. Further adding that the λ_{22} term is not obtainable from a free energy. Combining, the Langevin equation for the surface becomes

$$\begin{aligned} \frac{\partial \mathcal{Z}}{\partial t} &= \nu\nabla^2\mathcal{Z} - \gamma\nabla^4\mathcal{Z} + \frac{\sigma}{2}\nabla^2(\nabla\mathcal{Z})^2 + v\nabla^2\sin\left(2\pi\frac{z}{a}\right) + \mathbf{F}_0 + \eta, \\ \langle\eta(x,t)\eta(x',t')\rangle &= 2\mathcal{D}\delta^d(x-x')\delta(t-t'), \\ \mathcal{D} &= D_0 - D_1\nabla^2 + D_2\nabla^4. \end{aligned} \tag{6.9}$$

They explore different scaling regions and length scales for this equation, pointing out the EW region. Using a co-moving frame removes the constant $\mathbf{F}_0$, and puts a phase factor in front of the lattice pinning potential, which then ultimately averages to zero. The non-equilibrium terms were shown to reduce the extreme roughness caused by shot-noise that equilibrium diffusion could not, with finite growth destroying lattice pinning in the model. Though not of direct concern here, it is shown that at small length scales, and below the roughening temperature, a lattice pinning term would lead to formation of faceted areas, and that a vanishing beam intensity gives the equilibrium Sine-Gordon model.

A review of the physical motivations underlying the terms that appear in Langevin equations such as (6.1)-(6.4) can be found in a paper by Marsili, Maritan, Toigo and Banavar (Marsili *et al.*, 1996). They consider "local-growth processes in which the growth rate is a function of the local properties of the interface", and proceed by insisting, rather sensibly, that these considerations about the local geometry of the surface should be invariant under reparameterisation. They note that the usual Langevin equation form is Newton's law of motion with the inertial term neglected in comparison to the dissipative,

$$\frac{\partial \phi(x,t)}{\partial t} = G[\phi(x,t)] + \eta(x,t). \tag{6.10}$$

By inspection the inertial term would be irrelevant for scaling, going as ω^2 it is an order less relevant that the dissipative term for the asymptotic behaviour of height-height correlations as $\omega \to 0$. Any constant growth term can be rearranged away by going to a moving frame, $\phi' = \phi + G_0 t$. If thermal-like noise is replaced with quenched noise, $\eta(x, \phi)$, this stops the use of a co-moving frame to eliminate G_0. However, after some algebraic interrogation (Marsili *et al.*, 1996), it turns out that one can go back to a co-moving frame and keep using the thermal-like noise, $\eta(x,t)$. Using differential geometry to enforce reparameterisation invariance, a Langevin equation is derived from the minimisation of some potential $V(\phi)$ using a small gradient expansion,

$$G(x,t) = -\nu \frac{\partial V(\phi)}{\partial \phi}. \tag{6.11}$$

By examining the potentials arising from different physical considerations they conclude that the term $\nu_2 \nabla^2 \phi$ can result from surface tension, an orientation dependent potential, a geometric effect such as a height restriction in the model, or finite particle size. The $\nu_4 \nabla^4 \phi$ term can result from a curvature potential, of which surface tension would be the zeroth order. Surface diffusion, in the sense of the volume conserving surface dynamics of ideal MBE, is examined to leading order only, generating the ν_4 term; a ν_2 term is also shown to result from the case of gravity induced surface diffusion.

A small gradient expansion of any particular realisation of the potential loses the detail of whether the process conserves particle number or not. For example, if evaporation was allowed, this would also give a $\nu_2 \nabla^2 \phi$ term for the surface diffusion, even though the number of particles is not conserved by evaporation. It is only possible to distinguish between the physical origins of these couplings with higher order terms, terms that may be RG irrelevant.

Marsili et al. do not believe an invariant potential could lead to the λ_{22} coupling proposed for homoepitaxial growth. This would make the λ_{22} coupling a "true non-equilibrium term", like the KPZ term. The consequences of this are that it vanishes with the external flux, it gives the system a non-zero skewness and it violates detailed balance; this is presumably what Ortiz, Repetto and Si (Ortiz *et al.*, 1999) were objecting to when discarding the λ_{22} coupling as a possible term in a growth model. What Marsili et al. are driving at is that λ_{22} cannot be derived from a Hamiltonian, so they do not expect their method to be able to generate it.

A closer examination of the higher order terms that Marsili et al. do calculate shows that the coupling κ emerges with ν_4 from the curvature potential (see Eq. 34 (Marsili *et al.*, 1996)), although it is not in a particularly clear form:

$$\kappa \left[(\nabla^2 h)^2 - \sum_{i,j=1}^{d} (\partial_i \partial_j h)^2 \right]. \tag{6.12}$$

The equivalence is seen by taking a Fourier transform. They note that, for positive κ, large negative curvatures are favoured while positive ones are suppressed. The λ_{13} coupling also emerges as part of the surface tension, the zeroth order of the curvature potential, along with κ emerging from the first order terms of the curvature potential, and ν_4 from the second order terms.

All this provides some basis for understanding the physics captured by the different terms proposed in (6.1). Rejecting gravity induced surface diffusion in ideal MBE entails setting ν_2 to zero, with Lai and Das Sarma arguing it should be negligible small in experimental conditions. If λ_{13} is interpreted as a correction to ν_2, that it expresses essentially the same physics, it too can be argued to zero for the same reason; that the process is not relevant. The diffusive ν_4 term was known not to be enough on its own, including the λ_{22} coupling starts to capture the non-equilibrium aspects of ideal MBE.

Although there are solid physical reasons for including the κ coupling in a model, with hints of its existence there to be found, and although in principle it should warrant inclusion on symmetry grounds, it is passed by unnoticed.

6.1.2 Developments, lattice models, and the master equation method

The broad narrative present in the literature on surface growth is for the analysis of the VLDS equation (6.4), or some variant continuum equation postulated for the model in question, to be compared to numerical simulations performed on lattice models of the surface growth, with some attempt to reach out to experimental observations.

The analysis of the continuum equations has been carried out almost universally using the DRG, as introduced at the beginning of Chapter 4. Leading to the common conclusion that the λ_{22} coupling renormalises trivially to all orders, and that both the λ_{13} and ν_2 couplings must be set to zero, as λ_{13} generates ν_2. It was already clear that some of the RG analysis had been done poorly, as shown by Janssen in 1997 (Janssen, 1997). Indeed Janssen's analysis is the only example I have found of field theory being used instead of the DRG, however the results seem to have been of limited impact. The only dissent expressed as to whether the λ_{22} renormalises trivially to all orders, let alone whether anymore serious oversights had taken place, was done so by Tang and Nattermann (Tang and Nattermann, 1991) six years previously.

Perhaps the most popular lattice models of the surface behaviour are the Das Sarma-Tamborenea (DT) model (Das Sarma and Tamborenea, 1991) and the Wolf-Villain (WV) model (Wolf and Villain, 1990). In 1996 Das Sarma, Lanczycki, Kotlyar and Ghaisas (Das Sarma *et al.*, 1996) carried out a wide-ranging analysis with the aim to match up appropriate continuum models with microscopic models, as different hopping rule for the underlying model lead to different continuum equations. For example, the DT model has height driven rules, some curvature driven rules and some consistent with neither view. The ν_4 coupling alone does not fully describe the DT model, but it may need more than the addition of the λ_{22} non-linearity. They suggest that replacing the term $\lambda_{22}\nabla^2\left(\nabla\phi(x,t)\right)^2$ with an infinite sum: $\sum_n \lambda_{2\,2n}\nabla^2\left(\nabla\phi(x,t)\right)^{2n}$. It is also noted that placing restrictions on downward hopping is believed to give long transient behaviour, before EW behaviour ultimately sets in, dominated by the ν_2 coupling and governed by (6.2).

The narrative arc has been driven by improved computation, with better and bigger numerical simulations of lattice models showing the presence of mound instabilities and long transient behaviour in the models, and a disillusionment with the RG.

Throughout the late 90's, Das Sarma continued to work in the area. He published with Dasgupta, Kim and Dutta (Dasgupta *et al.*, 1997) on instabilities appearing in discrete versions of the continuum representations, where isolated towers are found to grow on an otherwise flat surface. Higher order non-linearities, such as the $\lambda_{2\,2n}$ sum above, were suggested as a solution to these instabilities. A link to intermittency in fluid dynamics, first mooted with Bhattacharjee and Kotlyar in 1996 (Bhattacharjee *et al.*, 1996), is further explored. Das Sarma published with Punyindu (Das Sarma and Punyindu, 1997) on long transient behaviour and anomalous scaling in lattice models, speculating that perhaps there arise from using an infinite series as the non-linearity. The work Das Sarma did with Kundagrami, Dasgupta and Punyindu (Kundagrami *et al.*, 1998) further developed the connection to fluid dynamics and intermittency, with the DT model showing slow transient behaviour. The Reynolds number would be equivalent to the correlation length, the energy dissipation rate to the nearest-neighbour height difference.

Jung, Kim and Kim introduced non-local models (Jung *et al.*, 1998) in 1998, studying the VLDS equation with a non-local λ_{22} term and conserved noise, using the DRG to analyse it.

Significantly, mounding was observed in two-dimensional lattice models studied by Chatraphorn, Toroczkai and Das Sarma (Chatraphorn *et al.*, 2001) in 2001.

In 2002 Katzav (Katzav, 2002) used a self-consistent expansion method to look at MBE with a spatially correlated noise. Significantly he notes that Das Sarma is not publicly enamoured with the DRG. Subsequently, Das Sarma, Chatraphorn and Toroczkai (Das Sarma *et al.*, 2002) used a technique called 'noise reduction' in numerical simulations of the WV and DT models. They found that there had been long crossover problems in the past. The DT model is found to be described by the VLDS equation for substrate dimension $d = 1$, for $d = 2$ it is described by the EW equation. The WV model is described by the EW equation in $d = 1$. However, in $d = 2$ the WV model is unstable, with the appearance of regular mound formations. They reach the conclusion that the "universality class concept ... [is] of limited usefulness", and RG methods in general are disowned. It is noted for later that, in the course of the analysis, they claimed that generalising hopping rates from one dimension to two dimensions is trivial.

Interest in analytic work persisted, with Escudero (Escudero, 2008) building on the work of Marsili et al. (Marsili *et al.*, 1996) to find a(n allegedly) new curvature term to go with ν_2 and ν_4 couplings, a term that captures the same behaviour as the VLDS equation.

The approach in the other branch of the literature, instead of postulating a continuum equation and comparing to the numerical simulation of lattice models, is to derive a continuum equation by considering the transition rules on a lattice model of the surface. Vvedensky, Zangwill, Luse and Willby (Vvedensky *et al.*, 1993) published in 1993 on going from a master equation description of the surface dynamics to a continuum Langevin equation. The model featured atomic deposition from a low density vapour, thermal desorption and surface diffusion, adding in the possible refinements of hot atom knockout effects and/or asymmetric energy barriers near step edges. Fairly non-rigorously, they find the generic behaviour is given by the KPZ equation. If evaporation is negligible they get the EW behaviour of (6.2). If hot atom knockouts and asymmetric step barriers are also eliminated, leaving only deposition and diffusion, then the VLDS equation (6.4), is derived. Tracking the physical sources of the different terms, they have desorption leading to the couplings ν_2, λ_{13} and λ_{22}, and surface diffusion leading to ν_4. This approach has been pursued by some others (Huang and Gu, 1996), and Vvedensky has continued to work on generating the Langevin equation systematically by considering the underlying lattice transition rules (Haselwandter and Vvedensky, 2007a).

6.2 Comparison of Analyses

Now that some perspective has been given on the literature, an interpretation of the calculation presented in Chapter 5 can be made. After performing a one loop RG analysis it is possible to answer questions about how height-height correlations scale in the system, about the roughness of the surface. In the bulk of the literature the 'width of the surface' is used to characterise its roughness (Lai and Das Sarma, 1991; Marsili *et al.*, 1996). The 'width' is an equal time correlator for the height at spatially separated points on a finite substrate, linear size L, dimension D, where the time is the total growth time from a flat distribution:

$$\begin{aligned} W(L,t) &= \left\langle \frac{1}{L^D} \int d^D x \, \left[\phi(x,t) - \bar{\phi}(L,t)\right]^2 \right\rangle^{1/2}, \\ \bar{\phi}(L,t) &= \frac{1}{L^D} \int d^D x \; \phi(x,t). \end{aligned} \tag{6.13a}$$

The surface width is sometimes expressed using the notation of the local difference from the average, $\delta\phi(x,t) = \phi(x,t) - \bar{\phi}(L,t)$. The surface width obeys a scaling relation (Family and Vicsek, 1985, 1991)

$$W(L,t) = a\,L^{\alpha} F\left(\frac{t}{bL^{z}}\right)$$
$$W(L,t) = \begin{cases} \tilde{a}\,t^{\alpha/z} & 1 \ll t \ll L^{z} \\ a\,L^{\alpha} & t \gg L^{z}, \end{cases} \tag{6.13b}$$

for some suitable metric factors a, $\tilde{a}$ and b. In the following a and b will be used as generically appropriate metric factors. The behaviour of the scaling function, $F(x)$, is seen to be such that it is a constant for large arguments, $F(x \to \infty) = 1$, and for small arguments goes as a simple power, $F(x \to 0) = x^{\alpha/z}$. This introduces the two scaling exponents, α and z, that are traditionally of primary interest. Instead, one could work directly with a scaling ansatz for the field and an equal-time height-height correlation function (Das Sarma *et al.*, 1996),

$$\phi(x,t) = s^{-\alpha}\phi(sx, s^{z}t)$$
$$\mathbf{G}(r,t) = a\,r^{2\alpha} g\left(\frac{r}{\xi(t)}\right) \qquad \xi(t) = b\,t^{1/z}, \tag{6.14a}$$

with the same behaviour for the scaling function $g(x)$, as for $F(x)$. Following on to, or alternatively skipping that first step and proudly asserting (Pimpinelli and Villain, 1998)

$$\mathbf{G}(r,t)\,\Big(= \langle(\phi(r',t) - \phi(r'+r,t))^{2}\rangle\Big) = \begin{cases} a\,r^{2\alpha} & r < \xi(t) \\ a\,t^{2\alpha/z} & r > \xi(t), \end{cases} \tag{6.14b}$$

where the initial conditions are understood to be a flat distribution: $\phi(r,0) = 0$ and $\mathbf{G}(r,0) = 0$. This is in contrast to using the more traditional dynamic scaling of correlations between space-time separated points, which is also generally concerned with correlations after the saturation of the interfacial roughness. Forming this into similar definition to the width (Marsili *et al.*, 1996) gives

$$C_L(x,t;x',t') = \langle[\delta\phi(x,t) - \delta\phi(x',t')]^{2}\rangle$$
$$C_L(x,t+\tau;x',t) = a\,|x-x'|^{2\alpha}\tilde{g}\left(\frac{\tau}{b|x-x'|^{z'}}\right), \tag{6.15}$$

in which the new dynamic exponent z' is not necessarily equal to the old z, though the roughness exponent α is necessarily the same, as can be seen by taking a spatial integral at $\tau = 0$. For the present purpose, it is safe to assume that the dynamic exponent is not affected, that z and z' are equal. The different attitudes to time in correlation functions (Pimpinelli and Villain, 1998) stem from wanting to run simulations, and track changes, from an initially flat distribution. In the field theory this is not particularly germane, as once the surface has become saturated, the one time scale available is the time-separation between fields, initial conditions do not raise themselves as an issue for the long-time behaviour, nor is finite system size a consideration. Instead of dealing with a surface-width-like definition, it is more convenient to use a scaling form for the two-point connected correlation function straightaway,

$$\hat{\mathbf{G}}^{20}(r,\tau) = a\, r^{2\alpha} \mathcal{G}\left(\frac{\tau}{br^z}\right), \tag{6.16}$$

with scaling function $\mathcal{G}(x)$. Fourier transforming, and making use of the relations derived between the connected correlation functions and that vertex functions in (3.31) gives

$$\begin{aligned} \hat{\mathbf{G}}^{20}(q,\omega) &= aq^{-(d+z+2\alpha)}\ \mathcal{G}\left(\frac{\omega}{bq^z}\right) \\ \hat{\mathbf{G}}^{20}(q,\omega) &= -\hat{\Gamma}^{20}(q,\omega)\left|\hat{\Gamma}^{11}(q,\omega)\right|^{-2} \\ &= 2\Gamma^2 \left| aq^{\gamma_4+4}\ \hat{\Gamma}^{11}\left(q, \frac{\omega}{bq^{\gamma_4+4}}\right)\right|^{-2}, \end{aligned} \tag{6.17a}$$

leading to the identification of the exponents

$$z = 4 + \gamma_4 \qquad \alpha = \frac{\epsilon + \gamma_4}{2}, \tag{6.17b}$$

where γ_4 is the γ-function as given in Section 5.2, describing the deviation from naïve engineering dimension of the coupling ν_4. From the scaling exponent definitions in (5.19), the relation between δ and α is

$$2\alpha + d = 4 + \delta. \tag{6.18}$$

6.2.1 The exponents and renormalisation schemes

For the EW model, these exponents can be re-derived with a quick and dirty, almost schematic, calculation:

$$\begin{aligned} \partial_t \phi(x,t) &= \nu_2 \nabla^2 \phi(x,t) + \eta(x,t) \\ \phi(q,\omega) &= \frac{\eta(q,\omega)}{-i\omega + \nu_2 q^2} \\ \langle \phi(q,\omega)\phi(q',\omega')\rangle &= \frac{\langle \eta(q,\omega)\eta(q',\omega')\rangle}{\omega^2 + \nu_2^2 q^4} \\ &= \frac{2\Gamma^2 \delta^d(q+q')\delta(\omega+\omega')}{\omega^2 + \nu_2^2 q^4} \end{aligned}$$

$$
\begin{aligned}
&= \hat{\mathbf{G}}^{20}(q,\omega)\bar{\delta}^d(q+q')\bar{\delta}(\omega+\omega') \\
\hat{\mathbf{G}}^{20}(q,\omega) &= \frac{2\Gamma^2}{\omega^2+\nu_2^2 q^4} \\
&= q^{-4}\left[\frac{2\Gamma^2}{\nu_2^2+\left(\frac{\omega}{q^2}\right)^2}\right] \\
&= a\, q^{-4}\cdot\tilde{\mathcal{G}}\left(\frac{\omega}{bq^2}\right), \\
z = 2 \qquad \alpha &= \frac{2-d}{2}.
\end{aligned} \tag{6.19a}
$$

Where, for the EW model, the upper critical dimension is $d_c = 2$, and the lower critical dimension is $d_{lc} = 0$ (Pimpinelli and Villain, 1998).

Similarly, this derivation can used to find the exponents for the MH model:

$$
\begin{aligned}
\partial_t\phi(x,t) &= \nu_4\nabla^4\phi(x,t)+\eta(x,t) \\
\phi(q,\omega) &= \frac{\eta(q,\omega)}{-i\omega+\nu_4 q^4} \\
\langle\phi(q,\omega)\phi(q',\omega')\rangle &= \frac{\langle\eta(q,\omega)\eta(q',\omega')\rangle}{\omega^2+\nu_4^2 q^8} \\
&= \frac{2\Gamma^2\bar{\delta}^d(q+q')\bar{\delta}(\omega+\omega')}{\omega^2+\nu_4^2 q^8} \\
&= \hat{\mathbf{G}}^{20}(q,\omega)\bar{\delta}^d(q+q')\bar{\delta}(\omega+\omega') \\
\hat{\mathbf{G}}^{20}(q,\omega) &= \frac{2\Gamma^2}{\omega^2+\nu_4^2 q^8} \\
&= q^{-8}\left[\frac{2\Gamma^2}{\nu_4^2+\left(\frac{\omega}{q^4}\right)^2}\right] \\
&= a\, q^{-8}\cdot\tilde{\mathcal{G}}\left(\frac{\omega}{bq^4}\right), \\
z = 4 \qquad \alpha &= \frac{4-d}{2}.
\end{aligned} \tag{6.19b}
$$

For the MH model the upper critical dimension is $d_c = 4$, and the lower critical dimension is $d_{lc} = 2$ (Pimpinelli and Villain, 1998). Using $\epsilon = 4 - d$ makes sense in this case, giving $\alpha = \epsilon/2$.

For the VLDS model (Lai and Das Sarma, 1991), the scaling exponents are

$$
z = \frac{8+d}{3} \qquad \alpha = \frac{4-d}{3} \qquad \text{or} \qquad z = 4-\frac{\epsilon}{3} \qquad \alpha = \frac{\epsilon}{3}. \tag{6.19c}
$$

Again, with upper critical dimension $d_c = 4$, and lower critical dimension $d_{lc} = 2$.

From the field theoretic analysis I performed, as seen in Chapter 5, there are three physically relevant fixed points to compare to the three standard models given above. First, at the critical point, $\nu_2 = \nu_2^c$, of the full theory

$$
\begin{aligned}
\partial_t\phi(x,t) = \nu_2\nabla^2\phi(x,t) - \nu_4\nabla^4\phi(x,t) + \lambda_{13}\nabla\left(\nabla\phi(x,t)\right)^3 \\
+ \lambda_{22}\nabla^2\Big(\nabla\phi\Big)^2 + \kappa\left[\nabla\left(\nabla\phi\nabla^2\phi\right) - \frac{1}{2}\nabla^2\Big(\nabla\phi\Big)^2\right] + \eta(x,t),
\end{aligned}
\tag{6.20a}
$$

there is a stable fixed point. The exponents at this fixed point, with the addition of α, are

$$
\begin{gathered}
z = 4 \qquad \alpha = \frac{\epsilon}{2} \qquad \nu = \frac{1}{2} + \frac{3\epsilon}{20} \qquad \delta = 0 \\
\pi_g = 0 \qquad \pi_\lambda = -\frac{4\epsilon}{5} \qquad \pi_\chi = 0 \\
\pi_{13} = 0 \qquad \pi_{22} = -\frac{2\epsilon}{5} \qquad \pi_\kappa = -\frac{2\epsilon}{5} \\
\eta = 0 \qquad\qquad \tilde{\eta} = 0.
\end{gathered}
\tag{6.20b}
$$

At first glance, the exponents for the stable fixed point of the full theory would make it the same theory as the MH model. The exponents α and z are the same as for MH, and the fixed point coupling values in (5.23) imply both λ_{22} and κ are driven to zero. However, there is more going than in the simple, linear MH model. The exponent ν, describing the approach to the fixed point, has an ϵ correction. The ν_2 renormalisation, from having $\lambda_{13} \neq 0$, means the critical point of the full theory is non-trivial, with $\nu_2^c \neq 0$. Although the non-linearities κ and λ_{22} are driven to zero, the exponents π_κ and π_{22} provide transient length scales at which to observe their presence.

If the full theory is adjusted to the limited case, so that $\lambda_{22} = 0$ from the start,

$$
\begin{aligned}
\partial_t\phi(x,t) = \nu_2\nabla^2\phi(x,t) - \nu_4\nabla^4\phi(x,t) + \lambda_{13}\nabla\left(\nabla\phi(x,t)\right)^3 \\
+ \kappa\left[\nabla\left(\nabla\phi\nabla^2\phi\right) - \frac{1}{2}\nabla^2\Big(\nabla\phi\Big)^2\right] + \eta(x,t).
\end{aligned}
\tag{6.21a}
$$

This changes the exponents at the stable fixed point to

$$
\begin{gathered}
z = 4 \qquad \alpha = \frac{\epsilon}{2} \qquad \nu = \frac{1}{2} + \frac{3\epsilon}{20} \qquad \delta = 0 \\
\pi_g = 0 \qquad \pi_{13} = 0 \qquad \pi_X = -\epsilon \qquad \pi_\kappa = -\frac{\epsilon}{2} \\
\eta = 0 \qquad\qquad \tilde{\eta} = 0.
\end{gathered}
\tag{6.21b}
$$

The key difference brought about by the absence of λ_{22} is an adjustment of π_κ. This alters the flow of κ to zero, and thus tweaks the transient length scale at which κ can be observed. At one loop, these two theories are otherwise indistinguishable.

Finally, there is the 'trivial' fixed point of the full theory. That is, the theory with $\lambda_{13} = 0$,

$$\begin{aligned}\partial_t \phi(x,t) =& \nu_2 \nabla^2 \phi(x,t) - \nu_4 \nabla^4 \phi(x,t) + \lambda_{22} \nabla^2 \Big(\nabla \phi\Big)^2 \\ &+ \kappa \left[\nabla \left(\nabla \phi \nabla^2 \phi \right) - \frac{1}{2} \nabla^2 \Big(\nabla \phi \Big)^2 \right] + \eta(x,t).\end{aligned} \tag{6.22a}$$

The exponents are

$$\begin{gathered} z = 4 - \frac{\epsilon}{3} \qquad \alpha = \frac{\epsilon}{3} \qquad \nu = \frac{1}{2} + \frac{\epsilon}{12} \qquad \delta = -\frac{\epsilon}{3} \\ \pi_\psi = 0 \qquad \pi_\chi = 0 \qquad \pi_{22} = 0 \qquad \pi_\kappa = 0 \\ \eta = -\frac{\epsilon}{6} \qquad \tilde{\eta} = \frac{\epsilon}{6}. \end{gathered} \tag{6.22b}$$

In the absence of λ_{13} there is no renormalisation of ν_2 at one loop, and so the critical point is trivially $\nu_2^c = 0$. The scaling exponents z and α are the same as the traditional one loop VLDS exponents. However, the exponent ν does not appear to have been calculated before, nor the field exponents. Interestingly, it was also found that though κ and λ_{22} are not renormalised at one loop order, renormalisation does imposes bounds on them, or the fixed point becomes unphysical. In order to get sensible results, at one loop the ratio of the non-linearities, $\chi = \dfrac{\kappa}{\lambda_{22}}$, is restricted to $\chi \in \left[\frac{3-\sqrt{19}}{5}, \frac{3+\sqrt{19}}{5}\right]$. Sufficiently large κ, or small λ_{22}, violates this. So it is not possible to have a theory with the coupling κ as the only non-linearity. It is possible to have $\kappa = 0$, and if λ_{22} is the only non-linearity present the theory becomes the original VLDS formulation (6.4). There is no transient length scale at one loop, κ and λ_{22} are present at all length scales.

Regarding the renormalisation more broadly, there was some discussion in Section 5.1.5 and Section 5.3. In keeping with the results in the literature, I found that the λ_{13} non-linearity generates ν_2 with the tadpole diagram Θ_1 in (5.1). Also in agreement with previous work, the λ_{22} coupling is found not to renormalises itself at one loop. Pleasingly the same holds for κ, so the statement could be strengthened to say that the vertex function $\hat{\Gamma}^{12}$ does not renormalise in its own right at one loop, the contributions from the diagrams neatly cancelling at this order.

The most significant disagreement with previous work is the generation of the new coupling κ, or even its inclusion. It should surely have been included in the original derivation, as can be seen just by writing down all the different terms that are fourth order in the gradient. Its existence should also have been revealed under renormalisation, when the interaction of λ_{22} and λ_{13} generates κ with the diagram Ξ_3 in (5.4).

The other $\hat{\Gamma}^{12}$ coupling, λ_{22}, is in principle generated by an interaction between λ_{13} and κ, however all the contributions cancel at one loop, so it would stay at $\lambda_{22} = 0$. The coupling λ_{13} cannot generate either of the $\hat{\Gamma}^{12}$ couplings on its own. Likewise, power counting shows that λ_{13} cannot be generated by them. Interestingly, the mixing of λ_{22} and κ in diagrams is particularly vital for κ, as $\hat{\Gamma}^{12}$ diagrams composed entirely of κ are not relevant, so κ cannot renormalise on its own.

The non-renormalisation of the λ_{22} and κ couplings does not hold beyond one loop order, as originally shown by Janssen (Janssen, 1997), in contrast to the initial beliefs of the community (Das Sarma and Kotlyar, 1994; Kim and Das Sarma, 1995). The theory Janssen studied had only the λ_{22} coupling, with κ unknown to the community it was implicitly set to zero. The two loop analysis led to corrections to the exponents, the inclusion of κ in a two loop calculation should lead to new corrections. Indeed, the presence of potential κ corrections in a model may explain why the corrections to scaling predicted by Janssen were found to be too small in numerical simulations (Yook *et al.*, 1997, 1998; Aarão Reis, 2004). That λ_{22} and κ will renormalise non-trivially above one loop order should not be controversial position to hold. Yet the attribution of these deviations to, and even the existence of, the two loop correction has been either ignored, or questioned (Katzav, 2002; Escudero, 2008). The accuracy of a correction when the dimension is far below the upper critical one and close to or possibly below the lower critical one is certainly a problem, though this was not the issue raised, for example, by Katzav. His application of a self-consistent expansion method agreed with the 'exactness' of the original one loop DRG results, and not the correction offered by Janssen.

It has been my observation that the renormalisation done in this part of the literature has been poor, symptomatic of this are the frequent arguments that make rather confusing uses of 'relevance' (Das Sarma and Kotlyar, 1994; Kshirsagar and Ghaisas, 1996; Bhattacharjee *et al.*, 1996; Dasgupta *et al.*, 1997; Das Sarma and Punyindu, 1997; Das Sarma *et al.*, 2002). This has a fairly precise technical meaning, about which terms can be ignored and under which conditions, about the renormalisability of a given theory.

6.3 Interpretation of κ

In interpreting the effects and influence of including the κ non-linearity in the Langevin equation, it is instructive to examine the work presented by Escudero in 2008 (Escudero, 2008), with a more recent follow up in 2012 (Escudero and Korutcheva, 2012). Escudero uses the same variational principle as Marsili et al. (Marsili *et al.*, 1996) in order to find an apparently new curvature term that, along with the ν_2 and ν_4 couplings, captures the same MBE behaviour as the VLDS equation. The new curvature term is referred to as the Monge-Ampère operator,

$$\partial_{xx}\phi \, \partial_{yy}\phi - (\partial_{xy}\phi)^2. \tag{6.23}$$

At the critical point, $\nu_2 = 0$, the model is claimed to be in the VLDS universality class, with the new non-linearity dominating and the ν_4 diffusive term becoming irrelevant. He rightly complements Vvedensky and Haselwandter on a "formidable" RG calculation, and agrees that the generic system behaviour moves transiently from MH, to VLDS, and ultimately to EW. Importantly, the non-linearity is shown to favour mound formation. It can also be easily interpreted in terms of microscopic processes, expressing optimal mass rearrangement, with newly deposited material moving to minimise the interface chemical potential. It is apparently also seen in atmospheric physics, in semigeostrophic flow (Cullen *et al.*, 1991).

In fact the formulation of the operator above is obfuscatory, and the pristine two-dimensional dressing is ultimately unhelpful. Fourier transforming to find a more convenient formulation, the Monge-Ampère operator capitulates to

$$\begin{aligned} \kappa\partial_{xx}\phi \, \partial_{yy}\phi - \kappa(\partial_{xy}\phi)^2 &\to -\frac{\kappa}{2}\nabla^2(\nabla\phi)^2 + \kappa(\nabla^2\phi)^2 + \kappa\nabla\phi\cdot\nabla^3\phi \\ &= -\frac{\kappa}{2}\nabla^2(\nabla\phi)^2 + \kappa\nabla(\nabla\phi\nabla^2\phi). \end{aligned} \tag{6.24}$$

It should then come as no surprise that it reproduces VLDS behaviour. A moments further thought on its lineage from Marsili et al., with Escudero studying the curvature derived terms that had been suggested, would link it straight to (6.12), and so again to κ and the VLDS equation. This would then suggest Escudero's derivation was halted before λ_{13} was produced.

Additionally, it is claimed by Escudero that the agreement with one loop DRG results is exact, that the result is not altered at higher orders. There is some truth to this, there is a good reason why the κ coupling on its own does not renormalise. This is because pursuing the calculation with only κ is especially straightforward, power counting reveals that diagrams for its renormalisation, diagrams constructed solely from the κ and the noise vertices, are always ultraviolet finite. In the absence of any other coupling κ is not renormalised at any order. The only renormalisation is of the propagator, with one diagram at one loop order. It is a peculiarity of having only the κ non-linearity that leads to non-renormalisation of the coupling, as opposed to a neat cancellation at one loop when λ_{22} is also present. However, while a model containing only κ will reproduce the VLDS exponents, that model's infrared stable fixed point is unphysical, and its behaviour thus not assessable by perturbation theory. The conclusion, that the κ-only theory reproduces VLDS behaviour and that the scaling laws are exact, is invalid. Perhaps other analyses finding exact scaling laws have inadvertently examined this case, rather than the VLDS equation.

The insight that the κ non-linearity favours mound formation provides a solid basis for interpretation. I would suggest that, in addition to Ehrlich-Schwoebel (ES) barriers expressed through ν_2 (Krug, 1997; Michely and Krug, 2004), it provides a natural mechanism at the level of the continuum equations for mound formation. Mound formation that is observed not only in numerical simulations of lattice models (Chatraphorn *et al.*, 2001; Das Sarma *et al.*, 2002), but experimentally (Lengel *et al.*, 1999; Kunkel *et al.*, 1990; van der Vegt *et al.*, 1992; Ernst *et al.*, 1994).

In terms of the one loop analysis this means that, at the critical point of the full theory, mounding is suppressed on the large scale, yet visible at and below (transient) length scales $\propto \kappa^{5/(2\epsilon)}$. Recent work on coupling flow in models of ideal MBE (Haselwandter and Vvedensky, 2007a) provides some theoretical justification for transient observation of mounding. Similarly in the case of the limited theory, the presence of the λ_{13} coupling suppresses mound formation, but it will be visible on a length scale $\propto \kappa^{2/\epsilon}$. Regarding the limited theory, it is not clear to me that is remains a good model of ideal MBE in the absence of the λ_{22} coupling. As the λ_{22} coupling was regarded as the 'true non-equilibrium term', with all the others derivable from a Hamiltonian, the limited theory may not satisfactorily capture the require non-equilibrium behaviour of MBE.

At the 'trivial' fixed point VLDS scaling applies generically, as with $\lambda_{13} = 0$, ν_2^c does not suffer an additive renormalisation, and mounding may be present on all length scales. This is important, as in the growth community there is almost universal antipathy to the idea of tuning parameters. There is no external field, so there would seem to be no mechanism to do so other than through the initial set up. However it is a natural procedure to use as a field theorist for a systematic analysis.

A connection may also be made to the dynamics used in lattice models of ideal MBE. There have been several investigations into the link between the rules of movement in lattice models and the terms in continuum equations that represent them, usually concentrating on one-dimensional models (Das Sarma *et al.*, 1996). Hugston and Ketterl (Hagston and Ketterl, 1999) showed that going from growth rules that seem intuitive, or even computationally convenient, to continuum equations is subtle and fraught with unintended consequences. Generalising the rules from a one-dimensional to a two-dimensional lattice is not a trivial process, in contrast to the belief of some (Das Sarma *et al.*, 2002). This does beg questions about what system would be described by just κ, and what lattice dynamics this term captures. One could speculate that the surface currents, or other two-dimensional effects such as the mounding that Chatraphorn, Toroczkai and Das Sarma (Chatraphorn *et al.*, 2001) looked at in simulations, may be captured by κ. Step edge diffusion (Chatraphorn *et al.*, 2001), appearing in two dimensions, has been proffered as an additional mechanism to ES barriers that leads to unstable mounding, for example in the two-dimensional WV model (Das Sarma *et al.*, 2002). The lattice rules for the DT model are slightly different, resulting in EW behaviour instead of mounding in two dimensions. The consequences of seemingly minor differences in lattice rule may also shed some light on why unexpected terms crop up now and then, such as in an analysis of the WV model (Kim, 2000) which throws up the $(\nabla^2\phi)^2$ part of the κ term. Differences in lattice rules will be the distinction between having mound formation and the coupling κ, or not, in the continuum equation for a lattice model.

The presence of the new coupling means that the paths through phase space and amplitudes, that are of particular interest to Vvedensky and Haselwandter (Haselwandter and Vvedensky, 2007a,b, 2008), must surely be altered. Indeed it was a stroke of misfortune on their part that they missed this term; as can be seen in a detailed, and otherwise excellent, appendix (Haselwandter and Vvedensky, 2007b), the diagram that generates κ was missed out.

6.3.1 Why κ was missed

It is important to offer some possible explanations as to why the κ interaction that I have introduced was missed in the literature over the last twenty years. To reiterate, the κ non-linearity should have been present for consideration in models of ideal MBE from the beginning; either from constructing fourth order terms in the gradient, ∇, that are conservative, or because it is generated anyway under renormalisation by an interaction between the λ_{22} and λ_{13} non-linearities. In speculating as to the reasons for the κ interaction being missed, rationalisations must be offered for the continuum equation lead analysis and for the lattice-based, master equation approach. Underlying both is a bias to one-dimensional systems that unfortunately concealed the presence of κ.

Regarding the use of master equations to generate continuum equations from lattice rules; the most telling sign of one-dimensional blindness is to be found in a paper by Huang and Gu (Huang and Gu, 1996). In developing a continuum equation for the WV model, derivatives are rearranged in one dimension, and the result generalised to higher dimensions where the rearrangement is invalid, e.g.: Eq.(23) from Eq.(21).[2] This paper also illustrates a second commonly committed sin. While the continuum equation they develop for the DT model just gives the VLDS terms ν_4 and λ_{22}, the WV model leads to the couplings ν_4, λ_{22} and λ_{13}. Under renormalisation this produces the ν_2 coupling so, from their point of view, in the absence of tuning the model will be generically in the EW class. They do not look at cross terms between the couplings, as ν_2 is the trump card, stopping the analysis once it appears. So there is no reason for them to suspect, find, or care about the presence of κ.

It is this general antipathy to tuning parameters that is, I suspect, why κ was not seen to be generated under renormalisation. It was not generated because analysis immediately focused on $\lambda_{13} = 0$ to prevent ν_2 generation. If ever $\lambda_{13} \neq 0$ then λ_{22} would be set to zero "without loss of generality", as λ_{13} was deemed "more relevant" (Das Sarma and Kotlyar, 1994), and further analysis halted by the generation of ν_2. One analysis (Haselwandter and Vvedensky, 2007a,b) did not do this, but unfortunately missed the diagram generating the coupling κ.

[2] Continuing the theme of close encounters with the coupling κ, in the 1993 paper by Vvedensky et al. (Vvedensky *et al.*, 1993), when investigating the role of left-right asymmetry in producing a Laplacian term, the missing κ term appears briefly as $\nabla\left[\ldots D\nabla\phi\nabla^2\phi\right]$ in Eq.(49) and Eq.(50).

As to the work driven by considering continuum equations, it must be supposed that κ was missed out initially by writing, and thinking of, one-dimensional derivatives. Where in one dimension there is no distinction between the λ_{22} and κ couplings. Thereafter κ does not appear under renormalisation as the two interactions, λ_{22} and λ_{13}, are never turned on at the same time.

There are, shamefully, some incorrect uses of vector identities scattered through the literature. For example, in the big computational review in 1996 by Das Sarma et al. (Das Sarma *et al.*, 1996), which aimed to match up appropriate continuum models with microscopic models, they use an incorrect identity:

$$\nabla(\nabla\phi)^3 = 3(\nabla\phi)^2\nabla^2\phi, \tag{6.25}$$

which is accidentally true in one dimension, but should in general involve a term more properly written as $(\nabla\phi\cdot\nabla)(\nabla\phi)^2$ (Stephenson, 1973). Fourier transforming such terms immediately reveals the problem. This is presumably symptomatic of a one-dimensional bias, and why κ was initially missed. One might wonder whether basic errors in multivariate calculus have thus far contributed to concealing the κ term.

6.4 Summary

In summary, I have shown that a field theoretic analysis of the original VLDS formulation, (6.1), generates a mounding term, κ, under a one loop renormalisation group calculation. This mounding term has been overlooked in the past, partly because of the deliberate omission of non-linearities, partly because the renormalisation schemes that were employed missed the generation of κ, and partly from an apparent misunderstanding of basic multivariate calculus.

In addition to the known Ehrlich-Schwoebel barriers captured by ν_2, the κ interaction provides another natural mechanism for mound formation at the level of the continuum equation (Escudero, 2008), mound formation that has been observed experimentally (Lengel *et al.*, 1999; Kunkel *et al.*, 1990; van der Vegt *et al.*, 1992; Ernst *et al.*, 1994). The κ interaction may effectively capture lattice rules that give rise to mound formation in computer simulations of lattice models (Chatraphorn *et al.*, 2001; Das Sarma *et al.*, 2002).

For models corresponding to the full theory, (6.20), mound formation will be suppressed, but will be observable on transient length scales $\propto \kappa^{5/(2\epsilon)}$. In the absence of the λ_{22} non-linearity this length scale is adjusted to $\propto \kappa^{2/\epsilon}$.

At the trivial fixed point of the full theory the traditional VLDS behaviour of (6.4) can be recovered, though equation (6.22) should more properly be called the full VLDS equation. At the fixed point of the full VLDS equation there is no transient length scale, and if the κ interaction is present, mound formation will occur at all length scales.

I have proposed a resolution to an apparent conflict in the literature. As inadvertently studying models with only the κ non-linearity present, instead of the λ_{22} non-linearity, may lead to the conclusion that scaling relations are in general exact, though such a model has no physical fixed point, and the conclusion is false. The presence of both κ and λ_{22} may account for the differences with expected scaling corrections found in simulation.

Chapter 7

Closing Remarks

The opening pair of chapters, in which I gave an introduction to the renormalisation group applied to equilibrium systems, were converted from a corresponding pair of seminars I have given. They were not intended to give a comprehensive account of the very rich fields of critical phenomena and phase transitions, and the renormalisation group. As such, it would be remiss of me not to acknowledge the incomplete treatment they were given, and point to the underlying work from which this thesis is constructed.

The renormalisation group as I know it owes everything to the revolutionary work done by Wilson in the early seventies. Building on the earlier work of Kadanoff and Widom (Kadanoff, 1966b; Widom, 1965b,a), along with Fisher he put it on firm theoretical footing (Wilson and Fisher, 1972), with early review papers by Wilson, by Wilson and Kogut, and by Fisher (Wilson, 1975a; Wilson and Kogut, 1974; Fisher, 1974). It was Brézin, Le Guillon and Zinn-Justin in Vol. 6 of the Domb and Green series (Domb *et al.*, 1976) who clearly laid out the foundations of the field theory approach to renormalisation. Of the many excellent books in the area, I have already mentioned those by Le Bellac and by Amit (Le Bellac, 1991; Amit, 1984), I would also like to mention those by Zinn-Justin, by Ma, by Collins, by Itzykson and Zuber, and by Itzykson and Drouffe (Zinn-Justin, 2002; Itzykson and Zuber, 1986; Itzykson and Drouffe, 1991; Collins, 1986; Ma, 1976).

Appendix A

Connected Correlation Functions and Vertex Functions

As presented in Section 3.3, the two fields defined in order to calculate the effective action are

$$\theta(x,t) = \frac{\delta \ln \mathcal{Z}\left[j,\tilde{j}\right]}{\delta j(x,t)}, \qquad \tilde{\theta}(x,t) = \frac{\delta \ln \mathcal{Z}\left[j,\tilde{j}\right]}{\delta \tilde{j}(x,t)}. \tag{A.1}$$

The effective action, $\Gamma[\tilde{\theta},\theta]$, is then the Legendre transform of the logarithm of the moment generating function, $\mathcal{Z}\left[j,\tilde{j}\right]$,

$$\Gamma[\tilde{\theta},\theta] = -\ln \mathcal{Z}\left[j,\tilde{j}\right] + \int d^d x \int dt \left(\tilde{j}(x,t)\tilde{\theta}(x,t) + j(x,t)\theta(x,t)\right). \tag{A.2}$$

As already shown, taking a derivative with respect to one of the new fields returns the associated source field:

$$\begin{aligned}
\frac{\delta\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(x,t)} &= \tilde{j}(x,t) + \int d^d x \int dt\, \tilde{\theta}(x,t)\frac{\delta\tilde{j}(x,t)}{\delta\tilde{\theta}(x,t)} \\
&\quad + \int d^d x \int dt\, \theta(x,t)\frac{\delta j(x,t)}{\delta\tilde{\theta}(x,t)} - \frac{\delta \ln \mathcal{Z}\left[j,\tilde{j}\right]}{\delta\tilde{\theta}(x,t)} \\
&= \tilde{j}(x,t) + \int d^d x \int dt \left(\tilde{\theta}(x,t)\frac{\delta\tilde{j}(x,t)}{\delta\tilde{\theta}(x,t)} + \theta(x,t)\frac{\delta j(x,t)}{\delta\tilde{\theta}(x,t)}\right) \\
&\quad - \int d^d x \int dt \left(\underbrace{\frac{\delta \ln \mathcal{Z}\left[j,\tilde{j}\right]}{\delta\tilde{j}(x,t)}}_{\tilde{\theta}(x,t)}\frac{\delta\tilde{j}(x,t)}{\delta\tilde{\theta}(x,t)} + \underbrace{\frac{\delta \ln \mathcal{Z}\left[j,\tilde{j}\right]}{\delta j(x,t)}}_{\theta(x,t)}\frac{\delta j(x,t)}{\delta\tilde{\theta}(x,t)}\right) \\
&= \tilde{j}(x,t).
\end{aligned} \tag{A.3}$$

Using the condensed notation, where a space-time point (x, t) in an argument is abbreviated to (y), the two derivatives at this order are

$$\frac{\delta\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)} = \tilde{j}(y), \qquad \frac{\delta\Gamma[\tilde{\theta},\theta]}{\delta\theta(y)} = j(y). \tag{A.4}$$

It was also shown in Section 3.3 that taking derivatives with respect to the source fields will generated relations between the connected correlation functions and the vertex functions,

$$\begin{aligned}
\frac{\delta^2\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\tilde{j}(\bar{y})} &= \delta^{(d+1)}(y-\bar{y}) \\
&= \int\limits_{y'} \frac{\delta^2\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\tilde{\theta}(y')}\frac{\delta\tilde{\theta}(y')}{\delta\tilde{j}(\bar{y})} + \frac{\delta^2\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\theta(y')}\frac{\delta\theta(y')}{\delta\tilde{j}(\bar{y})} \\
&= \int\limits_{y'} \frac{\delta^2\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\tilde{\theta}(y')}\frac{\delta^2\ln\mathcal{Z}[j,\tilde{j}]}{\delta\tilde{j}(y')\delta\tilde{j}(\bar{y})} + \frac{\delta^2\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\theta(y')}\frac{\delta^2\ln\mathcal{Z}[j,\tilde{j}]}{\delta j(y')\delta\tilde{j}(\bar{y})} \\
\left.\frac{\delta^2\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\tilde{j}(\bar{y})}\right|_{j,\tilde{j}=0} &= \int\limits_{y'} \frac{\delta^2\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\theta(y')}\left\langle\phi(y')\tilde{\phi}(\bar{y})\right\rangle \\
\delta^{d+1}(y-\bar{y}) &= \int\limits_{y'} \Gamma^{11}(y,y')\mathbf{G}^{11}(y',\bar{y}),
\end{aligned} \tag{A.5}$$

where a condensed notation for the integrals was introduced, so that $\int_y$ indicates is an integral over a space-time y; with an integral over the d-dimensional spatial measure, $d^d x$, and the one-dimensional time measure dt. The first term in the third line was dropped, as it contains a correlation function of response fields only, which is zero by definition. The notation for the vertex functions and the connected correlation functions was then defined as

$$\begin{aligned}
\mathbf{G}^{nm}(y'_1,\ldots,y'_n,\bar{y}_1,\ldots,\bar{y}_m) &= \left.\frac{\delta^{n+m}\ln\mathcal{Z}[j,\tilde{j}]}{\delta j(y'_n)\ldots\delta j(y'_1)\delta\tilde{j}(\bar{y}_m)\ldots\delta\tilde{j}(\bar{y}_1)}\right|_{j,\tilde{j}=0}, \\
\Gamma^{nm}(\bar{y}_1,\ldots,\bar{y}_n,y'_1,\ldots,y'_m) &= \frac{\delta^{n+m}\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(\bar{y}_n)\ldots\delta\tilde{\theta}(\bar{y}_1)\delta\theta(y'_m)\ldots\delta\theta(y'_1)}.
\end{aligned} \tag{A.6}$$

And the relationships, with a word of guidance on how to derive them, were then given as

$$\hat{\Gamma}^{11}(k) = \frac{1}{\hat{\mathbf{G}}^{11}(k)}, \tag{A.7a}$$

$$\hat{\Gamma}^{20}(k) = \frac{-\hat{\mathbf{G}}^{20}(k)}{\hat{\mathbf{G}}^{11}(k)\hat{\mathbf{G}}^{11}(-k)}, \tag{A.7b}$$

$$\hat{\Gamma}^{02}(k) = 0, \tag{A.7c}$$

$$\hat{\Gamma}^{12}(k_1,k_2) = \frac{-\hat{\mathbf{G}}^{12}(k_1,k_2)}{\hat{\mathbf{G}}^{11}(-k_1)\hat{\mathbf{G}}^{11}(-k_2)\hat{\mathbf{G}}^{11}(-k_1-k_2)}, \tag{A.7d}$$

$$\hat{\Gamma}^{13}(k_1,k_2,k_3) = \frac{-\hat{\mathbf{G}}^{13}(k_1,k_2,k_3)}{\hat{\mathbf{G}}^{11}(-k_1)\hat{\mathbf{G}}^{11}(-k_2)\hat{\mathbf{G}}^{11}(-k_3)\hat{\mathbf{G}}^{11}(-k_1-k_2-k_3)}. \tag{A.7e}$$

I shall now be a little more explicit.

A.1 The Functional Derivatives

Firstly, the functional derivatives need to be taken. There are four possible derivative that can taken with respect to a new field and then a source field, the first was given above in (A.5), the second is

$$\begin{aligned}\frac{\delta^2\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta j(\bar{y})} &= 0\\ &= \int\limits_{y'} \frac{\delta^2\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\tilde{\theta}(y')}\frac{\delta\tilde{\theta}(y')}{\delta j(\bar{y})} + \frac{\delta^2\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\theta(y')}\frac{\delta\theta(y')}{\delta j(\bar{y})}\\ &= \int\limits_{y'} \frac{\delta^2\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\tilde{\theta}(y')}\frac{\delta^2\ln\mathcal{Z}[j,\tilde{j}]}{\delta\tilde{j}(y')\delta j(\bar{y})} + \frac{\delta^2\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\theta(y')}\frac{\delta^2\ln\mathcal{Z}[j,\tilde{j}]}{\delta j(y')\delta j(\bar{y})},\end{aligned} \tag{A.8}$$

the third is

$$\begin{aligned}\frac{\delta^2\Gamma[\tilde{\theta},\theta]}{\delta\theta(y)\delta\tilde{j}(\bar{y})} &= 0\\ &= \int\limits_{y'} \frac{\delta^2\Gamma[\tilde{\theta},\theta]}{\delta\theta(y)\delta\tilde{\theta}(y')}\frac{\delta\tilde{\theta}(y')}{\delta\tilde{j}(\bar{y})} + \frac{\delta^2\Gamma[\tilde{\theta},\theta]}{\delta\theta(y)\delta\theta(y')}\frac{\delta\theta(y')}{\delta\tilde{j}(\bar{y})}\\ &= \int\limits_{y'} \frac{\delta^2\Gamma[\tilde{\theta},\theta]}{\delta\theta(y)\delta\tilde{\theta}(y')}\frac{\delta^2\ln\mathcal{Z}[j,\tilde{j}]}{\delta\tilde{j}(y')\delta\tilde{j}(\bar{y})} + \frac{\delta^2\Gamma[\tilde{\theta},\theta]}{\delta\theta(y)\delta\theta(y')}\frac{\delta^2\ln\mathcal{Z}[j,\tilde{j}]}{\delta j(y')\delta\tilde{j}(\bar{y})},\end{aligned} \tag{A.9}$$

and the fourth is

$$\begin{aligned}\frac{\delta^2\Gamma[\tilde{\theta},\theta]}{\delta\theta(y)\delta j(\bar{y})} &= \delta^{(d+1)}(y-\bar{y})\\ &= \int\limits_{y'} \frac{\delta^2\Gamma[\tilde{\theta},\theta]}{\delta\theta(y)\delta\tilde{\theta}(y')}\frac{\delta\tilde{\theta}(y')}{\delta j(\bar{y})} + \frac{\delta^2\Gamma[\tilde{\theta},\theta]}{\delta\theta(y)\delta\theta(y')}\frac{\delta\theta(y')}{\delta j(\bar{y})}\\ &= \int\limits_{y'} \frac{\delta^2\Gamma[\tilde{\theta},\theta]}{\delta\theta(y)\delta\tilde{\theta}(y')}\frac{\delta^2\ln\mathcal{Z}[j,\tilde{j}]}{\delta\tilde{j}(y')\delta j(\bar{y})} + \frac{\delta^2\Gamma[\tilde{\theta},\theta]}{\delta\theta(y)\delta\theta(y')}\frac{\delta^2\ln\mathcal{Z}[j,\tilde{j}]}{\delta j(y')\delta j(\bar{y})}.\end{aligned} \tag{A.10}$$

A third order derivative,

$$
\begin{aligned}
&\frac{\delta^3\Gamma[\tilde{\theta},\theta]}{\delta\theta(y)\delta\tilde{j}(\bar{y})\delta\tilde{j}(\bar{\bar{y}})} = 0 \\
&\quad = \int\limits_{y'} \frac{\delta^3\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\tilde{\theta}(y')\delta\tilde{j}(\bar{\bar{y}})}\frac{\delta\tilde{\theta}(y')}{\delta\tilde{j}(\bar{y})} + \frac{\delta^3\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\theta(y')\delta\tilde{j}(\bar{\bar{y}})}\frac{\delta\theta(y')}{\delta\tilde{j}(\bar{y})} \\
&\quad + \frac{\delta^2\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\tilde{\theta}(y')}\frac{\delta^2\tilde{\theta}(y')}{\delta\tilde{j}(\bar{y})\delta\tilde{j}(\bar{\bar{y}})} + \frac{\delta^2\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\theta(y')}\frac{\delta^2\theta(y')}{\delta\tilde{j}(\bar{y})\delta\tilde{j}(\bar{\bar{y}})} \\
&\quad = \int\limits_{y'} \frac{\delta^2\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\tilde{\theta}(y')}\frac{\delta^2\tilde{\theta}(y')}{\delta\tilde{j}(\bar{y})\delta\tilde{j}(\bar{\bar{y}})} + \frac{\delta^2\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\theta(y')}\frac{\delta^2\theta(y')}{\delta\tilde{j}(\bar{y})\delta\tilde{j}(\bar{\bar{y}})} \\
&\quad + \int\limits_{y',y''} \left[\frac{\delta^3\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\tilde{\theta}(y')\delta\tilde{\theta}(y'')}\frac{\delta\tilde{\theta}(y'')}{\delta\tilde{j}(\bar{\bar{y}})} + \frac{\delta^3\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\tilde{\theta}(y')\delta\theta(y'')}\frac{\delta\theta(y'')}{\delta\tilde{j}(\bar{\bar{y}})}\right]\frac{\delta\tilde{\theta}(y')}{\delta\tilde{j}(\bar{y})} \\
&\quad + \left[\frac{\delta^3\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\theta(y')\delta\tilde{\theta}(y'')}\frac{\delta\tilde{\theta}(y'')}{\delta\tilde{j}(\bar{\bar{y}})} + \frac{\delta^3\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\theta(y')\delta\theta(y'')}\frac{\delta\theta(y'')}{\delta\tilde{j}(\bar{\bar{y}})}\right]\frac{\delta\theta(y')}{\delta\tilde{j}(\bar{y})},
\end{aligned}
\tag{A.11}
$$

is required, and finally a fourth order derivative,

$$
\begin{aligned}
&\frac{\delta^4\Gamma[\tilde{\theta},\theta]}{\delta\theta(y)\delta\tilde{j}(\bar{y})\delta\tilde{j}(\bar{\bar{y}})\delta\tilde{j}(\bar{\bar{\bar{y}}})} = 0 \\
&\quad = \int\limits_{y'} \frac{\delta^3\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\tilde{\theta}(y')\delta\tilde{j}(\bar{\bar{\bar{y}}})}\frac{\delta^2\tilde{\theta}(y')}{\delta\tilde{j}(\bar{y})\delta\tilde{j}(\bar{\bar{y}})} + \frac{\delta^3\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\theta(y')\delta\tilde{j}(\bar{\bar{\bar{y}}})}\frac{\delta^2\theta(y')}{\delta\tilde{j}(\bar{y})\delta\tilde{j}(\bar{\bar{y}})} \\
&\quad + \frac{\delta^2\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\tilde{\theta}(y')}\frac{\delta^3\tilde{\theta}(y')}{\delta\tilde{j}(\bar{y})\delta\tilde{j}(\bar{\bar{y}})\delta\tilde{j}(\bar{\bar{\bar{y}}})} + \frac{\delta^2\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\theta(y')}\frac{\delta^3\theta(y')}{\delta\tilde{j}(\bar{y})\delta\tilde{j}(\bar{\bar{y}})\delta\tilde{j}(\bar{\bar{\bar{y}}})} \\
&\quad + \int\limits_{y',y''} \frac{\delta^3\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\tilde{\theta}(y')\delta\tilde{\theta}(y'')}\left(\frac{\delta^2\tilde{\theta}(y'')}{\delta\tilde{j}(\bar{\bar{y}})\delta\tilde{j}(\bar{\bar{\bar{y}}})}\frac{\delta\tilde{\theta}(y')}{\delta\tilde{j}(\bar{y})} + \frac{\delta\tilde{\theta}(y'')}{\delta\tilde{j}(\bar{\bar{y}})}\frac{\delta^2\tilde{\theta}(y')}{\delta\tilde{j}(\bar{y})\delta\tilde{j}(\bar{\bar{\bar{y}}})}\right) \\
&\quad + \frac{\delta^3\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\tilde{\theta}(y')\delta\theta(y'')}\left(\frac{\delta^2\theta(y'')}{\delta\tilde{j}(\bar{\bar{y}})\delta\tilde{j}(\bar{\bar{\bar{y}}})}\frac{\delta\tilde{\theta}(y')}{\delta\tilde{j}(\bar{y})} + \frac{\delta\theta(y'')}{\delta\tilde{j}(\bar{\bar{y}})}\frac{\delta^2\tilde{\theta}(y')}{\delta\tilde{j}(\bar{y})\delta\tilde{j}(\bar{\bar{\bar{y}}})}\right) \\
&\quad + \frac{\delta^3\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\theta(y')\delta\tilde{\theta}(y'')}\left(\frac{\delta^2\tilde{\theta}(y'')}{\delta\tilde{j}(\bar{\bar{y}})\delta\tilde{j}(\bar{\bar{\bar{y}}})}\frac{\delta\theta(y')}{\delta\tilde{j}(\bar{y})} + \frac{\delta\tilde{\theta}(y'')}{\delta\tilde{j}(\bar{\bar{y}})}\frac{\delta^2\theta(y')}{\delta\tilde{j}(\bar{y})\delta\tilde{j}(\bar{\bar{\bar{y}}})}\right) \\
&\quad + \frac{\delta^3\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\theta(y')\delta\theta(y'')}\left(\frac{\delta^2\theta(y'')}{\delta\tilde{j}(\bar{\bar{y}})\delta\tilde{j}(\bar{\bar{\bar{y}}})}\frac{\delta\theta(y')}{\delta\tilde{j}(\bar{y})} + \frac{\delta\theta(y'')}{\delta\tilde{j}(\bar{\bar{y}})}\frac{\delta^2\theta(y')}{\delta\tilde{j}(\bar{y})\delta\tilde{j}(\bar{\bar{\bar{y}}})}\right) \\
&\quad + \frac{\delta^4\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\tilde{\theta}(y')\delta\tilde{\theta}(y'')\delta\tilde{j}(\bar{\bar{\bar{y}}})}\frac{\delta\tilde{\theta}(y'')}{\delta\tilde{j}(\bar{\bar{y}})}\frac{\delta\tilde{\theta}(y')}{\delta\tilde{j}(\bar{y})} \\
&\quad + \frac{\delta^4\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\tilde{\theta}(y')\delta\theta(y'')\delta\tilde{j}(\bar{\bar{\bar{y}}})}\frac{\delta\theta(y'')}{\delta\tilde{j}(\bar{\bar{y}})}\frac{\delta\tilde{\theta}(y')}{\delta\tilde{j}(\bar{y})} \\
&\quad + \frac{\delta^4\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\theta(y')\delta\tilde{\theta}(y'')\delta\tilde{j}(\bar{\bar{\bar{y}}})}\frac{\delta\tilde{\theta}(y'')}{\delta\tilde{j}(\bar{\bar{y}})}\frac{\delta\theta(y')}{\delta\tilde{j}(\bar{y})}
\end{aligned}
$$

$$+\frac{\delta^4\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\theta(y')\delta\theta(y'')\delta\tilde{j}(\bar{\bar{y}})}\frac{\delta\theta(y'')}{\delta\tilde{j}(\bar{\bar{y}})}\frac{\delta\theta(y')}{\delta\tilde{j}(\bar{y})},$$

which is expanded to

$$\begin{aligned}
&\frac{\delta^4\Gamma[\tilde{\theta},\theta]}{\delta\theta(y)\delta\tilde{j}(\bar{y})\delta\tilde{j}(\bar{\bar{y}})\delta\tilde{j}(\bar{\bar{\bar{y}}})}=0\\
&=\int\limits_{y'}\frac{\delta^2\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\tilde{\theta}(y')}\frac{\delta^3\tilde{\theta}(y')}{\delta\tilde{j}(\bar{y})\delta\tilde{j}(\bar{\bar{y}})\delta\tilde{j}(\bar{\bar{\bar{y}}})}+\frac{\delta^2\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\theta(y')}\frac{\delta^3\theta(y')}{\delta\tilde{j}(\bar{y})\delta\tilde{j}(\bar{\bar{y}})\delta\tilde{j}(\bar{\bar{\bar{y}}})}\\
&+\int\limits_{y',y''}\left(\frac{\delta^3\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\tilde{\theta}(y')\delta\tilde{\theta}(y'')}\frac{\delta\tilde{\theta}(y'')}{\delta\tilde{j}(\bar{\bar{\bar{y}}})}+\frac{\delta^3\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\tilde{\theta}(y')\delta\theta(y'')}\frac{\delta\theta(y'')}{\delta\tilde{j}(\bar{\bar{\bar{y}}})}\right)\frac{\delta^2\tilde{\theta}(y')}{\delta\tilde{j}(\bar{y})\delta\tilde{j}(\bar{\bar{y}})}\\
&+\left(\frac{\delta^3\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\theta(y')\delta\tilde{\theta}(y'')}\frac{\delta\tilde{\theta}(y'')}{\delta\tilde{j}(\bar{\bar{\bar{y}}})}+\frac{\delta^3\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\theta(y')\delta\theta(y'')}\frac{\delta\theta(y'')}{\delta\tilde{j}(\bar{\bar{\bar{y}}})}\right)\frac{\delta^2\theta(y')}{\delta\tilde{j}(\bar{y})\delta\tilde{j}(\bar{\bar{y}})}\\
&+\frac{\delta^3\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\tilde{\theta}(y')\delta\tilde{\theta}(y'')}\left(\frac{\delta^2\tilde{\theta}(y'')}{\delta\tilde{j}(\bar{\bar{y}})\delta\tilde{j}(\bar{\bar{\bar{y}}})}\frac{\delta\tilde{\theta}(y')}{\delta\tilde{j}(\bar{y})}+\frac{\delta\tilde{\theta}(y'')}{\delta\tilde{j}(\bar{\bar{y}})}\frac{\delta^2\tilde{\theta}(y')}{\delta\tilde{j}(\bar{y})\delta\tilde{j}(\bar{\bar{\bar{y}}})}\right)\\
&+\frac{\delta^3\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\tilde{\theta}(y')\delta\theta(y'')}\left(\frac{\delta^2\theta(y'')}{\delta\tilde{j}(\bar{\bar{y}})\delta\tilde{j}(\bar{\bar{\bar{y}}})}\frac{\delta\tilde{\theta}(y')}{\delta\tilde{j}(\bar{y})}+\frac{\delta\theta(y'')}{\delta\tilde{j}(\bar{\bar{y}})}\frac{\delta^2\tilde{\theta}(y')}{\delta\tilde{j}(\bar{y})\delta\tilde{j}(\bar{\bar{\bar{y}}})}\right)\\
&+\frac{\delta^3\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\theta(y')\delta\tilde{\theta}(y'')}\left(\frac{\delta^2\tilde{\theta}(y'')}{\delta\tilde{j}(\bar{\bar{y}})\delta\tilde{j}(\bar{\bar{\bar{y}}})}\frac{\delta\theta(y')}{\delta\tilde{j}(\bar{y})}+\frac{\delta\tilde{\theta}(y'')}{\delta\tilde{j}(\bar{\bar{y}})}\frac{\delta^2\theta(y')}{\delta\tilde{j}(\bar{y})\delta\tilde{j}(\bar{\bar{\bar{y}}})}\right)\\
&+\frac{\delta^3\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\theta(y')\delta\theta(y'')}\left(\frac{\delta^2\theta(y'')}{\delta\tilde{j}(\bar{\bar{y}})\delta\tilde{j}(\bar{\bar{\bar{y}}})}\frac{\delta\theta(y')}{\delta\tilde{j}(\bar{y})}+\frac{\delta\theta(y'')}{\delta\tilde{j}(\bar{\bar{y}})}\frac{\delta^2\theta(y')}{\delta\tilde{j}(\bar{y})\delta\tilde{j}(\bar{\bar{\bar{y}}})}\right)\\
&+\int\limits_{y',y'',y'''}\left(\frac{\delta^4\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\tilde{\theta}(y')\delta\tilde{\theta}(y'')\delta\tilde{\theta}(y''')}\frac{\delta\tilde{\theta}(y''')}{\delta\tilde{j}(\bar{\bar{\bar{y}}})}\right.\\
&\qquad\left.+\frac{\delta^4\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\tilde{\theta}(y')\delta\tilde{\theta}(y'')\delta\theta(y''')}\frac{\delta\theta(y''')}{\delta\tilde{j}(\bar{\bar{\bar{y}}})}\right)\frac{\delta\tilde{\theta}(y'')}{\delta\tilde{j}(\bar{\bar{y}})}\frac{\delta\tilde{\theta}(y')}{\delta\tilde{j}(\bar{y})}\\
&+\left(\frac{\delta^4\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\tilde{\theta}(y')\delta\theta(y'')\delta\tilde{\theta}(y''')}\frac{\delta\tilde{\theta}(y''')}{\delta\tilde{j}(\bar{\bar{\bar{y}}})}+\frac{\delta^4\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\tilde{\theta}(y')\delta\theta(y'')\delta\theta(y''')}\frac{\delta\theta(y''')}{\delta\tilde{j}(\bar{\bar{\bar{y}}})}\right)\frac{\delta\theta(y'')}{\delta\tilde{j}(\bar{\bar{y}})}\frac{\delta\tilde{\theta}(y')}{\delta\tilde{j}(\bar{y})}\\
&+\left(\frac{\delta^4\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\theta(y')\delta\tilde{\theta}(y'')\delta\tilde{\theta}(y''')}\frac{\delta\tilde{\theta}(y''')}{\delta\tilde{j}(\bar{\bar{\bar{y}}})}+\frac{\delta^4\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\theta(y')\delta\tilde{\theta}(y'')\delta\theta(y''')}\frac{\delta\theta(y''')}{\delta\tilde{j}(\bar{\bar{\bar{y}}})}\right)\frac{\delta\tilde{\theta}(y'')}{\delta\tilde{j}(\bar{\bar{y}})}\frac{\delta\theta(y')}{\delta\tilde{j}(\bar{y})}\\
&+\left(\frac{\delta^4\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\theta(y')\delta\theta(y'')\delta\tilde{\theta}(y''')}\frac{\delta\tilde{\theta}(y''')}{\delta\tilde{j}(\bar{\bar{\bar{y}}})}+\frac{\delta^4\Gamma[\tilde{\theta},\theta]}{\delta\tilde{\theta}(y)\delta\theta(y')\delta\theta(y'')\delta\theta(y''')}\frac{\delta\theta(y''')}{\delta\tilde{j}(\bar{\bar{\bar{y}}})}\right)\frac{\delta\theta(y'')}{\delta\tilde{j}(\bar{\bar{y}})}\frac{\delta\theta(y')}{\delta\tilde{j}(\bar{y})}.
\end{aligned}\tag{A.12}$$

These derivative expressions need evaluated at vanishing source field, and then Fourier transformed. Fortunately, this massively simplifies the expressions.

A.2 The Fourier Transforms

The notation used is

$$\phi(y) = \phi(x,t) = \int đ^d q \int đ\omega \ \exp(iqx)\exp(-i\omega t)\phi(q,\omega) = \int đk \ \exp(-iky)\phi(k),$$
$$\delta\!\!\!^{-}{}^{d+1}(k) = \delta\!\!\!^{-}{}^{d}(q)\delta\!\!\!^{-}(\omega) = \int d^d x \int dt \ \exp(-iqx)\exp(i\omega t) = \int dy \ \exp(iky), \tag{A.13}$$
$$\frac{\delta}{\delta j(y)} = \int dk \ \exp(+iky)\frac{\delta}{\delta j(k)} \qquad \frac{\delta}{\delta j(k)} = \int đy \ \exp(-iky)\frac{\delta}{\delta j(y)}.$$

A.2.1 The propagator

The real-space expression, after eliminating correlation functions composed entirely of response fields, is as already seen,

$$\delta^{d+1}(y-\bar{y}) = \int\limits_{y'} \Gamma^{11}(y,y')\mathbf{G}^{11}(y',\bar{y}). \tag{A.14}$$

Fourier transforming the left hand side gives

$$\int dy d\bar{y} \exp\left(+iky + i\bar{k}\bar{y}\right)\delta(y-\bar{y}) = \int dy \exp\left(i\bar{y}(k+\bar{k})\right)$$
$$= \delta\!\!\!^{-}{}^{d+1}(k+\bar{k}). \tag{A.15}$$

The two functions on the right hand side can be written as

$$\Gamma^{11}(y,y') = \int đ\tilde{k}\, đ\tilde{k}'\Gamma^{11}(\tilde{k},\tilde{k}')\exp\left(-i\tilde{k}y - i\tilde{k}'y'\right)$$
$$\mathbf{G}^{11}(y',\bar{y}) = \int đ\tilde{\tilde{k}}'\, đ\bar{\tilde{k}}\mathbf{G}^{11}(\tilde{\tilde{k}}',\bar{\tilde{k}})\exp\left(-i\tilde{\tilde{k}}'y' - i\bar{\tilde{k}}\bar{y}\right). \tag{A.16}$$

Applying the Fourier transform, of the free y and $\bar{y}$ variables, to the these terms would give a total exponential on the right hand side of

$$\exp\left\{+i(ky+\bar{k}\bar{y}) - i(\tilde{k}y+\tilde{k}'y') - i(\tilde{\tilde{k}}'y' + \bar{\tilde{k}}\bar{y})\right\}. \tag{A.17}$$

Spending the three space-time integrals in turn would then generate three sets of delta functions,

$$dy \to \delta\!\!\!^{-}{}^{d+1}(k-\tilde{k}) \quad d\bar{y} \to \delta\!\!\!^{-}{}^{d+1}(\bar{k}-\bar{\tilde{k}}) \quad dy' \to \delta\!\!\!^{-}{}^{d+1}(\tilde{k}'+\tilde{\tilde{k}}'), \tag{A.18}$$

which can then be spent to eliminate three of the four momentum-frequency integrals, leaving

$$\delta\!\!\!^{-}{}^{d+1}(k+\bar{k}) = \int đ\tilde{k}' \ \Gamma^{11}(k,\tilde{k}')\mathbf{G}^{11}(-\tilde{k}',\bar{k}). \tag{A.19}$$

A.2.2 The noise vertex

From the second of the possible second order derivatives, (A.8),

$$0 = \int_{y'} \Gamma^{20}(y, y')\mathbf{G}^{11}(y', \bar{y}) + \Gamma^{11}(y, y')\mathbf{G}^{20}(y', \bar{y}), \tag{A.20}$$

the same procedure is followed. The first set of terms can be expressed as

$$\begin{aligned} \Gamma^{20}(y, y') &= \int đ\tilde{k}\, đ\tilde{k}'\Gamma^{20}(\tilde{k}, \tilde{k}') \exp\left(-i\tilde{k}y - i\tilde{k}'y'\right) \\ \mathbf{G}^{11}(y', \bar{y}) &= \int đ\tilde{\tilde{k}}'\, đ\tilde{\bar{k}}\mathbf{G}^{11}(\tilde{\tilde{k}}', \tilde{\bar{k}}) \exp\left(-i\tilde{\tilde{k}}'y' - i\tilde{\bar{k}}\bar{y}\right), \end{aligned} \tag{A.21}$$

and when the Fourier transform is applied the space-time integrals produce delta functions from the exponential terms, which can then be spent eliminating the momentum-frequency integrals. The second group of terms, which must be transformed separately, can similarly be written as

$$\begin{aligned} \Gamma^{11}(y, y') &= \int đ\tilde{k}\, đ\tilde{k}'\Gamma^{11}(\tilde{k}, \tilde{k}') \exp\left(-i\tilde{k}y - i\tilde{k}'y'\right) \\ \mathbf{G}^{20}(y', \bar{y}) &= \int đ\tilde{\tilde{k}}'\, đ\tilde{\bar{k}}\mathbf{G}^{20}(\tilde{\tilde{k}}', \tilde{\bar{k}}) \exp\left(-i\tilde{\tilde{k}}'y' - i\tilde{\bar{k}}\bar{y}\right). \end{aligned} \tag{A.22}$$

Leading to the expression

$$0 = \int đ\tilde{k}'\Gamma^{20}(k, \tilde{k}')\mathbf{G}^{11}(-\tilde{k}', \bar{k}) + \Gamma^{11}(k, \tilde{k}')\mathbf{G}^{20}(-\tilde{k}', \bar{k}). \tag{A.23}$$

A.2.3 Γ^{12} and $\mathbf{G}^{12}$

The third order derivative in (A.11) is much simpler after being evaluated at vanishing source field,

$$0 = \int_{y'} \Gamma^{11}(y, y')\mathbf{G}^{12}\left(y, \bar{y}, \bar{\bar{y}}\right) + \int_{y', y''} \mathbf{G}^{11}(y', \bar{y})\Gamma^{12}(y, y', y'')\mathbf{G}^{11}\left(y'', \bar{\bar{y}}\right). \tag{A.24}$$

This is Fourier transformed by applying

$$\int dy\; d\bar{y}\; d\bar{\bar{y}}\; \exp\left(+iky\right) \exp\left(+i\bar{k}\bar{y}\right) \exp\left(+i\bar{\bar{k}}\bar{\bar{y}}\right). \tag{A.25}$$

Treating the first group of terms,

$$\begin{aligned} \Gamma^{11}(y, y') &= \int đ\tilde{k}\, đ\tilde{k}'\; \Gamma^{11}\left(\tilde{k}, \tilde{k}'\right) \exp\left(-i\tilde{k}y\right) \exp\left(-i\tilde{k}'y'\right) \\ \mathbf{G}^{12}\left(y', \bar{y}, \bar{\bar{y}}\right) &= \int đ\tilde{\tilde{k}}'\, đ\tilde{\bar{k}}\, đ\tilde{\bar{\bar{k}}}\; \mathbf{G}^{12}\left(\tilde{\tilde{k}}', \tilde{\bar{k}}, \tilde{\bar{\bar{k}}}\right) \exp\left(-i\tilde{\tilde{k}}'y'\right) \exp\left(-i\tilde{\bar{k}}\bar{y}\right) \exp\left(-i\tilde{\bar{\bar{k}}}\bar{\bar{y}}\right), \end{aligned} \tag{A.26}$$

gives a combined exponential

$$\exp\left\{+i\left(ky+\bar{k}\bar{y}+\bar{\bar{k}}\bar{\bar{y}}\right)-i\left(\tilde{k}y+\tilde{k}'y'\right)-i\left(\tilde{\tilde{k}}'y'+\tilde{\bar{k}}\bar{y}+\tilde{\bar{\bar{k}}}\bar{\bar{y}}\right)\right\}. \tag{A.27}$$

Spending the space-time integrals produces the delta functions

$$dy\to \text{ẟ}^{d+1}\left(\tilde{k}-k\right)\quad d\bar{y}\to \text{ẟ}^{d+1}\left(\tilde{\bar{k}}-\bar{k}\right)\quad d\bar{\bar{y}}\to \text{ẟ}^{d+1}\left(\tilde{\bar{\bar{k}}}-\bar{\bar{k}}\right)\quad dy'\to \text{ẟ}^{d+1}\left(\tilde{k}'+\tilde{\tilde{k}}'\right). \tag{A.28}$$

Treating the second group of terms,

$$\begin{aligned}
\mathbf{G}^{11}(y',\bar{y}) &= \int \text{đ}\tilde{k}'\,\text{đ}\tilde{\bar{k}}\;\mathbf{G}^{11}\left(\tilde{k}',\tilde{\bar{k}}\right)\exp\left(-i\tilde{k}'y'\right)\exp\left(-i\tilde{\bar{k}}\bar{y}\right)\\
\Gamma^{12}(y,y',y'') &= \int \text{đ}\tilde{k}\,\text{đ}\tilde{\tilde{k}}'\,\text{đ}\tilde{\tilde{k}}''\;\Gamma^{12}\left(\tilde{k},\tilde{\tilde{k}}',\tilde{\tilde{k}}''\right)\exp\left(-i\tilde{k}y\right)\exp\left(-i\tilde{\tilde{k}}'y'\right)\exp\left(-i\tilde{\tilde{k}}''y''\right)\\
\mathbf{G}^{11}\left(y'',\bar{\bar{y}}\right) &= \int \text{đ}\tilde{k}''\,\text{đ}\tilde{\bar{\bar{k}}}\;\Gamma^{11}\left(\tilde{k}'',\tilde{\bar{\bar{k}}}\right)\exp\left(-i\tilde{k}''y''\right)\exp\left(-i\tilde{\bar{\bar{k}}}\bar{\bar{y}}\right),
\end{aligned} \tag{A.29}$$

leads to the total exponential

$$\exp\left\{+i\left(ky+\bar{k}\bar{y}+\bar{\bar{k}}\bar{\bar{y}}\right)-i\left(\tilde{k}'y'+\tilde{\bar{k}}\bar{y}\right)-i\left(\tilde{k}y+\tilde{\tilde{k}}'y'+\tilde{\tilde{k}}''y''\right)-i\left(\tilde{k}''y''+\tilde{\bar{\bar{k}}}\bar{\bar{y}}\right)\right\}, \tag{A.30}$$

and the delta functions

$$\begin{gathered}
dy\to \text{ẟ}^{d+1}\left(k-\tilde{k}\right)\quad d\bar{y}\to \text{ẟ}^{d+1}\left(\bar{k}-\tilde{\bar{k}}\right)\quad d\bar{\bar{y}}\to \text{ẟ}^{d+1}\left(\bar{\bar{k}}-\tilde{\bar{\bar{k}}}\right)\\
dy'\to \text{ẟ}^{d+1}\left(\tilde{k}'+\tilde{\tilde{k}}'\right)\quad dy''\to \text{ẟ}^{d+1}\left(\tilde{k}''+\tilde{\tilde{k}}''\right).
\end{gathered} \tag{A.31}$$

Spending the delta functions and combing the two groups of terms, the transformed expression is

$$\int \text{đ}\tilde{k}'\;\Gamma^{11}(k,\tilde{k}')\mathbf{G}^{12}\left(-\tilde{k}',\bar{k},\bar{\bar{k}}\right)+\iint \text{đ}\tilde{k}'\text{đ}\tilde{k}''\mathbf{G}^{11}(\tilde{k}',\bar{k})\Gamma^{12}(k,-\tilde{k}',-\tilde{k}'')\mathbf{G}^{11}\left(\tilde{k}'',\bar{\bar{k}}\right). \tag{A.32}$$

A.2.4 Γ^{13} and $\mathbf{G}^{13}$

The fourth order derivative in (A.12) is rather ungainly, even after is has been simplified:

$$\begin{aligned}
&\int_{y'}\Gamma^{11}(y,y')\mathbf{G}^{13}\left(y',\bar{y},\bar{y},\bar{\bar{y}}\right)+\int_{y',y'',y'''}\mathbf{G}^{11}(y',\bar{y})\mathbf{G}^{11}\left(y'',\bar{y}\right)\Gamma^{13}(y,y',y'',y''')\mathbf{G}^{11}\left(y''',\bar{\bar{y}}\right)\\
&+\int_{y',y''}\Gamma^{12}(y,y',y'')\Big(\mathbf{G}^{12}\left(y',\bar{y},\bar{\bar{y}}\right)\mathbf{G}^{11}\left(y'',\bar{y}\right)+\mathbf{G}^{12}\left(y'',\bar{y},\bar{\bar{y}}\right)\mathbf{G}^{11}\left(y',\bar{y}\right)\\
&\qquad\qquad+\mathbf{G}^{12}\left(y',\bar{y},\bar{y}\right)\mathbf{G}^{11}\left(y'',\bar{\bar{y}}\right)\Big).
\end{aligned} \tag{A.33}$$

Fourier transforming with

$$\int dy\, d\bar{y}\, d\bar{\bar{y}}\, d\bar{\bar{\bar{y}}} \exp\left(+iky\right) \exp\left(+i\bar{k}\bar{y}\right) \exp\left(+i\bar{\bar{k}}\bar{\bar{y}}\right) \exp\left(+i\bar{\bar{\bar{k}}}\bar{\bar{\bar{y}}}\right), \tag{A.34}$$

the first group of terms in the single integral is relatively straightforward,

$$\begin{aligned} \Gamma^{11}(y, y') &= \int đ\tilde{k}\, đ\tilde{k}' \Gamma^{11}(\tilde{k}, \tilde{k}') \exp\left(+i\tilde{k}y\right) \exp\left(+i\tilde{k}'y'\right) \\ \mathbf{G}^{13}\left(y', \bar{y}, \bar{\bar{y}}, \bar{\bar{\bar{y}}}\right) &= \int đ\tilde{\bar{k}}'\, đ\tilde{\bar{k}}\, đ\tilde{\bar{\bar{k}}}\, đ\tilde{\bar{\bar{\bar{k}}}} \mathbf{G}^{13}\left(\tilde{\bar{k}}', \tilde{\bar{k}}, \tilde{\bar{\bar{k}}}, \tilde{\bar{\bar{\bar{k}}}}\right) \exp\left(+i\tilde{\bar{k}}'y' + i\tilde{\bar{k}}\bar{y} + i\tilde{\bar{\bar{k}}}\bar{\bar{y}} + i\tilde{\bar{\bar{\bar{k}}}}\bar{\bar{\bar{y}}}\right), \end{aligned} \tag{A.35}$$

with the space-time integrals and exponential terms leading to

$$\begin{gathered} dy \to \delta^{d+1}\left(k - \tilde{k}\right) \quad d\bar{y} \to \delta^{d+1}\left(\bar{k} - \tilde{\bar{k}}\right) \quad d\bar{\bar{y}} \to \delta^{d+1}\left(\bar{\bar{k}} - \tilde{\bar{\bar{k}}}\right) \\ d\bar{\bar{\bar{y}}} \to \delta^{d+1}\left(\bar{\bar{\bar{k}}} - \tilde{\bar{\bar{\bar{k}}}}\right) \quad dy' \to \delta^{d+1}\left(\tilde{k}' + \tilde{\bar{k}}'\right), \end{gathered} \tag{A.36}$$

ultimately giving

$$\int đ\tilde{k}' \Gamma^{11}(k, \tilde{k}') \mathbf{G}^{13}\left(-\tilde{k}', \bar{k}, \bar{\bar{k}}, \bar{\bar{\bar{k}}}\right). \tag{A.37}$$

The second group of terms, inside the triple integral,

$$\begin{aligned} \mathbf{G}^{11}(y', \bar{y}) &= \int đ\tilde{k}'\, đ\tilde{\bar{k}}\ \mathbf{G}^{11}\left(\tilde{k}', \tilde{\bar{k}}\right) \exp\left(+i\tilde{k}'y'\right) \exp\left(+i\tilde{\bar{k}}\bar{y}\right) \\ \mathbf{G}^{11}\left(y'', \bar{\bar{y}}\right) &= \int đ\tilde{k}''\, đ\tilde{\bar{\bar{k}}}\ \mathbf{G}^{11}\left(\tilde{k}'', \tilde{\bar{\bar{k}}}\right) \exp\left(+i\tilde{k}''y''\right) \exp\left(+i\tilde{\bar{\bar{k}}}\bar{\bar{y}}\right) \\ \mathbf{G}^{11}\left(y''', \bar{\bar{\bar{y}}}\right) &= \int đ\tilde{k}'''\, đ\tilde{\bar{\bar{\bar{k}}}}\ \mathbf{G}^{11}\left(\tilde{k}''', \tilde{\bar{\bar{\bar{k}}}}\right) \exp\left(+i\tilde{k}'''y'''\right) \exp\left(+i\tilde{\bar{\bar{\bar{k}}}}\bar{\bar{\bar{y}}}\right) \\ \Gamma^{13}(y, y', y'', y''') &= \int đ\tilde{k}\, đ\tilde{\tilde{k}}'\, đ\tilde{\tilde{k}}''\, đ\tilde{\tilde{k}}''' \Gamma^{13}\left(\tilde{k}, \tilde{\tilde{k}}', \tilde{\tilde{k}}'', \tilde{\tilde{k}}'''\right) \exp\left(+i\tilde{\tilde{k}}y + i\tilde{\tilde{k}}'y' + i\tilde{\tilde{k}}''y'' + i\tilde{\tilde{k}}'''y'''\right), \end{aligned} \tag{A.38}$$

have rather more space-time integrals to spend,

$$\begin{gathered} dy \to \delta^{d+1}\left(k - \tilde{k}\right)\ d\bar{y} \to \delta^{d+1}\left(\bar{k} - \tilde{\bar{k}}\right)\ d\bar{\bar{y}} \to \delta^{d+1}\left(\bar{\bar{k}} - \tilde{\bar{\bar{k}}}\right)\ d\bar{\bar{\bar{y}}} \to \delta^{d+1}\left(\bar{\bar{\bar{k}}} - \tilde{\bar{\bar{\bar{k}}}}\right) \\ dy' \to \delta^{d+1}\left(\tilde{k}' + {}'\tilde{\tilde{k}}\right) \quad dy'' \to \delta^{d+1}\left(\tilde{k}'' + \tilde{\tilde{k}}''\right) \quad dy''' \to \delta^{d+1}\left(\tilde{k}''' + \tilde{\tilde{k}}'''\right), \end{gathered} \tag{A.39}$$

leading to

$$\int đ\tilde{k}'\, đ\tilde{k}''\, đ\tilde{k}''' \mathbf{G}^{11}(\tilde{k}', \bar{k}) \mathbf{G}^{11}(\tilde{k}'', \bar{\bar{k}}) \Gamma^{13}(k, -\tilde{k}', -\tilde{k}'', -\tilde{k}''') \mathbf{G}^{11}(\tilde{k}''', \bar{\bar{\bar{k}}}). \tag{A.40}$$

The third set of terms, in the double integral, can found by calculating one of the groups and then relabelling,

$$\begin{aligned} \Gamma^{12}(y, y', y'') &= \int đ\tilde{k}\, đ\tilde{k}'\, đ\tilde{k}'' \Gamma^{12}\left(\tilde{k}, \tilde{k}', \tilde{k}''\right) \exp\left(+i\tilde{k}y + i\tilde{k}'y' + i\tilde{k}''y''\right) \\ \mathbf{G}^{12}\left(y', \bar{y}, \bar{\bar{\bar{y}}}\right) &= \int đ\tilde{\tilde{k}}'\, đ\tilde{\bar{k}}\, đ\tilde{\bar{\bar{\bar{k}}}} \mathbf{G}^{12}\left(\tilde{\tilde{k}}', \tilde{\bar{k}}, \tilde{\bar{\bar{\bar{k}}}}\right) \exp\left(+i\tilde{\tilde{k}}'y' + i\tilde{\bar{k}}\bar{y} + i\tilde{\bar{\bar{\bar{k}}}}\bar{\bar{\bar{y}}}\right) \\ \mathbf{G}^{11}\left(y'', \bar{\bar{y}}\right) &= \int đ\tilde{\tilde{k}}''\, đ\tilde{\bar{\bar{k}}} \mathbf{G}^{11}\left(\tilde{\tilde{k}}'', \tilde{\bar{\bar{k}}}\right) \exp\left(+i\tilde{\tilde{k}}''y''\right) \exp\left(+i\tilde{\bar{\bar{k}}}\bar{\bar{y}}\right), \end{aligned} \tag{A.41}$$

leading to the delta functions

$$
\begin{aligned}
dy \to \not\delta^{d+1}(k-\tilde{k}) \quad d\bar{y} \to \not\delta^{d+1}\left(\bar{k}-\tilde{\bar{k}}\right) \quad d\bar{\bar{y}} \to \not\delta^{d+1}\left(\bar{\bar{k}}-\tilde{\bar{\bar{k}}}\right) \\
d\bar{\bar{\bar{y}}} \to \not\delta^{d+1}\left(\bar{\bar{\bar{k}}}-\tilde{\bar{\bar{\bar{k}}}}\right) \quad dy' \to \not\delta^{d+1}\left(\tilde{k}'+{}'\tilde{\tilde{k}}\right) \quad dy'' \to \not\delta^{d+1}\left(\tilde{k}''+\tilde{\tilde{k}}''\right),
\end{aligned}
\tag{A.42}
$$

and finally the terms

$$
\begin{aligned}
\int \text{đ}\tilde{k}'\,\text{đ}\tilde{k}'' \Big(&\Gamma^{12}(k,\tilde{k}',\tilde{k}'')\mathbf{G}^{12}(-\tilde{k}',\bar{k},\bar{\bar{\bar{k}}})\mathbf{G}^{11}(-\tilde{k}'',\bar{\bar{k}}) \\
&+\Gamma^{12}(k,\tilde{k}',\tilde{k}'')\mathbf{G}^{12}(-\tilde{k}'',\bar{\bar{k}},\bar{\bar{\bar{k}}})\mathbf{G}^{11}(-\tilde{k}',\bar{k}) \\
&+\Gamma^{12}(k,\tilde{k}',\tilde{k}'')\mathbf{G}^{12}(-\tilde{k}',\bar{k},\bar{\bar{y}})\mathbf{G}^{11}(-\tilde{k}'',\bar{\bar{\bar{k}}})\Big).
\end{aligned}
\tag{A.43}
$$

However, these terms would usually be dropped on the understanding that only 1PI diagrams are to be included from the other two terms.

A.2.5 Using translational invariance

The final step is to free these expressions from the remaining integral over the dummy momentum-frequency variables. This is done by using translational invariance, which becomes conservation of momentum and frequency, to drop an argument. The general argument goes along these lines; a translationally invariant function, f, can be written

$$
\begin{aligned}
f(x_1,\ldots,x_{n-1},x_n) &= f(x_1-a,\ldots,x_{n-1}-a,x_n-a) \\
&= f(x_1-x_n,\ldots,x_{n-1}-x_n,0),
\end{aligned}
\tag{A.44}
$$

Fourier transforming gives

$$
\begin{aligned}
f(k_1,\ldots,k_{n-1},k_n) &= \int dx_1\ldots dx_n f(x_1,\ldots,x_{n-1},x_n)\exp\left(i\sum_{p=1}^{n}x_pk_p\right) \\
&= \int dx_1\ldots dx_n f(x_1-x_n,\ldots,x_{n-1}-x_n,0)\exp\left(i\sum_{p=1}^{n-1}x_pk_p\right)\exp\left(ix_nk_n\right),
\end{aligned}
\tag{A.45}
$$

and using the linear transformations $x'_p = x_p - x_n$ leads to

$$
\begin{aligned}
f(k_1,\ldots,k_{n-1},k_n) &= \int dx_n \exp\left(ix_n\sum_{p=1}^{n}k_p\right) \\
&\quad\cdot\int dx'_1\ldots dx'_{n-1} f(x'_1,\ldots,x'_{n-1},0)\exp\left(i\sum_{p=1}^{n-1}x'_pk_p\right) \\
&= \hat{f}(k_1,\ldots,k_{n-1})\not\delta\left(\sum_{p=1}^{n}k_p\right).
\end{aligned}
\tag{A.46}
$$

For $\mathbf{G}^{11}$ and Γ^{11} this gives

$$\begin{aligned}
\langle \phi(x)\tilde{\phi}(\bar{x})\rangle &= \frac{\delta^2 \ln \mathcal{Z}}{\delta j(x)\delta \tilde{j}(\bar{x})} \\
\langle \phi(k)\tilde{\phi}(\bar{k})\rangle &= (2\pi)^{2d}\frac{\delta^2 \ln \mathcal{Z}}{\delta j^\dagger(k)\delta \tilde{j}^\dagger(\bar{k})} \\
&= \mathbf{G}^{11}(k,\bar{k}) = \hat{\mathbf{G}}^{11}(k)\delta\!\!\!^{-}{}^{d+1}(k+\bar{k}) \qquad \text{(A.47)}\\
\Gamma^{11}(x,\bar{x}) &= \frac{\delta^2\Gamma}{\delta\tilde{\theta}(x)\delta\theta(\bar{x})} \\
\Gamma^{11}(k,\bar{k}) &= (2\pi)^{2d}\frac{\delta^2\Gamma}{\delta\tilde{\theta}^\dagger(k)\delta\theta^\dagger(\bar{k})} = \hat{\Gamma}^{11}(k)\delta\!\!\!^{-}{}^{d+1}(k+\bar{k}).
\end{aligned}$$

This can be applied to the propagator identity,

$$\begin{aligned}
\delta\!\!\!^{-}{}^{d+1}(k+\bar{k}) &= \int d\!\!\!^{-}\tilde{k}'\ \Gamma^{11}(k,\tilde{k}')\mathbf{G}^{11}(-\tilde{k}',\bar{k}) \\
&= \int d\!\!\!^{-}\tilde{k}'\ \hat{\Gamma}^{11}(k)\delta\!\!\!^{-}{}^{d+1}(k+\tilde{k}')\hat{\mathbf{G}}^{11}(-\tilde{k}')\delta\!\!\!^{-}{}^{d+1}(\tilde{k}'-\bar{k}) \\
&= \hat{\Gamma}^{11}(k)\hat{\mathbf{G}}^{11}(k)\delta\!\!\!^{-}{}^{d+1}(k+\bar{k}) \qquad \text{(A.48)}\\
\hat{\Gamma}^{11}(k) &= \frac{1}{\hat{\mathbf{G}}^{11}(k)},
\end{aligned}$$

and similarly for the other identities.

This ties in pleasingly with what was found for the bare propagators of a field theory in (3.22). Given the identity

$$\langle \phi(k)\tilde{\phi}(\bar{k})\rangle_0 = \frac{\delta\!\!\!^{-}{}^{d+1}(k+\bar{k})}{-i\omega + \mathbf{f}(q)}, \qquad \text{(A.49)}$$

with $\mathbf{f}(q) = \nu_2 q^2 + \nu_4 q^4$ for the theory analysed in this thesis, this immediately gives the tree level

$$\begin{aligned}
\hat{\mathbf{G}}_0^{11}(q,\omega) &= \frac{1}{-i\omega + \nu_2 q^2 + \nu_4 q^4}, \qquad \text{(A.50)}\\
\hat{\Gamma}_0^{11}(q,\omega) &= -i\omega + \nu_2 q^2 + \nu_4 q^4.
\end{aligned}$$

Appendix B

The Diagrams

B.1 Integral identities

The identities necessary for calculating the diagrams listed below. They can be found, for example, in Appendix B of Quantum and Statistical Field Theory by M. Le Bellac (Le Bellac, 1991), or adapted from the celebrated 1972 Nuclear Physics B paper by G. 't Hooft and M. Veltman ('t Hooft and Veltman, 1972).

$$\int đ^d q \, \frac{1}{(q^2+m^2)^N} = \frac{\Gamma\,(N-d/2)}{(4\pi)^{d/2}\Gamma(N)} \frac{1}{(m^2)^{N-d/2}} \tag{B.1}$$

$$\int đ^d q \, \frac{q^2}{(q^2+m^2)^N} = \frac{\Gamma\,(N-1-d/2)}{2(4\pi)^{d/2}\Gamma(N)} \frac{D}{(m^2)^{N-1-d/2}} \tag{B.2}$$

$$\int đ^d q \, \frac{q_\mu q_\nu}{(q^2+m^2)^N} = \frac{\Gamma\,(N-1-d/2)}{2(4\pi)^{d/2}\Gamma(N)} \frac{\delta_{\mu\nu}}{(m^2)^{N-1-d/2}} \tag{B.3}$$

$$\int đ^d q \, q_\mu q_\nu f(q^2) = \frac{\delta_{\mu\nu}}{d} \int đ^d q \, q^2 f(q^2) \tag{B.4}$$

$$\begin{aligned}\int đ^d q \, \overbrace{q_\mu q_\nu q_\lambda q_\eta \ldots q_\alpha q_\beta}^{n \text{ factors of } k} f(q^2) = (\overbrace{\delta_{\mu\nu}\delta_{\lambda\eta}\ldots\delta_{\alpha\beta} + \text{permutations}}^{(n-1)!! \text{ terms}}) \frac{(d-2)!!}{(d+n-2)!!} \\ \cdot \int đ^d q \, (q^2)^{n/2} f(q^2)\end{aligned} \tag{B.5}$$

B.2 The frequency integrals

At one loop there are only a four basic structures to the diagrams, before the momentum products at the vertices are taken into account. It is these structures that the frequency integral is taken over, using the notation

$$f(q) = \nu_2 q^2 + \nu_4 q^4. \tag{B.6}$$

It will also be useful to note that

$$\begin{aligned} f\left(\bar{k}+q\right) &= \nu_2 \left(\bar{k}+q\right)^2 \nu_4 \left(\bar{k}+q\right)^4 \\ &= \nu_2 \left(\bar{k}^2 + 2\left(\bar{k}\cdot q\right) + q^2\right) \nu_4 \left(\bar{k}^2 + 2\left(\bar{k}\cdot q\right) + q^2\right)^2 \\ &= \nu_2 q^2 + \nu_4 q^4 + 2\nu_2 \left(\bar{k}\cdot q\right) + 4\nu_4 \left(\bar{k}\cdot q\right) q^2 + O(\bar{k}^2) \\ &= f(q) + 2\left(\bar{k}\cdot q\right)\left[2\nu_4 q^2 + \nu_2\right] + O(\bar{k}^2), \end{aligned} \tag{B.7}$$

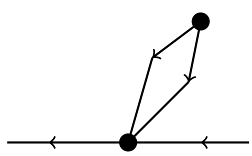
and

$$\begin{aligned} f\left(\bar{\bar{k}}-q\right) &= \nu_2 \left(\bar{\bar{k}}-q\right)^2 \nu_4 \left(\bar{\bar{k}}-q\right)^4 \\ &= \nu_2 \left(\bar{\bar{k}}^2 - 2\left(\bar{\bar{k}}\cdot q\right) + q^2\right) \nu_4 \left(\bar{\bar{k}}^2 - 2\left(\bar{\bar{k}}\cdot q\right) + q^2\right)^2 \\ &= \nu_2 q^2 + \nu_4 q^4 - 2\nu_2 \left(\bar{\bar{k}}\cdot q\right) - 4\nu_4 \left(\bar{\bar{k}}\cdot q\right) q^2 + O\left(\bar{\bar{k}}^2\right) \\ &= f(q) - 2\left(\bar{\bar{k}}\cdot q\right)\left[2\nu_4 q^2 + \nu_2\right] + O\left(\bar{\bar{k}}^2\right). \end{aligned} \tag{B.8}$$

Where, for the appendix, the notation for the external input momenta, k_i, and the internal momentum to be integrated over, q, has been adopted.

B.2.1 Tadpole

The tadpole structure,

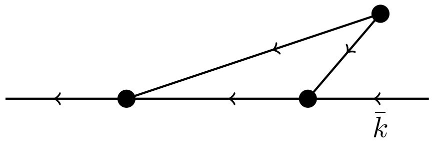


is a standard result:

$$\int_{\infty}^{\infty} d\omega \frac{1}{\omega^2 + f^2(q)} = \frac{\pi}{f(q)}. \tag{B.9}$$

B.2.2 Three-piece

The three propagator structure



gives the result

$$\begin{aligned}\int_{\infty}^{\infty} d\omega \frac{1}{\omega^2 + f^2(q)} \frac{1}{-i\omega + f\left(\bar{k}+q\right)} &= \frac{\pi}{f(q)\Big(f(q) + f\left(\bar{k}+q\right)\Big)} \\ &= \frac{\pi}{2f^2(q)} - \frac{\pi(\bar{k}\cdot q)\left[2\nu_4 q^2 + \nu_2\right]}{2f^3(q)} + O(\bar{k}^2),\end{aligned} \tag{B.10}$$

which requires a contour integral:

$$\begin{aligned}\int_{\infty}^{\infty} d\omega \frac{1}{\omega^2 + f^2(q)} \frac{1}{-i\omega + f\left(\bar{k}+q\right)} &= \int_{\infty}^{\infty} d\omega \frac{1}{-i\omega + f\,(q)} \frac{1}{+i\omega + f\,(q)} \frac{1}{-i\omega + f\left(\bar{k}+q\right)} \\ &= \int_{\infty}^{\infty} d\omega \frac{i}{\omega + if\,(q)} \frac{-i}{\omega - if\,(q)} \frac{i}{\omega + if\left(\bar{k}+q\right)} \\ i\int_{\infty}^{\infty} d\omega \frac{1}{\omega^2 + f^2(q)} \frac{1}{\omega + if\left(\bar{k}+q\right)} &= i\int_{\infty}^{\infty} d\omega \frac{1}{\omega + if\,(q)} \frac{1}{\omega - if\,(q)} \frac{1}{\omega + if\left(\bar{k}+q\right)} \\ &= i\cdot 2\pi i \cdot \mathrm{Res}\big(\omega = if(q)\big) \\ &= -2\pi \frac{1}{2if(q)} \cdot \frac{1}{i\Big(f(q) + f\left(\bar{k}+q\right)\Big)} \\ &= \frac{\pi}{f(q)\Big(f(q) + f\left(\bar{k}+q\right)\Big)}.\end{aligned} \tag{B.11a}$$

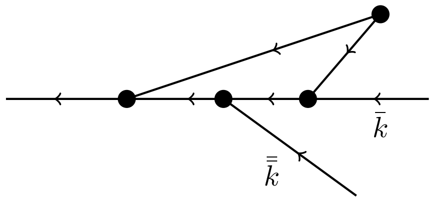
To identify the terms to first order in the external momentum $\bar{k}$, this is expanded as

$$\begin{aligned}\frac{\pi}{f(q)\Big(f(q) + f\left(\bar{k}+q\right)\Big)} &= \frac{\pi}{f(q)}\left[\frac{1}{f(q) + f(q) + 2\left(\bar{k}\cdot q\right)\left[2\nu_4 q^2 + \nu_2\right] + O(\bar{k}^2)}\right] \\ &= \frac{\pi}{f(q)}\left[\frac{1}{2f(q) + 2\left(\bar{k}\cdot q\right)\left[2\nu_4 q^2 + \nu_2\right] + O(\bar{k}^2)}\right] \\ &= \frac{\pi}{2f^2(q)}\left[\frac{1}{1 + \frac{\left(\bar{k}\cdot q\right)\left[2\nu_4 q^2 + \nu_2\right]}{f(q)} + O(\bar{k}^2)}\right]\end{aligned}$$

$$= \frac{\pi}{2f^2(q)} - \frac{\pi\left(\bar{k}\cdot q\right)\left[2\nu_4 q^2 + \nu_2\right]}{2f^3(q)} + O(\bar{k}^2). \tag{B.11b}$$

B.2.3 Asymmetric four-piece

The first of the arrangements of four propagators is the asymmetric



,

giving the result

$$\begin{aligned}\int_{\infty}^{\infty} d\omega \frac{1}{\omega^2 + f^2(q)} \frac{1}{-i\omega + f\left(\bar{k}+q\right)} \frac{1}{-i\omega + f\left(\bar{k}+\bar{\bar{k}}+q\right)} \\ = \frac{\pi}{f(q)\Big(f(q) + f\left(\bar{k}+q\right)\Big)\Big(f(q) + f\left(\bar{k}+\bar{\bar{k}}+q\right)\Big)} \\ = \frac{\pi}{4f^3(q)} + O(\bar{k}, \bar{\bar{k}}),\end{aligned} \tag{B.12}$$

via the calculation

$$\begin{aligned}\int_{\infty}^{\infty} d\omega \frac{1}{\omega^2 + f^2(q)} \frac{1}{-i\omega + f\left(\bar{k}+q\right)} \frac{1}{-i\omega + f\left(\bar{k}+\bar{\bar{k}}+q\right)} \\ = -\int_{\infty}^{\infty} d\omega \frac{1}{\omega^2 + f^2(q)} \frac{1}{\omega + if\left(\bar{k}+q\right)} \frac{1}{\omega + if\left(\bar{k}+\bar{\bar{k}}+q\right)} \\ = -2\pi i \cdot \mathrm{Res}\big(\omega = if(q)\big) \\ = -2\pi i \cdot \frac{1}{2if(q)} \frac{1}{i\Big(f(q) + f\left(\bar{k}+q\right)\Big)} \frac{1}{i\Big(f(q) + f\left(\bar{k}+\bar{\bar{k}}+q\right)\Big)}\end{aligned}$$

$$= \frac{\pi}{f(q)\Big(f(q) + f\left(\bar{k}+q\right)\Big)\Big(f(q) + f\left(\bar{k}+\bar{\bar{k}}+q\right)\Big)} \tag{B.13a}$$

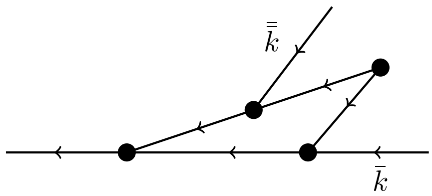
$$= \frac{\pi}{4f^3(q)} + O\left(\bar{k}, \bar{\bar{k}}\right), \tag{B.13b}$$

where only the lowest order contribution is required. This can be simplified to straightaway by observing

$$\begin{aligned}\frac{1}{f(q) + f\left(k+q\right)} &= \frac{1}{2f(q) + O(k)} \\ &= \frac{1}{2f(q)} + O(k).\end{aligned} \tag{B.14}$$

B.2.4 Symmetric four-piece

The symmetric arrangement of four propagators,



,

is by far the most algebraically challenging, with the result

$$\begin{aligned}
&\int_{\infty}^{\infty} d\omega \frac{1}{\omega^2+f^2(q)} \frac{1}{-i\omega+f\left(\bar{k}+q\right)} \frac{1}{+i\omega+f\left(\bar{\bar{k}}-q\right)} \\
&\quad= \frac{2\pi}{f\left(\bar{\bar{k}}-q\right)-f(q)} \\
&\quad\quad\cdot \frac{\left(f(q)+f\left(\bar{\bar{k}}-q\right)\right)\left(f\left(\bar{k}+q\right)+f\left(\bar{\bar{k}}-q\right)\right)-2f(q)\left(f(q)+f\left(\bar{k}+q\right)\right)}{2f(q)\left(f(q)+f\left(\bar{k}+q\right)\right)\left(f(q)+f\left(\bar{\bar{k}}-q\right)\right)\left(f\left(\bar{k}+q\right)+f\left(\bar{\bar{k}}-q\right)\right)} \\
&\quad= \frac{\pi}{2f^3(q)}+O(\bar{k},\bar{\bar{k}}).
\end{aligned} \tag{B.15}$$

The complexity resulting from the presence of multiple poles in either half-plane:

$$\begin{aligned}
&\int_{\infty}^{\infty} d\omega \frac{1}{\omega^2+f^2(q)} \frac{1}{-i\omega+f\left(\bar{k}+q\right)} \frac{1}{+i\omega+f\left(\bar{\bar{k}}-q\right)} \\
&\qquad\qquad= \int_{\infty}^{\infty} d\omega \frac{1}{\omega^2+f^2(q)} \frac{1}{\omega+if\left(\bar{k}+q\right)} \frac{1}{\omega-if\left(\bar{\bar{k}}-q\right)} \\
&\quad= 2\pi i\cdot \operatorname{Res}\left(\omega=if(q),\omega=if\left(\bar{\bar{k}}-q\right)\right) \\
&\quad= 2\pi i\left[\frac{1}{2if(q)} \frac{1}{i\left(f(q)+f\left(\bar{k}+q\right)\right)} \frac{1}{i\left(f(q)-f\left(\bar{\bar{k}}-q\right)\right)}\right. \\
&\qquad\left.+\frac{1}{i\left(f\left(\bar{\bar{k}}-q\right)+f(q)\right)} \frac{1}{i\left(f\left(\bar{\bar{k}}-q\right)-f(q)\right)} \frac{1}{i\left(f\left(\bar{k}+q\right)+f\left(\bar{\bar{k}}-q\right)\right)}\right] \\
&\quad= \frac{2\pi i}{i\left(f\left(\bar{\bar{k}}-q\right)-f(q)\right)}\left[\frac{1}{2f(q)\left(f(q)+f\left(\bar{k}+q\right)\right)}\right. \\
&\qquad\qquad\left.-\frac{1}{\left(f\left(\bar{\bar{k}}-q\right)+f(q)\right)\left(f\left(\bar{k}+q\right)+f\left(\bar{\bar{k}}-q\right)\right)}\right]
\end{aligned}$$

$$= \frac{2\pi}{f\left(\bar{\bar{k}} - q\right) - f(q)}$$

$$\cdot \frac{\left(f(q) + f\left(\bar{\bar{k}} - q\right)\right)\left(f\left(\bar{k} + q\right) + f\left(\bar{\bar{k}} - q\right)\right) - 2f(q)\left(f(q) + f\left(\bar{k} + q\right)\right)}{2f(q)\left(f(q) + f\left(\bar{k} + q\right)\right)\left(f(q) + f\left(\bar{\bar{k}} - q\right)\right)\left(f\left(\bar{k} + q\right) + f\left(\bar{\bar{k}} - q\right)\right)}. \quad \text{(B.16)}$$

Not only is this messy, but at first glance there is no term of zero order in the external momentum. The large denominator is fine,

$$\begin{aligned} &\frac{1}{2f(q)\left(f(q) + f\left(\bar{k} + q\right)\right)\left(f(q) + f\left(\bar{\bar{k}} - q\right)\right)\left(f\left(\bar{k} + q\right) + f\left(\bar{\bar{k}} - q\right)\right)} \\ &\qquad = \frac{1}{16f^4(q)} + O(\bar{k}, \bar{\bar{k}}). \end{aligned} \quad \text{(B.17a)}$$

However, the lowest order term in the numerator is linear in the external momentum,

$$\begin{aligned} &\left(f(q) + f\left(\bar{\bar{k}} - q\right)\right)\left(f\left(\bar{k} + q\right) + f\left(\bar{\bar{k}} - q\right)\right) - 2f(q)\left(f(q) + f\left(\bar{k} + q\right)\right) \\ &\quad = 4f^2(q)\left(1 - \frac{\left(\bar{\bar{k}} \cdot q\right)\left[2\nu_4 q^2 + \nu_2\right]}{f(q)} + O(\bar{\bar{k}}^2)\right) \\ &\quad\quad \cdot \left(1 + \frac{\left(\bar{k} \cdot q\right)\left[2\nu_4 q^2 + \nu_2\right] - \left(\bar{\bar{k}} \cdot q\right)\left[2\nu_4 q^2 + \nu_2\right]}{f(q)} + O(\bar{k}^2, \bar{\bar{k}}^2)\right) \\ &\quad\quad - 4f^2(q)\left(1 + \frac{\left(\bar{k} \cdot q\right)\left[2\nu_4 q^2 + \nu_2\right]}{f(q)} + O(\bar{k}^2)\right) \\ &\quad = -8f(q)\left(\bar{\bar{k}} \cdot q\right)\left[2\nu_4 q^2 + \nu_2\right] + O\left(\bar{k}\bar{\bar{k}}, \bar{\bar{k}}^2\right). \end{aligned} \quad \text{(B.17b)}$$

This apparent problem is solved by the smaller denominator term, which goes as

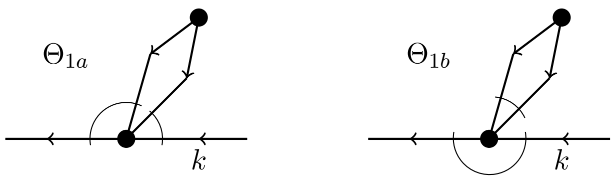
$$\begin{aligned} \frac{1}{f\left(\bar{\bar{k}} - q\right) - f(q)} &= \frac{1}{-2\left(\bar{\bar{k}} \cdot q\right)\left[2\nu_4 q^2 + \nu_2\right] + O\left(\bar{\bar{k}}^2\right)} \\ &= -\frac{1}{2\left(\bar{\bar{k}} \cdot q\right)\left[2\nu_4 q^2 + \nu_2\right]} \cdot \frac{1}{1 + O\left(\bar{\bar{k}}\right)}. \end{aligned} \quad \text{(B.17c)}$$

Combing these terms, the lowest order contribution is

$$\begin{aligned} &\frac{2\pi}{f\left(\bar{\bar{k}} - q\right) - f(q)} \frac{\left(f(q) + f\left(\bar{\bar{k}} - q\right)\right)\left(f\left(\bar{k} + q\right) + f\left(\bar{\bar{k}} - q\right)\right) - 2f(q)\left(f(q) + f\left(\bar{k} + q\right)\right)}{2f(q)\left(f(q) + f\left(\bar{k} + q\right)\right)\left(f(q) + f\left(\bar{\bar{k}} - q\right)\right)\left(f\left(\bar{k} + q\right) + f\left(\bar{\bar{k}} - q\right)\right)} \\ &\qquad = \frac{(2\pi) \cdot -8f(q)\left(\bar{\bar{k}} \cdot q\right)\left[2\nu_4 q^2 + \nu_2\right]}{16f^4(q) \cdot -2\left(\bar{\bar{k}} \cdot q\right)\left[2\nu_4 q^2 + \nu_2\right]} + O\left(\bar{k}, \bar{\bar{k}}\right) \\ &\qquad = \frac{\pi}{2f^3(q)} + O\left(\bar{k}, \bar{\bar{k}}\right). \end{aligned} \quad \text{(B.18)}$$

B.3 The $\hat{\Gamma}^{11}(k,0)$ diagrams

B.3.1 Θ_1



The integral for Θ_1 is the most straightforward to do. There are two arrangements to be considered; a, with a symmetry factor of 2, and b, with a symmetry factor of 1. This reduction for b is so as not to double count the permutation from the reversal of the noise vertex. The full integral is

$$\mathbf{I}_{\Theta_1} = \int \bar{d}^d q \, \frac{I_\omega}{2\pi} \cdot -2\Gamma^2 \lambda_{13} \left[2(k \cdot q)^2 + k^2 q^2 \right]. \tag{B.19}$$

Substituting in the ω integral for the tadpole,

$$\mathbf{I}_{\Theta_1} = -\frac{\Gamma^2 \lambda_{13}}{\nu_4} \int \bar{d}^d q \, \frac{2(k \cdot q)^2 + k^2 q^2}{q^2 \left(q^2 + \frac{\nu_2}{\nu_4} \right)}. \tag{B.20}$$

Performing both momentum integrals gives

$$\mathbf{I}_{\Theta_1} = -\frac{\Gamma^2 \lambda_{13}}{\nu_4} \left[\frac{2k^2}{d} \frac{\Gamma\left(1 - \frac{d}{2}\right)}{(4\pi)^{d/2} \Gamma(1)} \left(\frac{\nu_4}{\nu_2} \right)^{1-\frac{d}{2}} + k^2 \frac{\Gamma\left(1 - \frac{d}{2}\right)}{(4\pi)^{d/2} \Gamma(1)} \left(\frac{\nu_4}{\nu_2} \right)^{1-\frac{d}{2}} \right]. \tag{B.21}$$

Using $\epsilon = 4 - d$, the property of the gamma function $\Gamma(n+1) = n\Gamma(n)$, and keeping only terms to leading order in ϵ,

$$\Gamma\left(1 - \frac{d}{2}\right) = \Gamma\left(\frac{\epsilon}{2} - 1\right) = \frac{\Gamma\left(\frac{\epsilon}{2}\right)}{\frac{\epsilon}{2} - 1} = -\Gamma\left(\frac{\epsilon}{2}\right) + O(\epsilon), \tag{B.22}$$

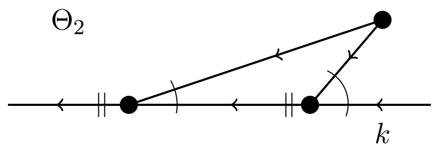
gives

$$\begin{aligned} \mathbf{I}_{\Theta_1} &= \frac{\Gamma^2 \lambda_{13}}{\nu_4^2 (4\pi)^2} \left(\frac{\nu_4}{\nu_2} \right)^{\frac{\epsilon}{2}} \Gamma\left(\frac{\epsilon}{2}\right) \nu_2 \left[\frac{k^2}{2} + k^2 \right] \\ &= \frac{3}{2} \frac{\Gamma^2 \lambda_{13}}{\nu_4^2 (4\pi)^2} \left(\frac{\nu_4}{\nu_2} \right)^{\frac{\epsilon}{2}} \Gamma\left(\frac{\epsilon}{2}\right) \nu_2 k^2 \\ &= \frac{3}{2} \lambda_{13} \mathcal{C} \nu_2 k^2, \end{aligned} \tag{B.23}$$

having used

$$\mathcal{C} = \frac{\Gamma^2}{(4\pi)^2 \nu_4^2} \left(\frac{\nu_4}{\nu_2} \right)^{\frac{\epsilon}{2}} \Gamma\left(\frac{\epsilon}{2}\right). \tag{B.24}$$

B.3.2 Θ_2



For Θ_2 there is one diagrammatic arrangement, with a symmetry factor of 4,

$$\mathbf{I}_{\Theta_2} = \int đ^d q \, \frac{I_\omega}{2\pi} \cdot 2\Gamma^2 \lambda_{22}^2 \cdot -4(k\cdot q)(k+q)^2 \left((k+q)\cdot q\right) k^2. \tag{B.25}$$

Which, as it stands, is analytically pretty intractable. However, it is only necessary to characterise how it diverges in the ultraviolet. Writing schematically as

$$\begin{aligned}
\mathbf{I} &\propto k^2 \int đ^d q \frac{(k\cdot q)(k+q)^2((k+q)\cdot q)}{(\nu_4 q^4 + \nu_2 q^2)\left((\nu_4 q^4 + \nu_2 q^2) + (\nu_4 (k+q)^4 + \nu_2 (k+q)^2)\right)} \\
&\propto \int đ^d q \frac{(k\cdot q)(q^4 + q^3 k + q^2 k^2 + q k^3)}{q^8 + q^7 k + q^6 k^2 + q^5 k^3 + q^4 k^4} \\
&= k^2\, k \cdot \int đ^d q \left[k^0 q^{-3} + k^1 q^{-4} + k^2 q^{-5} \dots\right],
\end{aligned} \tag{B.26}$$

the only term of interest in the expansion is the term linear in k. The term that goes as k^0 can be ignored by symmetry as not contributing, higher order terms are all ultraviolet finite. Collecting the momentum terms in the numerator of order k^3 and k^4,

$$k^2 (k\cdot q) q^4 + 3k^2 (k\cdot q)^2 q^2 + O(k^5), \tag{B.27}$$

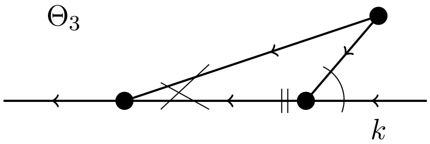
which combine with the first two terms from the ω-integral. Keeping only terms of order k^4, the integral is

$$\mathbf{I}_{\Theta_2} = \int đ^d q \, \frac{2\Gamma^2 \lambda_{22}^2}{\nu_4^2} \left[\frac{k^2 (k\cdot q)^2 q^4 [\nu_2 + 2q^2 \nu_4]}{\nu_4 q^6 \left(q^2 + \frac{\nu_2}{\nu_4}\right)^3} - \frac{3k^2 (k\cdot q)^2 q^2}{q^4 \left(q^2 + \frac{\nu_2}{\nu_4}\right)^2}\right], \tag{B.28}$$

which is evaluated as

$$\begin{aligned}
\mathbf{I}_{\Theta_2} &= \int đ^d q \, \frac{2\Gamma^2 \lambda_{22}^2}{\nu_4^2} k^2 \left[\frac{(1-3)(k\cdot q)^2}{q^2 \left(q^2 + \frac{\nu_2}{\nu_4}\right)^2} + \frac{(k\cdot q)^2}{\left(q^2 + \frac{\nu_2}{\nu_4}\right)^3}\right] \\
&= \frac{2\Gamma^2 \lambda_{22}^2}{\nu_4^2} k^2 \left[-\frac{2k^2}{d} \frac{1}{(4\pi)^2} \left(\frac{\nu_4}{\nu_2}\right)^{\frac{\epsilon}{2}} \Gamma\left(\frac{\epsilon}{2}\right) + \frac{k^2}{4} \frac{1}{(4\pi)^2} \left(\frac{\nu_4}{\nu_2}\right)^{\frac{\epsilon}{2}} \Gamma\left(\frac{\epsilon}{2}\right)\right] \\
&= -\frac{1}{2} \frac{\lambda_{22}^2}{\nu_4} \mathcal{C} \nu_4 k^4.
\end{aligned} \tag{B.29}$$

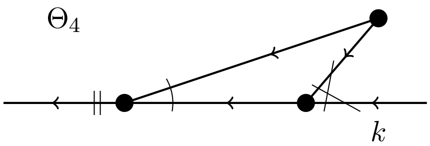
B.3.3 Θ_3



There are four different vertex compositions for the same diagrammatic structure premiered in Θ_2. All four have a symmetry factor of 4, with Θ_3, Θ_4, and Θ_5 having the κ vertex mixed in to replace the λ_{22} vertices. For Θ_3, the integral is

$$\begin{aligned}
\mathbf{I}_{\Theta_3} &= \int đ^d q\, \frac{I_\omega}{2\pi} \cdot 8\Gamma^2\kappa\lambda_{22} \cdot (k\cdot q)(k+q)^2 \left[k^2q^2 - (k\cdot q)^2\right] \\
&= \int đ^d q\, \frac{I_\omega}{2\pi} \cdot 8\Gamma^2\kappa\lambda_{22}\left[k^2(k\cdot q)q^4 - (k\cdot q)^3q^2 + 2k^2(k\cdot q)^2q^2 - 2(k\cdot q)^4 + O(k^5)\right] \\
&= \int đ^d q\, \frac{2\Gamma^2\kappa\lambda_{22}}{\nu_4^2}\left[-\frac{\left[k^2(k\cdot q)^2q^4 - (k\cdot q)^4q^2\right][\nu_2 + 2\nu_4 q^2]}{\nu_4 q^6\left(q^2 + \frac{\nu_2}{\nu_4}\right)^3} + \frac{2k^2(k\cdot q)^2q^2 - 2(k\cdot q)^4}{q^4\left(q^2+\frac{\nu_2}{\nu_4}\right)^2}\right] \\
&= \int đ^d q\, \frac{2\Gamma^2\kappa\lambda_{22}}{\nu_4^2}\left[k^2\frac{(k\cdot q)^2}{q^2\left(q^2+\frac{\nu_2}{\nu_4}\right)^2} - \frac{(k\cdot q)^4}{q^4\left(q^2+\frac{\nu_2}{\nu_4}\right)^2} + \frac{(k\cdot q)^4}{q^2\left(q^2+\frac{\nu_2}{\nu_4}\right)^3} - k^2\frac{(k\cdot q)^2}{\left(q^2+\frac{\nu_2}{\nu_4}\right)^3}\right] \\
&= \frac{2\Gamma^2\kappa\lambda_{22}}{\nu_4^2}\left[\frac{k^4}{d}\frac{1}{(4\pi)^2}\left(\frac{\nu_4}{\nu_2}\right)^{\frac{\epsilon}{2}}\Gamma\left(\frac{\epsilon}{2}\right) - \frac{k^4}{8}\frac{1}{(4\pi)^2}\left(\frac{\nu_4}{\nu_2}\right)^{\frac{\epsilon}{2}}\Gamma\left(\frac{\epsilon}{2}\right)\right. \\
&\qquad\left. + \frac{k^4}{8}\frac{1}{(4\pi)^2}\left(\frac{\nu_4}{\nu_2}\right)^{\frac{\epsilon}{2}}\Gamma\left(\frac{\epsilon}{2}\right) - \frac{k^4}{4}\frac{1}{(4\pi)^2}\left(\frac{\nu_4}{\nu_2}\right)^{\frac{\epsilon}{2}}\Gamma\left(\frac{\epsilon}{2}\right)\right] \\
&= 0.
\end{aligned} \tag{B.30}$$

B.3.4 Θ_4

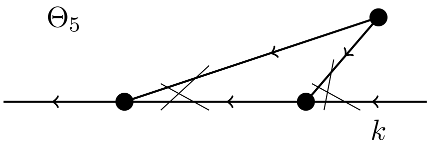


With the integral giving

$$\begin{aligned}
\mathbf{I}_{\Theta_4} &= \int đ^d q\, \frac{I_\omega}{2\pi} \cdot 8\Gamma^2\kappa\lambda_{22} \cdot -k^2\left((k+q)\cdot q\right)\left[k^2q^2 - (k\cdot q)^2\right] \\
&= \int đ^d q\, \frac{I_\omega}{2\pi} \cdot 8\Gamma^2\kappa\lambda_{22} \cdot \left[k^2(k\cdot q)^2q^2 - k^4q^4 + O(k^5)\right] \\
&= \int đ^d q\, \frac{2\Gamma^2\kappa\lambda_{22}}{\nu_4^2}\frac{k^2(k\cdot q)^2q^2 - k^4q^4}{q^2\left(q^2+\frac{\nu_2}{\nu_4}\right)^2}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2\Gamma^2\kappa\lambda_{22}}{\nu_4^2}\left[\frac{k^4}{d}\frac{1}{(4\pi)^2}\left(\frac{\nu_4}{\nu_2}\right)^{\frac{\epsilon}{2}}\Gamma\left(\frac{\epsilon}{2}\right) - k^4\frac{1}{(4\pi)^2}\left(\frac{\nu_4}{\nu_2}\right)^{\frac{\epsilon}{2}}\Gamma\left(\frac{\epsilon}{2}\right)\right] \\
&= -\frac{3}{2}\frac{\kappa\lambda_{22}}{\nu_4}\mathcal{C}\nu_4 k^4. \qquad \text{(B.31)}
\end{aligned}
$$

B.3.5 Θ_5

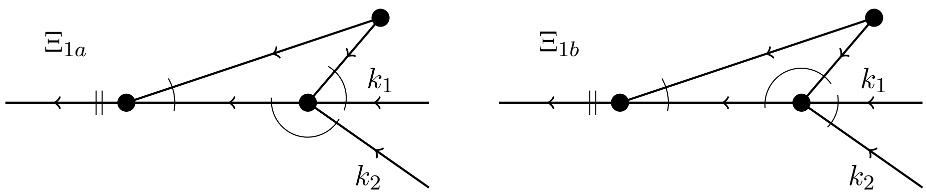


Leading to the contribution

$$
\begin{aligned}
\mathbf{I}_{\Theta_5} &= \int đ^d q\,\frac{I_\omega}{2\pi}\cdot 8\Gamma^2\kappa^2\cdot\left[k^2q^2 - (k\cdot q)^2\right]^2 \\
&= \int đ^d q\,\frac{I_\omega}{2\pi}\cdot 8\Gamma^2\kappa^2\cdot\left[k^4q^4 - 2k^2(k\cdot q)^2q^2 + (k\cdot q)^4\right] \\
&= \int đ^d q\,\frac{2\Gamma^2\kappa^2}{\nu_4^2}\left[k^4\frac{1}{\left(q^2+\frac{\nu_2}{\nu_4}\right)^2} - 2k^2\frac{(k\cdot q)^2}{q^2\left(q^2+\frac{\nu_2}{\nu_4}\right)^2} + \frac{(k\cdot q)^4}{q^4\left(q^2+\frac{\nu_2}{\nu_4}\right)^2}\right] \\
&= \frac{2\Gamma^2\kappa^2}{\nu_4^2}\left[k^4\frac{1}{(4\pi)^2}\left(\frac{\nu_4}{\nu_2}\right)^{\frac{\epsilon}{2}}\Gamma\left(\frac{\epsilon}{2}\right) - \frac{2k^4}{d}\frac{1}{(4\pi)^2}\left(\frac{\nu_4}{\nu_2}\right)^{\frac{\epsilon}{2}}\Gamma\left(\frac{\epsilon}{2}\right) + \frac{k^4}{8}\frac{1}{(4\pi)^2}\left(\frac{\nu_4}{\nu_2}\right)^{\frac{\epsilon}{2}}\Gamma\left(\frac{\epsilon}{2}\right)\right] \\
&= \frac{5}{4}\frac{\kappa^2}{\nu_4}\mathcal{C}\nu_4 k^4. \qquad \text{(B.32)}
\end{aligned}
$$

B.4 The $\hat{\Gamma}^{12}(k_1, k_2)$ diagrams

B.4.1 Ξ_1

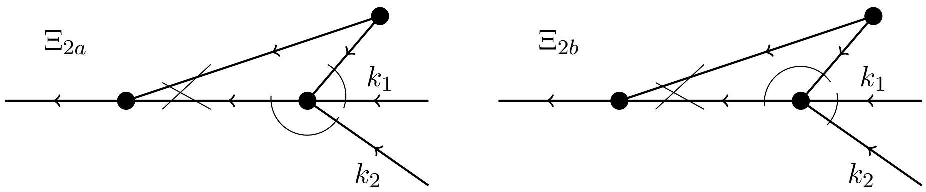


For the diagram Ξ_1 there two arrangements, a has a symmetry factor of 4, b has a symmetry factor of 2. Again, b's is reduced so as to avoid double counting, this time for the contribution from permutation of the external momenta. The third way of arranging the dot products is also equivalent to a permutation of the external momenta, and so is discounted. The contribution required from the frequency integral is of zeroth order,

$$
\begin{aligned}
\mathbf{I}_{\Xi_1} &= \int \bar{d}^d q \frac{I_\omega}{2\pi} \cdot 2\Gamma^2 \lambda_{13}\lambda_{22} \cdot \Big[-4(k_1 \cdot q)\big(k_2 \cdot (k_1 + k_2 + q)\big)(k_1 + k_2)^2 \big((k_1 + k_2 + q) \cdot q\big) \\
&\qquad\qquad - 2(k_1 \cdot k_2)(k_1 + k_2)^2 \big((k_1 + k_2 + q) \cdot q\big)^2 \Big] \\
&= \int \bar{d}^d q \frac{\Gamma^2 \lambda_{13}\lambda_{22}}{2\nu_4^2} \left[-\frac{4(k_1 + k_2)^2 (k_1 \cdot q)(k_2 \cdot q) q^2}{q^4 (q^2 + \frac{\nu_2}{\nu_4})^2} - \frac{2(k_1 \cdot k_2)(k_1 + k_2)^2 q^4}{q^4 (q^2 + \frac{\nu_2}{\nu_4})^2} \right] \\
&= \int \bar{d}^d q \frac{\Gamma^2 \lambda_{13}\lambda_{22}}{\nu_4^2} \left[-2(k_1 + k_2)^2 \frac{(k_1 \cdot q)(k_2 \cdot q)}{q^2 (q^2 + \frac{\nu_2}{\nu_4})^2} - (k_1 \cdot k_2)(k_1 + k_2)^2 \frac{1}{(q^2 + \frac{\nu_2}{\nu_4})^2} \right] \\
&= \frac{\Gamma^2 \lambda_{13}\lambda_{22}}{\nu_4^2} (k_1 \cdot k_2)(k_1 + k_2)^2 \left[-\frac{1}{2}\frac{1}{(4\pi)^2} \left(\frac{\nu_4}{\nu_2}\right)^{\frac{\epsilon}{2}} \Gamma\left(\frac{\epsilon}{2}\right) - \frac{1}{(4\pi)^2} \left(\frac{\nu_4}{\nu_2}\right)^{\frac{\epsilon}{2}} \Gamma\left(\frac{\epsilon}{2}\right) \right] \\
&= -\frac{3}{2} \lambda_{13} \mathcal{C} \lambda_{22} (k_1 \cdot k_2)(k_1 + k_2)^2, \qquad \text{(B.33)}
\end{aligned}
$$

with the result symmetric in the incoming k_1 and k_2 there is no need to permute the momenta and combine the results.

B.4.2 Ξ_2



Similar to the previous diagram, but with the replacement of λ_{22} in the nose with κ. A useful pair of identities to note for when the κ vertex is present are

$$
\begin{aligned}
([k_1 + k_2 + q] \wedge [-q])^2 &= ([k_1 + k_2 + q] \wedge q)^2 \\
&= (k_1 + k_2 + q)^2 q^2 - ((k_1 + k_2 + q) \cdot q)^2 \\
&= (k_1 + k_2)^2 q^2 + 2((k_1 + k_2) \cdot q) q^2 + q^4 \\
&\qquad - ((k_1 + k_2) \cdot q)^2 - 2((k_1 + k_2) \cdot q) q^2 - q^4 \\
&= (k_1 + k_2)^2 q^2 - ((k_1 + k_2) \cdot q)^2 \\
&= ([k_1 + k_2] \wedge [q])^2, \qquad \text{(B.34)}
\end{aligned}
$$

which could also be noted from

$$
([k_1 + k_2 + q] \wedge [-q])^2 = ([k_1 + k_2] \wedge q + q \wedge q)^2 = ([k_1 + k_2] \wedge [q])^2,
$$

and

$$
\begin{aligned}
([k_1+q]\wedge[k_2-q])^2 &= (k_1+q)^2(k_2-q)^2 - ((k_1+q)\cdot(k_2-q))^2 \\
&= (k_1^2+2(k_1\cdot q)+q^2)(k_2^2-2(k_2\cdot q)+q^2) \\
&\qquad\qquad - \big((k_1\cdot k_2)+(k_2\cdot q)-(k_1\cdot q)-q^2\big)^2 \\
&= \big(k_1^2k_2^2 - 4(k_1\cdot q)(k_2\cdot q)+q^4-2k_1^2(k_2\cdot q)+k_1^2q^2 \\
&\qquad +2k_2^2(k_1\cdot q)+2(k_1\cdot q)q^2+k_2^2q^2-2(k_2\cdot q)q^2\big) \\
&- \big((k_1\cdot k_2)^2+(k_2\cdot q)^2+(k_1\cdot q)^2+q^4+2(k_1\cdot k_2)(k_2\cdot q) \\
&\qquad -2(k_1\cdot k_2)(k_1\cdot q)-2(k_1\cdot k_2)q^2-2(k_1\cdot q)(k_2\cdot q) \\
&\qquad\quad -2(k_2\cdot q)q^2+2(k_1\cdot q)q^2\big) \\
&= k_1^2k_2^2-(k_1\cdot k_2)^2 \\
&\quad -2k_1^2(k_2\cdot q)+2k_2^2(k_1\cdot q)-2(k_1\cdot k_2)(k_2\cdot q)+2(k_1\cdot k_2)(k_1\cdot q) \\
&+(k_1^2+k_2^2)q^2+2(k_1\cdot k_2)q^2-(k_2\cdot q)^2-(k_1\cdot q)^2-2(k_1\cdot q)(k_2\cdot q) \\
&= (k_1+k_2)^2q^2-([k_1+k_2]\cdot q)^2+O(k^3) \\
&= ((k_1+k_2)\wedge q)^2+O(k^3),
\end{aligned}
\tag{B.35}
$$

which can also be derived by

$$
\begin{aligned}
([k_1+q]\wedge[k_2-q])^2 &= ([k_1\wedge -q]+[q\wedge k_2]+[k_1\wedge k_2])^2 \\
&= (k_1\wedge -q)^2+(q\wedge k_2)^2+2(k_1\wedge -q)\cdot(q\wedge k_2) \\
&\qquad +2(k_1\wedge -q)\cdot(k_1\wedge k_2)+2(q\wedge k_2)\cdot(k_1\wedge k_2)+(k_1\wedge k_2)^2 \\
&= (k_1\wedge q)^2+(k_2\wedge q)^2+2(k_1\wedge q)\cdot(k_2\wedge q)+O(k^3) \\
&= ([k_1+k_2]\wedge q)^2+O(k^3).
\end{aligned}
$$

So a κ vertex, with an external field entering it, will always provides a momentum factor of at least order k^2. Regarding the symmetrisation of external momenta, permuting the fields allows the following identities:

$$
k_1^2(k_2\cdot(k_1+k_2)) = \frac{1}{2}(k_1\cdot k_2)(k_1+k_2)^2+(k_1\wedge k_2)^2, \tag{B.36}
$$

$$
k_1\cdot(k_1+k_2)\; k_2\cdot(k_1+k_2) = (k_1\cdot k_2)(k_1+k_2)^2+(k_1\wedge k_2)^2, \tag{B.37}
$$

$$
(k_1\cdot k_2)(k_1\cdot(k_1+k_2)) = \frac{1}{2}(k_1\cdot k_2)(k_1+k_2)^2. \tag{B.38}
$$

Returning to the Ξ_2 diagram, by inspection in incoming legs will carry at least two power of the external momentum, and the κ vertex in the nose will provide another two powers of external momentum. So only the zeroth order contribution from the frequency integral will be required, and the lowest order internal power contribution, to capture the divergent contribution of the diagram. There are two arrangements, a has a symmetry factor of 4, b has a smaller factor 2. The reduction is not to double count the contribution from permuting the external momenta.

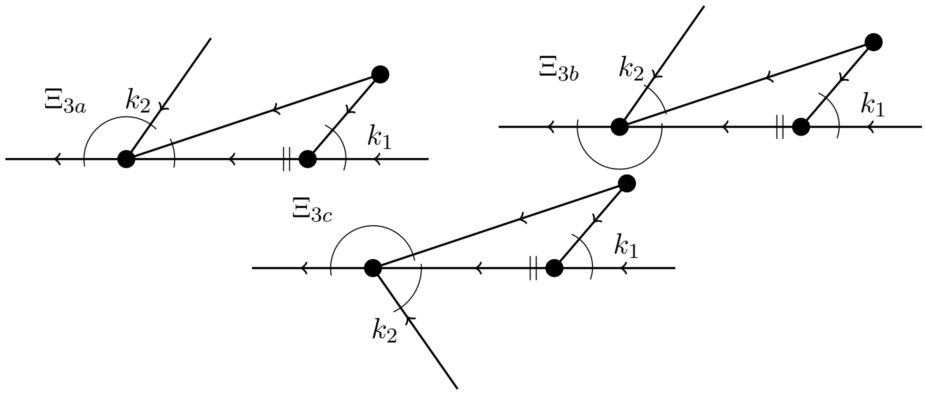
$$
\begin{aligned}
\mathbf{I}_{\Xi_2} &= \int đ^d q \frac{I_\omega}{2\pi} \cdot 2\Gamma^2 \lambda_{13}\kappa \cdot ([k_1+k_2+q]\wedge[-q])^2 \\
&\qquad \left[4(k_1\cdot q)(k_2\cdot(k_1+k_2+q)) + 2(k_1\cdot k_2)((k_1+k_2+q)\cdot q)^2\right] \\
&= \int đ^d q \frac{\Gamma^2\lambda_{13}\kappa}{\nu_4^2} \frac{([k_1+k_2]\wedge[q])^2\left[2(k_1\cdot q)(k_2\cdot q)+(k_1\cdot k_2)q^2\right]}{q^4(q^2+\frac{\nu_2}{\nu_4})^2} \\
&= \int đ^d q \frac{\Gamma^2\lambda_{13}\kappa}{\nu_4^2} \frac{\left[(k_1+k_2)^2q^2-((k_1+k_2)\cdot q)^2\right]\left[2(k_1\cdot q)(k_2\cdot q)+(k_1\cdot k_2)q^2\right]}{q^4(q^2+\frac{\nu_2}{\nu_4})^2} \\
&= \int đ^d q \frac{\Gamma^2\lambda_{13}\kappa}{\nu_4^2} \Bigg[(k_1\cdot k_2)(k_1+k_2)^2\frac{q^4}{q^4(q^2+\frac{\nu_2}{\nu_4})^2} - (k_1\cdot k_2)\frac{((k_1+k_2)\cdot q)^2q^2}{q^4(q^2+\frac{\nu_2}{\nu_4})^2} \\
&\qquad +(k_1+k_2)^2\frac{2(k_1\cdot q)(k_2\cdot q)q^2}{q^4(q^2+\frac{\nu_2}{\nu_4})^2} - \frac{2((k_1+k_2)\cdot q)^2(k_1\cdot q)(k_2\cdot q)}{q^4(q^2+\frac{\nu_2}{\nu_4})^2}\Bigg] \\
&= \int đ^d q \frac{\Gamma^2\lambda_{13}\kappa}{\nu_4^2} \Bigg[(k_1\cdot k_2)(k_1+k_2)^2\frac{1}{(q^2+\frac{\nu_2}{\nu_4})^2} - (k_1\cdot k_2)\frac{((k_1+k_2)\cdot q)^2}{q^2(q^2+\frac{\nu_2}{\nu_4})^2} \\
&\qquad +(k_1+k_2)^2\frac{2(k_1\cdot q)(k_2\cdot q)}{q^2(q^2+\frac{\nu_2}{\nu_4})^2} - \frac{2((k_1+k_2)\cdot q)^2(k_1\cdot q)(k_2\cdot q)}{q^4(q^2+\frac{\nu_2}{\nu_4})^2}\Bigg].
\end{aligned}
$$

The first three integrals are pretty straightforward, giving a contribution of $1-1/4+1/2$, with external momentum of the λ_{22} form $(k_1\cdot k_2)(k_1+k_2)^2$. The fourth term requires some more careful work,

$$
\begin{aligned}
\mathbf{I}_{\Xi_2} &= \int đ^d q \frac{\Gamma^2\lambda_{13}\kappa}{\nu_4^2} \Bigg[(k_1\cdot k_2)(k_1+k_2)^2\frac{1}{(q^2+\frac{\nu_2}{\nu_4})^2} - (k_1\cdot k_2)\frac{((k_1+k_2)\cdot q)^2}{q^2(q^2+\frac{\nu_2}{\nu_4})^2} \\
&\qquad +(k_1+k_2)^2\frac{2(k_1\cdot q)(k_2\cdot q)}{q^2(q^2+\frac{\nu_2}{\nu_4})^2}\Bigg] \\
&\qquad - \int đ^d q \frac{\Gamma^2\lambda_{13}\kappa}{\nu_4^2} \frac{2\big((k_1+k_2)\cdot q\big)^2(k_1\cdot q)(k_2\cdot q)}{q^4(q^2+\frac{\nu_2}{\nu_4})^2} \\
&= \lambda_{13}\kappa\mathcal{C}(k_1\cdot k_2)(k_1+k_2)^2 - \frac{1}{4}\lambda_{13}\kappa\mathcal{C}(k_1\cdot k_2)(k_1+k_2)^2 + \frac{1}{2}\lambda_{13}\kappa\mathcal{C}(k_1\cdot k_2)(k_1+k_2)^2 \\
&\qquad - \frac{\Gamma^2\lambda_{13}\kappa}{\nu_4^2}\frac{1}{12}\left[(k_1\cdot k_2)(k_1+k_2)^2 + 2k_1\cdot(k_1+k_2)\ k_2\cdot(k_1+k_2)\right]\int đ^d q\frac{1}{(q^2+\frac{\nu_2}{\nu_4})^2}
\end{aligned}
$$

$$= \frac{3}{4}\lambda_{13}\kappa\mathcal{C}(k_1 \cdot k_2)(k_1+k_2)^2 + \frac{1}{2}\lambda_{13}\kappa\mathcal{C}(k_1 \cdot k_2)(k_1+k_2)^2$$
$$- \frac{1}{12}\lambda_{13}\kappa\mathcal{C}(k_1 \cdot k_2)(k_1+k_2)^2 - \frac{1}{6}\lambda_{13}\kappa\mathcal{C}(k_1 \cdot k_2)(k_1+k_2)^2 - \frac{1}{6}\lambda_{13}\kappa\mathcal{C}(k_1 \wedge k_2)^2$$
$$= \frac{3}{4}\lambda_{13}\kappa\mathcal{C}(k_1 \cdot k_2)(k_1+k_2)^2 + \frac{1}{4}\lambda_{13}\kappa\mathcal{C}(k_1 \cdot k_2)(k_1+k_2)^2 - \frac{1}{6}\lambda_{13}\kappa\mathcal{C}(k_1 \wedge k_2)^2$$
$$= \frac{\lambda_{13}\kappa}{\lambda_{22}}\mathcal{C}\lambda_{22}(k_1 \cdot k_2)(k_1+k_2)^2 - \frac{1}{6}\lambda_{13}\mathcal{C}\kappa(k_1 \wedge k_2)^2. \tag{B.39}$$

B.4.3 Ξ_3



To leading order, it looks like this diagram shouldn't contribute from symmetry as the external momentum carried on the legs starts at order k^3. But it will have a relevant term in the sub-leading order of k^4. There are three phenotypes to consider, all with a symmetry factor of 4. The first two terms of the frequency integral will be required to construct terms of the relevant order,

$$\mathbf{I}_{\Xi_3} = \int \!\!d\!\!\!^{-}{}^d q \frac{\mathbf{I}_\omega}{2\pi} \cdot 8\Gamma^2\lambda_{13}\lambda_{22} \cdot -(k_1 \cdot q)(k_1+q)^2 \tag{B.40}$$
$$\cdot \Big[(k_2 \cdot [k_1+k_2])([k_1+q]\cdot q) + (k_2 \cdot q)([k_1+k_2]\cdot[k_1+q]) + (k_2 \cdot [k_1+q])([k_1+k_2]\cdot q)\Big].$$

Gathering terms of order k^3 and k^4:

$$\mathbf{I}_{\Xi_{3a}} = \int \!\!d\!\!\!^{-}{}^d q \frac{\mathbf{I}_\omega}{2\pi} - 8\Gamma^2\lambda_{13}\lambda_{22}\big[(k_2 \cdot [k_1+k_2])(k_1 \cdot q)q^4$$
$$+2(k_2\cdot[k_1+k_2])(k_1\cdot q)^2 q^2 + (k_2\cdot[k_1+k_2])(k_1\cdot q)^2 q^2 + O(k^5)\big], \tag{B.41a}$$

$$\mathbf{I}_{\Xi_{3b}} = \int \!\!d\!\!\!^{-}{}^d q \frac{\mathbf{I}_\omega}{2\pi} - 8\Gamma^2\lambda_{13}\lambda_{22}\big[(k_1\cdot q)(k_2\cdot q)([k_1+k_2]\cdot q)q^2 \tag{B.41b}$$
$$+2(k_1\cdot q)^2(k_2\cdot q)([k_1+k_2]\cdot q) + (k_1\cdot[k_1+k_2])(k_1\cdot q)(k_2\cdot q)q^2 + O(k^5)\big],$$

$$\mathbf{I}_{\Xi_{3c}} = \int \!\!d\!\!\!^{-}{}^d q \frac{\mathbf{I}_\omega}{2\pi} - 8\Gamma^2\lambda_{13}\lambda_{22}\big[(k_1\cdot q)(k_2\cdot q)([k_1+k_2]\cdot q)q^2 \tag{B.41c}$$
$$+2(k_1\cdot q)^2(k_2\cdot q)([k_1+k_2]\cdot q) + (k_1\cdot[k_1+k_2])(k_1\cdot q)(k_2\cdot q)q^2 + O(k^5)\big].$$

It can be seen at this stage that the b and c integrals will give the same contribution. Evaluating the first phenotype gives

$$\begin{aligned}
\mathbf{I}_{\Xi_{3a}} &= \int đ^d q - \frac{2\Gamma^2\lambda_{13}\lambda_{22}}{\nu_4^2}\left[(3-1)\frac{(k_2\cdot(k_1+k_2))(k_1\cdot q)^2}{q^2(q^2+\frac{\nu_2}{\nu_4})^2} - \frac{(k_2\cdot(k_1+k_2))(k_1\cdot q)^2}{(q^2+\frac{\nu_2}{\nu_4})^3}\right] \\
&= \int đ^d q - \frac{2\Gamma^2\lambda_{13}\lambda_{22}}{\nu_4^2}(k_2\cdot(k_1+k_2))\left[\frac{2(k_1\cdot q)^2}{q^2(q^2+\frac{\nu_2}{\nu_4})^2} - \frac{(k_1\cdot q)^2}{(q^2+\frac{\nu_2}{\nu_4})^3}\right] \\
&= -\frac{1}{2}\lambda_{13}\mathcal{C}\lambda_{22}k_1^2(k_2\cdot(k_1+k_2)). \qquad \text{(B.42a)}
\end{aligned}$$

It is from the symmetrisation of a, which is explicitly

$$\begin{aligned}
k_1^2(k_2\cdot(k_1+k_2)) &= \frac{1}{2}k_1^2(k_2\cdot(k_1+k_2)) + \frac{1}{2}k_2^2(k_1\cdot(k_1+k_2)) \\
&= \frac{1}{2}(k_1\cdot k_2)(k_1^2+k_2^2) + k_1^2k_2^2 \\
&= \frac{1}{2}(k_1\cdot k_2)(k_1+k_2)^2 + k_1^2k_2^2 - (k_1\cdot k_2)^2 \\
&= \frac{1}{2}(k_1\cdot k_2)(k_1+k_2)^2 + (k_1\wedge k_2)^2,
\end{aligned} \qquad \text{(B.42b)}$$

that a new coupling is seen to be generated. So the contribution of the phenotype is

$$\begin{aligned}
\mathbf{I}_{\Xi_{3a}} &= -\frac{1}{2}\lambda_{13}\mathcal{C}\lambda_{22}k_1^2(k_2\cdot(k_1+k_2)) \\
&= -\frac{1}{4}\lambda_{13}\mathcal{C}\lambda_{22}(k_1\cdot k_2)(k_1+k_2)^2 - \frac{1}{2}\frac{\lambda_{13}\lambda_{22}}{\kappa}\mathcal{C}\kappa(k_1\wedge k_2)^2. \qquad \text{(B.42c)}
\end{aligned}$$

In the second phenotype,

$$\begin{aligned}
\mathbf{I}_{\Xi_{3b}} &= \int đ^d q - \frac{2\Gamma^2\lambda_{13}\lambda_{22}}{\nu_4^2}\left[(2-1)\frac{(k_1\cdot q)^2(k_2\cdot q)((k_1+k_2)\cdot q)}{q^4(q^2+\frac{\nu_2}{\nu_4})^2}\right. \\
&\left. +(k_1\cdot(k_1+k_2))\frac{(k_1\cdot q)(k_2\cdot q)}{q^2(q^2+\frac{\nu_2}{\nu_4})^2} - \frac{(k_1\cdot q)^2(k_2\cdot q)((k_1+k_2)\cdot q)}{q^2(q^2+\frac{\nu_2}{\nu_4})^3}\right] \\
&= \int đ^d q - \frac{2\Gamma^2\lambda_{13}\lambda_{22}}{\nu_4^2}(k_1\cdot(k_1+k_2))\frac{(k_1\cdot q)(k_2\cdot q)}{q^2(q^2+\frac{\nu_2}{\nu_4})^2} \\
&= -\frac{1}{2}\lambda_{13}\mathcal{C}\lambda_{22}(k_1\cdot k_2)(k_1\cdot(k_1+k_2)) \\
&= -\frac{1}{4}\lambda_{13}\mathcal{C}\lambda_{22}(k_1\cdot k_2)(k_1+k_2)^2, \qquad \text{(B.42d)}
\end{aligned}$$

the first and third term cancel:

$$\begin{aligned}
\mathbf{I}_{\Xi_{3b1}} &= \int đ^d q - \frac{2\Gamma^2\lambda_{13}\lambda_{22}}{\nu_4^2}\frac{(k_1\cdot q)^2(k_2\cdot q)([k_1+k_2]\cdot q)}{q^4(q^2+\frac{\nu_2}{\nu_4})^2} \\
&= -\frac{2\Gamma^2\lambda_{13}\lambda_{22}}{\nu_4^2}\sum_{\mu\nu\lambda\eta}(k_1^\mu k_1^\nu k_2^\lambda k_1^\eta + k_1^\mu k_1^\nu k_2^\lambda k_2^\eta)\cdot\int đ^d q\frac{q^\mu q^\nu q^\lambda q^\eta}{q^4(q^2+\frac{\nu_2}{\nu_4})^2}
\end{aligned}$$

$$= -\frac{2\Gamma^2\lambda_{13}\lambda_{22}}{\nu_4^2}\sum_{\mu\nu\lambda\eta}(k_1^\mu k_1^\nu k_2^\lambda k_1^\eta + k_1^\mu k_1^\nu k_2^\lambda k_2^\eta)(\delta_{\mu\nu}\delta_{\lambda\eta}+\delta_{\mu\lambda}\delta_{\nu\eta}+\delta_{\mu\eta}\delta_{\nu\lambda})$$

$$\cdot\frac{1}{24}\int đ^d q\frac{1}{(q^2+\frac{\nu_2}{\nu_4})^2}$$

$$= -\frac{\Gamma^2\lambda_{13}\lambda_{22}}{12\nu_4^2}((k_1^2(k_1\cdot k_2)+k_1^2k_2^2)+2(k_1^2(k_1\cdot k_2)+(k_1\cdot k_2)^2))\int đ^d q\frac{1}{(q^2+\frac{\nu_2}{\nu_4})^2} \quad \text{(B.43a)}$$

$$\mathbf{I}_{\Xi_{3b3}} = \int đ^d q\frac{2\Gamma^2\lambda_{13}\lambda_{22}}{\nu_4^2}\frac{(k_1\cdot q)^2(k_2\cdot q)([k_1+k_2]\cdot q)}{q^2(q^2+\frac{\nu_2}{\nu_4})^3}$$

$$= \frac{\Gamma^2\lambda_{13}\lambda_{22}}{12\nu_4^2}((k_1^2(k_1\cdot k_2)+k_1^2k_2^2)+2(k_1^2(k_1\cdot k_2)+(k_1\cdot k_2)^2))\int đ^d q\frac{q^2}{(q^2+\frac{\nu_2}{\nu_4})^3}, \quad \text{(B.43b)}$$

which leads to

$$\int đ^d q\frac{q^2}{(q^2+\frac{\nu_2}{\nu_4})^3} - \int đ^d q\frac{1}{(q^2+\frac{\nu_2}{\nu_4})^2} = 0, \quad \text{(B.43c)}$$

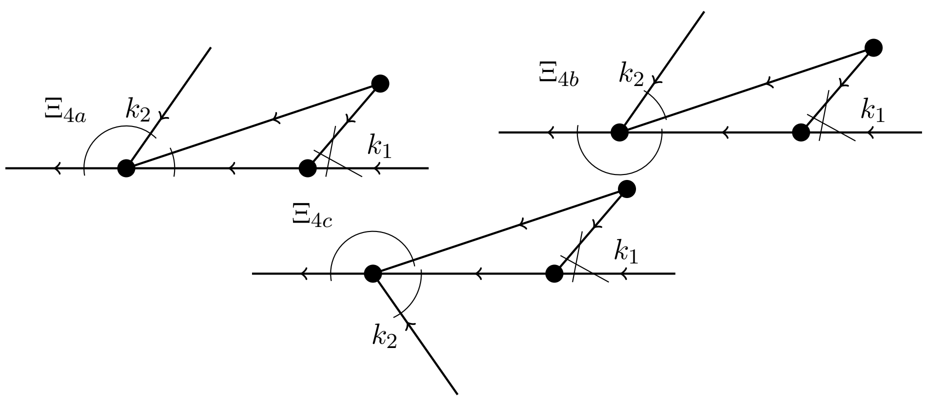
with, of course,

$$\mathbf{I}_{\Xi_{3c}} = \mathbf{I}_{\Xi_{3b}} = -\frac{1}{4}\lambda_{13}\mathcal{C}\lambda_{22}(k_1\cdot k_2)(k_1+k_2)^2.$$

In total this gives

$$\mathbf{I}_{\Xi_3} = -\frac{3}{4}\lambda_{13}\mathcal{C}(k_1\cdot k_2)(k_1+k_2)^2 - \frac{1}{2}\frac{\lambda_{13}\lambda_{22}}{\kappa}\mathcal{C}\kappa(k_1\wedge k_2)^2. \quad \text{(B.44)}$$

B.4.4 Ξ_4



The three phenotypes for this diagram all have a symmetry factor of 4. Four powers of external momentum are carried on the external legs as a minimum, so only the lowest order contribution is required from frequency integral and from the internal momentum contribution.

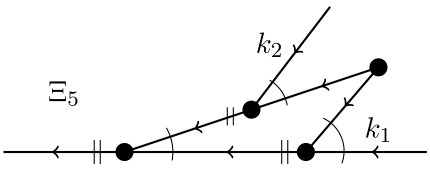
$$\mathbf{I}_{\Xi_4} = \int đ^d q \frac{I_\omega}{2\pi} \cdot 2\Gamma^2 \lambda_{13}\kappa \cdot -4\,(k_1 \wedge q)^2$$
$$\big[(k_2 \cdot [k_1 + k_2])([k_1 + q] \cdot q) + (k_2 \cdot q)([k_1 + k_2] \cdot [k_1 + q]) + (k_2 \cdot [k_1 + q])([k_1 + k_2] \cdot q)\big]$$
$$= \int đ^d q \frac{-8\Gamma^2 \lambda_{13}\kappa}{4\nu_4^2 q^4 (q^2 + \frac{\nu_2}{\nu_4})^2} \cdot \left[k_1^2 q^2 - (k_1 \cdot q)^2\right]$$
$$\big[(k_2 \cdot [k_1 + k_2])q^2 + (k_2 \cdot q)([k_1 + k_2] \cdot q) + (k_2 \cdot q)([k_1 + k_2] \cdot q)\big],$$

where again the phenotypes b and c give the same contribution,

$$\mathbf{I}_{\Xi_4} = \int đ^d q - \frac{2\Gamma^2 \lambda_{13}\kappa}{\nu_4^2} \left[\frac{k_1^2 (k_2 \cdot [k_1 + k_2])}{(q^2 + \frac{\nu_2}{\nu_4})^2} - \frac{(k_2 \cdot [k_1 + k_2])(k_1 \cdot q)^2}{q^2 (q^2 + \frac{\nu_2}{\nu_4})^2} \right.$$
$$\left. + 2\frac{k_1^2 (k_2 \cdot q)([k_1 + k_2] \cdot q)}{q^2 (q^2 + \frac{\nu_2}{\nu_4})^2} - 2\frac{(k_1 \cdot q)^2 (k_2 \cdot q)([k_1 + k_2] \cdot q)}{q^4 (q^2 + \frac{\nu_2}{\nu_4})^2} \right]$$
$$= -2\lambda_{13}\kappa \mathcal{C} k_1^2 (k_2 \cdot [k_1 + k_2]) + \frac{1}{2}\lambda_{13}\kappa \mathcal{C} k_1^2 (k_2 \cdot [k_1 + k_2])$$
$$- 2 \cdot \frac{1}{2}\lambda_{13}\kappa \mathcal{C} k_1^2 (k_2 \cdot [k_1 + k_2]) + 2 \cdot \frac{1}{12}\left(k_1^2 (k_2 \cdot [k_1 + k_2]) + 2(k_1 \cdot k_2)(k_1 \cdot [k_1 + k_2])\right)$$
$$= -\frac{3}{4}\lambda_{13}\kappa \mathcal{C} (k_1 \cdot k_2)(k_1 + k_2)^2 - \frac{3}{2}\lambda_{13}\kappa \mathcal{C} (k_1 \wedge k_2)^2$$
$$- 2 \cdot \frac{1}{8}\lambda_{13}\kappa \mathcal{C} (k_1 \cdot k_2)(k_1 + k_2)^2 - 2 \cdot \frac{5}{12}\lambda_{13}\kappa \mathcal{C} (k_1 \wedge k_2)^2$$
$$= -\frac{\lambda_{13}\kappa}{\lambda_{22}} \mathcal{C} \lambda_{22} (k_1 \cdot k_2)(k_1 + k_2)^2 - \frac{7}{3}\lambda_{13} \mathcal{C} \kappa (k_1 \wedge k_2)^2. \tag{B.45}$$

The first term cancels with the first term of Ξ_2, meaning that κ and λ_{13} do not generate λ_{22} at one loop, or generate it but with a value of zero.

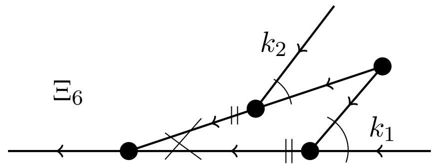
B.4.5 Ξ_5



In spite of the symmetric structure leading to a messy frequency integral, the result is pretty straightforward. With a symmetry factor of 4, the integral is

$$\mathbf{I}_{\Xi_5} = \int đ^d q \frac{\mathbf{I}_\omega}{2\pi} 2\Gamma^2 \lambda_{22}^3 \cdot -4(k_1 \cdot q)(k_2 \cdot q)(k_1 + q)^2 (k_2 - q)^2 \big((k_1 + q) \cdot (k_2 - q)\big)(k_1 + k_2)^2$$
$$= \int đ^d q - \frac{2\Gamma^2 \lambda_{22}^3}{\nu_4^3} (k_1 + k_2)^2 \cdot - \frac{(k_1 \cdot q)(k_2 \cdot q) q^6}{q^6 (q^2 + \frac{\nu_2}{\nu_4})^3}$$
$$= \int đ^d q \frac{2\Gamma^2 \lambda_{22}^3}{\nu_4^3} (k_1 + k_2)^2 \frac{(k_1 \cdot q)(k_2 \cdot q)}{(q^2 + \frac{\nu_2}{\nu_4})^3} = \frac{1}{2}\frac{\lambda_{22}^2}{\nu_4} \mathcal{C} \lambda_{22} (k_1 \cdot k_2)(k_1 + k_2)^2. \tag{B.46}$$

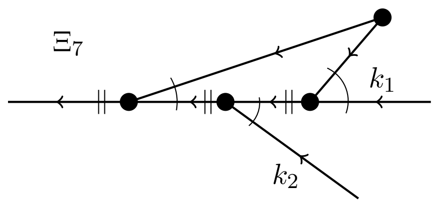
B.4.6 Ξ_6



The only κ variation possible on Ξ_5 is to have the κ vertex in the nose of the diagram, if it were on either of the incoming legs then diagram will go at a minimum as $O(k^5)$, and so not be relevant. With a symmetry factor of 4, the integral is

$$\begin{aligned}
\mathbf{I}_{\Xi_6} &= \int đ^d q \frac{I_\omega}{2\pi} \cdot -8\Gamma^2 \kappa \lambda_{22}^2 ([k_1+q] \wedge [k_2-q])^2 (k_1 \cdot q)(k_1+q)^2 (k_2 \cdot q)(k_2-q)^2 \\
&= \int đ^d q - \frac{8\Gamma^2 \kappa \lambda_{22}^2}{4\nu_4^3 q^6 (q^2 + \frac{\nu_2}{\nu_4})^3} \cdot (k_1 \cdot q)(k_2 \cdot q) q^4 \left[(k_1+k_2)^2 q^2 - ([k_1+k_2] \cdot q)^2\right] \\
&= \int đ^d q - \frac{2\Gamma^2 \kappa \lambda_{22}^2}{\nu_4^3} \cdot \frac{(k_1 \cdot q)(k_2 \cdot q)\left[(k_1+k_2)^2 q^2 - ([k_1+k_2] \cdot q)^2\right]}{q^2 (q^2 + \frac{\nu_2}{\nu_4})^3} \\
&= -\frac{1}{2} \kappa \lambda_{22}^2 \mathcal{C} (k_1 \cdot k_2)(k_1+k_2)^2 \\
&\qquad + \frac{1}{12} \kappa \lambda_{22}^2 \mathcal{C} \left[(k_1 \cdot k_2)(k_1+k_2)^2 + 2k_1 \cdot [k_1+k_2] k_2 \cdot [k_1+k_2]\right] \\
&= -\frac{1}{4} \kappa \lambda_{22} \mathcal{C} \lambda_{22} (k_1 \cdot k_2)(k_1+k_2)^2 + \frac{1}{6} \lambda_{22}^2 \mathcal{C} \kappa (k_1 \wedge k_2)^2.
\end{aligned} \tag{B.47}$$

B.4.7 Ξ_7

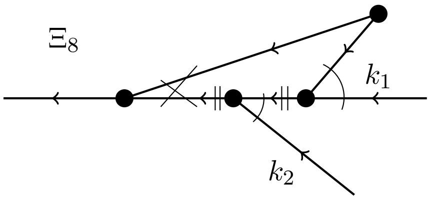


The asymmetric partner to Ξ_5 has a symmetry factor of 8,

$$\begin{aligned}
\mathbf{I}_{\Xi_5} &= \int đ^d q \frac{\mathbf{I}_\omega}{2\pi} 2\Gamma^2 \lambda_{22}^3 \cdot -8(k_1 \cdot q)(k_2 \cdot [k_1+q])(k_1+q)^2 (k_1+k_2+q)^2 \\
&\qquad \cdot \big((k_1+k_2+q) \cdot q\big)(k_1+k_2)^2 \\
&= \int đ^d q - \frac{2\Gamma^2 \lambda_{22}^3}{\nu_4^3} (k_1+k_2)^2 \frac{8(k_1 \cdot q)(k_2 \cdot q) q^6}{8q^6 (q^2 + \frac{\nu_2}{\nu_4})^3} \\
&= \int đ^d q - \frac{2\Gamma^2 \lambda_{22}^3}{\nu_4^3} (k_1+k_2)^2 \frac{(k_1 \cdot q)(k_2 \cdot q)}{(q^2 + \frac{\nu_2}{\nu_4})^3} = -\frac{1}{2} \frac{\lambda_{22}^2}{\nu_4} \mathcal{C} \lambda_{22} (k_1 \cdot k_2)(k_1+k_2)^2,
\end{aligned} \tag{B.48}$$

which cancels with Ξ_5.

B.4.8 Ξ_8



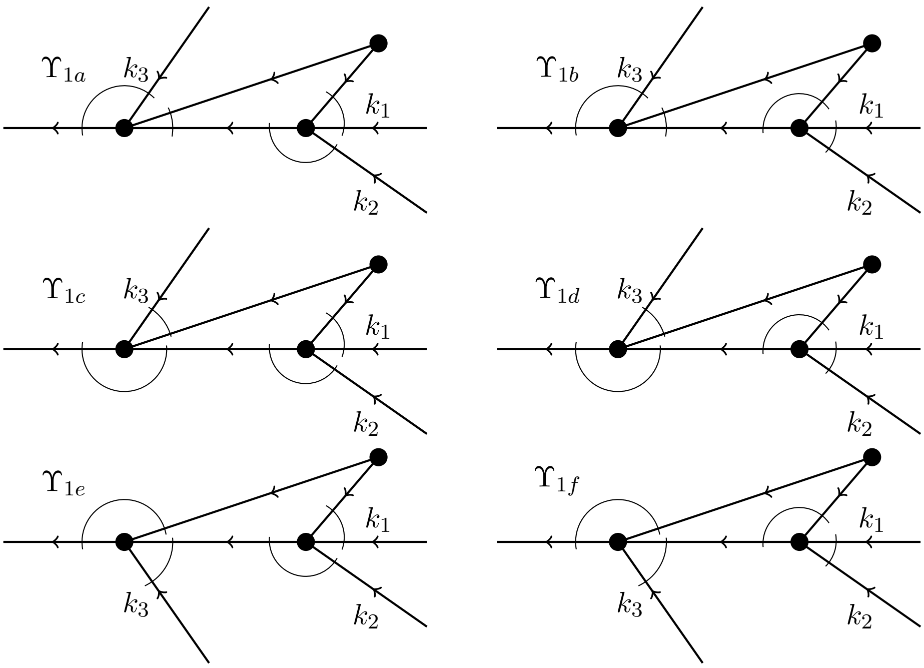
The asymmetric partner to Ξ_6, also has a symmetry factor of 8,

$$\begin{aligned}
\mathbf{I}_{\Xi_6} &= \int đ^d q \frac{I_\omega}{2\pi} \cdot 16\Gamma^2 \kappa \lambda_{22}^2 ([k_1 + k_2 + q] \wedge -q)^2 (k_1 \cdot q)(k_1 + q)^2 (k_2 \cdot [k_1 + q])(k_1 + k_2 + q)^2 \\
&= \int đ^d q \frac{16\Gamma^2 \kappa \lambda_{22}^2}{8\nu_4^3 q^6 (q^2 + \frac{\nu_2}{\nu_4})^3} \cdot (k_1 \cdot q)(k_2 \cdot q) q^4 \left[(k_1 + k_2)^2 q^2 - ([k_1 + k_2] \cdot q)^2 \right] \\
&= \int đ^d q \frac{2\Gamma^2 \kappa \lambda_{22}^2}{\nu_4^3} \cdot \frac{(k_1 \cdot q)(k_2 \cdot q) \left[(k_1 + k_2)^2 q^2 - ([k_1 + k_2] \cdot q)^2 \right]}{q^2 (q^2 + \frac{\nu_2}{\nu_4})^3} \\
&= \frac{1}{4} \kappa \lambda_{22} \mathcal{C} \lambda_{22} (k_1 \cdot k_2)(k_1 + k_2)^2 - \frac{1}{6} \lambda_{22}^2 \mathcal{C} \kappa (k_1 \wedge k_2)^2,
\end{aligned} \tag{B.49}$$

cancelling the Ξ_6 contribution.

B.5 The $\hat{\Gamma}^{13}(k_1, k_2, k_3)$ diagrams

B.5.1 Υ_1



There are six phenotypes, carrying symmetry factors alternating 2 and 4. All carry momentum of at least fourth order on the external legs, so only the zeroth order contribution from the frequency integral is needed, and a straightforward simplification of the internal momentum contribution,

$$\mathbf{I}_{\Upsilon_1} = \int đ^d q \frac{2\Gamma^2 \lambda_{13}^2 Q_{a+b+c+d+e+f}}{4\nu_4^2 q^4 (q^2 + \frac{\nu_2}{\nu_4})^2}, \tag{B.50}$$

where

$$\begin{aligned}
Q_a &= -4[k_3 \cdot (k_1 + k_2 + k_3)](k_1 \cdot q)[k_2 \cdot (k_1 + k_2 + q)][q \cdot (k_1 + k_2 + q)] \\
&= -4(k_3 \cdot [k_1 + k_2 + k_3])(k_1 \cdot q)(k_2 \cdot q)q^2 + O(k^5), \\
\\
Q_b &= -2(k_1 \cdot k_2)[k_3 \cdot (k_1 + k_2 + k_3)][q \cdot (k_1 + k_2 + q)]^2 \\
&= -2(k_1 \cdot k_2)(k_3 \cdot [k_1 + k_2 + k_3])q^4 + O(k^5), \\
\\
Q_c &= -4(k_1 \cdot q)(k_3 \cdot q)[k_2 \cdot (k_1 + k_2 + q)][(k_1 + k_2 + k_3) \cdot (k_1 + k_2 + q)] \\
&= -4([k_1 + k_2 + k_3] \cdot q)(k_3 \cdot q)(k_1 \cdot q)(k_2 \cdot q) + O(k^5), \\
\\
Q_d &= -2(k_1 \cdot k_2)(k_3 \cdot q)[(k_1 + k_2 + k_3) \cdot (k_1 + k_2 + q)][q \cdot (k_1 + k_2 + q)] \\
&= -2([k_1 + k_2 + k_3] \cdot q)(k_3 \cdot q)(k_1 \cdot k_2)q^2 + O(k^5), \\
\\
Q_e &= -4(k_1 \cdot q)[k_2 \cdot (k_1 + k_2 + q)][k_3 \cdot (k_1 + k_2 + q)][(k_1 + k_2 + k_3) \cdot q] \\
&= -4([k_1 + k_2 + k_3] \cdot q)(k_3 \cdot q)(k_1 \cdot q)(k_2 \cdot q) + O(k^5), \\
\\
Q_f &= -2(k_1 \cdot k_2)[k_3 \cdot (k_1 + k_2 + q)][(k_1 + k_2 + k_3) \cdot q][q \cdot (k_1 + k_2 + q)] \\
&= -2([k_1 + k_2 + k_3] \cdot q)(k_3 \cdot q)(k_1 \cdot k_2)q^2 + O(k^5).
\end{aligned} \tag{B.51}$$

Phenotypes d and f have the same divergent contribution, as do c and e.

$$\begin{aligned}
\mathbf{I}_{\Upsilon_{1a}} &= \int đ^d q - 2\Gamma^2 \lambda_{13}^2 \frac{4(k_3 \cdot [k_1 + k_2 + k_3])(k_1 \cdot q)(k_2 \cdot q)q^2}{4\nu_4^2 q^4 (q^2 + \frac{\nu_2}{\nu_4})^2} \\
&= \int đ^d q - \frac{2\Gamma^2 \lambda_{13}^2}{\nu_4^2} (k_3 \cdot [k_1 + k_2 + k_3]) \frac{(k_1 \cdot q)(k_2 \cdot q)}{q^2 (q^2 + \frac{\nu_2}{\nu_4})^2} \\
&= -\frac{1}{2} \lambda_{13}^2 \mathcal{C} (k_3 \cdot (k_1 + k_2 + k_3))(k_1 \cdot k_2),
\end{aligned} \tag{B.52a}$$

$$
\begin{aligned}
\mathbf{I}_{\Upsilon_{1b}} &= \int đ^d q - 2\Gamma^2 \lambda_{13}^2 \frac{2(k_1 \cdot k_2)(k_3 \cdot [k_1 + k_2 + k_3])q^4}{4\nu_4^2 q^4 (q^2 + \frac{\nu_2}{\nu_4})^2} \\
&= \int đ^d q - \frac{\Gamma^2 \lambda_{13}^2}{\nu_4^2}(k_3 \cdot [k_1 + k_2 + k_3])(k_1 \cdot k_2)\frac{1}{(q^2 + \frac{\nu_2}{\nu_4})^2} \\
&= -\lambda_{13}^2 \mathcal{C}(k_3 \cdot (k_1 + k_2 + k_3))(k_1 \cdot k_2),
\end{aligned} \tag{B.52b}
$$

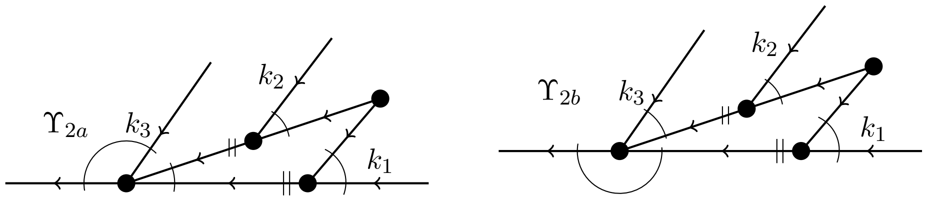
$$
\begin{aligned}
\mathbf{I}_{\Upsilon_{1d+f}} &= 2\int đ^d q - 2\Gamma^2 \lambda_{13}^2 \frac{2([k_1 + k_2 + k_3] \cdot q)(k_3 \cdot q)(k_1 \cdot k_2)q^2}{4\nu_4^2 q^4 (q^2 + \frac{\nu_2}{\nu_4})^2} \\
&= 2\int đ^d q - \frac{\Gamma^2 \lambda_{13}^2}{\nu_4^2}(k_1 \cdot k_2)\frac{([k_1 + k_2 + k_3] \cdot q)(k_3 \cdot q)}{q^2 (q^2 + \frac{\nu_2}{\nu_4})^2} \\
&= -2 \cdot \frac{1}{4}\lambda_{13}^2 \mathcal{C}(k_3 \cdot (k_1 + k_2 + k_3))(k_1 \cdot k_2),
\end{aligned} \tag{B.52c}
$$

$$
\begin{aligned}
\mathbf{I}_{\Upsilon_{1c+e}} &= 2\int đ^d q - 2\Gamma^2 \lambda_{13}^2 \frac{4([k_1 + k_2 + k_3] \cdot q)(k_3 \cdot q)(k_1 \cdot q)(k_2 \cdot q)}{4\nu_4^2 q^4 (q^2 + \frac{\nu_2}{\nu_4})^2} \\
&= 2\int đ^d q - \frac{2\Gamma^2 \lambda_{13}^2}{\nu_4^2}\frac{([k_1 + k_2 + k_3] \cdot q)(k_3 \cdot q)(k_1 \cdot q)(k_2 \cdot q)}{q^4 (q^2 + \frac{\nu_2}{\nu_4})^2} \\
&= -2\lambda_{13}^2 \mathcal{C}\frac{1}{12}\left\{\begin{array}{c} (k_1 \cdot k_2)(k_3 \cdot (k_1 + k_2 + k_3)) \\ +(k_1 \cdot k_3)(k_2 \cdot (k_1 + k_2 + k_3)) \\ +(k_2 \cdot k_3)(k_1 \cdot (k_1 + k_2 + k_3)) \end{array}\right\} \\
&= -2 \cdot \frac{1}{4}\lambda_{13}^2 \mathcal{C}(k_3 \cdot (k_1 + k_2 + k_3))(k_1 \cdot k_2).
\end{aligned} \tag{B.52d}
$$

The expression for the last pair of integrals, for Υ_{1c+e}, bring to the fore the issue of permutations of the external fields for $\hat{\Gamma}^{13}$. The momentum is symmetric with respect to two of the fields, but not all three. So three permutations need to be taken care of. Fortunately this is very simple, as a diagram can be calculated with one permutation, then the others two added on by relabelling. However, the expression for Υ_{1c+e} has a contribution to all three permutations. Of course, the other way of looking at it is that all three permutations will contribute equally to any one momentum arrangement. So the result may be stated

$$
\begin{aligned}
\mathbf{I}_{\Upsilon_1} &= -\frac{5}{2}\lambda_{13}\mathcal{C}\lambda_{13}(k_3 \cdot (k_1 + k_2 + k_3))(k_1 \cdot k_2) \\
&= -\frac{5}{2}\lambda_{13}\mathcal{C}\lambda_{13}\left\{\begin{array}{c} (k_1 \cdot k_2)(k_3 \cdot (k_1 + k_2 + k_3)) \\ +(k_1 \cdot k_3)(k_2 \cdot (k_1 + k_2 + k_3)) \\ +(k_2 \cdot k_3)(k_1 \cdot (k_1 + k_2 + k_3)) \end{array}\right\}.
\end{aligned} \tag{B.53}
$$

B.5.2 Υ_2



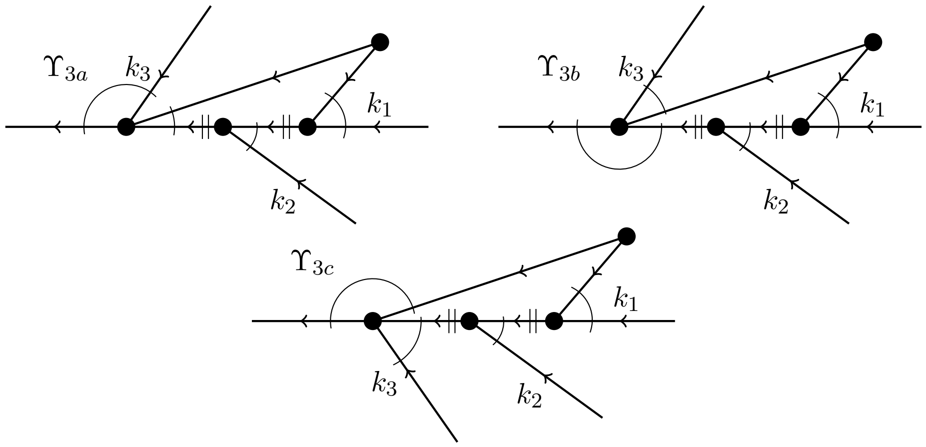
The symmetric body has two phenotypes, a has symmetry factor 4, b has symmetry factor 8. The momentum contribution from the vertices in the phenotypes is

$$
\begin{aligned}
Q_a &= -4(k_3 \cdot [k_1 + k_2 + k_3])(k_1 \cdot q)(k_2 \cdot q)(k_1 + q)^2(k_2 - q)^2([k_1 + q] \cdot [k_2 - q]) \\
&= 4(k_3 \cdot [k_1 + k_2 + k_3])(k_1 \cdot q)(k_2 \cdot q)q^6 \qquad \text{(B.54a)} \\
Q_b &= -8[(k_1 + k_2 + k_3) \cdot (k_1 + q)][k_3 \cdot (k_2 - q)](k_1 \cdot q)(k_2 \cdot q)(k_1 + q)^2(k_2 - q)^2 \\
&= 8(k_1 \cdot q)(k_2 \cdot q)(k_3 \cdot q)([k_1 + k_2 + k_3] \cdot q)q^4, \qquad \text{(B.54b)}
\end{aligned}
$$

with the integral

$$
\begin{aligned}
\mathbf{I}_{\Upsilon_2} &= \int đ^d q \frac{2\Gamma^2 \lambda_{13}\lambda_{22}^2 Q_{a+b}}{4\nu_4^3 q^6 (q^2 + \frac{\nu_2}{\nu_4})^3} \\
&= \int đ^d q \frac{2\Gamma^2 \lambda_{13}\lambda_{22}^2}{\nu_4^3} \left[(k_3 \cdot [k_1 + k_2 + k_3]) \frac{(k_1 \cdot q)(k_2 \cdot q)}{(q^2 + \frac{\nu_2}{\nu_4})^3} \right. \\
&\qquad \left. + 2 \frac{(k_1 \cdot q)(k_2 \cdot q)(k_3 \cdot q)([k_1 + k_2 + k_3] \cdot q)}{q^2 (q^2 + \frac{\nu_2}{\nu_4})^3} \right] \\
&= \frac{1}{2} \frac{\lambda_{13}\lambda_{22}^2}{\nu_4} \mathcal{C} (k_3 \cdot [k_1 + k_2 + k_3])(k_1 \cdot k_2) \\
&\qquad + \frac{1}{6} \frac{\lambda_{13}\lambda_{22}^2}{\nu_4} \mathcal{C} \left\{ \begin{array}{c} (k_1 \cdot k_2)(k_3 \cdot (k_1 + k_2 + k_3)) \\ +(k_1 \cdot k_3)(k_2 \cdot (k_1 + k_2 + k_3)) \\ +(k_2 \cdot k_3)(k_1 \cdot (k_1 + k_2 + k_3)) \end{array} \right\} \\
&= \frac{\lambda_{13}\lambda_{22}^2}{\nu_4} \mathcal{C} (k_3 \cdot (k_1 + k_2 + k_3))(k_1 \cdot k_2) \\
&= \frac{\lambda_{13}\lambda_{22}^2}{\nu_4} \mathcal{C} \left\{ \begin{array}{c} (k_1 \cdot k_2)(k_3 \cdot (k_1 + k_2 + k_3)) \\ +(k_1 \cdot k_3)(k_2 \cdot (k_1 + k_2 + k_3)) \\ +(k_2 \cdot k_3)(k_1 \cdot (k_1 + k_2 + k_3)) \end{array} \right\}. \qquad \text{(B.55)}
\end{aligned}
$$

B.5.3 Υ_3



The asymmetric body has three phenotypes, all with symmetry factors of 8. The internal momentum contribution for the phenotypes are

$$\begin{aligned}
Q_a &= -8(k_3 \cdot [k_1 + k_2 + k_3])(k_1 \cdot q)(k_2 \cdot [k_1 + q])(q \cdot [k_1 + k_2 + q])(k_1 + q)^2(k_1 + k_2 + q)^2 \\
&= -8(k_3 \cdot [k_1 + k_2 + k_3])(k_1 \cdot q)(k_2 \cdot q)q^6, && \text{(B.56a)} \\
Q_b &= -8(k_1 \cdot q)(k_3 \cdot q)[k_2 \cdot (k_1 + q)][(k_1 + k_2 + k_3) \cdot (k_1 + k_2 + q)](k_1 + q)^2(k_1 + k_2 + q)^2 \\
&= -8(k_1 \cdot q)(k_2 \cdot q)(k_3 \cdot q)([k_1 + k_2 + k_3] \cdot q)q^4, && \text{(B.56b)} \\
Q_c &= -8(k_1 \cdot q)[k_2 \cdot (k_1 + q)][(k_1 + k_2 + k_3) \cdot q][k_3 \cdot (k_1 + k_2 + q)](k_1 + q)^2(k_1 + k_2 + q)^2 \\
&= -8(k_1 \cdot q)(k_2 \cdot q)(k_3 \cdot q)([k_1 + k_2 + k_3] \cdot q)q^4. && \text{(B.56c)}
\end{aligned}$$

Phenotypes b and c have the same contribution. The integral is then

$$\begin{aligned}
\mathbf{I}_{\Upsilon_3} &= \int d\!\!{}^-\!{}^d q \frac{2\Gamma^2 \lambda_{13} \lambda_{22}^2 Q_{a+b+c}}{8\nu_4^3 q^6 (q^2 + \frac{\nu_2}{\nu_4})^3} \\
&= \int d\!\!{}^-\!{}^d q - \frac{2\Gamma^2 \lambda_{13} \lambda_{22}^2}{\nu_4^3} \Bigg[(k_3 \cdot [k_1 + k_2 + k_3]) \frac{(k_1 \cdot q)(k_2 \cdot q)}{(q^2 + \frac{\nu_2}{\nu_4})^3} \\
&\qquad + 2 \frac{(k_1 \cdot q)(k_2 \cdot q)(k_3 \cdot q)([k_1 + k_2 + k_3] \cdot q)}{q^2 (q^2 + \frac{\nu_2}{\nu_4})^3} \Bigg] \\
&= -\frac{\lambda_{13} \lambda_{22}^2}{\nu_4} \mathcal{C} (k_3 \cdot (k_1 + k_2 + k_3))(k_1 \cdot k_2) \\
&= -\frac{\lambda_{13} \lambda_{22}^2}{\nu_4} \mathcal{C} \left\{ \begin{array}{c} (k_1 \cdot k_2)(k_3 \cdot (k_1 + k_2 + k_3)) \\ +(k_1 \cdot k_3)(k_2 \cdot (k_1 + k_2 + k_3)) \\ +(k_2 \cdot k_3)(k_1 \cdot (k_1 + k_2 + k_3)) \end{array} \right\}, && \text{(B.57)}
\end{aligned}$$

which cancels the Υ_2 contribution.

Appendix C

β-functions, γ-functions, and the Fixed Points

Along with the reference results given for the exponents z and α in the linear theory, a quick, back of the envelope calculation can give the mean field value for the exponent ν. The linear theory,

$$\partial_t \phi(x,t) = \nu_2 \nabla^2 \phi(x,t) - \nu_4 \nabla^4 \phi(x,t) + \eta(x,t), \tag{C.1}$$

has the scaling exponents, as defined in (6.16), and given in (6.19):

	z	α	d_c	d_{lc}
$\nu_2 \neq 0$	2	$\frac{2-d}{2}$	2	0
$\nu_2 = 0$	4	$\frac{4-d}{2}$	4	2

The exponent ν, which characterises the approach to the critical point, $\nu_2 = 0$, is found by looking at the propagator,

$$\mathbf{G}_0(q,\omega) = \frac{1}{\nu_2 q^2 + \nu_4 q^4 - i\omega} \sim \begin{cases} \dfrac{1}{q^4} & \nu_2 = 0 \\ \dfrac{1}{q^2} & \nu_2 \neq 0, \end{cases}$$

and identifying the length scale at which the competing terms switch dominance:

$$\begin{gathered} \nu_2 {q*}^2 = \nu_4 {q*}^4 \\ \frac{1}{\xi} = q* = \sqrt{\frac{\nu_2}{\nu_4}} \qquad \rightarrow \qquad \xi = \sqrt{\frac{\nu_4}{\nu_2}} \sim \nu_2^{-1/2} \\ \nu = \frac{1}{2}. \end{gathered} \tag{C.2}$$

C.1 Theory with λ_{13}, κ and λ_{22}

As given in Chapter 5, the full theory

$$\begin{aligned}\partial_t\phi(x,t) = \nu_2\nabla^2\phi(x,t) - \nu_4\nabla^4\phi(x,t) + \lambda_{13}\nabla\left(\nabla\phi(x,t)\right)^3 \\ + \lambda_{22}\nabla^2\Big(\nabla\phi\Big)^2 + \kappa\left[\nabla\left(\nabla\phi\nabla^2\phi\right) - \frac{1}{2}\nabla^2\Big(\nabla\phi\Big)^2\right] + \eta(x,t),\end{aligned} \tag{C.3a}$$

with uncorrelated Gaussian white noise

$$\langle\eta(x,t)\eta(x',t')\rangle = 2\Gamma^2\delta^d(x-x')\delta(t-t'), \tag{C.3b}$$

has the one loop vertex functions:

$$\begin{aligned}\hat{\Gamma}^{11}(q) =&\nu_2 q^2 - \frac{3}{2}\lambda_{13}\left[\frac{\Gamma^2}{(4\pi)^2\nu_4^2}\left(\frac{\nu_4}{\nu_2}\right)^{\epsilon/2}\Gamma\left(\frac{\epsilon}{2}\right)\right]\nu_2 q^2 \\ &+ \nu_4 q^4 + \left[\frac{\lambda_{22}^2}{2\nu_4} + \frac{3\lambda_{22}\kappa}{2\nu_4} - \frac{5\kappa^2}{4\nu_4}\right]\left[\frac{\Gamma^2}{(4\pi)^2\nu_4^2}\left(\frac{\nu_4}{\nu_2}\right)^{\epsilon/2}\Gamma\left(\frac{\epsilon}{2}\right)\right]\nu_4 q^4\end{aligned} \tag{C.4a}$$

$$\begin{aligned}\frac{\hat{\Gamma}^{12}(q_1,q_2)}{2} =& -\lambda_{22}(q_1\cdot q_2)(q_1+q_2)^2 \\ &+ \frac{9}{4}\lambda_{22}\lambda_{13}\left[\frac{\Gamma^2}{(4\pi)^2\nu_4^2}\left(\frac{\nu_4}{\nu_2}\right)^{\epsilon/2}\Gamma\left(\frac{\epsilon}{2}\right)\right](q_1\cdot q_2)(q_1+q_2)^2 \\ &- \kappa(q_1\wedge q_2)^2 \\ &+ \frac{1}{2}\left(\lambda_{22}\lambda_{13} + 5\kappa\lambda_{13}\right)\left[\frac{\Gamma^2}{(4\pi)^2\nu_4^2}\left(\frac{\nu_4}{\nu_2}\right)^{\epsilon/2}\Gamma\left(\frac{\epsilon}{2}\right)\right](q_1\wedge q_2)^2\end{aligned} \tag{C.4b}$$

$$\begin{aligned}\frac{\hat{\Gamma}^{13}(q_1,q_2,q_3)}{2} =& -\lambda_{13}\left\{\begin{array}{c}(q_1\cdot q_2)(q_3\cdot(q_1+q_2+q_3)) \\ +(q_1\cdot q_3)(q_2\cdot(q_1+q_2+q_3)) \\ +(q_2\cdot q_3)(q_1\cdot(q_1+q_2+q_3))\end{array}\right\} \\ &+ \frac{5}{2}\lambda_{13}^2\left[\frac{\Gamma^2}{(4\pi)^2\nu_4^2}\left(\frac{\nu_4}{\nu_2}\right)^{\epsilon/2}\Gamma\left(\frac{\epsilon}{2}\right)\right]\left\{\begin{array}{c}(q_1\cdot q_2)(q_3\cdot(q_1+q_2+q_3)) \\ +(q_1\cdot q_3)(q_2\cdot(q_1+q_2+q_3)) \\ +(q_2\cdot q_3)(q_1\cdot(q_1+q_2+q_3))\end{array}\right\}.\end{aligned} \tag{C.4c}$$

Where a factor two, from the permutation of two of the incoming fields, has been divided out from $\hat{\Gamma}^{12}(q_1,q_2)$ and $\hat{\Gamma}^{13}(q_1,q_2,q_3)$. The renormalisation scheme uses the couplings

$$g = \frac{\Gamma^2}{(4\pi)^2}\frac{\lambda_{13}}{\nu_4^{2-\epsilon/2}}, \qquad \lambda = \frac{\lambda_{22}^2}{\nu_4\lambda_{13}}, \qquad \chi = \frac{\kappa}{\lambda_{22}}, \tag{C.5}$$

and an inverse length scale μ, along with the noise dimension B, and the ν_4 dimension A. Giving the dimensionless, one loop renormalised couplings, $\bar{a}$, and the Z-factors, Z_a:

$$\begin{aligned}
\bar{\nu}_2 =& Z_2 \nu_2 A^{-1} \mu^{-2} & Z_2 =& 1 - \frac{3}{\epsilon} g \nu_2^{-\epsilon/2} \\
\bar{\nu}_4 =& Z_4 \nu_4 A^{-1} & Z_4 =& 1 - \frac{\lambda}{\epsilon} \left[\frac{5}{2}\chi^2 - 3\chi - 1\right] g \nu_2^{-\epsilon/2} \\
\bar{\lambda}_{22} =& Z_{22} \lambda_{22} A^{-3/2} \mu^{-\epsilon/2} B & Z_{22} =& 1 - \frac{9}{2\epsilon} g \nu_2^{-\epsilon/2} \\
\bar{\kappa} =& Z_\kappa \kappa A^{-3/2} \mu^{-\epsilon/2} B & Z_\kappa =& 1 - \frac{1}{\epsilon}\left(5 + \frac{1}{\chi}\right) g \nu_2^{-\epsilon/2} \\
\bar{\lambda}_{13} =& Z_{13} \lambda_{13} A^{-2} \mu^{-\epsilon} B^2 & Z_{13} =& 1 - \frac{5}{\epsilon} g \nu_2^{-\epsilon/2} \\
\bar{g} =& Z_g g A^{-\epsilon/2} \mu^{-\epsilon} & Z_g =& 1 - \frac{1}{\epsilon}\left(5 - \left(2 - \frac{\epsilon}{2}\right)\lambda\left[\frac{5}{2}\chi^2 - 3\chi - 1\right]\right) g \nu_2^{-\epsilon/2} \\
\bar{\lambda} =& Z_\lambda \lambda & Z_\lambda =& 1 - \frac{1}{\epsilon}\left(4 - \lambda\left[\frac{5}{2}\chi^2 - 3\chi - 1\right]\right) g \nu_2^{-\epsilon/2} \\
\bar{\chi} =& Z_\chi \chi & Z_\chi =& 1 - \frac{1}{\epsilon}\left(\frac{1}{2} + \frac{1}{\chi}\right) g \nu_2^{-\epsilon/2}
\end{aligned} \tag{C.6}$$

As was run through in Chapter 5, the β-functions and γ-functions are calculated as

$$\beta_{\bar{a}} = \frac{d\bar{a}}{d\ln\mu}, \qquad \gamma_{\bar{a}} = \frac{d\,\ln Z_{\bar{a}}}{d\ln\mu}, \tag{C.7}$$

where the derivatives are taken with respect to constant bare parameters. The form of the Z-factors is $Z_{\bar{a}} = 1 - X_{\bar{a}}$, so the γ-functions can be simplified at this point to

$$\gamma_{\bar{a}} = -\frac{d\,X_{\bar{a}}}{d\ln\mu} + O(g^2). \tag{C.8}$$

The composite Z-factors are calculated using the first order expansion

$$\frac{1}{(1-X)^\alpha} \sim 1 + \alpha X, \tag{C.9}$$

giving

$$\begin{aligned}
Z_g &= \frac{Z_{13}}{Z_4^{2-\epsilon/2}} \\
&= 1 - X_{13} + \left(2 - \frac{\epsilon}{2}\right) X_4 + O(g^2),
\end{aligned} \tag{C.10a}$$

where the factor of $\epsilon/2$ can be dropped. And, finally,

$$\begin{aligned}
Z_\lambda &= \frac{Z_{22}^2}{Z_4 Z_{13}} \\
&= 1 - 2X_{22} + X_4 + X_{13} + O(g^2),
\end{aligned} \tag{C.10b}$$

and

$$
\begin{aligned}
Z_\chi &= \frac{Z_\kappa}{Z_{22}} \\
&= 1 - X_\kappa + X_{22} + O(g^2).
\end{aligned} \tag{C.10c}
$$

Generally care should be taken about when bare and renormalised parameters are being used, however a laissez-faire approach is permissible at one loop. Looking more closely at the renormalisation of g, with bare coupling g_0,

$$
\begin{aligned}
g &= g_0 + \frac{1}{\epsilon}\left(5 - 2\lambda\left[\frac{5}{2}\chi - 3\chi - 1\right]\right)\nu_2^{-\frac{\epsilon}{2}} g_0^2 + g_0^3 \\
&= Z_{g_0} g_0.
\end{aligned} \tag{C.11}
$$

At the renormalisation point this is

$$
\nu_2^{-\frac{\epsilon}{2}} g_0 \Big|_{\nu_2=1} = \bar{g}_0. \tag{C.12}
$$

To this order the couplings can be interchanged without concern for the second order corrections, as

$$
g_0 = g + O\left(g_0^2\right), \tag{C.13}
$$

giving

$$
\begin{aligned}
Z_{g_0} &= 1 + \frac{1}{\epsilon}\left(5 - 2\lambda\left[\frac{5}{2}\chi - 3\chi - 1\right]\right)\bar{g}_0 + O\left(\bar{g}_0^2\right) \\
&= 1 + \frac{1}{\epsilon}\left(5 - 2\bar{\lambda}\left[\frac{5}{2}\bar{\chi} - 3\bar{\chi} - 1\right]\right)\bar{g} + O\left(\bar{g}^2\right) \\
&= Z_{\bar{g}}.
\end{aligned} \tag{C.14}
$$

The β-functions emerge as

$$
\begin{aligned}
\beta_{\bar{g}} &= \frac{\partial}{\partial \ln \mu} g\mu^{-\epsilon} = -\epsilon \bar{g} + \mu^{-\epsilon}\frac{\partial g}{\partial \ln \mu} \\
\frac{\partial g}{\partial \ln \mu} &= g_0 \frac{\partial Z_{g_0}}{\partial \ln \mu} = g_0 Z_{g_0} \frac{\partial \ln Z_{g_0}}{\partial \ln \mu} \\
&= g\frac{\partial \ln Z_{\bar{g}}}{\partial \ln \mu} + O\left(\bar{g}^2\right) \\
\beta_{\bar{g}} &= -\epsilon\bar{g} + \bar{g}\gamma_{\bar{g}},
\end{aligned} \tag{C.15a}
$$

$$
\beta_{\bar{\lambda}} = \frac{\partial \bar{\lambda}}{\partial \ln \mu} = \bar{\lambda}\frac{\partial \ln Z_\lambda}{\partial \ln \mu} + O\left(\bar{g}^2\right) = \bar{\lambda}\gamma_{\bar{\lambda}}, \tag{C.15b}
$$

$$
\beta_{\bar{\chi}} = \bar{\chi}\gamma_{\bar{\chi}}, \tag{C.15c}
$$

with the γ-functions

$$\gamma_{\bar{g}} = -\frac{1}{\epsilon}\left(5 - 2\lambda\left[\frac{5}{2}\chi - 3\chi - 1\right]\right)\beta_{\bar{g}} + O\left(\bar{g}\beta_{\bar{\lambda}}, \bar{g}\beta_{\bar{\chi}}\right)$$
$$= \frac{1}{\epsilon}\left(5 - 2\lambda\left[\frac{5}{2}\chi - 3\chi - 1\right]\right)\epsilon\bar{g} + O\left(\bar{g}\gamma_{\bar{g}}, \bar{g}\gamma_{\bar{\lambda}}, \bar{g}\gamma_{\bar{\chi}}\right)$$
$$= \left(5 - 2\lambda\left[\frac{5}{2}\chi - 3\chi - 1\right]\right)\bar{g} + O\left(\bar{g}^2\right) \tag{C.16a}$$
$$\gamma_{\bar{\lambda}} = -\frac{1}{\epsilon}\left(4 - \lambda\left[\frac{5}{2}\chi^2 - 3\chi - 1\right]\right)\beta_{\bar{g}} + O\left(\bar{g}\beta_{\bar{\lambda}}, \bar{g}\beta_{\bar{\chi}}\right)$$
$$= \frac{1}{\epsilon}\left(4 - \lambda\left[\frac{5}{2}\chi^2 - 3\chi - 1\right]\right)\epsilon\bar{g} + O\left(\bar{g}\gamma_{\bar{g}}\bar{g}\gamma_{\bar{\lambda}}, \bar{g}\gamma_{\bar{\chi}}\right)$$
$$= \left(4 - \lambda\left[\frac{5}{2}\chi^2 - 3\chi - 1\right]\right)\bar{g} + O\left(\bar{g}^2\right) \tag{C.16b}$$
$$\gamma_{\bar{\chi}} = -\frac{1}{\epsilon}\left(\frac{1}{2} + \frac{1}{\chi}\right)\beta_{\bar{g}} + O\left(\bar{g}\beta_{\bar{\chi}}\right)$$
$$= \frac{1}{\epsilon}\left(\frac{1}{2} + \frac{1}{\chi}\right)\epsilon\bar{g} + O\left(\bar{g}\gamma_{\bar{g}}, \bar{g}\gamma_{\bar{\chi}}\right)$$
$$= \left(\frac{1}{2} + \frac{1}{\chi}\right)\bar{g} + O\left(\bar{g}^2\right). \tag{C.16c}$$

Combining this gives

$$\tilde{\beta}_g = -\epsilon + \left(5 - (2 - \epsilon/2)\lambda\left[\frac{5}{2}\chi^2 - 3\chi - 1\right]\right)g, \tag{C.17a}$$
$$\tilde{\beta}_\lambda = \left(4 - \lambda\left[\frac{5}{2}\chi^2 - 3\chi - 1\right]\right)g, \tag{C.17b}$$
$$\tilde{\beta}_\chi = \left(\frac{1}{2} + \frac{1}{\chi}\right)g, \tag{C.17c}$$

and

$$\begin{aligned}
\gamma_2 &= 3g \qquad \gamma_4 = \lambda\left(\frac{5}{2}\chi^2 - 3\chi - 1\right)g \\
\gamma_g &= \left(5 - (2 - \epsilon/2)\lambda\left[\frac{5}{2}\chi^2 - 3\chi - 1\right]\right)g \\
\gamma_\lambda &= \left(4 - \lambda\left[\frac{5}{2}\chi^2 - 3\chi - 1\right]\right)g \\
\gamma_\chi &= \left(\frac{1}{2} + \frac{1}{\chi}\right)g \\
\gamma_{22} &= \frac{9}{2}g \quad \gamma_\kappa = \left(5 + \frac{1}{\chi}\right)g \quad \gamma_{13} = 5g.
\end{aligned} \tag{C.17d}$$

The simultaneous roots of the β-functions give three fixed points for the theory:

$$g = 0 \tag{C.18a}$$

$$\chi = -2 \qquad \lambda = 0 \qquad g = \epsilon/5 \tag{C.18b}$$

$$\chi = -2 \qquad \lambda = \frac{4}{15} \qquad g = \frac{\epsilon}{2\epsilon - 3}. \tag{C.18c}$$

The stability matrix, M, is

$$M = \begin{pmatrix} \partial_g\beta_g & \partial_\lambda\beta_g & \partial_\chi\beta_g \\ \partial_g\beta_\lambda & \partial_\lambda\beta_\lambda & \partial_\chi\beta_\lambda \\ \partial_g\beta_\chi & \partial_\lambda\beta_\chi & \partial_\chi\beta_\chi \end{pmatrix} \tag{C.19}$$

$$= \begin{pmatrix} -\epsilon + \left(10 - 4\lambda\left[\frac{5}{2}\chi^2 - 3\chi - 1\right]\right) g & -2\left[\frac{5}{2}\chi^2 - 3\chi - 1\right] g^2 & -2\left[5\chi - 3\right]\lambda g^2 \\ 4\lambda - \left[\frac{5}{2}\chi^2 - 3\chi - 1\right]\lambda^2 & 4g - 2\left[\frac{5}{2}\chi^2 - 3\chi - 1\right] g\lambda & \left[5\chi - 3\right] g\lambda^2 \\ 1 + \frac{\chi}{2} & 0 & \frac{g}{2} \end{pmatrix}.$$

For the fixed point $\chi = -2$, $\lambda = 0$, $g = \epsilon/5$, this gives the eigenvalues

$$\rho = \begin{cases} \epsilon \\ \frac{4\epsilon}{5} \\ \frac{\epsilon}{10}, \end{cases} \tag{C.20}$$

which is infrared stable. The γ-functions are

$$\begin{aligned} \gamma_2 &= \frac{3\epsilon}{5} \qquad \gamma_4 = 0 \\ \gamma_g = \epsilon \qquad \gamma_\lambda &= \frac{4\epsilon}{5} \qquad \gamma_\chi = 0 \\ \gamma_{13} = \epsilon \qquad \gamma_{22} &= \frac{9\epsilon}{10} \qquad \gamma_\kappa = \frac{9\epsilon}{10}, \end{aligned} \tag{C.21}$$

and, from (5.20), the exponents are

$$\begin{aligned} z = 4 \qquad \nu &= \frac{1}{2} + \frac{3\epsilon}{20} \qquad \delta = 0 \\ \pi_g = 0 \qquad \pi_\lambda &= -\frac{4\epsilon}{5} \qquad \pi_\chi = 0 \\ \pi_{13} = 0 \qquad \pi_{22} &= -\frac{2\epsilon}{5} \qquad \pi_\kappa = -\frac{2\epsilon}{5} \\ \eta = 0 &\qquad \tilde{\eta} = 0. \end{aligned} \tag{C.22}$$

For the fixed point $\chi = -2$, $\lambda = \frac{4}{15}$, $g = \frac{\epsilon}{2\epsilon - 3}$, the eigenvalues of the eigenvalues are

$$\rho = \begin{cases} \frac{-4\epsilon}{5} \\ \frac{\epsilon}{10} \\ \frac{-11\epsilon}{5}, \end{cases} \tag{C.23}$$

which is infrared unstable. The γ-functions are

$$\begin{aligned} &\gamma_2 = -\epsilon \qquad \gamma_4 = \frac{-4\epsilon}{3} \\ &\gamma_g = \epsilon \qquad \gamma_\lambda = 0 \qquad \gamma_\chi = 0 \\ &\gamma_{13} = \frac{-5\epsilon}{3} \qquad \gamma_{22} = \frac{-3\epsilon}{2} \qquad \gamma_\kappa = \frac{-3\epsilon}{2}, \end{aligned} \tag{C.24}$$

leading to the exponents

$$\begin{aligned} &z = 4 - \frac{4\epsilon}{3} \qquad \nu = \frac{1}{2} + \frac{\epsilon}{12} \qquad \delta = -\frac{4\epsilon}{3} \\ &\pi_g = 0 \qquad \pi_\lambda = 0 \qquad \pi_\chi = 0 \\ &\pi_{13} = 0 \qquad \pi_{22} = 0 \qquad \pi_\kappa = 0 \\ &\eta = -\frac{2\epsilon}{3} \qquad \tilde{\eta} = \frac{2\epsilon}{3}. \end{aligned} \tag{C.25}$$

For $g = 0$, the trivial fixed point, the stability matrix gives

$$\rho = \begin{cases} -\epsilon \\ 0 \\ 0. \end{cases} \tag{C.26}$$

However, this fixed point requires more work, and is dealt with in the next section.

C.2 Theory with κ and λ_{22}

The trivial fixed point of the full theory has the coupling $\lambda_{13} = 0$. It is then properly understood as the equation for the full VLDS theory, as opposed to the original VLDS equation in the absence of the κ coupling:

$$\begin{aligned} \partial_t \phi(x,t) =& \nu_2 \nabla^2 \phi(x,t) - \nu_4 \nabla^4 \phi(x,t) + \lambda_{22} \nabla^2 \Big(\nabla\phi\Big)^2 \\ &+ \kappa \left[\nabla\left(\nabla\phi\nabla^2\phi\right) - \frac{1}{2}\nabla^2\Big(\nabla\phi\Big)^2\right] + \eta(x,t). \end{aligned} \tag{C.27}$$

In order to replace the couplings g and λ, a new coupling needs to be defined:

$$\psi := g\lambda = \frac{\Gamma^2}{(4\pi)^2}\frac{\lambda_{22}^2}{\nu_4^{3-\epsilon/2}} \qquad [\psi] = A^{\epsilon/2}\mu^{\epsilon}. \tag{C.28}$$

The diagrams and Z-factors face a cull:

$$\begin{aligned}
\bar{\nu}_2 =& \nu_2 A^{-1}\mu^{-2} & Z_2 =& 1 \\
\bar{\nu}_4 =& Z_4\nu_4 A^{-1} & Z_4 =& 1 - \frac{1}{\epsilon}\left(\frac{5}{2}\chi^2 - 3\chi - 1\right)\psi\nu_2^{-\epsilon/2} \\
\bar{\lambda}_{22} =& \lambda_{22}A^{-3/2}\mu^{-\epsilon/2}B & Z_{22} =& 1 \\
\bar{\kappa} =& \kappa A^{-3/2}\mu^{-\epsilon/2}B & Z_\kappa =& 1 \\
\bar{\psi} =& Z_\psi \psi A^{-\epsilon/2}\mu^{-\epsilon} & Z_\psi =& 1 + \frac{3}{\epsilon}\left(\frac{5}{2}\chi^2 - 3\chi - 1\right)\psi\nu_2^{-\epsilon/2} \\
\chi =& \bar{\chi} & Z_\chi =& 1.
\end{aligned} \tag{C.29}$$

The β-function and two γ-functions are

$$\begin{aligned}
\beta_\psi &= -\epsilon\psi - \frac{3}{\epsilon}\left(\frac{5}{2}\chi^2 - 3\chi - 1\right)\psi^2 \\
\gamma_\psi = -3\left(\frac{5}{2}\chi^2 - 3\chi - 1\right)\psi & \qquad \gamma_4 = \left(\frac{5}{2}\chi^2 - 3\chi - 1\right)\psi.
\end{aligned} \tag{C.30}$$

The fixed point of the β-function gives

$$\psi = \frac{-\epsilon}{3\left(\frac{5}{2}\chi^2 - 3\chi - 1\right)}, \tag{C.31}$$

as well as the trivial fixed point $\psi = 0$. The stability is given by

$$\partial_\psi\beta_\psi = -\epsilon - \frac{6}{\epsilon}\left[\frac{5}{2}\chi^2 - 3\chi - 1\right]\psi, \tag{C.32}$$

at the trivial fixed point this is $-\epsilon$, which is unstable. At the other fixed point it gives the stable ϵ, with the γ-functions

$$\gamma_\psi = \epsilon \qquad \gamma_4 = -\frac{\epsilon}{3}. \tag{C.33}$$

By inspection, the dimension of the new coupling, ψ, gives the scaling form of the new coupling as $\psi l^{\gamma_\psi - \frac{\epsilon\gamma_4}{2} - \epsilon}$, and the associated scaling exponent as $\pi_\psi = \frac{\epsilon\gamma_4}{2} + \epsilon - \gamma_\psi$.

More formally, reworking the scaling arguments made for the full theory gives

$$
\begin{aligned}
&\hat{\Gamma}^{nm}\left(\{q\}\{\omega\};\nu_2,\nu_4,\Gamma^2,\psi,\chi\right)\\
&\quad= A^{1+\frac{m-n}{2}}\mu^{\frac{m\epsilon}{2}-n(4-\frac{\epsilon}{2})+8-\epsilon} \qquad \text{(C.34)}\\
&\qquad\cdot\bar{\hat{\Gamma}}^{nm}\left(\left\{\frac{q}{\mu}\right\}\left\{\frac{\omega}{A\mu^4}\right\};\frac{\nu_2}{A\mu^2},\frac{\nu_4}{A},\Gamma^2,\frac{\psi}{A^{\epsilon/2}\mu^{\epsilon}},\chi\right)\\
&\quad= l^{\gamma_4(1+\frac{m-n}{2})+\frac{m\epsilon}{2}-n(4-\frac{\epsilon}{2})+8-\epsilon} \qquad \text{(C.35)}\\
&\qquad\cdot\hat{\Gamma}^{nm}\left(\left\{\frac{q}{l}\right\}\left\{\frac{\omega}{l^{\gamma_4+4}}\right\};\nu_2 l^{\gamma_2-\gamma_4-2},\nu_4,\Gamma^2,\psi l^{\gamma_\psi-\frac{\epsilon\gamma_4}{2}-\epsilon},\chi l^{\gamma_\chi}\right),
\end{aligned}
$$

defining

$$
\begin{aligned}
&\hat{\Gamma}^{(n,m)}\left(k,\omega;\nu_2,\nu_4,\Gamma^2,[\psi,\chi/\lambda_{22},\kappa]\right)\\
&\quad= l^{-\frac{n}{2}(2\eta+8-\epsilon)-\frac{m}{2}(2\tilde{\eta}-\epsilon)+8-\epsilon+\delta} \qquad \text{(C.36)}\\
&\qquad\hat{\Gamma}^{(n,m)}\left(\frac{k}{l},\frac{\omega}{l^z};\frac{\nu_2}{l^{1/\nu}},\nu_4,\Gamma^2,\left[\frac{\psi}{l^{\pi_\psi}},\frac{\chi}{l^{\pi_\chi}}\bigg/\frac{\lambda_{22}}{l^{\pi_{22}}},\frac{\kappa}{l^{\pi_\kappa}}\right]\right),
\end{aligned}
$$

and identifying

$$
\begin{gathered}
z = 4+\gamma_4 \qquad \nu = \frac{1}{2+\gamma_4-\gamma_2} \qquad \delta = \gamma_4\\
\pi_\psi = \epsilon\left(1+\frac{\gamma_4}{2}\right)-\gamma_\psi \qquad \pi_\chi = -\gamma_\chi \qquad \pi_{22} = \frac{\epsilon}{2}+\frac{3\gamma_4}{2}-\gamma_{22} \qquad \pi_\kappa = \frac{\epsilon}{2}+\frac{3\gamma_4}{2}-\gamma_\kappa\\
\eta = \frac{\gamma_4}{2} \qquad \tilde{\eta} = \frac{-\gamma_4}{2},
\end{gathered} \qquad \text{(C.37)}
$$

the exponents at the stable fixed point of the full VLDS equation are then

$$
\begin{gathered}
z = 4-\frac{\epsilon}{3} \qquad \nu = \frac{1}{2}+\frac{\epsilon}{12} \qquad \delta = -\frac{\epsilon}{3}\\
\pi_\psi = 0 \qquad \pi_\chi = 0 \qquad \pi_{22} = 0 \qquad \pi_\kappa = 0\\
\eta = -\frac{\epsilon}{6} \qquad \tilde{\eta} = \frac{\epsilon}{6}.
\end{gathered} \qquad \text{(C.38)}
$$

Renormalisation imposes bounds on χ, or the fixed point of ψ would correspond to an unphysical ν_4. To one loop, in order to get sensible results $(5/2)\chi^2 - 3\chi - 1$ needs to be negative, $\chi \in [\frac{3-\sqrt{19}}{5}, \frac{3+\sqrt{19}}{5}]$, sufficiently large κ, or small λ_{22}, violates this.

At the trivial fixed point the exponents are

$$
\begin{gathered}
z = 4 \qquad \nu = \frac{1}{2} \qquad \delta = 0\\
\pi_\psi = \epsilon \qquad \pi_\chi = 0 \qquad \pi_{22} = \frac{\epsilon}{2} \qquad \pi_\kappa = \frac{\epsilon}{2}\\
\eta = 0 \qquad \tilde{\eta} = 0.
\end{gathered} \qquad \text{(C.39)}
$$

C.3 Theory with λ_{13} and κ

The full theory can be adjusted to have $\lambda_{22} = 0$ at the start, this limited case with the λ_{13} and κ couplings is consistent, at least to one loop order, as λ_{22} is not finitely generated by the other two at one loop order. The Langevin equation for this limited theory is

$$\begin{aligned}\partial_t \phi(x,t) =& \nu_2 \nabla^2 \phi(x,t) - \nu_4 \nabla^4 \phi(x,t) + \lambda_{13} \nabla \left(\nabla \phi(x,t)\right)^3 \\ &+ \kappa \left[\nabla \left(\nabla \phi \nabla^2 \phi\right) - \frac{1}{2} \nabla^2 \Big(\nabla \phi\Big)^2\right] + \eta(x,t).\end{aligned} \tag{C.40}$$

Eliminating the any diagrams featuring the λ_{22} vertex, or indeed setting $\lambda_{22} = 0$ in (C.4), gives the vertex functions:

$$\begin{aligned}\hat{\Gamma}^{11}(q) =& \nu_2 q^2 - \frac{3}{2}\lambda_{13}\left[\frac{\Gamma^2}{(4\pi)^2 \nu_4^2}\left(\frac{\nu_4}{\nu_2}\right)^{\epsilon/2}\Gamma\left(\frac{\epsilon}{2}\right)\right]\nu_2 q^2 \\ &+ \nu_4 q^4 - \frac{5\kappa^2}{4\nu_4}\left[\frac{\Gamma^2}{(4\pi)^2 \nu_4^2}\left(\frac{\nu_4}{\nu_2}\right)^{\epsilon/2}\Gamma\left(\frac{\epsilon}{2}\right)\right]\nu_4 q^4\end{aligned} \tag{C.41a}$$

$$\frac{\hat{\Gamma}^{12}(q_1,q_2)}{2} = -\kappa (q_1 \wedge q_2)^2 + \frac{5}{2}\kappa\lambda_{13}\left[\frac{\Gamma^2}{(4\pi)^2 \nu_4^2}\left(\frac{\nu_4}{\nu_2}\right)^{\epsilon/2}\Gamma\left(\frac{\epsilon}{2}\right)\right](q_1 \wedge q_2)^2 \tag{C.41b}$$

$$\begin{aligned}\frac{\hat{\Gamma}^{13}(q_1,q_2,q_3)}{2} =& -\lambda_{13}\left\{\begin{matrix}(q_1\cdot q_2)(q_3\cdot(q_1+q_2+q_3)) \\ +(q_1\cdot q_3)(q_2\cdot(q_1+q_2+q_3)) \\ +(q_2\cdot q_3)(q_1\cdot(q_1+q_2+q_3))\end{matrix}\right\} \\ &+ \frac{5}{2}\lambda_{13}^2\left[\frac{\Gamma^2}{(4\pi)^2 \nu_4^2}\left(\frac{\nu_4}{\nu_2}\right)^{\epsilon/2}\Gamma\left(\frac{\epsilon}{2}\right)\right]\left\{\begin{matrix}(q_1\cdot q_2)(q_3\cdot(q_1+q_2+q_3)) \\ +(q_1\cdot q_3)(q_2\cdot(q_1+q_2+q_3)) \\ +(q_2\cdot q_3)(q_1\cdot(q_1+q_2+q_3))\end{matrix}\right\}.\end{aligned} \tag{C.41c}$$

The renormalisation can be conducted using the couplings

$$g = \frac{\Gamma^2}{(4\pi)^2}\frac{\lambda_{13}}{\nu_4^{2-\epsilon/2}}, \qquad X = \frac{\kappa^2}{\nu_4 \lambda_{13}}, \tag{C.42}$$

giving the Z-factors:

$$\begin{aligned}
\bar{\nu}_2 &= Z_2\nu_2 A^{-1}\mu^{-2} & Z_2 &= 1-\frac{3}{\epsilon}g\nu_2^{-\epsilon/2} \\
\bar{\nu}_4 &= Z_4\nu_4 A^{-1} & Z_4 &= 1-\frac{5X}{2\epsilon}g\nu_2^{-\epsilon/2} \\
\bar{\kappa} &= Z_\kappa\kappa A^{-3/2}\mu^{-\epsilon/2}B & Z_\kappa &= 1-\frac{5}{\epsilon}g\nu_2^{-\epsilon/2} \\
\bar{\lambda}_{13} &= Z_{13}\lambda_{13}A^{-2}\mu^{-\epsilon}B^2 & Z_{13} &= 1-\frac{5}{\epsilon}g\nu_2^{-\epsilon/2} \\
\bar{g} &= Z_g g A^{-\epsilon/2}\mu^{-\epsilon} & Z_g &= 1-\frac{5}{\epsilon}\left(1-X\right)g\nu_2^{-\epsilon/2} \\
\bar{X} &= Z_X X & Z_X &= 1-\frac{5}{\epsilon}\left(1-\frac{X}{2}\right)g\nu_2^{-\epsilon/2}.
\end{aligned} \tag{C.43}$$

The β-functions and γ-functions can be calculated as

$$\begin{aligned}
\beta_g &= -\epsilon g + g\gamma_g \\
\beta_X &= X\gamma_X \\
\gamma_g &= -\frac{5}{\epsilon}\left(1-X\right)\beta_g + \frac{5}{\epsilon}g\beta_X \\
\gamma_X &= -\frac{5}{\epsilon}\left(1-\frac{X}{2}\right)\beta_g + \frac{5}{2\epsilon}g\beta_X.
\end{aligned} \tag{C.44}$$

Substituting back in gives the β-functions

$$\begin{aligned}
\beta_X &= \frac{X-\frac{5}{\epsilon}\left(1-\frac{X}{2}\right)\beta_g}{1-\frac{5g}{2\epsilon}} \\
&= -\frac{5}{\epsilon}\left(1-\frac{X}{2}\right)X\beta_g + O\left(g^2\right) \\
&= 5X\left(1-\frac{X}{2}\right)g,
\end{aligned} \tag{C.45}$$

and

$$\begin{aligned}
\beta_g &= -\epsilon g - \frac{5}{\epsilon}\left(1-X\right)g\beta_g + O\left(g^3\right) \\
&= \frac{-\epsilon g}{1+\frac{5}{\epsilon}\left(1-X\right)g} \\
&= -\epsilon g + 5\left(1-X\right)g^2,
\end{aligned} \tag{C.46}$$

and the γ-functions

$$\begin{gathered}
\gamma_g = 5\left(1-X\right)g \qquad \gamma_X = 5\left(1-\frac{X}{2}\right)g \\
\gamma_2 = 3g \qquad \gamma_4 = \frac{5X}{2}g \qquad \gamma_{13} = 5g \qquad \gamma_\kappa = 5g.
\end{gathered} \tag{C.47}$$

The fixed points are either the trivial $g = 0$ with arbitrary X, or

$$g = \frac{\epsilon}{5\,(1 - X)} \qquad X = \begin{cases} 2 \\ 0. \end{cases} \tag{C.48}$$

The stability matrix, M, is

$$\begin{aligned} M &= \begin{pmatrix} \partial_X \beta_X & \partial_g \beta_X \\ \partial_X \beta_g & \partial_g \beta_g \end{pmatrix} \\ &= \begin{pmatrix} 5g\,(1 - X) & 5X\left(1 - \frac{X}{2}\right) \\ -5g^2 & -\epsilon + 10\,(1 - X)\,g \end{pmatrix}. \end{aligned} \tag{C.49}$$

For the trivial fixed point, $g = 0$, this has the eigenvalues:

$$\rho = \begin{cases} 0 \\ -\epsilon \end{cases} \tag{C.50}$$

Additionally, the trivial fixed point would lead to a theory with only the κ coupling, which is known to be unphysical.

For the fixed point $X = 2$, $g = -\frac{\epsilon}{5}$, the stability matrix has eigenvalues

$$\rho = \begin{cases} \epsilon \\ \epsilon. \end{cases} \tag{C.51}$$

The second fixed point is stable. However it is also unphysical. This can be seen by considering the definition of g and X, which with these values would imply a negative ν_4.

For the fixed point $X = 0$, $g = \frac{\epsilon}{5}$, the stability matrix eigenvalues are

$$\rho = \begin{cases} \epsilon \\ \epsilon, \end{cases} \tag{C.52}$$

which is both stable and physical.

Reworking the scaling arguments from (5.19) gives

$$\begin{aligned} &\hat{\Gamma}^{nm}\left(\{q\}\{\omega\};\nu_2,\nu_4,\Gamma^2,g,X\right) \\ &\quad = A^{1+\frac{m-n}{2}}\mu^{\frac{m\epsilon}{2}-n(4-\frac{\epsilon}{2})+8-\epsilon} \qquad \text{(C.53a)} \\ &\qquad \cdot \bar{\hat{\Gamma}}^{nm}\left(\left\{\frac{q}{\mu}\right\}\left\{\frac{\omega}{A\mu^4}\right\};\frac{\nu_2}{A\mu^2},\frac{\nu_4}{A},\Gamma^2,\frac{g}{A^{\epsilon/2}\mu^{\epsilon}},X\right) \\ &\quad = l^{\gamma_4(1+\frac{m-n}{2})+\frac{m\epsilon}{2}-n(4-\frac{\epsilon}{2})+8-\epsilon} \qquad \text{(C.53b)} \\ &\qquad \cdot \hat{\Gamma}^{nm}\left(\left\{\frac{q}{l}\right\}\left\{\frac{\omega}{l^{\gamma_4+4}}\right\};\nu_2 l^{\gamma_2-\gamma_4-2},\nu_4,\Gamma^2,g l^{\gamma_g-\frac{\epsilon\gamma_4}{2}-\epsilon},X l^{\gamma_X}\right), \end{aligned}$$

with the original non-linear couplings scaling as

$$\lambda_{13} l^{2\gamma_\Gamma - 2\gamma_4 - \epsilon + \gamma_{13}} \quad \kappa l^{\gamma_\kappa + \gamma_\Gamma - \frac{3\gamma_4}{2} - \frac{\epsilon}{2}}, \tag{C.53c}$$

defining the exponents

$$\begin{aligned} &\hat{\Gamma}^{(n,m)}\left(k, \omega; \nu_2, \nu_4, \Gamma^2, [g, X/\lambda_{13}, \kappa]\right) \\ &\quad = l^{-\frac{n}{2}(2\eta+8-\epsilon) - \frac{m}{2}(2\tilde{\eta}-\epsilon)+8-\epsilon+\delta} \\ &\qquad \hat{\Gamma}^{(n,m)}\left(\frac{k}{l}, \frac{\omega}{l^z}; \frac{\nu_2}{l^{1/\nu}}, \nu_4, \Gamma^2, \left[\frac{g}{l^{\pi_g}}, \frac{X}{l^{\pi_X}} \Big/ \frac{\lambda_{13}}{l^{\pi_{13}}}, \frac{\kappa}{l^{\pi_\kappa}}\right]\right), \end{aligned} \tag{C.54}$$

which can be read off as

$$\begin{gathered} z = 4 + \gamma_4 \qquad \nu = \frac{1}{2 + \gamma_4 - \gamma_2} \qquad \delta = \gamma_4 \\ \pi_g = \epsilon\left(1 + \frac{\gamma_4}{2}\right) - \gamma_g \qquad \pi_{13} = \epsilon + 2\gamma_4 - \gamma_{13} \qquad \pi_X = -\gamma_X \qquad \pi_\kappa = \frac{\epsilon}{2} + \frac{3\gamma_4}{2} - \gamma_\kappa \\ \eta = \frac{\gamma_4}{2} \qquad \tilde{\eta} = \frac{-\gamma_4}{2}. \end{gathered} \tag{C.55}$$

From the γ-functions at the stable fixed point,

$$\gamma_g = \epsilon \quad \gamma_X = \epsilon \quad \gamma_2 = \frac{3\epsilon}{5} \quad \gamma_4 = 0 \quad \gamma_{13} = \epsilon \quad \gamma_\kappa = \epsilon, \tag{C.56}$$

this gives the exponents at the stable fixed point of the limited theory as

$$\begin{gathered} z = 4 \qquad \nu = \frac{1}{2} + \frac{3\epsilon}{20} \qquad \delta = 0 \\ \pi_g = 0 \qquad \pi_{13} = 0 \qquad \pi_X = -\epsilon \qquad \pi_\kappa = -\frac{\epsilon}{2} \\ \eta = 0 \qquad \tilde{\eta} = 0. \end{gathered} \tag{C.57}$$

The γ-functions at the unstable fixed point,

$$\gamma_g = \epsilon \quad \gamma_X = 0 \quad \gamma_2 = -\frac{3\epsilon}{5} \quad \gamma_4 = -\epsilon \quad \gamma_{13} = -\epsilon \quad \gamma_\kappa = -\epsilon \tag{C.58}$$

give the exponents

$$\begin{gathered} z = 4 - \epsilon \qquad \nu = \frac{1}{2} + \frac{\epsilon}{10} \qquad \delta = -\epsilon \\ \pi_g = 0 \qquad \pi_{13} = -2\epsilon \qquad \pi_X = 0 \qquad \pi_\kappa = -2\epsilon \\ \eta = -\frac{\epsilon}{2} \qquad \tilde{\eta} = \frac{\epsilon}{2}. \end{gathered} \tag{C.59}$$

The γ-functions at the trivial fixed point,

$$\gamma_g = 0 \quad \gamma_X = 0 \quad \gamma_2 = 0 \quad \gamma_4 = 0 \quad \gamma_{13} = 0 \quad \gamma_\kappa = 0, \tag{C.60}$$

give the exponents

$$\begin{gathered} z = 4 \qquad \nu = \frac{1}{2} \qquad \delta = 0 \\ \pi_g = \epsilon \qquad \pi_{13} = \epsilon \qquad \pi_X = 0 \qquad \pi_\kappa = \frac{\epsilon}{2} \\ \eta = 0 \qquad \tilde{\eta} = 0. \end{gathered} \tag{C.61}$$

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