LONDON MATHEMATICAL SOCIETY
EST. 1865

IOP Publishing

PAPER • OPEN ACCESS

An implementation of Hasselmann's paradigm for stochastic climate modelling based on stochastic Lie transport[*]

View the article online for updates and enhancements.

An implementation of Hasselmann's paradigm for stochastic climate modelling based on stochastic Lie transport[*]

D Crisan[1], D D Holm[1] and P Korn[2,]**

[1] Department of Mathematics, Imperial College, London SW7 2AZ, London, United Kingdom
[2] Climate Variability Department, Max Planck Institute for Meteorology, Bunndesstr. 53, 20146 Hamburg, Hamburg, Germany

E-mail: peter.korn@mpimet.mpg.de

Recommended by Dr Jean-Luc Thiffeault

Abstract

A generic approach to stochastic climate modelling is developed for the example of an idealised Atmosphere-Ocean model that rests upon Hasselmann's paradigm for stochastic climate models. Namely, stochasticity is incorporated into the fast moving atmospheric component of an idealised coupled model by means of stochastic Lie transport, while the slow moving ocean model remains deterministic. More specifically the stochastic model stochastic advection by Lie transport (SALT) is constructed by introducing stochastic transport into the material loop in Kelvin's circulation theorem. The resulting stochastic model preserves circulation, as does the underlying deterministic climate model. A variant of SALT called Lagrangian-averaged (LA)-SALT is introduced in this paper. In LA-SALT, we replace the drift velocity of the stochastic vector field by its expected value. The remarkable property of LA-SALT is that the evolution of its higher moments are governed by deterministic equations. Our modelling approach is substantiated by establishing local existence results, first, for the deterministic climate model

---

[*] Dedicated to the memory of Charlie Doering.
[**] Author to whom any correspondence should be addressed.

that couples compressible atmospheric equations to incompressible ocean equation, and second, for the two stochastic SALT and LA-SALT models.

Contents

1. Introduction

Prediction of climate dynamics is one of the great societal and intellectual challenges of our time. The complexity of this task has prompted the formulation of idealised climate models that target the representation of selected spatio-temporal characteristics, instead of representing the full bandwidth of physical processes ranging from seconds to millennia and from centimetres to thousands of kilometres. A climate model of full complexity would couple, for example, an atmospheric model, described by the compressible three-dimensional Navier–Stokes equations and a set of advection-diffusion equations for temperature and humidity, to an oceanic model, given by the three-dimensional incompressible Navier–Stokes equations and advection-diffusion equations for temperature and salinity. Each model would completed by thermodynamic relationships and physical parametrisations to account for non-resolved processes, and the boundary conditions would represent the physics of the air-sea interface and the ocean's mixed layer. In contrast, idealised models tend to simplify these equations by reducing the number of state variables, terms in the equations, or spatial dimensions. The

simplification is not the result of a systematic model reduction but only rationalised by heuristic scaling considerations. The amount of simplification needed in each climate model is dictated by the climate processes under investigation.

In this paper we formulate a framework for deriving stochastic idealised climate models. The deterministic climate model for which we derive here two stochastic versions belongs to a family of idealised climate models of intermediate complexity that were developed to study specific phenomena, with El-Nino-Southern Oscillation (ENSO) as a prominent example that has played an important role in the evolution of climate science. ENSO is an instability of the coupled atmosphere-ocean system that occurs with quasi-periodic frequency of 5–7 years. This instability is absent in (uncoupled) atmosphere and ocean models and lives from the feedback and exchange between atmosphere and ocean. The attempt to model this phenomenon has culminated in the famous *Cane–Zebiak model* [5] that was able to predict successfully an ENSO event. Although this paper is not about ENSO we review in later section of this paper (see section 1.1) some of these models and we do this for three reasons: first, to emphasize a certain degree of formal similarity between these models and our deterministic model that, according to our experience, supports the conjecture that the similarity between these models is sufficient to create similarities between their solutions. Second, we emphasize a fundamental difference, namely that our deterministic idealised coupled model is of variational nature and this is of decisive importance in our approach to incorporate stochasticity in the coupled atmosphere-ocean system. Third, ENSO provides an important example of a climate phenomenon that is subject to Hasselmann's paradigm.

A conceptual picture of the integration of stochasticity into a climate model was formulated by Hasselmann [20]. In Hasselmann's paradigm, the atmosphere acts with high frequency on short time scales, represented as a stochastic white-noise forcing of the ocean. The integration of the atmosphere's stochastic white-noise forcing over long time scales produces a low-frequency response in the ocean. As a result of the back-reaction, a red spectrum of the atmosphere's climate fluctuation is produced which complies with a variety of observations of the internal variability of the climate system [20]. For a description of Hasselmanns program in probabilistic terms we refer to [2]. It is common practice in climates modelling to incorporate Hasselmann's paradigm as a stochastic perturbation of the initial conditions, or as a stochastic forcing to the right-hand side of the dynamical equations of a deterministic climate model, then model the range of stochastic effects by creating an ensemble of simulations (see e.g. [36]). The particular choice of the stochastic perturbation must be based on the modelling objectives in the case at hand. A concise mathematical framework for stochastic climate modelling was developed in [32]. This approach relies also on scale separation in fast and slow dynamics following Hasselmann's paradigm but it differs methodically from ours in modelling the nonlinear self-interaction of the fast variables by means of a linear stochastic model such as an Ornstein–Uhlenbeck process and incorporates multiplicative noise while at the same maintaining energy conservation.

While following Hasselmann's view in a general sense, in this paper we incorporate the stochasticity in a novel way, applied in two stages which both deviate from the established practice. First, the path of a fluid element in the Lagrangian sense is assumed to be stochastic. This assumption injects stochasticity directly into the transport velocity of the atmospheric fluid dynamics, thereby transforming the governing equations into stochastic PDEs. Second, although our stochasticity is introduced *ab initio* and not a posteriori via external forcing, both stages of our stochastic models are transparently related to the deterministic model by the Kelvin Circulation Theorem. This fundamental connection facilitates the physical interpretation of the two stages of the stochastic models. Our modelling approach could be seen as an implementation of Hasselmann's program, since we couple the fast and stochastic

atmosphere model to the slow and deterministic ocean model and we implement this through a new coupling mechanism that passes the *expectation* of the atmospheric wind forcing to the ocean. However, Hasselmann's paradigm discussed in [17, 20] now has more than three thousand citations, so the present paper could equally well be considered as a footnote to Hasselmann's program.

The two stages of our stochastic approach are called stochastic advection by Lie transport (SALT) and Lagrangian averaged SALT (LA-SALT). The two stages of our approach represent two different viewpoints or modelling philosophies depending on the time scales of the intended application. For SALT, atmospheric 'weather' produces uncertainty in advection arising from motion on unresolved time scales. In LA-SALT, atmospheric 'climate' is taken as the baseline, and the atmospheric 'weather' is treated as a field of fluctuations around the climate baseline, as discussed in Ed Lorenz's famous lecture [30]. The LA-SALT approach brings us back to Hasselmann's paradigm, which decomposes a general climate model into deterministic and stochastic parts. In LA-SALT, in addition, the ideas of McKean, Vlasov and Kac [23, 34, 41][3] are applied to the deterministic climate description. Namely, the LA-SALT approach results in deterministic linear fluctuation equations that govern the dynamics of the climate statistics themselves, including variance, covariance and higher statistical moments. Within our framework these higher order statistical moments are governed by linear equations. This result offers potential computational advantages and opens new perspectives for the theoretical analysis of these moments.

In summary, this paper formulates two complementary stochastic idealised climate models called SALT and LA-SALT. The SALT climate model couples a stochastic PDE for the atmospheric circulation to a deterministic PDE for the circulation of the ocean. The stochasticity is incorporated by assuming that Lagrangian particles in the atmosphere follow a stochastic path given by a Stratonovich process which appears in the motion of the material loop in Kelvin's circulation theorem. The stochastic Lagrangian path of the material loop is a semimartingale stochastic process in the SALT approach and is a McKean-Vlasov process in the LA SALT approach. Both the SALT and LA-SALT approaches are related to an underlying deterministic model via Kelvin's circulation theorem. We substantiate our modelling choices by anchoring them within an established class of idealised climate models, as well as providing a mathematical analysis that demonstrates by proving a local well-posedness theorem that the proposed stochastic climate models rest on a firm mathematical basis.

The numerical simulations that would demonstrate the capabilities of the SALT and LA-SALT stochastic models, however, are beyond the scope of the present paper. The key element of such a numerical experiment is the sensible specification of the stochastic process. For the purpose of mathematical analysis carried out here, though, it is sufficient to assume that the stochastic process is of Stratonovich type. In contrast, a numerical experiment would require one to choose a specific Stratonovich process by incorporating externally obtained information either from observations or from high-resolution simulations. For an example of the latter procedure in the context of the Euler fluid equations in two dimensions, we refer to [7].

In the remainder of the introduction we detail our modelling approach for the deterministic model in section 1.1 and for the stochastic model in section 1.2.

1. Main content of the paper
    (a) Adaptation of the deterministic Gill–Matsuno [18, 31] class of ocean-atmosphere climate model (OACM) to the geometric variational framework. This adaptation produces a Kelvin circulation theorem which retains the transformation properties which are

---

[3] See also the seminal discussion in Sznitman [40] of the 'propagation of chaos' introduced in [23].

the basis for the remainder of the paper. These transformation properties are inherited from the variational framework. They enable the formulation of the deterministic and stochastic models in terms of the same type of Kelvin circulation theorem.
  (b) Derivations of the SALT and LA-SALT stochastic versions of the OACM, whose flows all possess the same geometric transformation properties. This shared geometric structure enables the analysis to develop sequentially from deterministic to stochastic models.
  (c) Mathematical analysis for the deterministic, SALT and LA-SALT versions of the OACM. Specifically we prove existence and uniqueness of local solutions for the deterministic OACM, the existence of a martingale solution for the SALT version of the OACM and existence and uniqueness of local solution for the LA-SALT version.
  (d) Outlook—open problems, including further pusuit of the predictive equations derived here for the dynamics of OACM statistics.
  (e) Two appendices provide details of the derivations of the deterministic and stochastic models using Hamilton's variational principle.
2. Plan of the paper
  (1) The introduction in section 1 explains that the present work is based on Hasselmann's program of fast-slow decomposition of the climate into deterministic and stochastic components. It also introduces the deterministic climate model upon which we implement Hasselmann's program and it compares the deterministic and stochastic models we treat in terms of their individual Kelvin circulation theorems.
  (2) Section 2 proves the local existence and uniqueness properties of our variational geometric adaptation of the deterministic Gill–Matsuno climate model.
  (3) Section 3 also discusses the analytical properties of the SALT in section 3.1 and LA-SALT in section 3.2 stochastic models.
  (4) Section 4 provides a summary conclusion and specification of open problems for the SALT and LA-SALT OACM.

1.1. The deterministic climate model

The model of the atmospheric component of our idealised climate model consists of the compressible 2D Navier–Stokes equation coupled to an advection-diffusion equation for temperature θ^a. This model is a modification of the model suggested in [26]. The atmospheric velocity field $\mathbf{u}^a$ transports the temperature that provides the gradient term of the velocity equation. The ocean component of the coupled system consists of a 2D incompressible Navier–Stokes equations and an equation for the oceanic temperature variable θ^o that is passively advected by the ocean velocity field $\mathbf{u}^o$. Here the pressure acts here as a Lagrange multiplier to impose incompressibility. More specifically, the deterministic coupled PDE's for the ocean and the atmosphere are given by

$$\text{Atmosphere:} \quad \frac{\partial \mathbf{u}^a}{\partial t} + (\mathbf{u}^a \cdot \nabla)\mathbf{u}^a + \frac{1}{Ro^a}\mathbf{u}^{a\perp} + \frac{1}{Ro^a}\nabla\theta^a = \frac{1}{Re^a}\triangle\mathbf{u}^a, \qquad (1)$$

$$\frac{\partial \theta^a}{\partial t} + (\mathbf{u}^a \cdot \nabla)\theta^a = \gamma(\theta^a - \theta^o) + \frac{1}{Pe^a}\triangle\theta^a. \qquad (2)$$

$$\text{Ocean:} \quad \frac{\partial \mathbf{u}^o}{\partial t} + (\mathbf{u}^o \cdot \nabla)\mathbf{u}^o + \frac{1}{Ro^o}\mathbf{u}^{o\perp} + \frac{1}{Ro^o}\nabla(p^o + q^a)$$

$$= \sigma(\mathbf{u}^o - \bar{\mathbf{u}}^a_{sol}) + \frac{1}{Re^o}\triangle\mathbf{u}^o, \qquad (3)$$

$$\frac{\partial \theta^o}{\partial t} + (\mathbf{u}^o \cdot \nabla)\theta^o = \frac{1}{Pe^o}\triangle\theta^o, \tag{4}$$

$$div(\mathbf{u}^o) = 0, \tag{5}$$

with initial conditions

$$\mathbf{u}^a(t_0) = \mathbf{u}_0^a, \ \theta^a(t_0) = \theta_0^a, \ \mathbf{u}^o(t_0) = \mathbf{u}_0^o, \ \theta^o(t_0) = \theta_0^o.$$

In these equations, the ocean velocity $\mathbf{u}^o$ is coupled to the atmospheric velocity $\mathbf{u}^a$ and the atmospheric temperature θ^a is coupled to the oceanic temperature θ^o. The *coupling constants* $\gamma, \sigma < 0$ regulate the strength of the interaction between the two components.

The velocity coupling between the compressible atmosphere and the incompressible ocean model deserves some consideration. To preserve the incompressibility of the oceanic velocity field during the coupling we apply the Leray–Helmholtz Theorem to decompose the atmospheric velocity $\mathbf{u}^a$ into a solenoidal component $\mathbf{u}_{sol}^a$ and a gradient term q^a such that $\mathbf{u}^a = \mathbf{u}_{sol}^a + \nabla q^a$. The gradient part is combined with the oceanic pressure. In a second step we remove the space average via $\bar{\mathbf{u}} := \mathbf{u} - \frac{1}{|\Omega|}\int_\Omega \mathbf{u}dx$ such that the oceanic velocity fields remains in the space of periodic flows with vanishing average. This property allows to determine the oceanic pressure. Physically, this step removes the rapid mean velocity of the atmosphere relative to the slower ocean velocity in the frame of motion of the Earth's rotation. This means the ocean momentum responds to the shear force, which is proportional to the difference between the local ocean velocity at a given time and the local deviation of the atmospheric velocity away from its mean velocity.

The model above belongs to the class of coupled models of intermediate complexity. These models are more complex than energy-balance models but simpler than the coupled general circulation models of the atmosphere-ocean system that are used for climate research. Models of intermediate complexity allow one to study fundamental aspects of the atmosphere-ocean interaction. The most prominent example is ENSO in the tropical Pacific. As originally hypothesized by Bjerknes [4] this climate phenomenon crucially depends on the coupled interaction of both ocean and atmosphere.

The story of intermediate coupled models began with (uncoupled) models to study equatorial waves and their response to external forcing. Matsuno [31] investigated an (uncoupled) divergent barotropic model (single layer of incompressible fluid of homogeneous density, with a free surface, on the beta plane)

$$\begin{aligned}&\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{Ro^a}\mathbf{u}^{a\perp} + \frac{1}{Ro^a}\nabla\theta = 0,\\&\frac{\partial \theta}{\partial t} + Hdiv(\mathbf{u}) = Q.\end{aligned} \tag{6}$$

Matsuno [31] refers to θ as surface elevation above a mean depth H, and in this context Q appears as a source/sink of mass. Gill [18] studied the steady response to heating anomalies of a tropical atmosphere, as described by the Matsuno model. Systems of equations in the following class are often called *Gill models*

$$\begin{aligned}&\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{Ro^a}\mathbf{u}^{a\perp} + \frac{1}{Ro^a}\nabla\theta + a\mathbf{u} = 0,\\&\frac{\partial \theta}{\partial t} + Hdiv(\mathbf{u}) + b\theta = Q,\end{aligned} \tag{7}$$

where a, b are Raleigh friction and Newtonian cooling and where Q is a heating term. In Gill's work θ is proportional to the surface pressure. Since surface hydrostatic pressure is proportional to surface height, this identification is consistent with Matsuno's interpretation.

Atmospheric models of Gill–Matsuno type are often used to understand the atmospheric response during an El Niño to observed sea surface temperature anomalies. Zebiak [42] parametrised the heat flux Q from the ocean to the atmosphere in terms of the ocean sea surface temperature SST $Q = \alpha$ SST. This relation can be motivated by a linearisation of Clausius-Clapeyron relation (see [43] and also [19]).

As a next step, intermediate coupled models were constructed with atmospheric model either of Gill–Matsuno type [18] or as a statistical model of the atmosphere, that is for example constructed through an statistical analysis of atmospheric data (e.g. by Empirical Orthogonal Functions) and that provides some random forcing of the ocean. Then the atmospheric model is coupled to a one or two-layer ocean model. Nonlinear terms are omitted. The famous *Cane–Zebiak model* [5] applied a steady state atmosphere following Gill (7) and a two-layer ocean model, with two equations for layer thickness and two equations for temperature. This model was used to issue the first ENO forecast [6]. An overview can be found in chapter 7 of [12], or in [35].

We have modified the equations in the model above by including the nonlinear terms in the velocity and temperature equations. First, we have replaced the damping terms due to Raleigh friction and Newtonian cooling in (7) by Laplace operators for velocity and temperature. Furthermore, we interpret atmospheric $\theta = h$ as surface elevation or equivalent depth and combined this with the Charles Law of thermodynamics (see e.g. [15]) according to which the volume V is proportional to temperature T, $V = cT$, with constant $c > 0$. Since the volume is also proportional to the surface elevation $V = ch$ one obtains $h = cT$. This identification allows one to interpret $\theta = T$ as the atmospheric temperature variable. This interpretation also relates the Matsuno and Gill equations (6) and (7) to the atmospheric θ^a-equation and allows one to interpret θ^a as a temperature variable[4].

The pronounced irregularity of ENSO, with extremes occurring irregularly in time and vastly differing amplitudes of sea surface temperature anomalies, has motivated stochastic modelling approaches from early on. In the following section we suggest a new modelling approach that is based on stochastic transport along the Lagrangian paths of advected fluid properties and provide a mathematical analysis of this modelling approach.

1.2. The stochastic atmospheric climate model

The fundamental principle in modelling stochastic fluid advection is the Kelvin circulation theorem. As we shall see, each component of the deterministic atmosphere-ocean model in equations (1)–(5) above possesses its own Kelvin theorem, and the two components are coupled together by their relative velocity. The model (1)–(5) describes their interaction as the exchange of circulation between the atmosphere and ocean. Later we treat the atmospheric component of the model as being stochastic either in the sense of weather (SALT) or in the sense of climate (LA-SALT). In either case, the stochastic modification of the atmospheric dynamics will retain a Kelvin circulation theorems.

Theorem 1.1 (Kelvin theorem for the deterministic atmospheric model in (1)–(5)). *The deterministic model for atmospheric dynamics satisfies the following Kelvin theorem for circulation around a loop $c(u^a)$ moving with the flow of the atmospheric velocity $\mathbf{u}^a$. Namely,*

$$\frac{d}{dt} \oint_{c(u^a)} \left(\mathbf{u}^a + \frac{1}{Ro^a} \mathbf{R}(\mathbf{x}) \right) \cdot d\mathbf{x} = \frac{1}{Re^a} \oint_{c(u^a)} \triangle \mathbf{u}^a \cdot d\mathbf{x},$$

---

[4] For horizontal 2D models, the difference between potential and absolute temperature disappears.

where curl$\boldsymbol{R} = 2\hat{z}\,\Omega(\boldsymbol{x})$ *is the Coriolis parameter in nondimensional units.*

Proof. By direct calculation, one shows that the deterministic atmospheric dynamics in the model above satisfies the relation in the Kelvin circulation theorem,

$$\frac{d}{dt}\oint_{c(u^a)}\left(\boldsymbol{u}^a + \frac{1}{Ro^a}\boldsymbol{R}(\boldsymbol{x})\right)\cdot d\boldsymbol{x} = \oint_{c(u^a)}(\partial_t + \mathcal{L}_{u^a})\left(\left(\boldsymbol{u}^a + \frac{1}{Ro^a}\boldsymbol{R}(\boldsymbol{x})\right)\cdot d\boldsymbol{x}\right)$$

$$= \oint_{c(u^a)}\left(\partial_t\boldsymbol{u}^a + (\boldsymbol{u}^a\cdot\nabla)\boldsymbol{u}^a + u^a_j\nabla u^{aj}\right.$$

$$\left. -\boldsymbol{u}^a\times\text{curl}\frac{1}{Ro^a}\boldsymbol{R}(\boldsymbol{x}) + \nabla\left(\boldsymbol{u}^a\cdot\frac{1}{Ro^a}\boldsymbol{R}\right)\right)\cdot d\boldsymbol{x}$$

$$\text{By the model} = \oint_{c(u^a)}\left(-\frac{1}{Ro^a}\nabla\theta^a + \frac{1}{2}\nabla|\boldsymbol{u}^a|^2 + \nabla\left(\boldsymbol{u}^a\cdot\frac{1}{Ro^a}\boldsymbol{R}\right) + \triangle\mathbf{u}^a\right)\cdot d\boldsymbol{x}$$

$$= \frac{1}{Re^a}\oint_{c(u^a)}\triangle\mathbf{u}^a\cdot d\boldsymbol{x}\,.$$

$\square$

Remark. In the proof above, $\mathcal{L}_u$ represents Lie derivative with respect to the vector field $u^a = \boldsymbol{u}^a\cdot\nabla$ with components $\boldsymbol{u}^a(\boldsymbol{x},t)$ and $c(u^a)$ denotes a material loop moving with the atmospheric Lagrangian transport velocity $\boldsymbol{u}^a(\boldsymbol{x},t)$. Consequently, in the absence of viscosity, atmospheric circulation is conserved by the deterministic model because the viscous term is absent then and the loop integrals of gradients such as $u_j\nabla u^j = \frac{1}{2}\nabla|\boldsymbol{u}|^2$ vanish on the right-hand side of the equation in the proof.

Likewise, the dynamics of the ocean component of the model above satisfies the following Kelvin circulation theorem.

Theorem 1.2 (Kelvin theorem for the deterministic oceanic model in (1)–(5)). *The circulation dynamics around a loop $c(u^o)$ moving with the flow of the oceanic velocity $\boldsymbol{u}^o$ is given by*

$$\frac{d}{dt}\oint_{c(u^o)}\left(\boldsymbol{u}^o + \frac{1}{Ro^o}\boldsymbol{R}(\boldsymbol{x})\right)\cdot d\boldsymbol{x} = \oint_{c(u^o)}\left(\sigma(\mathbf{u}^o - \bar{\mathbf{u}}^a) + \frac{1}{Re^o}\triangle\mathbf{u}^o\right)\cdot d\boldsymbol{x}\,.$$

Proof. The proof follows analogously to the proof of theorem 1.1. $\square$

1.3. SALT atmospheric model

Let $(\Xi, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ be a filtered probability space on which we have defined a sequence of independent Brownian motions $(W^i)_i$. Let $(\xi_i)_i$ be a given sequence of sufficiently smooth vector fields that satisfies the condition in (69) below. In this work we assume the vector fields $(\xi_i)_i$ to be given. For numerical simulations one defines these vector fields by extracting information from observational data. For an example we refer to [10]. The derivation of the SALT atmospheric model introduces the stochastic Lagrangian path

$$d\mathbf{x}_t := \mathbf{u}^a(\boldsymbol{x},t)dt - \sum_i \xi^a_i(\boldsymbol{x})\circ dW_i(t)\,. \tag{8}$$

Following [21], appendix '2D SALT stochastic atmospheric model (SAM)' discusses the introduction of the stochastic Lagrangian paths in (8) into Hamilton's variational principle

for the atmospheric model equations. This step leads to the SALT version of the idealised deterministic climate model comprising equations (1)–(5). Namely, the SALT model is specified by the system of stochastic differential equations below:

Atmosphere[5]:

$$d\mathbf{u}^a + (d\mathbf{x}_t^a \cdot \nabla)\mathbf{u}^a + \frac{1}{Ro^a}d\mathbf{x}_t^{a\perp} + \sum_i \left(u_j^a \nabla \xi_i^j + \frac{1}{Ro^a}\nabla\left(R_j(\mathbf{x})\xi_i^j\right) \right) \circ dW_t^i$$

$$+ \frac{1}{Ro^a}\nabla\theta^a = \frac{1}{Re^a}\triangle\mathbf{u}^a, \tag{9}$$

$$d\theta^a + d\mathbf{x}_t^a \cdot \nabla\theta^a = -\gamma(\theta^o - \theta^a) + \frac{1}{Pe^a}\triangle\theta^a, \tag{10}$$

$$d\mathbf{x}_t^a = \mathbf{u}^a dt + \sum_i \xi_i \circ dW_l^i \tag{11}$$

Ocean:

$$\frac{\partial\mathbf{u}^o}{\partial t} + (\mathbf{u}^o \cdot \nabla)\mathbf{u}^o + \frac{1}{Ro^o}\mathbf{u}^{o\perp} + \frac{1}{Ro^o}\nabla p^o = \sigma(\mathbf{u}^o - \mathbb{E}\bar{\mathbf{u}}^a) + \frac{1}{Re^o}\triangle\mathbf{u}^o, \tag{12}$$

$$\frac{\partial\theta^o}{\partial t} + (\mathbf{u}^o \cdot \nabla)\theta^o = \frac{1}{Pe^o}\triangle\theta^o, \tag{13}$$

$$div(\mathbf{u}^o) = 0, \tag{14}$$

Theorem 1.3. *Kelvin theorem for the SALT version of the atmospheric model in equations (9)–(11)*

$$d\oint_{c(d\mathbf{x}_t)} \left(\mathbf{u}^a + \frac{1}{Ro^a}\mathbf{R}(\mathbf{x}) \right) \cdot d\mathbf{x} = \frac{1}{Re}\oint_{c(d\mathbf{x}_t)} \triangle\mathbf{u}^a \, dt \cdot d\mathbf{x}, \tag{15}$$

where $c(d\mathbf{x}_t)$ denotes any closed material loop whose line elements follow stochastic Lagrangian paths as in (8).

Proof. Upon suppressing the superscript *a* in the velocity $\mathbf{u}^a$ for brevity of notation, we calculate

$$d\oint_{c(d\mathbf{x}_t)} (\mathbf{u} + \mathbf{R}(\mathbf{x})) \cdot d\mathbf{x} = \oint_{c(d\mathbf{x}_t)} (d + \mathcal{L}_{d\mathbf{x}_t})\big((\mathbf{u} + \mathbf{R}(\mathbf{x})) \cdot d\mathbf{x}\big)$$

$$= \oint_{c(d\mathbf{x}_t)} \Big(d\mathbf{u} + (d\mathbf{x}_t \cdot \nabla)\mathbf{u} + u_j\nabla d\mathbf{x}_{tj}$$

$$- d\mathbf{x}_t \times \mathrm{curl}\mathbf{R}(\mathbf{x}) + \nabla(d\mathbf{x}_t \cdot \mathbf{R}) \Big) \cdot d\mathbf{x}$$

$$[\textit{By motion equation (9)}] = \oint_{c(d\mathbf{x}_t)} \bigg(-\nabla\theta dt + \frac{1}{2}\nabla|\mathbf{u}|^2 dt - d\mathbf{x}_t \times \mathrm{curl}\mathbf{R}(\mathbf{x}) + \nabla(\mathbf{u} \cdot \mathbf{R})dt$$

$$+ u_j\nabla\sum\xi^j \circ dW(t) + \sum\nabla\big(\boldsymbol{\xi} \circ dW(t) \cdot \mathbf{R}\big) \bigg) \cdot d\mathbf{x}$$

$$= \frac{1}{Re}\oint_{c(d\mathbf{x}_t)} \triangle\mathbf{u}\, dt \cdot d\mathbf{x}.$$

$\square$

---

[5] As in the deterministic case we will write the Coriolis parameter as $\mathrm{curl}\,\mathbf{R}(\mathbf{x}) = 2\Omega(\mathbf{x})$..

Remark. The stochastic equation for the potential temperature θ^a in the atmospheric model inherits the stochasticity of the Lagrangian trajectories $\mathrm{d}\mathbf{x}_t$ in (8), as a scalar tracer transport equation,

$$\mathrm{d}\theta^a + (\mathrm{d}\boldsymbol{x}_t \cdot \nabla)\theta^a = \left[\gamma(\theta^a - \theta^o) + \frac{1}{Pe^a}\triangle\theta^a\right] dt. \tag{16}$$

1.4. LA-SALT atmospheric model

We next modify the SALT approach to the two-dimensional atmospheric component of the climate system in the previous section to make it non-local in probability space, in the sense that the expected velocity will replace the drift velocity in the semimartingale for the SALT transport velocity of the stochastic fluid flow. This stochastic fluid model is derived by exploiting a novel idea introduced in [13] and developed further in [1, 14], of applying LA in probability space to the fluid equations governing SALT which were introduced in [21].

The LA-SALT approach achieves three results of potential interest in climate modelling. These results address three different components of the climate science paradigm.

- First, the LA-SALT approach introduces a sense of determinism into climate science, by replacing the drift velocity of the stochastic vector field for material transport by its expected value in equation (8). In this step, the expected fluid velocity becomes deterministic.
- Second, the LA-SALT approach reduces the dynamical equations for the fluctuations to a linear stochastic transport problem with a deterministic drift velocity. Such problems are well-posed. We prove here that the LA-SALT version of the SALT climate model possesses local weak solutions.
- Third, the LA-SALT approach addresses the dynamics of the variances of the fluctuations. In particular, it enables the variances and higher moments of the fluctuation statistics to be found deterministically, as they are driven by a certain set of correlations of the fluctuations among themselves.

In summary, the first LA-SALT result makes the distinction between climate and weather for the case at hand. Namely, the LA-SALT fluid equations for the 2D atmosphere-ocean climate model system may be regarded as a dissipative system akin to the Navier–Stokes equations for the expected motion (climate) which is embedded into a larger conservative system which includes the statistics of the fluctuation dynamics (weather). The second result provides a set of linear stochastic transport equations for predicting the fluctuations (weather) of the physical variables, as they are driven by the deterministic expected motion. The third result produces closed deterministic evolutionary equations for the dynamics of the variances and covariances of the stochastic fluctuations. Thus, the LA-SALT approach to investigating the 2D atmosphere-ocean climate model system treated here reveals that its statistical properties are fundamentally dynamical. Specifically, the LA-SALT analysis of the 2D atmosphere-ocean model presented here defines its climate, climate change, weather, and change of weather statistics, in the context of a hierarchical systems of PDEs and SPDEs with unique local weak solutions.

1.4.1. LA-SALT atmospheric equations in Stratonovich form.   Below the Stratonovich LA-SALT equations are given in the standard notation for stochastic fluid dynamics. In

appendix '2D LA-SALT stochastic atmospheric model (LASAM)' we show the details of the derivation of the LA-SALT equations and its relation to an atmospheric model assuming an isothermal ideal gas.

$$
\mathrm{d}\mathbf{u}^a + (\mathrm{d}\mathbf{X}_t^{a} \cdot \nabla)\mathbf{u}^a + \frac{1}{Ro^a}\mathrm{d}\mathbf{X}_t^{a\perp} + \sum_i \left(u_j^a \nabla \xi_i^j + \frac{1}{Ro^a}\nabla\left(R_j(\mathbf{x})\xi_i^j\right)\right) \circ dW_t^i
$$

$$
+ u_j^a \nabla \mathbb{E}[u^{aj}]dt + \frac{1}{Ro^a}\nabla(\mathbb{E}[\mathbf{u}^a]\cdot\mathbf{R})dt + \frac{1}{Ro^a}\nabla\theta^a\, dt
$$

$$
= \frac{1}{Re^a}\triangle\mathbf{u}^a\, dt, \mathrm{d}\theta^a + \mathrm{d}\mathbf{X}_t^{a}\cdot\nabla\theta^a = -\gamma(\theta^o - \theta^a)\,dt + \frac{1}{Pe^a}\triangle\theta^a\, dt. \tag{17}
$$

where the *Stratonovich* stochastic Lagrangian trajectory for LA-SALT is given by

$$
\mathrm{d}\mathbf{X}_t^a := \mathbb{E}[\mathbf{u}^a](\boldsymbol{x},t)dt + \sum_i \xi_i^a(\boldsymbol{x})\circ dW_i(t). \tag{18}
$$

The expectation terms in LA-SALT induce another modification of the model which preserves the Kelvin circulation theorem, whose expectation yields a deterministic equation,

Theorem 1.4 (Kelvin theorem for the LA-SALT atmospheric model).

$$
\mathrm{d}\oint_{c(\mathrm{d}\mathbf{X}_t^a)} \left(\boldsymbol{u}^a + \boldsymbol{R}(\boldsymbol{x})\right)\cdot d\boldsymbol{x} = \frac{1}{Re}\oint_{c(\mathrm{d}\mathbf{X}_t^a)}\triangle\mathbf{u}^a\, dt\cdot d\boldsymbol{x}, \tag{19}
$$

where $\mathbf{X}_t^a$ *satisfies (18).*

Proof. The proof of the Kelvin theorem for LA-SALT follows the same lines as for SALT. □

1.4.2. LA-SALT atmospheric equations in Itô form. Likewise, the *Itô* LA-SALT equations are given in the standard notation for stochastic fluid dynamics in appendix '2D LA-SALT stochastic atmospheric model (LASAM)' by

$$
\mathrm{d}\mathbf{u}^a + (\mathrm{d}\widehat{\mathbf{X}}_t^{a}\cdot\nabla)\mathbf{u}^a + \frac{1}{Ro^a}\mathrm{d}\widehat{\mathbf{X}}_t^{a\perp} + \sum_i\left(u_j^a\nabla\xi_i^j + \frac{1}{Ro^a}\nabla\left(R_j(\mathbf{x})\xi_i^j\right)\right)dW_t^i
$$

$$
+ \frac{1}{2}\left[\hat{\mathbf{z}}\times\xi\left(\mathrm{div}\left(\xi\left(\hat{\mathbf{z}}\cdot\mathrm{curl}\left(\mathbb{E}[\mathbf{u}^a]+\frac{1}{Ro^a}\mathbf{R}(\mathbf{x})\right)\right)\right)\right) - \nabla\left(\xi\cdot\nabla\left(\xi\cdot\left(\mathbb{E}[\mathbf{u}^a]+\frac{1}{Ro^a}\mathbf{R}(\mathbf{x})\right)\right)\right)\right]dt
$$

$$
+ u_j^a\nabla\mathbb{E}[u^{aj}]dt + \frac{1}{Ro^a}\nabla(\mathbb{E}[\mathbf{u}^a]\cdot\mathbf{R})dt + \frac{1}{Ro^a}\nabla\theta^a\, dt = \frac{1}{Re^a}\triangle\mathbf{u}^a\, dt, \tag{20}
$$

$$
\mathrm{d}\theta^a + \mathrm{d}\widehat{\mathbf{X}}_t^{a}\cdot\nabla\theta^a - \frac{1}{2}\left(\xi\cdot\nabla(\xi\cdot\nabla\theta^a)\right)dt = -\gamma(\theta^o - \theta^a)\,dt + \frac{1}{Pe^a}\triangle\theta^a\, dt, \tag{21}
$$

where the *Itô* stochastic Lagrangian trajectory for LA-SALT is given by

$$
\mathrm{d}\widehat{\mathbf{X}}_t^a := \mathbb{E}[\mathbf{u}^a](\boldsymbol{x},t)dt + \sum_i\xi_i^a(\boldsymbol{x})dW_i(t). \tag{22}
$$

Expected LA-SALT atmospheric equations. Taking the expectation of equations (20) and (21) yields a closed set of deterministic PDE for the expectations $\mathbb{E}[\mathbf{u}^a]$ and $\mathbb{E}[\theta^a]$. Subtracting the expectations from equations (20) and (21) yields *linear equations* for the differences,

$$
\mathbf{u}^{a\prime} := \mathbf{u}^a - \mathbb{E}[\mathbf{u}^a] \quad\text{and}\quad \theta^{a\prime} := \theta^a - \mathbb{E}[\theta^a]. \tag{23}
$$

Since $\mathbf{u}^{a'}$ and $\theta^{a'}$ satisfy $\mathbb{E}[\mathbf{u}^{a'}] = 0$ and $\mathbb{E}[\theta^{a'}] = 0$, one may regard these difference variables as fluctuations of $\mathbf{u}^a$ and θ^a away from their expected values. From here one can calculate the dynamical equations for the statistics of the atmospheric model, e.g. its variances and its other tensor moments, as detailed in [1, 13, 14]. Further details of these equations can be found in section 3.2.

1.4.3. Oceanic part of the LA-SALT model. The oceanic part of the LA-SALT model *coincides* with the oceanic part of the SALT model (12)–(14).

2. Local existence and uniqueness of the deterministic climate model

2.1. Functional analytic framework

In the treatment below, we compactify the notation for the dynamics of the two-component system (1)–(5), as follows. The state of the system is described by a state vector $\psi := (\psi^a, \psi^o)$ with atmospheric component $\psi^a := (\mathbf{u}^a, \theta^a)$ and oceanic component $\psi^o := (\mathbf{u}^o, \theta^o)$. The initial state is denoted by $\psi(t_0) = \psi_0$. where $\psi_0 = (\mathbf{u}_0^a, \theta_0^a, \mathbf{u}_0^o, \theta_0^o)$.

In this notation, equations (1)–(5) take the operator form

$$d_t\psi + B(\psi, \psi) + C\psi + D(\psi^a, \psi^o) = L\psi, \tag{24}$$

where one defines

- $B := (B^a, B^o)$ is the usual bilinear transport operator, with $B^a(\psi^a, \psi^a) := (\mathbf{u}^a \cdot \nabla \mathbf{u}^a, \mathbf{u}^a \cdot \nabla \theta^a)$ and $B^o(\psi^o, \psi^o) := (\mathbf{u}^o \cdot \nabla \mathbf{u}^o, \mathbf{u}^o \cdot \nabla \theta^o)$.
- $C := (C^a, C^o)$ with $C^a\psi^a := (\frac{1}{Ro^a}\mathbf{u}^{a\perp} + \nabla\theta^a, 0)$ and $C^o\psi^o := (\frac{1}{Ro^o}\mathbf{u}^{o\perp} + \nabla p^o, 0)$
- $L := (L^a, L^o)$ denotes the dissipation/diffusion operator for velocity and temperature with $L^a\psi^a := (\frac{1}{Re^a}\triangle\mathbf{u}^a, \frac{1}{Pe^a}\triangle\theta^a)$ and $L^o\psi^o := (\frac{1}{Re^o}\triangle\mathbf{u}^o, \frac{1}{Pe^o}\triangle\theta^o)$
- $D(\psi^a, \psi^o) := (0, \gamma(\theta^a - \theta^o), \sigma(\mathbf{u}^o - \bar{\mathbf{u}}_{sol}^a), 0)$ is the coupling operator.

Domain and boundary conditions: The spatial domain is a two dimensional square $\Omega := [0, L] \times [0, L]$ with $L \in \mathbb{R}^+$. We assume periodic boundary conditions.

Operators and spaces: By $W^s(\Omega)$ we denote the L^2-Sobolev space of order $s \in \mathbb{Z}_+ \cup \{0\}$ that is defined as the set of functions $f \in L^2(\Omega)$ such that its derivatives in the distributional sense $\mathcal{D}^\alpha f(x,y) = \partial_x^{\alpha_1}\partial_y^{\alpha_2}f(x,y)$ are in $L^2(\Omega)$ for all $|\alpha| \leqslant s$, with multi-index $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$, and degree $|\alpha| := \alpha_1 + \alpha_2$. The scalar product in $W^s(\Omega)$ is defined by

$$\langle f, g\rangle_{W^s} := \sum_{|\alpha|\leqslant s}\int_\Omega \mathcal{D}^\alpha f \cdot \mathcal{D}^\alpha g\,dx. \tag{25}$$

The vectorial counterpart of the Sobolev space $W^s(\Omega)$ is denoted by $\mathbf{W}^s(\Omega)$. More information about Sobolev spaces can be found for example in [16, 33]. We define the scalar space

$$V := \{f : \mathbb{R}^2 \to \mathbb{R} : f \text{ is a trigonometric polynomial with period L}\}, \tag{26}$$

and its vector-valued equivalents for atmosphere and ocean component

$$\mathbf{V}^a := \{u : \mathbb{R}^2 \to \mathbb{R}^2 : u \text{ is a vector-valued trigonometric polynomial with period L}\}$$

$$\mathbf{V}^o := \{u : \mathbb{R}^2 \to \mathbb{R}^2 : u \text{ is a vector-valued trigonometric polynomial with period L}$$

$$\text{and } \int_\Omega u\,dx = 0\}. \tag{27}$$

We define now the following function spaces

$$H^s(\Omega) := \text{the closure of } V \text{ in } \mathbf{W}^s(\Omega), \quad \mathbf{H}^{s,a}(\Omega) := \text{the closure of } \mathbf{V}^a \text{ in } \mathbf{W}^s(\Omega),$$

$$\mathbf{H}^{s,o}(\Omega) := \text{the closure of } \mathbf{V}^o \text{ in } \mathbf{W}^s(\Omega), \quad \mathbf{H}^s_{div}(\Omega) := \{u \in \mathbf{H}^{s,o}(\Omega) : div(u) = 0\}. \tag{28}$$

In this notation, we define for $s \in \mathbb{N} \cup \{0\}$ the *Sobolev space of state vectors* by

$$\mathcal{H}^s(\Omega) := \mathbf{H}^{s,a}(\Omega) \times H^s(\Omega) \times \mathbf{H}^{s,o}_{div}(\Omega) \times H^s(\Omega), \tag{29}$$

in which the norm of $\psi = (\mathbf{u}^a, \theta^a, \mathbf{u}^o, \theta^o) \in \mathcal{H}^s$ is given by

$$\|\psi\|_{\mathcal{H}^s} := (\|\mathbf{u}^a\|^2_{\mathbf{H}^s} + \|\theta^a\|^2_{H^s} + \|\mathbf{u}^o\|^2_{\mathbf{H}^s} + \|\theta^o\|^2_{H^s})^{1/2}. \tag{30}$$

We use an analogous notation for the Lebesgue spaces and denote by $L^2, \mathbf{L}^2, \mathcal{L}^2$ the sets of square-integrable scalar functions, vector fields and state vectors, respectively.

Definition 2.1. Let $s \in \mathbb{N}$. A state vector $\psi := (\mathbf{u}^a, \theta^a, \mathbf{u}^o, \theta^o)$ is said to be a local regular solution of (1)–(5) on the time interval $T := [t_0, t_1]$ if it satisfies (1)–(5) with initial condition $\psi(t_0) = \psi_0$ and if

$$\psi \in C(T, \mathcal{H}^s(\Omega)) \cap L^2(T, \mathcal{H}^{s+1}(\Omega)), \qquad \frac{d\psi}{dt} \in \mathcal{H}^0(T, \mathcal{H}^{s-1}(\Omega)). \tag{31}$$

Define a cut-off function as follows

$$g_R(x) := \begin{cases} 1, & \text{if } 0 \leqslant x \leqslant R, \\ 0, & \text{if } x \geqslant R + \delta, \\ \text{smoothly decaying} & \text{if } R < x < R + \delta. \end{cases} \tag{32}$$

Next, we define a finite-dimensional approximate system of equations for (24). Because of our assumption of periodic boundary conditions we may write the finite-dimensional approximation in terms of the Fourier basis $w_{\mathbf{n}}(\mathbf{x}) := e^{\frac{2\pi i \mathbf{n} \cdot \mathbf{x}}{L}}$. We remark that the specific form of the basis does not play a role in our proofs. We must take into account the incompressibility of the ocean flow. This is imposed through the Leray projection, which projects the ocean equation onto the space of divergence-free vector fields. For periodic boundary conditions, the Leray projection commutes with the Laplace operator, so the Stokes operator coincides with the Laplacian.

The Galerkin approximations for the atmospheric component $\psi^a = (\mathbf{u}^a, \theta^a)$ of the state vector are given by

$$\mathbf{P}^a_m \mathbf{u}^a(x,t) := \sum_{\mathbf{n} \in \{\mathbb{Z}^2, |\mathbf{n}| \leqslant m\}} \widehat{\mathbf{u}}^a_{\mathbf{n}}(t) w_{\mathbf{n}}(x)$$

$$\text{and } P^a_m \theta^a(x,t) := \sum_{\mathbf{n} \in \{\mathbb{Z}^2, |\mathbf{n}| \leqslant m\}} \hat{\theta}^a_{\mathbf{n}}(t) w_{\mathbf{n}}(x), \tag{33}$$

with $\widehat{\mathbf{u}}^a_{\mathbf{n}} := \int_\Omega \mathbf{u}^a(x,t) w_{\mathbf{n}}(x) \, dx$, $\hat{\theta}^a_{\mathbf{n}} := \int_\Omega \theta^a(x,t) w_{\mathbf{n}}(x) \, dx$. Incompressibility must be into account for the oceanic component $\psi^o = (\mathbf{u}^o, \theta^o)$, so the basis combines the Galerkin approximation with the Leray projection onto the space of divergence-free vector fields

$$\mathbf{P}^o_m \mathbf{u}^o(x,t) := \sum_{\mathbf{n} \in \{\mathbb{Z}^2 \setminus \{0\}, |\mathbf{n}| \leqslant m\}} \left(\widehat{\mathbf{u}}^o_{\mathbf{n}}(t) - \frac{\widehat{\mathbf{u}}^o_{\mathbf{n}}(t) \cdot \mathbf{n}}{|\mathbf{n}|^2} \mathbf{n} \right) w_{\mathbf{n}}(x),$$

$$\text{and } P^o_m \theta^o(x,t) := \sum_{\mathbf{n} \in \{\mathbb{Z}^2, |n| \leqslant m\}} \hat{\theta}^o_{\mathbf{n}}(t) w_{\mathbf{n}}(x), \tag{34}$$

where $\widehat{\mathbf{u}}^o_{\mathbf{n}} := \int_\Omega \mathbf{u}^o(x,t) w_{\mathbf{n}}(x) \, dx$, $\hat{\theta}^o_{\mathbf{n}} := \int_\Omega \theta^o(x,t) w_{\mathbf{n}}(x) \, dx$.

The Galerkin approximation of the state vector $\psi = (\psi^a, \psi^o)$ is defined as

$$P_m\psi := (\mathbf{P}_m^a\psi^a, \mathbf{P}_m^o\psi^o) := (\mathbf{P}_m^a\mathbf{u}^a, P_m^a\theta^a, \mathbf{P}_m^o\mathbf{u}^o, P_m^o\theta^o). \tag{35}$$

We also use the notation

$$\psi_m := P_m\psi, \text{ with } \mathbf{u}_m^a := \mathbf{P}_m^a\mathbf{u}^a, \theta_m^a := P_m^a\theta, \text{ and } \mathbf{u}_m^o := \mathbf{P}_m^o\mathbf{u}^o, \theta_m^o := P_m^o\theta^o. \tag{36}$$

In preparation for establishing the local existence of the stochastic version of the coupled model, we prove the following theorem on the global existence in time of the truncated approximation to the coupled model.

Theorem 2.2 (Global well-posedness of the truncated coupled model). *For the time interval* $[0, T]$, *let* $s \geqslant 2$ *and suppose the initial conditions of (38) satisfy* $\psi_0 = (\mathbf{u}^a, \theta_0^a, \mathbf{u}_0^o, \theta_0^o) \in \mathcal{H}^s(\Omega)$. *The truncation of the coupled model (24) is given by*

$$d_t\psi + g_R(\|\psi\|_{H^s})B(\psi, \psi) + C\psi + D(\psi^a, \psi^o) = L\psi. \tag{37}$$

Then there exists a unique solution to (37) in the sense of definition 2.1. This solution depends continuously with respect to the $\mathcal{L}^2$*-norm on the initial conditions.*

Proof. We first show the local existence in time of solutions for the truncated Galerkin system. Next, we prove the global existence via $\mathcal{H}^s$-estimates. Then, we pass to the limit and prove the corresponding assertions for the truncated system (37). Finally, we show uniqueness and continuous dependency on the initial condition.

The truncated Galerkin system is given by

$$d_t\psi_m + g_R(\|\psi_m\|_{H^s})\mathbb{P}_mB(\psi_m, \psi_m) + C\psi_m + D(\psi_m^a, \psi_m^o) = L\psi_m, \tag{38}$$

where P_m was defined in (35).

Step 1: Local existence of the truncated Galerkin approximation.
The truncated Galerkin system can be written as

$$d_t\psi_m = K(\psi_m)$$
$$\text{with } K(\psi_m) := -g_R(\|\psi_m\|_{H^s})P_mB(\psi_m, \psi_m) - C\psi_m - D(\psi_m^a, \psi_m^o) + L\psi_m. \tag{39}$$

The right-hand side K of (39) is a Lipschitz continuous mapping from $\mathcal{H}^s$ into itself. It follows from the Picard Theorem that a unique solution $\psi_m \in C^1([t_0, t_1^m], \mathcal{H}^s)$ of (38) exists on time intervals $[t_0^m, t_1^m]$ that depend on m.

Step 2: Global existence of the truncated Galerkin approximation.
We show now that a solution exists globally in time.

We apply the derivative $\mathcal{D}^\alpha$ to (38) and then take L^2-scalar product of equation (38) with $\mathcal{D}^\alpha\psi_m$. This yields

$$\frac{1}{2}d_t\|\mathcal{D}^\alpha\psi_m\|_{L^2}^2 + \left\langle g_R(\|\psi_m\|_{H^1})\mathcal{D}^\alpha P_mB(\psi_m, \psi_m), \mathcal{D}^\alpha\psi_m \right\rangle_{L^2}$$
$$+ \left\langle C\mathcal{D}^\alpha\psi_m, \mathcal{D}^\alpha\psi_m \right\rangle_{L^2} + \left\langle D(\mathcal{D}^\alpha\psi_m^a, \mathcal{D}^\alpha\psi_m^o), \mathcal{D}^\alpha\psi_M \right\rangle_{L^2} - \left\langle L\mathcal{D}^\alpha\psi_m, \mathcal{D}^\alpha\psi_m \right\rangle_{L^2} = 0. \tag{40}$$

2.2. The nonlinear transport operator B

The term with the operator B can with Leibniz' rule and the inequalities of Hölder, Sobolev, Poincare and Young be estimated as follows

$$|\langle g_R(\|\psi_m\|_{H^s})\mathcal{D}^\alpha P_m(\psi_m\cdot\nabla)\psi_m, \mathcal{D}^\alpha\psi_m\rangle_{L^2}|$$

$$= g_R(\|\psi_m\|_{H^s})|\sum_{|k|\leqslant|\alpha|}\langle P_m(\mathcal{D}^{\alpha-k}\psi_m\cdot\nabla)\mathcal{D}^k\psi_m, \mathcal{D}^\alpha\psi_m\rangle_{L^2}|$$

$$\leqslant g_R(\|\psi_m\|_{H^s})\sum_{|k|\leqslant|\alpha|}\|\mathcal{D}^{\alpha-k}\psi_m\|_{L^4}\|\mathcal{D}^k\nabla\psi_m\|_{L^2}\|\mathcal{D}^\alpha\psi_m\|_{L^4}$$

$$\leqslant g_R(\|\psi_m\|_{H^s})\sum_{|k|\leqslant|\alpha|}\|\mathcal{D}^{\alpha-k}\psi_m\|_{L^2}^{1/2}\|\mathcal{D}^{\alpha-k+1}\psi_m\|_{L^2}^{1/2}\|\mathcal{D}^k\nabla\psi_m\|_{L^2}\|\mathcal{D}^\alpha\psi_m\|_{L^2}^{1/2}\|\mathcal{D}^{\alpha+1}\psi_m\|_{L^2}^{1/2}$$

$$\leqslant g_R(\|\psi_m\|_{H^s})\|\mathcal{D}^{\alpha-1}\psi_m\|_{L^2}^{1/2}\|\mathcal{D}^\alpha\psi_m\|_{L^2}^{1/2}\|\mathcal{D}^\alpha\nabla\psi_m\|_{L^2}\|\mathcal{D}^{\alpha+1}\nabla\psi_m\|_{L^2}^{3/2}$$

$$\leqslant \frac{c}{\epsilon}g_R^4(\|\psi_m\|_{H^s})\|\mathcal{D}^{\alpha-1}\psi_m\|_{L^2}^2\|\mathcal{D}^\alpha\psi_m\|_{L^2}^4 + \frac{\epsilon}{2}\|\mathcal{D}^{\alpha+1}\nabla\psi_m\|_{L^2}^2. \tag{41}$$

2.3. The linear operator C

We obtain using the inequalities of Cauchy–Schwarz and Young that

$$\int_\Omega (C\mathcal{D}^\alpha\psi_m)\cdot\mathcal{D}^\alpha\psi_m\,dx = \int_\Omega \left(C^a\mathcal{D}^\alpha\psi_m^a\right)\cdot\mathcal{D}^\alpha\psi_m^a\,dx + \int_\Omega (C^o\mathcal{D}^\alpha\psi_m^o)\cdot\mathcal{D}^\alpha\psi_m^o\,dx$$

$$= \int_\Omega \left(\frac{1}{Ro^a}\mathcal{D}^\alpha\mathbf{u}_m^{a\perp} + \nabla\mathcal{D}^\alpha\theta_m^a\right)\cdot\mathcal{D}^\alpha\mathbf{u}_m^a\,dx$$

$$+ \int_\Omega \left(\frac{1}{Ro^o}\mathcal{D}^\alpha\mathbf{u}_m^{o\perp} + \nabla\mathcal{D}^\alpha p_m^o\right)\cdot\mathcal{D}^\alpha\mathbf{u}_m^o\,dx$$

$$\leqslant \left|\int_\Omega \nabla\mathcal{D}^\alpha\theta_m^a\cdot\mathcal{D}^\alpha\mathbf{u}_m^a\,dx\right| \leqslant \frac{1}{Ro^a}\|\mathcal{D}^\alpha\theta_m^a\|_{L^2}\|\mathcal{D}^\alpha\nabla\mathbf{u}_m^a\|_{L^2}$$

$$\leqslant \frac{1}{2\epsilon_4 Ro^a}\|\mathcal{D}^\alpha\theta_m^a\|_{L^2}^2 + \frac{\epsilon_4}{2}\|\mathcal{D}^{\alpha+1}\mathbf{u}_m^a\|_{L^2}^2, \tag{42}$$

where the ocean component of the pressure term has vanished, due to the incompressibility of the ocean flow.

2.4. The coupling operator D

For the coupling term in the atmospheric temperature equation we find by using the inequalities of Cauchy–Schwarz and Young that

$$|\gamma\int_\Omega \mathcal{D}^\alpha\left(\theta_m^o - \theta_m^a\right)\cdot\mathcal{D}^\alpha\theta_m^a\,dx| \leqslant |\gamma|\left(\|\mathcal{D}^\alpha\theta_m^o\|_{L^2} + \|\mathcal{D}^\alpha\theta_m^a\|_{L^2}\right)\|\mathcal{D}^\alpha\theta_m^a\|_{L^2}$$

$$\leqslant \frac{3|\gamma|}{2}(\|\mathcal{D}^\alpha\theta_m^o\|_{L^2}^2 + \|\mathcal{D}^\alpha\theta_m^a\|_{L^2}^2). \tag{43}$$

The coupling term in the oceanic velocity equation can be estimated as follows

$$|\sigma\int_\Omega \mathcal{D}^\alpha(\mathbf{u}_m^o - \bar{\mathbf{u}}_{sol,m}^a)\cdot\mathcal{D}^\alpha\mathbf{u}_m^o\,dx| = \left|\sigma\int_\Omega \mathcal{D}^\alpha\left(\mathbf{u}_m^o - \mathbf{u}_{sol,m}^a + \frac{1}{|\Omega|}\int_\Omega \mathbf{u}_{sol,m}^a\,dx\right)\cdot\mathcal{D}^\alpha\mathbf{u}_m^o\,dx\right|$$

$$\leqslant |\sigma|\left(\|\mathcal{D}^\alpha\mathbf{u}_m^o\|_{\mathbf{L}^2}^2 + \|\mathcal{D}^\alpha\mathbf{u}_{sol,m}^o\|_{\mathbf{L}^2}\|\mathbf{u}_{sol,m}^a\|_{\mathbf{L}^2} + \int_\Omega \frac{1}{|\Omega|}\int_\Omega |\mathcal{D}^\alpha\mathbf{u}_{sol,m}^a(x)|\,dx\right)|\mathcal{D}^\alpha\mathbf{u}_m^o(y)|\,dy$$

$$\leqslant |\sigma|\left(\|\mathcal{D}^\alpha\mathbf{u}_m^o\|_{\mathbf{L}^2}^2 + \|\mathcal{D}^\alpha\mathbf{u}_m^o\|_{\mathbf{L}^2}^2 + \|\mathbf{u}_{sol,m}^a\|_{\mathbf{L}^2}^2 + \int_\Omega \frac{1}{|\Omega|}\|\mathcal{D}^\alpha\mathbf{u}_{sol,m}^a\|_{L^2}|\Omega||\mathcal{D}^\alpha\mathbf{u}_m^o|\,dx\right)$$

$$\leqslant C|\sigma|(\|\mathcal{D}^\alpha\mathbf{u}_m^o\|_{L^2}^2 + \|\mathcal{D}^\alpha\mathbf{u}_m^a\|_{L^2}^2). \tag{44}$$

This estimate implies for the coupling operator

$$\int_{\Omega} \big(D(\mathcal{D}^{\alpha}\psi_m^a, \mathcal{D}^{\alpha}\psi_m^o) \big) \cdot \mathcal{D}^{\alpha}\psi \, dx \leqslant C(|\gamma| + |\sigma|)(\|\mathcal{D}^{\alpha}\psi_m^o\|_{L^2}^2 + \|\mathcal{D}^{\alpha}\psi_m^a\|_{L^2}^2). \tag{45}$$

From (41)–(45) follows after summing over $|\alpha|$ up to s for (40)

$$\frac{1}{2} d_t \|\psi_m\|_{\mathcal{H}^s}^2 + \frac{1}{P} \|\nabla \psi_m\|_{\mathcal{H}^s}^2 \leqslant C g_R^4(\|\psi_m\|_{\mathcal{H}^s}) \|\psi_m\|_{\mathcal{H}^s}^6 + C(|\gamma| + |\sigma|)(\|\psi_m^o\|_{\mathcal{H}^s}^2 + \|\psi_m^a\|_{\mathcal{H}^s}^2). \tag{46}$$

where $\frac{1}{P} := \min\{\frac{1}{Re^a}, \frac{1}{Re^o}, \frac{1}{Pe^a}, \frac{1}{Pe^o}\}$. Upon using the truncation g_R it follows

$$\frac{1}{2} d_t \|\psi_m\|_{\mathcal{H}^s}^2 + \frac{1}{P} \|\nabla \psi_m\|_{\mathcal{H}^s}^2 \leqslant CR \|\psi_m\|_{\mathcal{H}^s}^2. \tag{47}$$

With Gronwall's inequality it follows that

$$\|\psi_m(t)\|_{\mathcal{H}^s}^2 \leqslant \|\psi_m(t_0)\|_{\mathcal{H}^s}^2 e^{CR(t-t_0)} \leqslant \|\psi(t_0)\|_{\mathcal{H}^s}^2 e^{CR(t-t_0)}, \tag{48}$$

where $\psi(t_0)$ denotes the initial condition of the coupled equations (1)–(5). This estimate implies in particular that $\|\psi_m(t)\|_{\mathcal{H}^s}^2$ is bounded uniformly in m. Integrating (47) over the time interval $[t_0, t]$ yields with (48)

$$\frac{1}{R} \int_{t_0}^{t} \|\nabla \psi_m(s)\|_{\mathcal{H}^s}^2 ds \leqslant CR \int_{t_0}^{t} \|\psi_m(s)\|_{\mathbf{H}^s}^2 ds + \|\psi_m(t_0)\|_{\mathcal{H}^s}^2$$
$$\leqslant CR \|\psi(t_0)\|_{\mathcal{H}^s}^2 e^{2CR(t-t_0)} + \|\psi(t_0)\|_{\mathcal{H}^s}^2. \tag{49}$$

From (48) and (49) it follows that $(\psi_m)_m$ is uniformly bounded in $L^\infty(T, \mathcal{H}^s) \cap L^2(T, \mathcal{H}^{s+1})$ with time derivative $(\frac{d\psi_m}{dt})_m$ which is according to (47) uniformly bounded in $L^2(T, \mathcal{H}^s)$.

The oceanic pressure p can be recovered analogously to the Navier–Stokes equations by solving the elliptic equation

$$\Delta p_m = div\big((\mathbf{u}_m^o \cdot \nabla)\mathbf{u}_m^o + \nabla q_m^a\big), \tag{50}$$

where q_m^a is the gradient part of the Leray–Helmholtz decomposition of the atmospheric velocity $\mathbf{u}_m^a$.

Step 3: Passage to the limit.
The uniform boundedness of $(\psi_m)_m$ in $L^2(T, \mathcal{H}^{s+1})$ implies with the compact embedding of $L^2(T, \mathcal{H}^{s+1})$ into $L^2(T, \mathcal{H}^s)$ that a subsequence $(\psi_k)_k$ exists that converges strongly to $\psi \in L^2(T, \mathcal{H}^s)$. This subsequence converges also weakly in $L^\infty(T, \mathcal{H}^s)$. We show now that the limit ψ satisfies the truncated equation (37). For the coupling term it holds for all $\phi \in [H^2(\Omega)]^6$

$$\lim_{k \to \infty} \int_T \big\langle D(\psi_k^a, \psi_k^o) - D(\psi^a, \psi^o), \phi \big\rangle_{L^2} dt$$
$$= \lim_{k \to \infty} \int_T \big\langle \gamma\big((\theta_k^a - \theta^a) + (\theta^o - \theta_k^o)\big) + \sigma\big((\mathbf{u}_k^o - \mathbf{u}^o) + (\bar{\mathbf{u}}_{sol}^a - \bar{\mathbf{u}}_{sol,k}^a)\big), \phi \big\rangle_{L^2} dt \tag{51}$$

For the velocity coupling involving the solenoidal part of the atmospheric velocity field in the ocean component it follows by using the Cauchy–Schwarz inequality that

$$\Big| \int_T \big\langle \sigma\big(\bar{\mathbf{u}}_{sol}^a - \bar{\mathbf{u}}_{sol,k}^a\big), \phi \big\rangle_{L^2} dt \Big| = |\sigma \int_T \int_\Omega \big(\bar{\mathbf{u}}_{sol}^a(x,t) - \bar{\mathbf{u}}_{sol,k}^a(x,t)\big) \cdot \phi(x) \, dx dt|$$
$$= |\sigma \int_T \int_\Omega \big(\mathbf{u}_{sol}^a(x,t) - \mathbf{u}_{sol,k}^a(x,t)$$
$$+ \frac{1}{|\Omega|} \int_\Omega \mathbf{u}_{sol,k}^a(z,t) - \mathbf{u}_{sol}^a(z,t) \, dz\big) \cdot \phi(x) \, dx dt|$$

$$
\leqslant |\sigma| \int_T \int_\Omega |\mathbf{u}_{sol}^a(x,t) - \mathbf{u}_{sol,k}^a(x,t)| \, |\phi(x)| \, dx dt
$$

$$
+ |\sigma| \frac{1}{|\Omega|} \int_\Omega \left(\int_T \int_\Omega |\mathbf{u}_{sol,k}^a(z,t) - \mathbf{u}_{sol}^a(z,t)| \, dz dt \right) |\phi(x)| \, dx
$$

$$
\leqslant |\sigma| \int_T \|\mathbf{u}_{sol}^a(t) - \mathbf{u}_{sol,k}^a(t)\|_{L^2} \|\phi\|_{L^2} dt
$$

$$
+ |\sigma| \int_\Omega \left(\int_T \|\mathbf{u}_{sol,k}^a(t) - \mathbf{u}_{sol}^a(t)\|_{L^2} dt \right) |\phi(x)| \, dx. \tag{52}
$$

From (52) and the convergence of $(\psi_k)_k$ in $\psi \in L^2(T, \mathcal{H}^s)$, there follows the convergence of the integral in (51). The convergence of the remaining linear terms in the equations is obvious. Next, we focus on the nonlinear terms for which we have to show that

$$
\lim_{k \to \infty} \int_T \left\langle g_R(\|\psi_k\|_{H^s}) P_m B(\psi_k, \psi_k) - g_R(\|\psi\|_{H^s}) P_m B(\psi, \psi), \phi \right\rangle_{L^2} dt = 0, \quad \text{for all } \phi \in [C^\infty(\Omega)]^6. \tag{53}
$$

The integral above can be written as

$$
\int_T \left\langle g_R(\|\psi_k\|_{\mathcal{H}^s}) P_m B(\psi_k, \psi_k) - g_R(\|\psi\|_{\mathcal{H}^s}) P_m B(\psi, \psi), \phi \right\rangle_{L^2} dt
$$

$$
= \int_T \left(g_R(\|\psi_k\|_{H^s}) - g_R(\|\psi\|_{\mathcal{H}^s}) \right) \left\langle P_m B(\psi_k, \psi_k), \phi \right\rangle_{L^2} dt
$$

$$
+ \int_T g_R(\|\psi\|_{H^s}) \left\langle P_m B(\psi_k, \psi_k) - P_m B(\psi, \psi), \phi \right\rangle_{L^2} dt \tag{54}
$$

For the first integral on the right-hand side it follows with the Hölder inequality

$$
\int_T \left(g_R(\|\psi_k\|_{\mathcal{H}^s}) - g_R(\|\psi\|_{\mathcal{H}^s}) \right) \left\langle P_m B(\psi_k, \psi_k), \phi \right\rangle_{L^2} dt
$$

$$
\leqslant \int_T \left(g_R(\|\psi_k\|_{\mathcal{H}^s}) - g_R(\|\psi\|_{\mathcal{H}^s}) \right) \|\psi_k\|_{L^3} \|\nabla \psi_k\|_{L^2} \|\phi\|_{L^6} dt
$$

$$
\leqslant c \int_T \left(g_R(\|\psi_k\|_{\mathcal{H}^s}) - g_R(\|\psi\|_{\mathcal{H}^s}) \right) \|\psi_k\|_{H^1}^2 \|\phi\|_{H^1} dt
$$

$$
\leqslant c \sup_{t \in T} \left(g_R(\|\psi_k(t)\|_{\mathcal{H}^s}) - g_R(\|\psi(t)\|_{\mathcal{H}^s}) \right) \int_T \|\psi_k\|_{H^1}^2 \|\phi\|_{H^1} dt. \tag{55}
$$

The sequence $(\psi_k)_k$ converges weakly to ψ in $L^\infty(T, \mathcal{H}^s)$, i.e. $\|\psi_k\|_{H^s}$ converges to $\|\psi\|_{\mathcal{H}^s}$ and with the continuity of the truncation function g_R follows that the first term on the right-hand side converges to zero. Since $(\psi_k)_k$ is bounded in $L^\infty(T, \mathcal{H}^s) \cap L^2(T, \mathcal{H}^{s+1})$ the right-hand side of (55) converges for $k \to \infty$ to zero.

For the second integral in (54) it follows with Hölder's inequality that

$$
\int_T g_R(\|\psi\|_{\mathcal{H}^s}) \left\langle P_m B(\psi_k, \psi_k) - P_m B(\psi, \psi), \phi \right\rangle_{L^2} dt
$$

$$
= \int_T g_R(\|\psi\|_{\mathcal{H}^s}) \left\langle P_m B(\psi_k - \psi, \psi_k) + P_m B(\psi, \psi_k - \psi), \phi \right\rangle_{L^2} dt
$$

$$
\leqslant \int_T g_R(\|\psi\|_{\mathcal{H}^s}) \|\psi_k - \psi\|_{L^4} \|\nabla \psi_k\|_{L^2} \|\phi\|_{L^4} dt + \int_T g_R(\|\psi\|_{\mathcal{H}^s}) \|\psi\|_{L^4} \|\nabla(\psi_k - \psi)\|_{L^2} \|\phi\|_{L^4} dt
$$

$$
\leqslant \int_T g_R(\|\psi\|_{\mathcal{H}^s}) \|\psi_k - \psi\|_{H^1} \|\nabla \psi_k\|_{L^2} \|\phi\|_{H^1} dt + \int_T g_R(\|\psi\|_{\mathcal{H}^s}) \|\psi\|_{H^1} \|\psi_k - \psi\|_{H^1} \|\phi\|_{H^1} dt, \tag{56}
$$

where the right-hand side tends to zero for $k \to \infty$ as a consequence of the boundedness of the sequence $(\psi_k)_k$ in $L^\infty(T, \mathcal{H}^s) \cap L^2(T, \mathcal{H}^{s+1})$ and its converges in $L^2(T, \mathcal{H}^s)$.

Step 4: Uniqueness of solutions of the truncated system

Let ψ_1, ψ_2 be two solutions of (37) with respective initial conditions $\psi_1(t_0 = 0)$ and $\psi_2(t_0 = 0)$. We assume that $\|\psi_1(t)\|_{\mathcal{H}^s}, \|\psi_2(t)\|_{\mathcal{H}^s} \leqslant R$ for $t \in T$. This implies for the difference by $\widehat{\psi} := \psi_1 - \psi_2$ that $\|\psi(t)\|_{\mathcal{H}^s} \leqslant R$.

The difference $\widehat{\psi}$ satisfies the following equation

$$d_t\widehat{\psi} + g_R(\|\psi_1\|_{\mathcal{H}^s})\big(B(\psi_1, \psi_1) - B(\psi_2, \psi_2)\big)$$
$$+ B(\psi_2, \psi_2)\big(g_R(\|\psi_1\|_{\mathcal{H}^s}) - g_R(\|\psi_2\|_{\mathcal{H}^s})\big) + C\widehat{\psi} + D(\widehat{\psi}^a, \widehat{\psi}^o) = L\widehat{\psi}. \quad (57)$$

Taking the L^2-inner product with $\widehat{\psi}$ yields

$$d_t\|\widehat{\psi}\|_{\mathcal{L}^2}^2 + \int_\Omega g_R(\|\psi_1\|_{\mathcal{H}^s})\big(B(\psi_1, \psi_1) - B(\psi_2, \psi_2)\big) \cdot \widehat{\psi} dx$$
$$+ \int_\Omega B(\psi_2, \psi_2) \cdot \widehat{\psi}\big(g_R(\|\psi_1\|_{\mathcal{H}^s}) - g_R(\|\psi_2\|_{\mathcal{H}^s})\big) dx$$
$$+ \int_\Omega \big(C\widehat{\psi} + D(\widehat{\psi}^a, \widehat{\psi}^o)\big) \cdot \widehat{\psi} dx = \int_\Omega (L\widehat{\psi}) \cdot \widehat{\psi} dx. \quad (58)$$

For the difference of the two nonlinear terms we have

$$\int_\Omega g_R(\|\psi_1\|_{\mathcal{H}^s})\big(B(\psi_1, \psi_1) - B(\psi_2, \psi_2)\big) \cdot \widehat{\psi} dx = \int_\Omega g_R(\|\psi_1\|_{H^s})\big(B(\widehat{\psi}, \psi_1) + B(\psi_2, \widehat{\psi})\big) \cdot \widehat{\psi} dx \quad (59)$$

In order to estimate the right-hand side consider the atmospheric velocity component of (59), with the inequalities of Hölder, Agmon and Young follows

$$\int_\Omega g_R(\|\mathbf{u}_1^a\|_{\mathbf{H}^s})\big((\widehat{\mathbf{u}}^a \cdot \nabla)\mathbf{u}_1^a) + (\mathbf{u}_2^a \cdot \nabla)\widehat{\mathbf{u}}^a)\big) \cdot \widehat{\mathbf{u}}^a \, dx$$
$$\leqslant c g_R(\|\mathbf{u}_1^a\|_{\mathbf{H}^s})\|\nabla\mathbf{u}_1^a\|_{\mathbf{L}^6}\|\widehat{\mathbf{u}}^a\|_{\mathbf{L}^3}\|\widehat{\mathbf{u}}^a\|_{\mathbf{L}^2} + \|\mathbf{u}_2^a\|_{\mathbf{L}^\infty}\|\nabla\widehat{\mathbf{u}}^a\|_{\mathbf{L}^2}\|\widehat{\mathbf{u}}^a\|_{\mathbf{L}^2}$$
$$\leqslant c g_R(\|\mathbf{u}_1^a\|_{\mathbf{H}^s})\|\mathbf{u}_1^a\|_{\mathbf{H}^2}\|\widehat{\mathbf{u}}^a\|_{\mathbf{H}^1}\|\widehat{\mathbf{u}}^a\|_{\mathbf{L}^2} + c\|\mathbf{u}_2^a\|_{\mathbf{H}^2}\|\widehat{\mathbf{u}}^a\|_{\mathbf{H}^1}\|\widehat{\mathbf{u}}^a\|_{\mathbf{L}^2}$$
$$\leqslant \frac{c g_R(\|\mathbf{u}_1^a\|_{\mathbf{H}^s})}{2\epsilon_1}\|\mathbf{u}_1^a\|_{\mathbf{H}^2}^2\|\widehat{\mathbf{u}}^a\|_{\mathbf{L}^2}^2 + \frac{c}{\epsilon_2}\|\mathbf{u}_2^a\|_{\mathbf{H}^2}^2\|\widehat{\mathbf{u}}^a\|_{\mathbf{L}^2}^2 + \left(\frac{\epsilon_1}{2} + \frac{\epsilon_2}{2}\right)\|\widehat{\mathbf{u}}^a\|_{\mathbf{H}^1}^2$$
$$\leqslant M_0\|\widehat{\mathbf{u}}^a\|_{\mathbf{L}^2}^2 + \left(\frac{\epsilon_1}{2} + \frac{\epsilon_2}{2}\right)\|\widehat{\mathbf{u}}^a\|_{\mathbf{H}^1}^2, \quad (60)$$

where $M_0 = M_0(\|\mathbf{u}_1^a\|_{\mathbf{H}^2}^2, \|\mathbf{u}_2^a\|_{\mathbf{H}^2}^2)$. Similarly we derive for the oceanic velocity component

$$\int_\Omega \big((\widehat{\mathbf{u}}^o \cdot \nabla)\mathbf{u}_1^o) + (\mathbf{u}_2^o \cdot \nabla)\widehat{\mathbf{u}}^o)\big) \cdot \widehat{\mathbf{u}}^o \, dx \leqslant M_1\|\widehat{\mathbf{u}}^o\|_{\mathcal{L}^2}^2, \quad (61)$$

where $M_1 = M_1(\|\mathbf{u}_1^o\|_{H^2}^2)$, because the dependency on $\|\mathbf{u}_2^o\|_{H^s}$ vanishes due to the incompressibility. Analogous estimates hold for the atmospheric and oceanic temperature transport terms, such that the nonlinear operator difference in (59) can be bounded by

$$\int_\Omega g_R(\|\psi_1\|_{\mathcal{H}^s})\big(B(\psi_1, \psi_1) - B(\psi_2, \psi_2)\big) \cdot \widehat{\psi} dx \leqslant K_0\|\widehat{\psi}\|_{\mathcal{L}^2}^2, \quad (62)$$

where $K_0 = K_0(\|\psi_1\|^2_{\mathcal{H}^s}, (\|\psi_2\|^2_{\mathcal{H}^s})$. The second term in (58) can be estimated analogously as above

$$\int_\Omega B(\psi_2, \psi_2) \cdot \widehat{\psi}\big(g_R(\|\psi_1\|_{H^s}) - g_R(\|\psi_2\|_{H^s})\big) dx$$
$$\leqslant c\|\psi_2\|_{\mathcal{H}^2}\|\psi_2\|_{\mathcal{H}^1}\|\widehat{\psi}\|_{\mathcal{L}^2}\big|\|\psi_1\|_{\mathcal{H}^s} - \|\psi_2\|_{\mathcal{H}^s}\big| \leqslant K_1\|\widehat{\psi}\|^2_{\mathcal{L}^2}, \tag{63}$$

where $K_1 = K_1(\|\psi_2\|_{\mathcal{H}^2})$ and where we have used the reverse triangle inequality in the last step.

For the (linear) coupling operator it holds that

$$|\int_\Omega D(\widehat{\psi^a}, \widehat{\psi^o}) \cdot \widehat{\psi}\, dx| = |\int_\Omega \gamma(\hat{\theta}^a - \hat{\theta}^o)^2 + \sigma(\widehat{\mathbf{u}}^o - \overline{\widehat{\mathbf{u}}^a})^2\, dx| \leqslant K_3\|\widehat{\psi}\|^2_{\mathcal{L}^2} \tag{64}$$

where we have used that $\bar{\mathbf{u}}^a_1 - \bar{\mathbf{u}}^a_2 = \overline{\mathbf{u}^a_1 - \mathbf{u}^a_2} = \overline{\widehat{\mathbf{u}}^a}$ and where K_3 depends on the coupling constants. This implies the following estimate for the difference equation (58)

$$\frac{1}{2}d_t\|\widehat{\psi}\|^2_{\mathcal{L}^2} + \frac{1}{R}\|\nabla\widehat{\psi}\|^2_{\mathcal{L}^2} \leqslant K\|\widehat{\psi}\|^2_{\mathcal{L}^2}, \tag{65}$$

where $K = K(\|\psi_1\|_{\mathcal{H}^2}, \|\psi_2\|_{\mathcal{H}^2}), \sigma, \gamma)$. From Gronwall's inequality we obtain

$$\|\widehat{\psi}(t)\|^2_{\mathcal{L}^2} \leqslant \|\widehat{\psi}(t_0)\|^2_{\mathcal{L}^2} e^{\int_{t_0}^t K(s)ds}. \tag{66}$$

Since $\psi_1\psi_2 \in L^2(T, \mathcal{H}^2(\Omega))$ the function K is integrable and the right-hand side is bounded. This proves the continuous dependency on the initial condition. If the two solutions have the same initial conditions, then the solutions coincide on T and uniqueness follows. $\qquad\square$

The following theorem is the main result for the deterministic version of the coupled model.

Theorem 2.3 (Local well-posedness of the coupled model). *Let $s \geqslant 2$ and suppose the initial condition of the coupled equations* (1)–(5) *satisfy* $\psi_0 = (\mathbf{u}^a_0, \theta^a_0, \mathbf{u}^o_0, \theta^o_0) \in \mathcal{H}^s(\Omega)$. *Then there exists a unique time* $t^*_1 \in (t_0, \infty]$ *such that a local regular solution ψ of* (1)–(5) *in the sense of definition* 2.1 *exists and is unique on any interval* $T := [t_0, t_1]$, *where* $t_0 < t_1 < t^*_1$ *and that, if* $t^*_1 < \infty$, *then*

$$\lim_{t \nearrow t^*_1} \|\psi(t)\|_{\mathcal{H}^s} = \infty. \tag{67}$$

Proof. We define $t_R := \inf\{t \geqslant t_0 : \|\psi(t)\|_{H^s} > R\}$, for $R > 0$ and $\tau := \lim_{R\to\infty} t_R$. By ψ_R we denote the solution of (37) with initial condition ψ_0. We define $\psi(t) := \psi_R(t)$ for $t \in [t_0, t_R]$. On any time interval $[t_0, t_1]$ with $t_0 < t_1 < t_R$ the solutions ψ and ψ_R coincide, as a consequence of the uniqueness of solutions of the truncated equation (37). If $\tau = \infty$ then ψ is a global solution of (24). If $\tau < \infty$ then $\|\psi(\tau)\|_{H^s} = R$ and $[t_0, \tau]$ is the maximal interval of existence of the solution ψ. $\qquad\square$

3. The stochastic idealised atmospheric climate model

3.1. SALT atmospheric climate model

Recall that the state of the system is described by a state vector $\psi := (\psi^a, \psi^o)$ with atmospheric component $\psi^a := (\mathbf{u}^a, \theta^a)$ and oceanic component $\psi^o := (\mathbf{u}^o, \theta^o)$. The initial state is denoted by $\psi(t_0) = \psi_0$. where $\psi_0 = (\mathbf{u}^a_0, \theta^a_0, \mathbf{u}^o_0, \theta^o_0)$, with six entries.

We summarise equations (9)–(13) for the SALT version of the idealised climate model as

$$\mathrm{d}\psi_t + (B(\psi_t, \psi_t) + C(\psi_t) + D\mathbb{E}[\bar{\psi}])dt + \sum_{i=1}^{\infty} E_i(\psi_t) \circ dW_t^i = \nu\Delta\psi_t dt, \quad (68)$$

where

- The process ψ gathers all variables in (9)–(13), i.e.

$$\psi_t = \{\psi_t^i\}_{i=1}^6 := (\mathbf{u}_t^a, \theta_t^a, \mathbf{u}_t^o, \theta_t^o) = (u_t^{a,1}, u_t^{a,2}, \theta_t^a, u_t^{o,1}, u_t^{o,2}, \theta_t^o),$$

  as in the deterministic case.
- B is the usual bilinear transport operator,
- C comprises all the linear terms (including the pressure term in the Equation for the components of ψ corresponding to $\bar{\mathbf{u}}^o$ as well as the term $-\gamma(\theta^a - \theta^o)$ from (10)).
- $D = \{D_{ij}\}_{i,j=1}^6$ is a 6×6-matrix that captures the influence of $\mathbb{E}[\bar{\psi}]$ on the various components of ψ_t. More precisely, D_{ij} is the coefficient appearing in front of $\mathbb{E}[\bar{\psi}^j]$ in the equation satisfied by ψ_t^i. For the SALT equations (9)–(13), we have $D_{41} = D_{52} = -\sigma$, with all the other entries equal to 0. Therefore the pair $(u_t^{o,1}, u_t^{o,2})$ is affected by $(\mathbb{E}[\bar{u}_t^{a,1}], \mathbb{E}[\bar{u}_t^{a,2}])$.
- E_i are diagonal operators given by

$$E_i(\mathbf{u}^a, \theta^a, \mathbf{u}^o, \theta^o) = \mathrm{diag}\left(\xi_i \cdot \nabla\mathbf{u}^a + \frac{1}{Ro^a}\xi_i + u_j^a\nabla\xi_i^j + \frac{1}{Ro^a}\nabla(R_j(\mathbf{x})\xi_i^j), \xi_i \cdot \nabla\theta^a, 0, 0\right)$$

- $\mathrm{curl}R(\mathbf{x}) = 2\Omega(\mathbf{x})$.
- $\bar{\psi} := \psi - \frac{1}{|\Omega|}\int_\Omega \psi\, dx.$[6]

We start by giving a rigorous definition of the solution of (68). Let $(\Xi, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P}, (W^i)_i)$ be a fixed stochastic basis. In addition to the Sobolev spaces defined above we introduce $C^m(\Omega; \mathbb{R}^p)$ to be the (vector) space of all $\mathbb{R}^p$-valued functions f which are continuous on Ω with continuous partial derivatives $D^\alpha f$ of orders $|\alpha| \leqslant m$, for fixed $m \geqslant 0$. Notice that on the torus all continuous functions are bounded. The space $C^m(\Omega; \mathbb{R}^p)$ is a Banach space when endowed with the usual supremum norm

$$\|f\|_{m,\infty} = \sum_{|\alpha|\leqslant m} \|D^\alpha f\|_\infty.$$

The space $C^\infty(\Omega; \mathbb{R}^p)$ is regarded as the intersection of all spaces $C^m(\Omega; \mathbb{R}^p)$. Let $(\xi_i)_i$ a sequence of vector fields which satisfy the following condition:

$$\sum_{i=1}^{\infty} \|\xi_i\|_{s+3,\infty}^2 < \infty. \quad (69)$$

We will work directly with the Itô version of (68). In this version, the Stratonovich integrals in (68) are recast as Itô integrals with the required Itô correction added in the drift term of the equation. The equivalence between the two versions is straightforward, see e.g. [11] for details.

---

[6] Recall that subtraction of the mean $\frac{1}{|\Omega|}\int_\Omega \psi\, dx$ places the oceanic and atmospheric variables all into the same frame of motion relative to the Earth's rotation. Note that this subtraction is only applied to $\mathbf{u}^o$. This is ensured by the multiplication by the matrix D.

Definition 3.1.

a. A pathwise *local* solution of the system (68) is given by a pair (ψ, τ), where $\tau : \Xi \to [t_0, \infty]$ is a strictly positive stopping time and $\psi : \Omega \times [t_0, \infty] \to \mathcal{H}^s(\Omega)$, is an $\mathcal{F}_t$-adapted process with initial condition $\psi_{t_0} \in \mathcal{H}^s(\Omega)$ such that

$$\psi \in L^2\left(\Xi; C\left([t_0, T]; \mathcal{H}^s(\Omega)\right)\right) \quad \psi 1_{[t_0, \tau]} \in L^2\left(\Xi; L^2\left([t_0, T]; \mathcal{H}^{s+1}(\Omega)\right)\right)$$

for any $T \geqslant t_0$ and the system (68) is satisfied locally i.e. the following identity

$$\psi_t = \psi_{t_0} - \int_{t_0}^{t \wedge \tau} \left(F(\psi_s) + D\mathbb{E}\left[\bar{\psi}_s\right]\right) ds - \sum_{i=1}^{\infty} \int_{t_0}^{t \wedge \tau} E_i(\psi_s) dW_s^i, \tag{70}$$

holds $\mathbb{P}$-almost surely, as an identity in $L^2\left(\Xi; \mathcal{H}^0(\Omega)\right)$ for any $t \in [t_0, \infty)$. In (70), the mapping $F(\psi_s)$ is defined as

$$F(\psi_s) = B(\psi_s, \psi_s) + C(\psi_s) - \frac{1}{2} \sum_{i=1}^{\infty} E_i^2(\psi_s) - \nu \Delta \psi_s. \tag{71}$$

b. A *martingale local* solution of equation (68) is a triple $(\check{\Omega}, \check{\mathcal{F}}, \check{\mathbb{P}}), (\check{\mathcal{F}}_t)_t, (\check{\psi}, \check{\tau}, (\check{W}^i)_i)$ such that $(\check{\Omega}, \check{\mathcal{F}}, \check{\mathbb{P}})$ is a probability space, $(\check{\mathcal{F}}_t)_t$ is a filtration defined on this space, where $\check{\tau} : \Xi \to [t_0, \infty]$ is a strictly positive $\check{\mathcal{F}}_t$-stopping time and $\psi : \Omega \times [t_0, \infty] \to \mathcal{H}^s(\Omega)$, is an $\check{\mathcal{F}}_t$-adapted process with initial condition $\psi_{t_0} \in \mathcal{H}^s(\Omega)$ such that

$$\check{\psi} \in L^2\left(\Xi; C\left([t_0, T]; \mathcal{H}^s(\Omega)\right)\right) \quad \check{\psi} 1_{[t_0, \check{\tau}]} \in L^2\left(\Xi; L^2\left([t_0, T]; \mathcal{H}^{s+1}(\Omega)\right)\right)$$

for any $T \geqslant t_0$ and which satisfies equations (70)+(71) with ψ replaced by $\check{\psi}$.[7]

c. If $\tau = \infty$, then we say that the system (68) has a *global* solution. In this case can remove the usage of the stopping time from equation (70). In other words, we have that

$$\psi_t = \psi_{t_0} - \int_{t_0}^{t} \left(F(\psi_s) + D\mathbb{E}[\bar{\psi}_s]\right) ds - \sum_{i=1}^{\infty} \int_{t_0}^{t} E_i(\psi_s) dW_s^i, \tag{72}$$

holds as an identity in $L^2\left(\Xi; \mathcal{H}^0(\Omega)\right)$.

Remark. Observe that $\psi_t = \psi_\tau$ for any $t \geqslant \tau$, it other words the solution remains constant once it hits the defining stopping time. We require this to be able to make sense of the quantity $\mathbb{E}[\bar{\psi}_s]$ even for temporal values s larger than τ. In fact, equation (70) can be re-written as

$$\psi_t = \psi_{t_0} - \int_{t_0}^{t \wedge \tau} \left(F(\psi_s) + D\mathbb{E}\left[\bar{\psi}_{s \wedge \tau}\right]\right) ds - \sum_{i=1}^{\infty} \int_{t_0}^{t \wedge \tau} E_i(\psi_s) dW_s^i, \tag{73}$$

Roadmap of the section

In the following we show that equation (68) has a martingale solution provided the additional condition (87) is satisfied. To do so, we follow the same route as in the deterministic case. In

---

[7] We use the 'check' notation (ˇ) in the description of the various components of a martingale solution, to emphasize that the existence of a martingale solution does not guarantee that, for a *given* set of Brownian motions $(W^i)_i$ defined on the (possibly different) probability space $(\Omega, \mathcal{F}, \mathbb{P})$ a solution of (68) will exist. Clearly the existence of a strong solution implies the existence of a martingale solution.

theorem 3.2 we show that the Galerkin approximations of a truncated version of equation (70) are well defined globally. Moreover we show that we can control the Sobolev norms of these approximations uniformly in the level of approximation, see (76) and (77). The truncation is done by multiplying each the coefficients of (70) with the function $g_{R,\delta}(||\cdot||_{\mathcal{H}^s(\Omega)})$, where, as in the deterministic case, we use the cut-off function $g_{R,\delta} : \mathbb{R}_+ \to [0,1]$

$$
g_{R,\delta}(x) := \left\{
\begin{array}{ll}
1, & \text{if } 0 \leqslant x \leqslant R \\
0, & \text{if } x \geqslant R + \delta \\
\text{smoothly decaying} & \text{if } R < x < R + \delta
\end{array}
\right. .
$$

for arbitrary $R \geqslant 0$ and $\delta \in [0,1]$. We then show that the laws of the Galerkin approximations are relatively compact. Using this we deduce that equation (78) which is the truncated version of equation (70) has a global solution, see theorem 3.3. We also show that (78) has a unique solution. In the deterministic case, the existence of a solution of the truncated equation would immediately imply the existence of a local solution of the original equation on the interval $[t_0, \tau_R]$, where τ_R is the first time when $||\psi_s^{R,\delta}||_{\mathcal{H}^s(\Omega)}$ reaches the value R. We cannot do this here as the solution of the truncated equation (78) as well as that of the original equation (70) depend on temporal values s larger than τ_R through the quantity $\mathbb{E}[\bar{\psi}_s^{R,\delta}]$, respectively, $\mathbb{E}[\bar{\psi}_s]$. A final convergence argument is required: We consider a sequence ψ^{R,δ_n} with δ_n tending to 0. Then the laws of the elements of this sequence are relatively compact and we deduce from here that any limit point of the sequence that satisfies the additional property (87) is a martingale solution of the original equation (68).

With regards to the uniqueness of the solutions of (68): If (ψ_1, τ_1), and (ψ_2, τ_2), are local solutions that are defined with different stopping times, then we cannot deduce that $\psi_1 = \psi_2$ on the common interval of existence $[t_0, \tau_1 \wedge \tau_2]$. The reason for this is that, in contrast with the deterministic case, the choice of the stopping time influences ψ_s even for values $s \leqslant \tau$ as a result of the expectation term in (68). This lack of consistency between local solutions deters us to construct a pathwise local solution of equation (68) and also a corresponding maximal solution for (68).

Remark. The assumption $\psi \in L^2(\Xi; C(([t_0, T]; \mathcal{H}^s(\Omega))))$ for any $T \geqslant 0$ insures that the term

$$
\int_{t_0}^{t \wedge \tau} \mathbb{E}[\bar{\psi}_s] \, ds
$$

is well defined as an element of $\mathcal{H}^0(\Omega)$. Observe that, since $\bar{\psi} := \psi - \frac{1}{|\Omega|} \int_\Omega \psi \, dx$, we have that, for $t \in [t_0, T]$,

$$
\left| \int_{t_0}^{t \wedge \tau} \mathbb{E}[\bar{\psi}_s] \, ds \right| \leqslant (t - t_0) \mathbb{E}\left[\sup_{s \in [t_0, t \wedge \tau]} ||\bar{\psi}_s||_{\mathcal{H}^0(\Omega)} \right]
$$

$$
\leqslant 2(T - t_0) \mathbb{E}\left[\sup_{s \in [t_0, T]} ||\psi_s||_{\mathcal{H}^s(\Omega)} \right] < \infty.
$$

Moreover, for $0 < |\alpha| \leqslant s$, we have that

$$
\mathcal{D}^\alpha \mathbb{E}[\bar{\psi}_s] = \mathbb{E}[\mathcal{D}^\alpha \psi_s]
$$

$$
||\mathbb{E}[\mathcal{D}_s^\alpha \psi]||_{\mathcal{H}^0(\Omega)}^2 \leqslant \mathbb{E}\left[||\mathcal{D}_s^\alpha \psi||_{\mathcal{H}^0(\Omega)}^2 \right] \leqslant \mathbb{E}\left[||\psi_s||_{\mathcal{H}^s(\Omega)}^2 \right]
$$

as the centralizing term vanishes when differentiated.

The proof of the existence and uniqueness of a local solution for the system (68) shares many of the steps with that the existence and uniqueness of the corresponding deterministic case. It uses the same truncated procedure as in the deterministic case and the same Galerkin approximations. However several technical difficulties need to be overcome. In the arguments below we will emphasize these difficulties and the methodology used to resolve them and omit the arguments that coincide with the deterministic case.

We begin by introducing the Galerkin approximation to a truncated version of the system (68). More precisely, let $\psi^{m,R,\delta} = \left\{ \psi_t^{m,R,\delta}, t \geqslant 0 \right\}$ be the solution of the following stochastic differential system

$$\psi_t^{m,R,\delta} = P_m[\psi_{t_0}] - \int_{t_0}^t F^{m,R,\delta}(\psi_s^{m,R,\delta})ds - \sum_{i=1}^{\infty} \int_{t_0}^t g_{R,\delta}(\|\psi_s^{m,R,\delta}\|_{\mathcal{H}^s(\Omega)})P_m E_i(\psi_s^{m,R,\delta})dW_s^i.$$

(74)

Just as in (70), the identity in (74) is assumed to hold, $\mathbb{P}$-almost surely, in $L^2\left(\Xi;\mathcal{H}^0(\Omega)\right)$. In (74), the mapping $F^{m,R,\delta}(\psi_s^{m,R,\delta})$ is defined as

$$F^{m,R,\delta}(\psi_s^{m,R,\delta}) = g_{R,\delta}(\|\psi_s^{m,R,\delta}\|_{\mathcal{H}^s(\Omega)})\Bigg(P_m\left[B\left(\psi_s^{m,R,\delta},\psi_s^{m,R,\delta}\right)\right] + C\left(\psi_s^{m,R,\delta}\right)$$

$$+ \frac{1}{2}\sum_{i=1}^{\infty} P_m E_i^2(\psi_s^{m,R,\delta}) - \nu\Delta\psi_s^{m,R,\delta} + D\mathbb{E}\left[\bar{\psi}_s^{m,R,\delta}\right] \Bigg). \quad (75)$$

The projection operator P_m is defined as in (33)+(34). Recall that, when defining the projection corresponding to the ocean velocity component, we have taken the incompressibility into account and projected onto the space of divergence-free vector fields.

We then have the following:

Theorem 3.2. *Assume that $\psi_{t_0} \in \mathcal{H}^s(\Omega)$. Then the stochastic differential system (74) admits a unique global solution with values in the space*

$$L^2\left(\Xi;C(([t_0,T];\mathcal{H}^s(\Omega)))\right) \cap L^2\left(\Xi;L^2(([t_0,T];\mathcal{H}^{s+1}(\Omega)))\right).$$

for any $T > 0$. Moreover, there exists a constant $C = C(R,T)$ independent of m and δ such that

$$\mathbb{E}\left[\sup_{t\in[t_0,T]}\|\psi_s^{m,R,\delta}\|_{\mathcal{H}^s(\Omega)}^2\right] + \mathbb{E}\left[\int_{t_0}^T \|\psi_s^{m,R,\delta}\|_{\mathcal{H}^{s+1}(\Omega)}^2\right] \leqslant C \quad (76)$$

for any $T > 0$.

Proof. Similar to the deterministic case, the system (74) is equivalent to a finite dimensional system of stochastic differential equations of McKean–Vlasov type with Lipschitz continuous coefficients. The same holds true for the system satisfied by $\left(\mathcal{D}^\alpha\psi^{m,R,\delta}\right)_{|\alpha|\leqslant s}$ which involves $\psi^{m,R,\delta}$ as well as all of its partial derivatives up to order s. The existence and uniqueness of a solution of the system system (74) follows for example from [40]. The bound (76) is obtained, as in the deterministic case, via a Gronwall type argument. □

Remark. In addition, one can prove that there exists a constant $C = C(p,R,T)$ independent of m such that

$$\mathbb{E}\left[\sup_{t\in[t_0,T]}\|\psi_s^{m,R,\delta}\|_{\mathcal{H}^s(\Omega)}^p\right] \leqslant C \quad (77)$$

for any $T > 0$ and $p \geqslant 2$. This bound is useful to show the continuity of the limit of the Galerkin approximation.

Introduce next $\psi^{R,\delta} = \left\{ \psi_t^{R,\delta}, t \geqslant 0 \right\}$ to be the solution of the following stochastic differential system

$$\psi_t^{R,\delta} = \psi_{t_0} - \int_{t_0}^t F^{R,\delta}(\psi_s^{R,\delta}) ds - \sum_{i=1}^{\infty} \int_{t_0}^t g_{R,\delta}(\|\psi_s^{R,\delta}\|_{\mathcal{H}^s(\Omega)}) E_i(\psi_s^{R,\delta}) dW_s^i, \quad (78)$$

Just as in (70), the identity (78) is assumed to hold, $\mathbb{P}$-almost surely, in $L^2\left(\Xi; \mathcal{H}^0(\Omega)\right)$. In (78), the mapping $F^{R,\delta}(\psi_s^{R,\delta})$ is defined as

$$F^{R,\delta}(\psi_s^{R,\delta}) = g_{R,\delta}(\|\psi_s^{R,\delta}\|_{\mathcal{H}^s(\Omega)}) \Bigg(B\left(\psi_s^{R,\delta}, \psi_s^{R,\delta}\right) + C\left(\psi_s^{R,\delta}\right)$$

$$+ \frac{1}{2} \sum_{i=1}^{\infty} E_i^2(\psi_s^{R,\delta}) - \nu\Delta\psi_s^{R,\delta} + D\mathbb{E}\left[\bar{\psi}_s^{R,\delta}\right] \Bigg). \quad (79)$$

In the following, we will also need the weak version of the systems (74) and (78). These are standard: For example, the weak version of (78) reads as

$$\left\langle \psi_t^{R,\delta}, \varphi \right\rangle_{\mathcal{H}^0(\Omega)^6} = \langle \psi_{t_0}, \varphi \rangle_{\mathcal{H}^0(\Omega)^6} - \int_{t_0}^t \left(\left\langle F^{R,\varphi}\left(\psi_s^{R,\delta}\right), \varphi \right\rangle_{\mathcal{H}^0(\Omega)^6} \right) ds$$

$$- \sum_{i=1}^{\infty} \int_{t_0}^t g_{R,\delta}(\|\psi_s^{R,\delta}\|_{\mathcal{H}^s(\Omega)}) \left\langle \psi_s^{R,\delta}, E_i^*\varphi \right\rangle_{\mathcal{H}^0(\Omega)^6} dW_s^i, \quad \varphi \in \left(\mathcal{H}^2(\Omega)\right)^6,$$

$$(80)$$

where, for $\psi_t = (\mathbf{u}_t^a, \theta_t^a, \mathbf{u}_t^o, \theta_t^o)$ and $\varphi = (\varphi_i)_{i=1}^6$, $\varphi_i \in \mathcal{H}^2(\Omega)$ we have the composite inner product

$$\langle \psi_t, \varphi \rangle_{\mathcal{H}^0(\Omega)^6} = \left\langle \mathbf{u}_t^{a,1}, \varphi_1 \right\rangle_{\mathcal{H}^0(\Omega)} + \left\langle \mathbf{u}_t^{a,2}, \varphi_2 \right\rangle_{\mathcal{H}^0(\Omega)} + \langle \theta_t^a, \varphi_3 \rangle_{\mathcal{H}^0(\Omega)}$$

$$+ \left\langle \mathbf{u}_t^{o,1}, \varphi_4 \right\rangle_{\mathcal{H}^0(\Omega)} + \left\langle \mathbf{u}_t^{o,2}, \varphi_5 \right\rangle_{\mathcal{H}^0(\Omega)} + \langle \theta_t^o, \varphi_6 \rangle_{\mathcal{H}^0(\Omega)}.$$

As the ocean velocity component takes values in the the space of divergence-free vector fields, we will take the corresponding pair of test function (φ_4, φ_5) to take value in the same space. The operators E_i^* are adjoint operators corresponding to the operators E_i, so that

$$\left\langle E_i\psi_s^{R,\delta}, \varphi \right\rangle_{\mathcal{H}^0(\Omega)^6} = \left\langle \psi_s^{R,\delta}, E_i^*\varphi \right\rangle_{\mathcal{H}^0(\Omega)^6}.$$

In other words, E_i^* are diagonal operators given by

$$E_i^*\varphi := \mathrm{diag}(-\bar{E}^1(\varphi_1, \varphi_2), -\bar{E}^2(\varphi_1, \varphi_2), -\mathrm{div}(\xi_i\varphi_3), 0, 0, 0)$$

$$\bar{E}^k(\varphi_1, \varphi_2) := \mathrm{div}(\xi_i\varphi_k) + \frac{1}{Ro^a}\xi_i^k + \varphi_j^a\partial_k\xi_i^j + \frac{1}{Ro^a}\partial^k(R_j(\mathbf{x})\xi_i^j).$$

The identity in (80) holds, $\mathbb{P}$-almost surely, in $L^2(\Xi; \mathbb{R})$.

For the following theorem we need to introduce the additional space $\mathcal{W}^{\alpha,p}\left([t_0, T]; \mathcal{H}^0(\Omega)\right)$, where $\beta \in (0,1)$ and $p > 2$ with $\beta p > 1$, defined as

$$\mathcal{W}^{\beta,p}\left([t_0, T]; \mathcal{H}^0(\Omega)\right) := \left\{ a \in \mathcal{L}_p(([t_0, T]; \mathcal{H}^0(\Omega)) \mid \|a\|_{\mathcal{W}^{\beta,p}(0,T;\mathcal{H}^0(\Omega))} < \infty \right\}$$

where the norm $\| \cdot \|_{\mathcal{W}^{\beta,p}(0,T;\mathcal{H}^0(\Omega))}$ is defined as

$$\|a\|^p_{\mathcal{W}^{\beta,p}(0,T;\mathcal{H}^0(\Omega))} := \int_0^T \|a_t\|^p_{\mathcal{H}^0(\Omega))} \, dt + \int_0^T \int_0^T \frac{\|a_t - a_s\|^p_{\mathcal{H}^0(\Omega))}}{|t-s|^{1+\beta p}} \, dt ds.$$

Theorem 3.3. *Assume that* $\psi_{t_0} \in \mathcal{H}^s(\Omega)$. *Then the stochastic differential system* (78) *admits a unique global solution (in the sense of definition* 3.1*) with values in the space*

$$C\left(\Xi; C(([t_0,T];\mathcal{H}^s(\Omega))) \cap L^2\left(\Xi; L^2(([t_0,T];\mathcal{H}^{s+1}(\Omega)))\right)\right)$$

for any $T > 0$ *and any* $p \geqslant 2$. *Moreover, there exists a constant* $C = C(R,T)$ *independent of* δ *such that*

$$\mathbb{E}\left[\sup_{t \in [t_0,T]} \|\psi_s^{R,\delta}\|^p_{\mathcal{H}^s(\Omega)}\right] + \mathbb{E}\left[\int_{t_0}^T \|\psi_s^{R,\delta}\|^2_{\mathcal{H}^{s+1}(\Omega)}\right] \leqslant C \tag{81}$$

for any $T > 0$.

Proof.

Existence. We follow the same steps as in [11], underlying the main differences below. Let $\{Q^m\}$ be the family of the probability laws of the processes $\{\psi^{m,R,\delta}\}$. These laws supported on the space

$$\begin{aligned}
\mathcal{E}_0 &:= \mathcal{E}_1 \cap \mathcal{E}_2 \\
\mathcal{E}_1 &:= L^p\left(\Xi; \mathcal{W}^{\beta,p}\left([t_0,T];\mathcal{H}^0(\Omega)\right)\right) \\
\mathcal{E}_2 &:= L^p\left(\Xi; C(([t_0,T];\mathcal{H}^s(\Omega))) \cap L^2\left(\Xi; L^2(([t_0,T];\mathcal{H}^{s+1}(\Omega)))\right),
\end{aligned}$$

where β is an arbitrary positive constant such that $\beta < 1/2 - 1/p$ and $p > 2$. Since $\mathcal{E}_0$ is compactly embedded in $L^p\left(\Xi; C(([t_0,T];\mathcal{H}^0(\Omega)))\right)$, we deduce that these laws are relatively compact in the space of probability measures $L^p\left(\Xi; C(([t_0,T];\mathcal{H}^0(\Omega)))\right)$. We add to the processes $\{\psi^{m,R,\delta}\}$ the driving Brownian motions $\mathcal{W} = \{W^i\}_{i=1}^\infty$. Then the pairs $\{\psi^{m,R,\delta}, \mathcal{W}\}$ have probability laws $\{\widetilde{Q}^m\}$ that are relatively compact in the space of probability measures over the state space

$$L^p\left(\Xi; C(([t_0,T];\mathcal{H}^0(\Omega)))\right) \times L^2\left(\Xi; C(([t_0,T];\mathbb{R})^\infty)\right).$$

Let $\widetilde{Q}$ be a limit point and $\{\widetilde{Q}^{m_n}\}$ be a subsequence of measures converging to $\widetilde{Q}$. Also let $\{\varphi_k\}_k$ be a countable dense set of $\left(\mathcal{H}^2(\Omega)\right)^6$. By theorem 2.2 in [27], it follows that the processes

$$\left\{\psi^{m_n,R,\delta}, \int_{t_0}^{\cdot} \left\langle \psi_s^{m_n,R,\delta}, E_i^* P_{m_n} \varphi_k \right\rangle_{\mathcal{H}^0(\Omega)^6} dW_s^i, i,k = 1, \dots \infty, \mathcal{W}\right\}$$

converge in distribution. By using the Skorohod representation theorem, there exists a probability space $\left(\widetilde{\Xi}, \widetilde{\mathcal{F}}, \widetilde{P}\right)$ on which we can find

- a sequence

$$\widetilde{A}^{m_n} = \left\{\widetilde{\psi}^{m_n,R,\delta}, \int_{t_0}^{\cdot} \left\langle \widetilde{\psi}_s^{m_n,R,\delta}, E_i^* P_{m_n} \varphi_k \right\rangle_{\mathcal{H}^0(\Omega)^6} d\widetilde{W}_s^{m_n,i}, \ i,k = 1, \dots \infty, \ \widetilde{\mathcal{W}}^{m_n}\right\}$$

such that $\left\{\widetilde{\psi}^{m_n,R,\delta}, \widetilde{\mathcal{W}}^{m_n}\right\}$ has law $\widetilde{Q}^{m_n}$

- a process

$$\widetilde{A} = \left\{ \widetilde{\psi}^{R,\delta}, \int_{t_0}^{\cdot} \left\langle \widetilde{\psi}_s^{R,\delta}, E_i^* \varphi_k \right\rangle_{\mathcal{H}^0(\Omega)^6} d\widetilde{W}_s^i, \quad i,k = 1,\dots\infty, \ \widetilde{\mathcal{W}} \right\}$$

with values in the product space

$$\widetilde{E} := L^2 \left(\widetilde{\Xi}; C(([t_0,T];\mathcal{H}^0(\Omega))) \right) \times L^2 \left(\widetilde{\Xi}; C(([t_0,T];\mathbb{R})^{\mathbb{N}\times\mathbb{N}}) \right) \times L^2 \left(\widetilde{\Xi}; C(([t_0,T];\mathbb{R})^{\mathbb{N}}) \right).$$

such that the component $\left\{ \widetilde{\psi}^R, \widetilde{\mathcal{W}} \right\}$ from $\widetilde{A}$ has law $\widetilde{Q}$.

- The sequence $\left(\widetilde{A}^{m_n} \right)_n$ converges to $\widetilde{A}$ as elements in the space $\widetilde{E}$.
- Since Q^{m_m}, the law of $\widetilde{\psi}^{m_n,R,\delta}$ is supported on the space

$$L^p \left(\widetilde{\Xi}; L^{\infty}(([t_0,T];\mathcal{H}^s(\Omega))) \right) \cap L^2 \left(\widetilde{\Xi}; L^2(([t_0,T];\mathcal{H}^{s+1}(\Omega))) \right),$$

the law of $\widetilde{\psi}^{R,\delta}$ has the same property.

Next we take limits of all the terms in the weak version of the equation (74) and show that $\widetilde{\psi}^R$ satisfies (80)[8] for any φ_k in a countable dense set of $\left(\mathcal{H}^2(\Omega)\right)^6$, and therefore, by a density argument, for an arbitrary $\varphi \in \left(\mathcal{H}^2(\Omega)\right)^6$. The convergence of all the linear terms is straightforward. We only discuss the convergence of the nonlinear term, in other words, the limit

$$\lim_{n\to\infty} \int_{t_0}^t \left\langle g_R(\|\widetilde{\psi}_s^{m_n,R,\delta}\|_{\mathcal{H}^s(\Omega)}) P_{m_n} B\left(\widetilde{\psi}_s^{m_n,R,\delta}, \widetilde{\psi}_s^{m_n,R,\delta} \right), \varphi_k \right\rangle_{\mathcal{H}^0(\Omega)^6} ds$$

$$= \int_{t_0}^{t\wedge\bar{\tau}} \left\langle g_R(\|\widetilde{\psi}_s^{R,\delta}\|_{\mathcal{H}^s(\Omega)}) B\left(\widetilde{\psi}_s^{R,\delta}, \widetilde{\psi}_s^{R,\delta} \right), \varphi_k \right\rangle_{\mathcal{H}^0(\Omega)^6} ds.$$

Recall that we took the corresponding pair of test function (φ_4, φ_5) to be divergence free so we do not need to apply the Leray projection on the corresponding ocean velocity component of the bilinear form $B\left(\widetilde{\psi}_s^{R,\delta}, \widetilde{\psi}_s^{R,\delta} \right)$.

The convergence follows via a Sobolev space interpolation argument from the following

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{s\in[t_0,T]} \left\| \widetilde{\psi}_s^{m_n,\delta} - \widetilde{\psi}_s^{R,\delta} \right\|_{\mathcal{H}^0(\Omega)^6}^2 \right] = 0$$

$$\mathbb{E}\left[\sup_{t\in[t_0,T]} \|\widetilde{\psi}_s^{m_n,R,\delta}\|_{\mathcal{H}^s(\Omega)}^p \right] + \mathbb{E}\left[\int_{t_0}^T \|\widetilde{\psi}_s^{m_n,R,\delta}\|_{\mathcal{H}^{s+1}(\Omega)}^2 \right] \leqslant C, \ T \geqslant t_0$$

$$\mathbb{E}\left[\sup_{t\in[t_0,T]} \|\widetilde{\psi}_s^{R,\delta}\|_{\mathcal{H}^s(\Omega)}^p \right] + \mathbb{E}\left[\int_{t_0}^T \|\widetilde{\psi}_s^{R,\delta}\|_{\mathcal{H}^{s+1}(\Omega)}^2 \right] \leqslant C, \ T \geqslant t_0,$$

where $C = C(p,R,T)$ is independent of m_n.

The above argument justifies the existence of a solution of (78) which is weak in probability sense. From this and the pathwise uniqueness (see the argument below) of (78), by Yamada-Watanabe theorem, see e.g. [39] we deduce the existence of a solution of in the original space

---

[8] In (80), the set of original Brownian motions $\mathcal{W} = \left\{ W^i \right\}_{i=1}^{\infty}$ is replaced by $\widetilde{\mathcal{W}} = \left\{ \widetilde{W}^i \right\}_{i=1}^{\infty}$ ..

driven by the original set of Brownian motions $\mathcal{W} = \left\{W^i\right\}_{i=1}^{\infty}$. Note that both the solution of the equation (78) on the original space and $\widetilde{\psi}^{R,\delta}$ have the same distribution. Since the law of $\widetilde{\psi}^{R,\delta}$ has support on the space

$$L^p\left(\widetilde{\Xi}; C(([t_0,T]; \mathcal{H}^s(\Omega))\right) \cap L^2\left(\widetilde{\Xi}; L^2(([t_0,T]; \mathcal{H}^{s+1}(\Omega))\right),$$

it follows that the solution of the equation (78) satisfies (76).

Uniqueness. Let $\psi^{R,\delta,1}$ and $\psi^{R,\delta,2}$ be two solutions of (78), in other words,

$$\psi_t^{R,\delta,i} = \psi_{t_0} - \int_{t_0}^t F^{R,\delta,i}(\psi_s^{R,\delta,i})ds - \sum_{i=1}^{\infty}\int_{t_0}^t g_{R,\delta,i}(\|\psi_s^{R,\delta,i}\|_{\mathcal{H}^s(\Omega)})E_i(\psi_s^{R,\delta,i})dW_s^i, \;\; i=1,2$$

where $F^{R,\delta,i}(\psi_s^{R,\delta,i})$ are defined as

$$F^{R,\delta,i}(\psi_s^{R,\delta,i}) = g_{R,\delta}(\|\psi_s^{R,\delta,i}\|_{\mathcal{H}^s(\Omega)})\Bigg(B\left(\psi_s^{R,\delta,i}, \psi_s^{R,\delta,i}\right) + C\left(\psi_s^{R,\delta,i}\right)$$
$$+\frac{1}{2}\sum_{i=1}^{\infty}E_i^2(\psi_s^{R,\delta,i}) - \nu\Delta\psi_s^{R,\delta,i} + D\mathbb{E}\left[\bar{\psi}_s^{R,\delta,i}\right]\Bigg). \tag{82}$$

The uniqueness argument is now standard: We use a Gronwall argument. We introduce the following notation

$$\psi^{R,\delta,1,2} = \psi^{R,\delta,1} - \psi^{R,\delta,2}, \quad \bar{F}^{R,\delta,1,2} = \bar{F}^R(\psi_s^{R,\delta,1}) - \bar{F}^R(\psi_s^{R,\delta,2})$$
$$\bar{\psi}_s^{R,\delta,1,2} = \bar{\psi}_s^{R,\delta,1} - \bar{\psi}_s^{R,\delta,2}.$$

Then

$$\psi_t^{R,\delta,1,2} = \psi_{t_0}^{1,2} - \int_{t_0}^t\left(\bar{F}^{R,\delta,1,2} + \frac{1}{2}\sum_{i=1}^{\infty}E_i^2(\psi_s^{R,\delta,1,2}) + D\mathbb{E}\left[\bar{\psi}_s^{R,\delta,1,2}\right]\right)ds$$
$$-\sum_{i=1}^{\infty}\int_{t_0}^t E_i(\psi_s^{R,\delta,1,2})dW_s^i,$$

from which we deduce that

$$\mathbb{E}\left[\|\psi_t^{R,\delta,1,2}\|_{\mathcal{H}^0(\Omega)}^2\right] = \mathbb{E}\left[\|\psi_{t_0}^{R,\delta,1,2}\|_{\mathcal{H}^0(\Omega)}^2\right] - \int_{t_0}^t\mathbb{E}\left[\left\langle\psi_s^{R,\delta,1,2}, \bar{F}^{R,\delta,1,2} + D\mathbb{E}\left[\bar{\psi}_s^{R,\delta,1,2}\right]\right\rangle_{\mathcal{H}^0(\Omega)}\right]ds$$
$$+\sum_{i=1}^{\infty}\frac{1}{2}\int_{t_0}^t\mathbb{E}\left[\left\langle\psi_s^{R,\delta,1,2}, E_i^2(\psi_s^{R,\delta,1,2})\right\rangle_{\mathcal{H}^0(\Omega)} + \|E_i(\psi_s^{R,\delta,1,2})\|_{\mathcal{H}^0(\Omega)}^2\right]ds. \tag{83}$$

Observe that

$$\sum_{i=1}^{\infty}\left(\left\langle\psi_s^{R,\delta,1,2}, E_i^2(\psi_s^{R,\delta,1,2})\right\rangle_{\mathcal{H}^0(\Omega)} + \|E_i(\psi_s^{R,\delta,1,2})\|_{\mathcal{H}^0(\Omega)}^2\right) \leqslant C\|\psi_s^{R,\delta,1,2}\|_{\mathcal{H}^0(\Omega)}^2 \tag{84}$$

$$\mathbb{E}\left[\left(\left\langle\psi_s^{R,\delta,1,2}, D\mathbb{E}\left[\bar{\psi}_s^{R,\delta,1,2}\right]\right\rangle_{\mathcal{H}^0(\Omega)}\right)\right] \leqslant C\mathbb{E}\left[\|\psi_s^{R,\delta,1,2}\|_{\mathcal{H}^0(\Omega)}\|\mathbb{E}\left[\bar{\psi}_s^{R,\delta,1,2}\right]\|_{\mathcal{H}^0(\Omega)}\right]$$
$$\leqslant C\mathbb{E}\left[\|\psi_s^{R,\delta,1,2}\|_{\mathcal{H}^0(\Omega)}\right]^2$$
$$\leqslant C\mathbb{E}\left[\|\psi_s^{R,\delta,1,2}\|_{\mathcal{H}^0(\Omega)}^2\right] \tag{85}$$

Finally, similar to the deterministic case, we deduce that

$$\langle \psi_s^{R,\delta,1,2}, \overline{F}^{R,\delta,1,2} \rangle_{\mathcal{H}^0(\Omega)} \leqslant C(R) \|\psi_s^{R,\delta,1,2}\|_{\mathcal{H}^0(\Omega)}^2. \tag{86}$$

From (83)–(86), we deduce that there exists a constant $C(R,T)$ such that

$$\mathbb{E}\left[\|\psi_t^{R,\delta,1,2}\|_{\mathcal{H}^0(\Omega)}^2\right] \leqslant \mathbb{E}\left[\|\psi_{t_0}^{R,\delta,1,2}\|_{\mathcal{H}^0(\Omega)}^2\right] + C\int_{t_0}^t \mathbb{E}\left[\|\psi_s^{R,\delta,1,2}\|_{\mathcal{H}^0(\Omega)}^2\right] ds, \quad t \in [t_0, T],$$

and, by Gronwall's inequality, we deduce that

$$\mathbb{E}\left[\|\psi_t^{R,\delta,1,2}\|_{\mathcal{H}^0(\Omega)}^2\right] \leqslant e^{Ct}\mathbb{E}\left[\|\psi_{t_0}^{R,\delta,1,2}\|_{\mathcal{H}^0(\Omega)}^2\right] \quad t \in [t_0, T].$$

The continuous dependence of the initial condition implies the uniqueness of the solution of (78). □

We choose next a sequence $\psi^{R,\delta_n} = \left\{ \psi_t^{R,\delta_n}, t \geqslant 0 \right\}$ of solutions of the truncated equation (78) such that $\lim \delta_n = 0$. Using arguments similar to those applied to the sequence of Galerkin approximations $\psi^{m,R,\delta}$ one shows that these laws of the elements of the sequence are relatively compact in the space of probability measures $L^p\left(\Xi; C(([t_0, T]; \mathcal{H}^0(\Omega)))\right)$. Via a Skorohod representation theorem, there exists a probability space $\left(\widetilde{\Xi}, \widetilde{\mathcal{F}}, \widetilde{P}\right)$ on which we can find a sequence $\{\widetilde{\psi}^{R,\delta_n}\}$ with the same law as the original sequence which converges in $L^p\left(\Xi; C(([t_0, T]; \mathcal{H}^0(\Omega)))\right)$ to a process $\{\widetilde{\psi}^R\}$. that satisfies

$$\mathbb{E}\left[\sup_{t \in [t_0, T]} \|\widetilde{\psi}_s^{R,\delta}\|_{\mathcal{H}^s(\Omega)}^p\right] + \mathbb{E}\left[\int_{t_0}^T \|\widetilde{\psi}_s^{R,\delta}\|_{\mathcal{H}^{s+1}(\Omega)}^2\right] \leqslant C, \quad T \geqslant t_0,$$

where $C = C(p, R, T)$. Via a Sobolev interpolation argument we can also deduce that

$$\lim_{n \to \infty} \mathbb{E}\left[\sup_{s \in [t_0, T]} \left\|\widetilde{\psi}_s^{R,\delta_n} - \widetilde{\psi}_s^R\right\|_{\mathcal{H}^{s-1}(\Omega)^6}^p\right] = 0$$

$$\lim_{n \to \infty} \mathbb{E}\left[\int_{t_0}^T \left\|\widetilde{\psi}_s^{R,\delta_n} - \widetilde{\psi}_s^R\right\|_{\mathcal{H}^s(\Omega)^6}^p ds\right] = 0.$$

Next let $g_R : \mathbb{R}_+ \to [0, 1]$ be the cut-off function as follows

$$g_R(x) := \begin{cases} 1, & \text{if } 0 \leqslant x \leqslant R \\ 0, & \text{if } x \geqslant R \end{cases}.$$

and assume that

$$\lim_{n \to \infty} \mathbb{E}\left[\int_{t_0}^T \left(g_{R,\delta_n}(\|\widetilde{\psi}_s^{R,\delta_n}\|_{\mathcal{H}^s(\Omega)^6}) - g_R(\|\widetilde{\psi}_s^R\|_{\mathcal{H}^s(\Omega)^6})\right)^p dt\right] = 0. \tag{87}$$

If condition (87) is satisfied then, by taking the limit of each term in equation (80), we can deduce that the limiting process $\widetilde{\psi}^R$ solves the following stochastic differential system

$$\widetilde{\psi}_t^R = \widetilde{\psi}_{t_0} - \int_{t_0}^t F^R(\widetilde{\psi}_s^R)ds - \sum_{i=1}^\infty \int_{t_0}^t g_R(\|\widetilde{\psi}_s^R\|_{\mathcal{H}^s(\Omega)})E_i(\widetilde{\psi}_s^R)dW_s^i. \tag{88}$$

In (88), the mapping $F^R(\widetilde{\psi}_s^R)$ is defined as

$$
\begin{aligned}
F^R(\widetilde{\psi}_s^R) = g_R(\|\widetilde{\psi}_s^R\|_{\mathcal{H}^s(\Omega)})\bigg(& B\left(\widetilde{\psi}_s^R, \widetilde{\psi}_s^R\right) + C\left(\widetilde{\psi}_s^R\right) \\
& + \frac{1}{2}\sum_{i=1}^{\infty} E_i^2(\widetilde{\psi}_s^R) - \nu\Delta\widetilde{\psi}_s^R + D\mathbb{E}\left[\overline{\widetilde{\psi}}_s^R\right]\bigg).
\end{aligned}
\tag{89}
$$

It is then immediate that equation (88) is equivalent to (70) where we choose the stopping time

$$
\tau_R := \inf\{t \geqslant t_0|\ \|\widetilde{\psi}_s^R\|_{\mathcal{H}^s(\Omega)^6} \geqslant R\}
$$

Remark. By using standard Sobolev interpolation results, one can prove that

$$
\lim_{n\to\infty} \mathbb{E}\left[\int_{t_0}^{T}\left|\|\widetilde{\psi}_s^{R,\delta_n}\|_{\mathcal{H}^s(\Omega)^6} - \|\widetilde{\psi}_s^R\|_{\mathcal{H}^s(\Omega)^6}\right|dt\right] = 0.
\tag{90}
$$

This implies that the sequence $\widetilde{\psi}^{R,\delta_n}$ has a subsequence that converges to $\widetilde{\psi}^R$ on a set

$$
\{(\xi \times t) \in \Xi \times [t_0, T]\}
$$

of full $\mathbb{P} \otimes \ell_{[t_0,T]}$-measure[9], where $\ell_{[t_0,T]}$ is the Lebesgue measure on the interval $[t_0, T]$. From the definition of $\widetilde{\psi}_s^{R,\delta_n}$ we can deduce that

$$
\{(\xi,t) \in \Xi \times [t_0,T]|\|\widetilde{\psi}_t^{R,\delta_n}(\xi)\|_{\mathcal{H}^s(\Omega)^6} \leqslant R + \delta_n\}
$$

and therefore that

$$
\begin{aligned}
\Xi \times [t_0,T] \supseteq \ & \{(\xi,t)|\|\widetilde{\psi}_t^R(\xi)\|_{\mathcal{H}^s(\Omega)^6} \leqslant R\} \\
= \ & \{(\xi,t)|\|\widetilde{\psi}_t^R(\xi)\|_{\mathcal{H}^s(\Omega)^6} < R\} \cup \{(\xi,t)|\|\widetilde{\psi}_t^R(\xi)\|_{\mathcal{H}^s(\Omega)^6} = R\} \\
=: \ & \mathcal{A}_{<R} \cup \mathcal{A}_{=R}
\end{aligned}
$$

has full $\mathbb{P} \otimes \ell_{[t_0,T]}$-measure. We can deduce from here, via Egorov's theorem that (87) holds true provided $\mathcal{A}_{=R}$ is a set of null $\mathbb{P} \otimes \ell_{[t_0,T]}$-measure[10].

Remark. The argument so far only shows the existence of a martingale solution (equivalently, a probabilistically weak solution). To prove the existence of a (probabilistically) strong solution one would need to show the uniqueness of equation (88). This cannot be done as the cut-off function is no longer Lipschitz over the positive half-line. One can try to control the difference between two solutions $\psi^{R,1}$ and $\psi^{R,2}$ up to the minimum of their corresponding hitting times $\tau_R^1 \wedge \tau_R^1$. On the interval $[t_0, \tau_R^1 \wedge \tau_R^1]$, $g_R(\|\widetilde{\psi}_s^{R,1}\|_{\mathcal{H}^s(\Omega)}) = g_R(\|\widetilde{\psi}_s^{R,2}\|_{\mathcal{H}^s(\Omega)})$ so we can avoid the difficulty raised by g_R being non-Lipschitz. However, due to the the expectation terms in (88). the solution depends on temporal values beyond $\tau_R^1 \wedge \tau_R^1$ which we cannot control. Overcoming this difficulty is beyond the scope of the current paper.

3.2. LA-SALT climate model

Using the same notation as in section 3.1, we describe the state of the system is described by a state vector $\psi := (\psi^a, \psi^o)$ with atmospheric component $\psi^a := (\mathbf{u}^a, \theta^a)$ and

---

[9] That is the complement of the set has null measure.

[10] We thank Tom Kurtz for pointing out this remark.

oceanic component $\psi^o := (\mathbf{u}^o, \theta^o)$ with initial state is denoted by $\psi(t_0) = \psi_0$. where $\psi_0 = (\mathbf{u}_0^a, \theta_0^a, \mathbf{u}_0^o, \theta_0^o)$. The LA-SALT equations for the oceanic component ψ^o are the same as the corresponding SALT equations, in other words $(\mathbf{u}^o, \theta^o)$ satisfy equations (12)+(13). The LA-SALT equations for the atmospheric component ψ^a differ from the the corresponding SALT equations. More precisely, $\psi^a := (\mathbf{u}^a, \theta^a)$ satisfy equations (20)+(21).

In the following we work with the same stochastic basis $(\Xi, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P}, (W^i)_i)$ and use the same sequence of vector fields $(\xi_i)_i$ as in section 3.1. Similar to (68), we summarise the LA-CMSE equations (12), (13), (20) and (21) as

$$\mathrm{d}\psi_t + (B^L(\psi_t, \psi_t) + C^L(\psi_t))dt + \sum_{i=1}^{\infty} E_i^L(\psi_t) \circ dW_t^i = \nu \Delta \psi_t dt, \tag{91}$$

where the process ψ gathers all variables in the LASALT model, i.e. $\psi_t := (\mathbf{u}_t^a, \theta_t^a, \mathbf{u}_t^o, \theta_t^o)$ (as in the deterministic and SALT cases), $\mathrm{curl}R(\mathbf{x}) = 2\Omega(\mathbf{x})$, $\bar{\psi} := \psi - \frac{1}{|\Omega|}\int_\Omega \psi\, dx$ and[11]:

- B^L is the bilinear transport operator

$$B^L(\psi_t, \psi_t) = (\mathbb{E}[\mathbf{u}^a] \cdot \nabla \mathbf{u}^a + u_j^a \nabla \mathbb{E}[\mathbf{u}^a]^j, \ \mathbb{E}[\mathbf{u}^a] \cdot \nabla \theta^a, \ \mathbf{u}^o \cdot \nabla \mathbf{u}^o, \ \mathbf{u}^o \cdot \nabla \theta^o)^T.$$

- C^L comprises all the linear terms (including the pressure term in the equation for the components of ψ corresponding to $\bar{\mathbf{u}}^o$)

$$C^L(\psi_t) = \left(\frac{1}{Ro^a}\mathbb{E}[\mathbf{u}^a] + \frac{1}{Ro^a}\nabla(\mathbb{E}[\mathbf{u}^a] \cdot R(\mathbf{x})) + \frac{1}{Ro^a}\nabla\theta^a, \ -\gamma(\theta^o - \theta^a), \right.$$
$$\left. \frac{1}{Ro^o}(\mathbf{u}^o)^\perp + \frac{1}{Ro^o}\nabla p^o + \sigma(\mathbf{u}^o - \mathbb{E}[\bar{\mathbf{u}}^a]), \ 0 \right)^T$$

- E_i^L are operators given by

$$E_i^L(\psi_t) = \left(\xi_i \cdot \nabla \mathbf{u}^a + \frac{1}{Ro^a}\xi_i + u_j^a \nabla \xi_i^j + \frac{1}{Ro^a}\nabla(R_j(\mathbf{x})\xi_i^j), \ \xi_i \cdot \nabla\theta^a, \ 0, \ 0 \right)^T.$$

The treatment of equation (91) differs slightly from that of (68). The reason is that the expected value of state vector ψ satisfies a closed form equation. More precisely, as the oceanic component ψ^o is not random, we only need to take expectation in the equations satisfied by the atmospheric component and deduce that $\widehat{\psi}^a := \mathbb{E}[\psi^a] = (\mathbb{E}[\mathbf{u}^a], \mathbb{E}[\theta^a])^T =: (\widehat{\mathbf{u}}^a, \widehat{\theta}^a)^T$ satisfies

$$d_t\widehat{\psi}_t^a + B^{L,a}(\widehat{\psi}_t^a, \widehat{\psi}_t^a) + C^L(\widehat{\psi}_t^a) = \frac{1}{2}\sum_{i=1}^{\infty} E_i^{L,a,2}(\widehat{\psi}_t^a) + \nu\Delta\widehat{\psi}_t^a, \tag{92}$$

where

$$B^{L,a}\left(\widehat{\psi}_t^a, \widehat{\psi}_t^a\right) = (\widehat{\mathbf{u}}^a \cdot \nabla\widehat{\mathbf{u}}^a + \widehat{u}_j^a\nabla(\widehat{\mathbf{u}}^a)^j, \ \widehat{\mathbf{u}}^a \cdot \nabla\widehat{\theta}^a)^T.$$

$$C^L\left(\widehat{\psi}_t\right) = \left(\frac{1}{Ro^a}\widehat{\mathbf{u}}^a + \frac{1}{Ro^a}\nabla(\widehat{\mathbf{u}}^a \cdot R(\mathbf{x})) + \frac{1}{Ro^a}\nabla\widehat{\theta}^a], \ -\gamma\left(\theta^o - \widehat{\theta}^a\right) \right)^T$$

---

[11] As in section 3.1, the notation $(\cdot)^T$ indicates that the operators B^L, C^L, E_i^L are column vectors.

$$E_i^{L,a,2}(\widehat{\psi}_t^a) = \left(\frac{1}{2}\widehat{\mathbf{z}} \times \xi \left(\mathrm{div}\left(\xi \left(\widehat{\mathbf{z}} \cdot \mathrm{curl}\left(\widehat{\mathbf{u}}^a + \frac{1}{Ro^a}\mathbf{R}(\mathbf{x}) \right) \right) \right) \right) \right.$$
$$- \nabla\left(\xi \cdot \nabla\left(\xi \cdot \left(\widehat{\mathbf{u}}^a + \frac{1}{Ro^a}\mathbf{R}(\mathbf{x}) \right) \right) \right),$$
$$\left. - \xi \cdot \nabla(\xi_i \cdot \nabla\widehat{\theta}^a) \right) \right)^T.$$

It is immediate that the pair $(\widehat{\psi}^a, \psi^o)$ satisfies a system of equations of the form (1)–(5). More precisely, the only difference between the system of equations satisfied by the pair $(\widehat{\psi}^a, \psi^o)$ and the system of equations (1)–(5) is the linear term $\frac{1}{2}\sum_{i=1}^{\infty} E_i^{L,a,2}(\widehat{\psi}_t^a)$. This term does not hinder (or help) the analysis of the system (92)+(12)+(13), where, in (12)+(13) we replace $\mathbb{E}[\bar{\psi}^a]$ by $\overline{\widehat{\psi}^a} := \widehat{\psi} - \frac{1}{|\Omega|}\int_\Omega \widehat{\psi}dx$.

As for the deterministic case in theorem 2.3 we have the following

Theorem 3.4. *Let $s \geqslant 2$ and suppose the initial condition of the system (92)+(12)+(13) satisfies $\psi_0 = (\widehat{\psi}_0^a, \psi_0^o) = (\psi_0^a, \psi_0^o) \in \mathcal{H}^s(\Omega)$. Then there exists a unique time $t_{e,1}^* \in (t_0, \infty]$ such that a local regular solution $(\widehat{\psi}^a, \widehat{\psi}^o)$ of (92)+(12)+(13) in the sense of definition 2.1 exists and is unique on any interval $T := [t_0, t_1]$, where $t_0 < t_1 < t_{e,1}^*$ and that, if $t_{e,1}^* < \infty$, then*

$$\lim_{t \nearrow t_{e,1}^*} \|(\widehat{\psi}^a, \widehat{\psi}^o)\|_{\mathcal{H}^s} = \infty. \tag{93}$$

The fact that the equation satisfied by the coupled system $(\mathbb{E}[\psi^a], \psi^o)$ has a closed form and, following theorem 3.4, has a local regular solution enables us to show that there exists a solution of the system (91) up to a time $t_{e,2}^* \in (t_0, \infty]$. We have the following

Theorem 3.5. *Let $s \geqslant 2$ and suppose the initial condition of the system (91) satisfies $\psi_0 \in \mathcal{H}^s(\Omega)$. Then there exists a unique time $t_{e,2}^* \in (t_0, \infty]$, such that on any interval $T := [t_0, t_1]$, where $t_0 < t_1 < t_{e,2}^*$, the system (91) has a unique solution with the property that*

$$\psi^a \in L^2\left(\Xi; C(T; \mathbf{H}^{s-2,a}(\Omega) \times H^{s-2}(\Omega)) \cup L^2\left(\Xi; L^2(T; \mathbf{H}^{s-1,a}(\Omega) \times H^{s-1}(\Omega)\right)\right), \tag{94}$$

$$(\mathbb{E}[\psi^a], \psi^o) \in C(T; \mathcal{H}^s(\Omega)) \cup L^2(T; \mathcal{H}^{s+1}(\Omega)). \tag{95}$$

and that, if $t_{e,2}^* < \infty$, then

$$\lim_{t \nearrow t_{e,2}^*} \|(\mathbb{E}[\psi^a], \psi^o)\|_{\mathcal{H}^s} = \infty. \tag{96}$$

Remark. The loss of regularity in the atmospheric component ψ^a as compared to the regularity of the pair $(\mathbb{E}[\psi^a], \psi^o)$ is an artifact of our proof. We use theorems 1 and 2 in [38], chapter 4, to justify the existence of ψ^a which require additional regularity on the coefficients. This can only be ensured by defining the solution in a lower Sobolev space.

Proof.
Existence. From theorem 3.4, we have that there exists a solution of the system of equations (92)+(12)+(13) up to $t_{e,1}^* \in (t_0, \infty]$ and that, if $t_{e,1}^* < \infty$, then (93) holds.

We consider next the system of equations (91) where we replace $\mathbb{E}[\psi^a]$ by the 'atmospheric' component $\widehat{\psi}^a$ which is part of the solution of the system (92)+(12)+(13). The resulting system $(\check{\psi}^a, \check{\psi}^o)$ is linear in the stochastic component $\check{\psi}^a$ and nonlinear in the deterministic component $\check{\psi}^o$. Moreover, the oceanic component $\check{\psi}^o$ is decoupled from the atmospheric component as we have replaced the dependence on the atmospheric component $\mathbb{E}[\bar{\psi}^a]$ by

$\overline{\widehat{\psi^a}} := \widehat{\psi} - \frac{1}{|\Omega|} \int_\Omega \widehat{\psi} dx$. Similar to the proof of theorem 2.3 we deduce that there exists a unique time $t^*_{e,2} \in (t_0, \infty]$, $t^*_{e,2} \leqslant t^*_{e,1}$ such that on any interval $T := [t_0, t_1]$, where $t_0 < t_1 < t^*_{e,2}$, the equation satisfied by the oceanic component $\check{\psi}^o$ has a unique solution with the property that

$$\check{\psi}^o \in C(T; \mathbf{H}^{s,o}_{div}(\Omega) \times H^s(\Omega)) \cup L^2(T; \mathbf{H}^{s+1,o}_{div}(\Omega) \times H^{s+1}(\Omega)).$$

and, if $t^*_{e,2} < t^*_{e,1}$, then

$$\lim_{t \nearrow t^*_{e,2}} \|\check{\psi}^o\|_{\mathbf{H}^{s,o}_{div}(\Omega) \times H^s(\Omega)} = \infty. \tag{97}$$

Crucially, we have that $t^*_{e,2} \leqslant t^*_{e,1}$ (beyond $t^*_{e,1}$ the coefficient of the system satisfied by $(\check{\psi}^a, \check{\psi}^o)$ may not be defined, because of blowup exhibited in (93).

   The linear equation satisfied by the *stochastic component* $\check{\psi}^a$ is a particular case of the equations (1.1) and (1.2) in chapter 4, section 4.1, p 129 in [38]. It is easy to check that all assumptions required by theorems 1 and 2 in [38], chapter 4, are fulfilled. Therefore the equation satisfied by the stochastic component $\check{\psi}^a$ has a unique solution defined on the same interval $[t_0, t^*_{e,2})$ such that on any interval $T := [t_0, t_1]$, where $t_0 < t_1 < t^*_{e,2}$, we have $\check{\psi}^a$ belongs to the space stated in (94).

   Because of the linearity of the equation, no blow-up of $\check{\psi}^a$ is possible before $t^*_{e,2}$. Moreover $\mathbb{E}[\check{\psi}^a]$ satisfies a deterministic linear equation which will have a unique solution on the interval $[t_0, t^*_{e,2})$ in the same space as $\check{\psi}^a$. However this equations is also satisfied by $\widehat{\psi}^a$. It follows that $\mathbb{E}[\check{\psi}^a] \equiv \widehat{\psi}^a$. Moreover the pair $(E[\check{\psi}^a], \check{\psi}^o)$ is a solution of the system (91) and the pair $(\check{\psi}^a, \check{\psi}^o)$ also satisfies (95) (because $(\widehat{\psi}^a, \widehat{\psi}^o)$ does).

   If $t^*_{e,2} < t^*_{e,1}$, the blow-up at $t^*_{e,2}$ holds because of (97). If $t^*_{e,2} = t^*_{e,1} < \infty$, then the blow-up at $t^*_{e,2}$ holds because of (93). Hence (96) holds if $t^*_{e,2} < \infty$.

Uniqueness. Assume that we have another time $\tilde{t}^*_{e,2} \in (t_0, \infty]$, such that on any interval $T := [t_0, t_1]$, where $t_0 < t_1 < t^*_{e,2}$, the system (91) has a unique solution $\tilde{\psi}$ with the property that

$$\tilde{\psi}^a \in L^2\left(\Xi; C(T; \mathbf{H}^{s-2,a}(\Omega) \times H^{s-2}(\Omega))\right) \cup L^2\left(\Xi; L^2(T; \mathbf{H}^{s-1,a}(\Omega) \times H^{s-1}(\Omega))\right), \tag{98}$$

$$(\mathbb{E}[\tilde{\psi}^a], \tilde{\psi}^o) \in C(T; \mathcal{H}^s(\Omega) \cup L^2(T; \mathcal{H}^{s+1}(\Omega)). \tag{99}$$

and that, if $\tilde{t}^*_{e,2} < \infty$, then

$$\lim_{t \nearrow \tilde{t}^*_{e,2}} \|(\mathbb{E}[\tilde{\psi}^a], \tilde{\psi}^o)]\|_{\mathcal{H}^s} = \infty. \tag{100}$$

If $\tilde{t}^*_{e,2} < t^*_{e,2}$, then the system (91) has a unique solution in the interval $[t_0, \tilde{t}^*_{e,2}]$, hence $\tilde{\psi} = \psi$ on $[t_0, \tilde{t}^*_{e,2}]$ and it must be that

$$\lim_{t \nearrow \tilde{t}^*_{e,2}} \|(\mathbb{E}[\tilde{\psi}^a], \tilde{\psi}^o)]\|_{\mathcal{H}^s} = \lim_{t \nearrow \tilde{t}^*_{e,2}} \|(\mathbb{E}[\psi^a], \psi^o)]\|_{\mathcal{H}^s} < \infty. \tag{101}$$

which contradicts (100). Similarly we cannot have $\tilde{t}^*_{e,2} > t^*_{e,2}$, so we must have $\tilde{t}^*_{e,2} = t^*_{e,2}$ and, by the local uniqueness of the system (91), we also get that $\tilde{\psi} = \psi$ on the maximal interval of existence. □

   From (91) and (92), we can deduce the equation satisfied by the fluctuations of the system, i.e.

$$\widetilde{\psi}_t := \psi_t - \mathbb{E}[\psi_t] = \psi_t - \widehat{\psi}_t, \quad t_0 \leqslant t < t^*_{e,1}.$$

Then

$$\widetilde{\psi}_t = \widetilde{\psi}_{t_0} - \int_{t_0}^t \widetilde{F}^L(\psi_s)ds - \sum_{i=1}^\infty \int_{t_0}^t E_i^L(\psi_s)dW_s^i, \tag{102}$$

where $\widetilde{F}^L(\psi_s) = F^L(\psi_s) - \mathbb{E}[(F^L(\psi_s))]$.

Since only the atmospheric component is random, we can restrict (102) to $\widetilde{\psi}_t^a := (\widetilde{\mathbf{u}}_t^a, \, \widetilde{\theta}_t^a)^T$ to deduce that

$$\mathrm{d}\widetilde{\psi}_t^a = -\widetilde{F}^{L,a}\widetilde{\psi}_t^a dt - \sum_{i=1}^\infty E_i^{L,a}\psi_t^a dW_s^i$$

where

$$\widetilde{F}^{L,a}\widetilde{\psi}_t^a = B^{L,a}(\widetilde{\mathbf{u}}^a, \, \widetilde{\theta}^a) + C^{L,a}(\widetilde{\mathbf{u}}^a, \, \widetilde{\theta}^a) - \frac{1}{2}\sum_{i=1}^\infty \left(E_i^{L,a}\right)^2 (\widetilde{\mathbf{u}}^a, \, \widetilde{\theta}^a) - \nu\Delta(\widetilde{\mathbf{u}}^a, \, \widetilde{\theta}^a)^T$$

$$B^{L,a}\widetilde{\psi}_t^a = \left(\mathbb{E}[\mathbf{u}^a]\cdot\nabla\widetilde{\mathbf{u}}^a + \widetilde{u}_j^a\nabla\mathbb{E}[\mathbf{u}^a]^j, \, \mathbb{E}[\mathbf{u}^a]\cdot\nabla\widetilde{\theta}^a\right)^T$$

$$\simeq \left(\mathcal{L}_{\mathbb{E}[\mathbf{u}^a]}(\widetilde{\mathbf{u}}^a\cdot d\mathbf{x}), \mathcal{L}_{\mathbb{E}[\mathbf{u}^a]}\widetilde{\theta}^a\right)^T$$

$$C^{L,a}\widetilde{\psi}_t^a = \left(\frac{1}{Ro^a}\nabla\widetilde{\theta}^a, \, \gamma\widetilde{\theta}^a\right)^T \simeq \left(\frac{1}{Ro^a}d\widetilde{\theta}^a, \, \gamma\widetilde{\theta}^a\right)^T$$

$$E_i^{L,a}\psi_t^a = (\xi_i\cdot\nabla\mathbf{u}^a + u_j^a\nabla\xi_i^j, \, \xi_i\cdot\nabla\theta^a)^T \simeq \left(\mathcal{L}_{\xi_i}(\mathbf{u}^a\cdot d\mathbf{x}), \mathcal{L}_{\xi_i}\theta^a\right)^T$$

$$\left(E_i^{L,a}\right)^2\widetilde{\psi}_t^a = \left(\xi_i\cdot\nabla\left(\xi_i\cdot\nabla\widetilde{\mathbf{u}}^a + \widetilde{u}_j^a\nabla\xi_i^j\right) + \left(\xi_i\cdot\nabla\widetilde{u}_j^a + \widetilde{u}_k^a\partial_j\xi_i^k\right)\nabla\xi_i^j, \, \xi_i\cdot\nabla(\xi_i\cdot\nabla\widetilde{\theta}^a)\right)^T$$

$$\simeq \left(\mathcal{L}_{\xi_i}(\mathcal{L}_{\xi_i}(\widetilde{\mathbf{u}}^a\cdot d\mathbf{x})), \mathcal{L}_{\xi_i}(\mathcal{L}_{\xi_i}\widetilde{\theta}^a)\right)^T.$$

Here the symbol $\simeq$ recalls the geometric meanings of the coefficients appearing in the fluid equations above. Namely, the left component in the pairs $(\cdot\cdot)$ above is understood as the Lie derivative of a 1-form, while the right component is understood as a scalar function. For more discussion of the geometric meanings of these equations, see equation (125) in appendix '2D LA-SALT stochastic atmospheric model (LASAM)'.

Define next the variance of the atmospheric component $\Theta^a = \{z\Theta_t^a, t \geqslant 0\}$ as

$$\Theta_t = \mathbb{E}\left[\left|\left|\widetilde{\psi}_t^a\right|\right|_{\mathcal{H}^0}^2\right].$$

Then the previous set of equations implies the following dynamics of the variance

$$\frac{d\Theta_t}{dt} = 2\mathbb{E}\left[\left\langle\widetilde{\psi}_t^a, \widetilde{F}^{L,a}\widetilde{\psi}_t^a\right\rangle_{\mathcal{H}^0}\right] + \sum_{i=1}^\infty \mathbb{E}\left[\left|\left|E_i^{L,a}\psi_t^a\right|\right|_{\mathcal{H}^0}^2\right]. \tag{103}$$

Formula (103), along with the definitions from above, is important as it can be used to simulate the dynamics of the variance of the models. In other words, equation (103) enables one to compute the statistical dynamics of the deviations of the fluctuations of the weather that are consistent with the climatological expectation dynamics. Note that all of the quantities in the original equation combine to influence the fluctuations of the atmospheric component.

In spite of the brevity of formula (103), we observe that the variance of the fluctuations of the atmospheric component is influenced by all the components of the coupled system ψ as can be seen from the explicit description of the operators $\widetilde{F}^{L,a}$ and $E_i^{L,a}$.

This section has distinguished between the stochastic atmospheric weather model described by the SALT equations in (9)–(14) as summarised formally in equation (68) and the atmospheric expectation-fluctuation climate model described by the LA-SALT dynamical equations which are formulated in (91). The latter system of equations has led to the evolution equation (103) which predicts how the evolution of the variances of the atmospheric fluctuations are affected by statistical correlations in their fluctuating dynamics.

Our analysis for both the SALT and LA-SALT models has determined the local well-posedness properties for the dynamics of the corresponding physical variables. In this well-posed mathematical setting, we have shown that the LA-SALT expectation dynamics can combine with an intricate array of correlations in the fluctuation dynamics to determine the evolution of the mean statistics of the LA-SALT atmospheric climate model.

4. Summary conclusion and outlook

(1) We have shown that the OACM in equation set (1)–(5) analysed here for the well-known Gill–Matsuno class of models is simple enough to successfully admit the mathematical analysis required to prove the local well-posedness of these models. The physics underlying these models can be improved, of course. For example, one could naturally include heating by the Greenhouse Effect, and this heating would drive the statistical properties of the climate model.

(2) In addition to proving well-posedness for both deterministic and stochastic OACM, we have developed a new tool for climate science for predicting the evolution of climate statistics such as the variance. Indeed, the application of LA-SALT to the OACM here has established a method for also predicting the evolution of climate statistics such as the expectation of tensor moments of the fluctuations. We believe the new tools introduced here in the context of LA-SALT show considerable promise for future applications.

(3) We also expect that the shared geometric structure of these OACM will facilitate the parallel development of their numerical simulations; for example, in undertaking stochastic ENSO simulations. These stochastic simulations would be excellent candidates for the Data Analysis, Uncertainty Quantification and Particle Filtering methods for Data Assimilation which are already under development for SALT. See, e.g. [7–10].

Data availability statement

No new data were created or analysed in this study.

Acknowledgments

Appendix. Variational derivations of the deterministic and stochastic atmospheric models

Summary

In this appendix we explain the Euler–Poincaré variational principle and use it to re-derive the equations of the standard ideal 2D compressible atmospheric and incompressible oceanic

flows. These ideal models separately conserve their corresponding energy and potential vorticity. Next, we couple these models using the reduced Lagrange-d'Alembert method. Finally, we introduce stochasticity into the atmospheric flow and derive the full stochastic SALT and LA-SALT climate models which we analyse in the text in both their Stratonovich and Itô forms.

Mathematical setting

Definition A.1 (Fluid trajectory). A fluid trajectory starting from $X \in M$ in the flow domain manifold M at time $t = 0$ is given by $x(t) = g_t(X) = g(X,t)$, with $g : M \times \mathbb{R}^+ \to M$ being a smooth one-parameter submanifold (i.e. a curve parameterised by time t) in the manifold of diffeomorphisms acting on M, denoted $\mathrm{Diff}(M)$. In the deterministic case, computing the time derivative, i.e. the tangent to the curve with initial data $g(X,0) = X$ along $g_t(X)$, leads to the following *reconstruction equation*, given by

$$\partial_t g_t(X) = u(g_t(X),t), \tag{104}$$

where $u_t(\cdot) = u(\cdot,t)$ is a time-dependent vector field whose flow, $g_t(\cdot) = g(\cdot,t)$, is defined by the characteristic curves of the vector field $u_t(\cdot) \in \mathfrak{X}(M)$. The vector fields in $\mathfrak{X}(M)$ comprise the Lie algebra associated to the class of time-dependent maps $g_t \in \mathrm{Diff}(M)$.

Definition A.2 (Advected quantities and Lie derivatives). A fluid variable $a \in V^*$ defined in a vector space V^* is said to be *advected*, if it keeps its value $a_t = a_0$ along the fluid trajectories. Advected quantities are sometimes called *tracers*, because the histories of scalar advected quantities with different initial values (labels) trace out the Lagrangian trajectories of each label, or initial value, via the *push-forward* by the flow group, i.e. $a_t = g_{t*}a_0 = a_0 g_t^{-1}$, where $g_t \in \mathrm{Diff}(M)$ is the time-dependent curve on the manifold of diffeomorphisms whose action represents the evolution of the fluid trajectory by push-forward. An advected quantity a_t satisfies an evolutionary partial differential equation (PDE) obtained from the time derivative of the pull-back relation $a_0 = g_t^* a_t$, as follows

$$0 = \partial_t a_0 = \partial_t(g_t^* a_t) = g_t^*\big(\partial_t a_t + \mathcal{L}_u a_t\big) \quad \Longrightarrow \quad \partial_t a_t + \mathcal{L}_u a_t = 0,$$

where $\mathcal{L}_{u_t}(\cdot)$ denotes *Lie derivative* by the vector field u_t whose characteristic curves comprise the fluid trajectories.

Definition A.3 (Stochastic advection by Lie transport (SALT) [21]). In the setting of stochastic advection by Lie transport (SALT) the deterministic reconstruction equation in (104) is replaced by the semimartingale

$$\mathrm{d}g(X,t) = u(g_t(X),t)dt + \sum_{i=1}^{M} \xi_i(g_t(X)) \circ dW_t^i, \tag{105}$$

where the symbol $\circ$ means that the stochastic integral is taken in the Stratonovich sense. The initial data is given by $g(X,0) = X$. The W_t^i are independent, identically distributed Brownian motions, defined with respect to the standard stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geqslant 0}, \mathbb{P})$. The $\xi_i(\cdot) \in \mathfrak{X}$ are prescribed vector fields which are meant to represent uncertainty due to effects on advection of unknown rapid time dependence.

A stochastically advected quantity a_t satisfies an evolutionary stochastic partial differential equation (SPDE) obtained as a semimartingale relation via the pull-back relation $a_0 = g_t^* a_t$, as follows

$$0 = \mathrm{d}a_0 = \mathrm{d}(g_t^* a_t) = g_t^*\big(\mathrm{d}a_t + \mathcal{L}_{\mathrm{d}x_t} a_t\big) \quad \Longrightarrow \quad \mathrm{d}a_t + \mathcal{L}_{\mathrm{d}x_t} a_t = 0, \tag{106}$$

where $\mathcal{L}_{u_t}(\cdot)$ denotes *Lie derivative* by the vector field $\mathrm{d}x_t$ whose characteristic curves comprise the stochastic fluid trajectories in equation (105).

Kunita–Itô–Wentzell (KIW) formula. Equation (106) is called the *KIW formula*, after its discovery by Kunita as an extension of the Itô-Wentzell formula to define a stochastic Lie derivative for tensors and differential *k*-forms. For references and a discussion of its recent role in stochastic advection for fluid dynamics, see [3].

The deterministic limit. In what follows, any of the stochastic fluid equations derived from the Euler–Poincaré variational approach will reduce to the corresponding deterministic fluid equations by simply setting $\xi_i \to 0$ in the reconstruction equation for the stochastic fluid trajectory in (105).

Definition A.4 (The diamond operator). The *diamond operator* is defined for $a \in V^*$, $u \in \mathfrak{X}$ and fixed $b \in V$ as

$$\langle b \diamond a, u \rangle_{\mathfrak{X}^* \times \mathfrak{X}} := -\langle b, \mathcal{L}_u a \rangle_{V^* \times V}. \tag{107}$$

Here, $\langle \cdot, \cdot \rangle_{\mathfrak{X}^* \times \mathfrak{X}}$ and $\langle \cdot, \cdot \rangle_{\mathfrak{X}^* \times \mathfrak{X}}$ denote the real-valued, non-degenerate, symmetric pairings between corresponding dual spaces, which can be defined on a case-by-case basis. The diamond operator provides a map dual to the Lie derivative, as $\mathcal{L}_{(\cdot)} b : \mathfrak{X} \to V$ and $b \diamond (\cdot) : V^* \to \mathfrak{X}^*$. This duality is crucial in defining the Euler–Poincaré variational principle.

Definition A.5 (The variational derivative). The variational derivative of a functional $F : \mathcal{B} \to \mathbb{R}$, where $\mathcal{B}$ is a Banach space, is denoted $\delta F/\delta \rho$ with $\rho \in \mathcal{B}$. The variational derivative $\delta F/\delta \rho$ can be defined via the linearisation of the functional F with respect to the following infinitesimal deformation

$$\delta F[\rho] := \frac{d}{d\epsilon}\Big|_{\epsilon=0} F[\rho + \epsilon \delta \rho] = \int \frac{\delta F}{\delta \rho}(x) \delta \rho(x)\, dx =: \left\langle \frac{\delta F}{\delta \rho}, \delta \rho \right\rangle. \tag{108}$$

In the definition above, $\epsilon \ll 1 \in \mathbb{R}$ is a parameter, $\delta \rho \in \mathcal{B}$ is an arbitrary function and the variation can be understood as a Fréchet derivative. With the definition of the functional derivative in place, the following lemma can be formulated.

Theorem A.6 (Stochastic Euler–Poincaré theorem). *With the notation as above, the following statements are equivalent.*

i) The constrained variational principle

$$\delta \int_{t_1}^{t_2} \ell(u, a)\, dt = 0 \tag{109}$$

holds on $\mathfrak{X} \times V^*$, using variations δu and δa of the form

$$\delta u = dw - [dx_t, w], \qquad \delta a = -\mathcal{L}_w a, \tag{110}$$

where $w(t) \in \mathfrak{X}$ is arbitrary and vanishes at the endpoints in time for arbitrary times t_1, t_2.
ii) The stochastic Euler–Poincaré equations

$$d\frac{\delta \ell}{\delta u} + \mathcal{L}_{dx_t} \frac{\delta \ell}{\delta u} = \frac{\delta \ell}{\delta a} \diamond a\, dt, \tag{111}$$

hold on $\mathfrak{X}^*$ and the stochastic advection equations

$$da + \mathcal{L}_{dx_t} a = 0, \tag{112}$$

hold on $\times V^*$.

Proof. Using integration by parts and the endpoint conditions $w(t_1) = 0 = w(t_2)$, the variation can be computed to be

$$
\begin{aligned}
\delta \int_{t_1}^{t_2} \ell(u,a)\,dt &= \int_{t_1}^{t_2} \left\langle \frac{\delta\ell}{\delta u}, \delta u \right\rangle + \left\langle \frac{\delta\ell}{\delta a}, \delta a \right\rangle dt \\
&= \int_{t_1}^{t_2} \left\langle \frac{\delta\ell}{\delta u}, dw - [dx_t, w] \right\rangle + \left\langle \frac{\delta\ell}{\delta a}\,dt, -\mathcal{L}_w a \right\rangle \\
&= \int_{t_1}^{t_2} \left\langle -d\frac{\delta\ell}{\delta u} - \mathcal{L}_{dx_t}\frac{\delta\ell}{\delta u} + \frac{\delta\ell}{\delta a}\diamond a\,dt, w \right\rangle \\
&= 0.
\end{aligned} \tag{113}
$$

Since the vector field w is arbitrary, one obtains the stochastic Euler–Poincaré equations. Finally, the advection equation (112) follows by applying the KIW formula to $a(t) = g_{t*}a_0$. $\qquad\square$

Remark. This version of the stochastic Euler–Poincaré theorem is equivalent to the version presented in [21], which uses stochastic Clebsch constraints. In [21] one can also find an investigation of the Itô formulation of the stochastic Euler–Poincaré equation.

Theorem A.7 (Stochastic Kelvin–Noether theorem). *Let c denote a compact embedded one-dimensional smooth submanifold of M and denote $c_t = g_t(c)$ for all $t \in [0,T]$. If the mass density D_0 (a top-form) is initially non-vanishing, then*

$$
d\oint_{c_t} \frac{1}{D_t}\frac{\delta\ell}{\delta u}(u_t, a_t) = \oint_{c_t} \frac{1}{D_t}\frac{\delta\ell}{\delta a}(u_t, a_t) \diamond a_t.
$$

The integrated form of this relation is

$$
\oint_{c_t} \frac{1}{D_t}\frac{\delta\ell}{\delta u}(u_t, a_t) = \oint_{c_0} \frac{1}{D_0}\frac{\delta\ell}{\delta u}(u_0, a_0) + \int_0^t \oint_{c_s} \frac{1}{D_s}\frac{\delta\ell}{\delta a}(u_s, a_s) \diamond a_s ds.
$$

Proof. The KIW formula (106)—also known as the *Lie chain rule*—implies that

$$
\begin{aligned}
d\oint_{c_t} \frac{1}{D_t}\frac{\delta\ell}{\delta u}(u_t, a_t) &= \oint_{c_0} dg_t^*\left(\frac{1}{D_t}\frac{\delta\ell}{\delta u}\right) = \oint_{c_0} g_t^*\left(\left(d + \mathcal{L}_{dx_t}\right)\left(\frac{1}{D_t}\frac{\delta\ell}{\delta u}\right)\right) \\
&= \oint_{c_t} \left(d + \mathcal{L}_{dx_t}\right)\left(\frac{1}{D_t}\frac{\delta\ell}{\delta u}\right) = \oint_{c_t} \frac{1}{D_t}\frac{\delta\ell}{\delta a}(u_t, a_t) \diamond a_t,
\end{aligned}
$$

where the first step also uses the stochastic advection equation for D_t in (111) and the final step follows by substituting the stochastic Euler–Poincaré equations in (112). $\qquad\square$

2D SALT stochastic atmospheric model (SAM)

Summary. This part of the appendix derives the SALT atmospheric model (SAM) for the isothermal ideal gas in equations (15) and (16).

In the present notation, the Lagrangian for the deterministic 2D Compressible Atmospheric Model (DAM) in Eulerian (x,y) coordinates is,

$$
\ell[\boldsymbol{u}, D, \theta] = \int_{\Omega} \frac{D}{2}|\boldsymbol{u}|^2 + D\boldsymbol{u}\cdot\boldsymbol{R}(\boldsymbol{x}) - c_v D\theta\Pi\,dxdy, \tag{114}
$$

where $\boldsymbol{u}$ denotes 2D fluid velocity, D is mass density, θ denotes the potential temperature, the function $\boldsymbol{R}(\boldsymbol{x})$ with $\mathrm{curl}\boldsymbol{R} = 2\boldsymbol{\Omega}$ denotes the vector potential for the Coriolis parameter, c_v is specific heat at constant volume, and Π is the well-known Exner function, given by

$$\Pi = \left(\frac{p}{p_0}\right)^{R/c_p},$$

in which p_0 is a reference pressure level, c_p is specific heat at constant pressure and $R = c_p - c_v$ is the gas constant. In these variables, the equation of state for an ideal gas in 2D with n degrees of freedom is expressed as

$$\Pi = \left(\frac{RD\theta}{p_0}\right)^{R/c_p} = \left(\frac{RD\theta}{p_0}\right)^{1-\gamma^{-1}} = \left(\frac{RD\theta}{p_0}\right)^{2/(n+2)},$$

since the specific heat ratio $\gamma = c_p/c_v = 1 + 2/n$ for ideal gases whose molecules possess n degrees of freedom, comprising spatial translations, rotations and oscillations. Ideal gases of diatomic molecules in 3D have three translations, plus rotations and oscillations, so $n = 5$ and $\gamma = 7/5$ in that case.

Accordingly, the Lagrangian in (114) specialises for a ideal gas in 2D to

$$\ell\big[\boldsymbol{u},D,\theta\big] = \int_\Omega \frac{D}{2}|\boldsymbol{u}|^2 + D\boldsymbol{u}\cdot\boldsymbol{R}(\boldsymbol{x}) - \kappa(D\theta)^\alpha\,dx\,dy, \tag{115}$$

where the constants (κ,α) take the values,

$$\kappa = c_v(R/p_0)^{2/(n+2)} \quad \text{and} \quad \alpha = \frac{n+4}{n+2} = 1 + \frac{2}{n+2} = 2 - \gamma^{-1}.$$

We obtain the following variational derivatives of the Lagrangian in (114),

$$\frac{1}{D}\frac{\delta\ell}{\delta\boldsymbol{u}} = \boldsymbol{u} + \boldsymbol{R}(\boldsymbol{x}),$$
$$\frac{\delta\ell}{\delta D} = \frac{1}{2}|\boldsymbol{u}|^2 + \boldsymbol{u}\cdot\boldsymbol{R}(\boldsymbol{x}) - \kappa\alpha(D\theta)^{\alpha-1}\theta,$$
$$\frac{\delta\ell}{\delta\theta} = -\kappa\alpha(D\theta)^{\alpha-1}D. \tag{116}$$

Substitution of the variational derivatives (116) of the Lagrangian (114) into the stochastic Euler–Poincaré equations with SALT in (111) gives the SAM system

$$(\mathrm{d} + \mathcal{L}_{\mathrm{d}x_t})\big((\boldsymbol{u} + \boldsymbol{R}(\boldsymbol{x}))\cdot d\boldsymbol{x}\big) = d\left(\frac{1}{2}|\boldsymbol{u}|^2 + \boldsymbol{u}\cdot\boldsymbol{R}(\boldsymbol{x}) - (\kappa/\gamma)(D\theta)^\alpha\right)dt,$$
$$(\mathrm{d} + \mathcal{L}_{\mathrm{d}x_t})(D\theta\,dx\,dy) = 0. \tag{117}$$

Consequently, we recover the Kelvin circulation conservation law for the SAM in the compact form

$$\mathrm{d}\oint_{c_t}(\boldsymbol{u} + \boldsymbol{R}(\boldsymbol{x}))\cdot d\boldsymbol{x} = 0, \tag{118}$$

where $c_t = g_t(c_0)$ for all $t \in [0,T]$ denotes the push-forward by the SAM flow of the initial c_0, a compact embedded one-dimensional smooth submanifold of M.

Corollary A.8. *The system of SAM equations in (117) implies that potential vorticity* $q := \omega/(D\theta)$ *is conserved along flow lines of the stochastic fluid trajectory* $\mathrm{d}x_t$,

$$\mathrm{d}q + \mathrm{d}\boldsymbol{x}_t\cdot\nabla q = 0 \quad \text{with potential vorticity } q := \omega/(D\theta) \quad \text{and} \quad \omega := \widehat{\boldsymbol{z}}\cdot\mathrm{curl}\big(\boldsymbol{u} + \boldsymbol{R}(\boldsymbol{x})\big) \tag{119}$$

In turn, this formula implies that the following infinite family of integral quantities is conserved

$$C_\Phi = \int_\Omega (D\theta)\Phi(q)\,dxdy\,, \tag{120}$$

for any differentiable function Φ.

Corollary A.9. *The Deterministic AM equations in (117) with* $\boldsymbol{\xi}_i \to 0$ *are Hamiltonian, with conserved energy*[12]

$$E = \int_\Omega \frac{D}{2}|\boldsymbol{u}|^2 + \kappa(D\theta)^\alpha\,dxdy. \tag{121}$$

The Lie-Poisson Hamiltonian structure of deterministic fluid equations is discussed in [22] from the viewpoint of the Euler–Poincaré of Hamilton's principle for fluid dynamics.

Remark. The system of SAM equations in (117) may also be written equivalently in *standard* fluid dynamics notation as

$$d\boldsymbol{u} + d\boldsymbol{x}_t \cdot \nabla\boldsymbol{u} - d\boldsymbol{x}_t \times 2\boldsymbol{\Omega} + \sum_{i=1}^M \left(u_j \nabla \xi_i^j + \nabla\left(\boldsymbol{\xi}_i \cdot \boldsymbol{R}\right) \right) \circ dW_t^i = -(\kappa/\gamma)\nabla(D\theta)^\alpha dt,$$

$$d(D\theta) + \nabla \cdot \left(D\theta\,d\boldsymbol{x}_t \right) = 0. \tag{122}$$

If one assumes low Mach number, so that $D \approx 1$, and then adds viscosity and diffusion of heat, this set of equations will reproduce the SALT atmospheric model in equations (15) and (16) when one also sets $\alpha = 1 = \gamma$, which holds for the isothermal case of the ideal gas.

2D LA-SALT stochastic atmospheric model (LASAM)

Summary.   Here we provide a geometric derivation of the Lagrangian Averaged Stochastic Advection by Lie Transport (LA-SALT) atmospheric model (LASAM) for the isothermal ideal gas studied in section 3.2.

The simplest way to derive the 2D LA-SALT Stochastic Atmospheric Model (LASAM) is to alter the *Stratonovich* stochastic Lagrangian trajectory in equation (117) to the *Stratonovich* LA-SALT stochastic path, which reads

$$dx_t^a \to \ d\mathbf{X}_t^a := \mathbb{E}[\mathbf{u}^a](\boldsymbol{x},t)dt + \sum_i \xi_i^a(\boldsymbol{x}) \circ dW_i(t). \tag{123}$$

The LASAM system then emerges in *Stratonovich* stochastic geometric form as

$$\left(d + \mathcal{L}_{d\mathbf{X}_t^a}\right)\left(\boldsymbol{u} + \frac{1}{Ro^a}\boldsymbol{R}(\boldsymbol{x})\right) \cdot d\boldsymbol{x} = d\left(\frac{1}{2}|\boldsymbol{u}|^2 + \frac{1}{Ro^a}\boldsymbol{u} \cdot \boldsymbol{R}(\boldsymbol{x}) - \frac{1}{Ro^a}(\kappa/\gamma)(D\theta)^\alpha\right)dt,$$

$$\left(d + \mathcal{L}_{d\mathbf{X}_t^a}\right)(D\theta\,dxdy) = 0\,, \tag{124}$$

where $\mathcal{L}_{d\mathbf{X}_t^a}$ denotes Lie derivative with respect to *Stratonovich* LA-SALT stochastic path in (123). The stochastic geometric LASAM system in equation (124) is given in its corresponding *Itô* form by

---

[12] The energy E in (121) is not conserved for $\boldsymbol{\xi}_i \neq 0$, though, because $\boldsymbol{\xi}_i \neq 0$ injects the explicit time dependence of stochastic Lagrangian trajectories into the Euler–Poincaré variations (110) in Hamilton's principle.

$$\left(\mathrm{d} + \mathcal{L}_{\mathrm{d}\widehat{\mathbf{X}}_t^a}\right)\left(\left(\boldsymbol{u} + \frac{1}{Ro^a}\boldsymbol{R}(\boldsymbol{x})\right)\cdot d\boldsymbol{x}\right) - \frac{1}{2}\sum_i \mathcal{L}_{\xi_i^a}\left(\mathcal{L}_{\xi_i^a}\left(\left(\boldsymbol{u} + \frac{1}{Ro^a}\boldsymbol{R}(\boldsymbol{x})\right)\cdot d\boldsymbol{x}\right)\right)$$

$$= d\left(\frac{1}{2}|\boldsymbol{u}|^2 + \frac{1}{Ro^a}\boldsymbol{u}\cdot\boldsymbol{R}(\boldsymbol{x}) - \frac{1}{Ro^a}(\kappa/\gamma)(D\theta)^\alpha\right)dt, \left(\mathrm{d} + \mathcal{L}_{\mathrm{d}\widehat{\mathbf{X}}_t^a}\right)(D\theta\, dx dy)$$

$$- \frac{1}{2}\sum_i \mathcal{L}_{\xi_i^a}\left(\mathcal{L}_{\xi_i^a}(D\theta\, dx dy)\right) = 0, \tag{125}$$

where the *Itô* Lagrangian trajectory for the LA-SALT, reads

$$\mathrm{d}\widehat{\mathbf{X}}_t^a := \mathbb{E}[\mathbf{u}^a](\boldsymbol{x},t)dt + \sum_i \xi_i^a(\boldsymbol{x})dW_i(t).$$

In standard notation for fluid dynamics and with $\alpha = 1 = \gamma$, the *Stratonovich* LASAM equations in (124) become

$$\mathrm{d}\mathbf{u}^a + (\mathrm{d}\mathbf{X}_t^a\cdot\nabla)\mathbf{u}^a + \frac{1}{Ro^a}\mathrm{d}\mathbf{X}_t^{a\perp} + \sum_i\left(u_j^a\nabla\xi_i^j + \frac{1}{Ro^a}\nabla\left(R_j(\mathbf{x})\xi_i^j\right)\right)\circ dW_t^i$$

$$+ u_j^a\nabla\mathbb{E}[u^{aj}]dt + \frac{1}{Ro^a}\nabla(\mathbb{E}[\mathbf{u}^a]\cdot\mathbf{R})dt + \frac{1}{Ro^a}\nabla\theta^a\, dt = \frac{1}{Re^a}\triangle\mathbf{u}^a\, dt,$$

$$\mathrm{d}\theta^a + \mathrm{d}\mathbf{X}_t^a\cdot\nabla\theta^a = -\gamma(\theta^o - \theta^a)\, dt + \frac{1}{Pe^a}\triangle\theta^a\, dt. \tag{126}$$

In the previous equation, we have used the continuity equation to eliminate the areal density D.

Likewise, in standard notation for fluid dynamics and with $\alpha = 1 = \gamma$, the *Itô* LASAM equations in (125) become

$$\mathrm{d}\mathbf{u}^a + (\mathrm{d}\widehat{\mathbf{X}}_t^a\cdot\nabla)\mathbf{u}^a + \frac{1}{Ro^a}\mathrm{d}\widehat{\mathbf{X}}_t^{a\perp} + \sum_i\left(u_j^a\nabla\xi_i^j + \frac{1}{Ro^a}\nabla\left(R_j(\mathbf{x})\xi_i^j\right)\right)dW_t^i$$

$$+ \frac{1}{2}\left[\hat{\mathbf{z}}\times\xi\left(\mathrm{div}\left(\xi\left(\hat{\mathbf{z}}\cdot\mathrm{curl}\left(\mathbb{E}[\mathbf{u}^a] + \frac{1}{Ro^a}\mathbf{R}(\mathbf{x})\right)\right)\right)\right) - \nabla\left(\xi\cdot\nabla\left(\xi\cdot\left(\mathbb{E}[\mathbf{u}^a] + \frac{1}{Ro^a}\mathbf{R}(\mathbf{x})\right)\right)\right)\right]dt$$

$$+ u_j^a\nabla\mathbb{E}[u^{aj}]dt + \frac{1}{Ro^a}\nabla(\mathbb{E}[\mathbf{u}^a]\cdot\mathbf{R})dt + \frac{1}{Ro^a}\nabla\theta^a\, dt = \frac{1}{Re^a}\triangle\mathbf{u}^a\, dt, \tag{127}$$

$$\mathrm{d}\theta^a + \mathrm{d}\widehat{\mathbf{X}}_t^a\cdot\nabla\theta^a - \frac{1}{2}\left(\xi\cdot\nabla(\xi\cdot\nabla\theta^a)\right)dt = -\gamma(\theta^o - \theta^a)\, dt + \frac{1}{Pe^a}\triangle\theta^a\, dt. \tag{128}$$

In the final equation, we have again used the continuity equation to eliminate the areal density D.

Expectation equations. Taking the expectation of equations (127) and (128) yields a closed set of deterministic PDE for the expectations $\mathbb{E}[\mathbf{u}^a]$ and $\mathbb{E}[\theta^a]$. Subtracting the expectations from equations (127) and (128) yields linear equations for the differences,

$$\mathbf{u}^{a'} := \mathbf{u}^a - \mathbb{E}[\mathbf{u}^a] \quad \text{and} \quad \theta^{a'} := \theta^a - \mathbb{E}[\theta^a].$$

Since $\mathbf{u}^{a'}$ and $\theta^{a'}$ satisfy $\mathbb{E}[\mathbf{u}^{a'}] = 0$ and $\mathbb{E}[\theta^{a'}] = 0$, one may regard these difference variables as fluctuations of $\mathbf{u}^a$ and θ^a away from their expected values.

References

[1] Alonso-Orán D, Bethencourt de León A, Holm D D and Takao S 2020 Modelling the climate and weather of a 2D Lagrangian-averaged Euler–Boussinesq equation with transport noise *J. Stat. Phys.* **179** 1267–303

[2] Arnold L 2001 Hasselmann's program revisited: the analysis of stochasticity in deterministic climate models *Stochastic Climate Models. Progress in Probability* vol 49, ed P Imkeller and J S von Storch (Birkhäuser) pp 141–57

[3] Bethencourt de León A, Holm D D, Luesink E and Takao S 2020 Implications of Kunita–Itô–Wentzell formula for *k*-forms in stochastic fluid dynamics *J. Nonlinear Sci.* **30** 1421–54

[4] Bjerknes J 1969 Atmospheric teleconnections from the equatorial pacific *Mon. Weather Rev.* **97** 163–72

[5] Cane M A and Zebiak S E 1985 A theory for El Nino and the Southern Oscillation *Science* **228** 1084–7

[6] Cane M A, Zebiak S E and Dolan S C 1986 Experimental forecasts of El Nino *Nature* **321** 827–32

[7] Cotter C, Crisan D, Holm D D, Pan W and Shevchenko I 2019 Numerical modelling stochastic Lie transport in fluid dynamics *Multiscale Model. Simul.* **17** 192–232

[8] Cotter C, Crisan D, Holm D D, Pan W and Shevchenko I 2020 A particle filter for stochastic advection by Lie transport: a case study for the damped and forced incompressible two-dimensional Euler equation *SIAM/ASA J. Uncertain. Quantif.* **8** 1446–92

[9] Cotter C, Crisan D, Holm D D, Pan W and Shevchenko I 2020 Data assimilation for a quasi-geostrophic model with circulation-preserving stochastic transport noise *J. Stat. Phys.* **179** 1186–221

[10] Cotter C, Crisan D, Holm D D, Pan W and Shevchenko I 2020 Modelling uncertainty using stochastic transport noise in a 2-layer quasi-geostrophic model *Found. Data Sci.* **2** 173

[11] Crisan D, Flandoli F and Holm D 2019 Solution properties of a 3D stochastic Euler fluid equation *J. Nonlinear Sci.* **29** 813–70

[12] Dijkstra H A 2005 *Nonlinear Physical Oceanography* (Springer) (https://doi.org/10.1007/978-94-015-9450-9)

[13] Drivas T D and Holm D D 2020 Circulation and energy theorem preserving stochastic fluids *Proc. R. Soc.* A **150** 2776–814

[14] Drivas T D, Holm D D and Leahy J 2020 Lagrangian averaged stochastic advection by Lie transport for fluids *J. Stat. Phys.* **179** 1304–42

[15] Dutton J A 1986 *The Ceaseless Wind* (Dover)

[16] Evans L C 1998 *Partial Differential Equations* (American Mathematical Society)

[17] Frankignoul C and Hasselmann K 1977 Stochastic climate models. Part II: application to sea-surface temperature anomalies and thermocline variability *Tellus* A **29** 289–305

[18] Gill A E 1980 Some simple solutions for heat-induced tropical circulation *Q. J. R. Meteorol. Soc.* **106** 447–62

[19] Haney R L 1971 Surface thermal boundary conditions for ocean circulation models *J. Phys. Oceanogr.* **1** 241–8

[20] Hasselmann K 1976 Stochastic climate models Part I. Theory *Tellus* **28** 473–85

[21] Holm D D 2015 Variational principles for stochastic fluid dynamics *Proc. R. Soc.* A **471** 20140963

[22] Holm D D, Marsden J E and Ratiu T S 1998 The Euler–Poincaré equations and semidirect products with applications to continuum theories *Adv. Math.* **137** 1–81

[23] Kac M 1959 *Probability and Related Topics in Physical Sciences* vol 1 (*Lectures in Applied Mathematics Series, Vol 1A*) (American Mathematical Society)

[24] Klainermann S and Majda A 1981 Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids *Commun. Pure Appl. Math.* **34** 481–524

[25] Kleeman R 2008 Stochastic theories for the irregularity of ENSO *Phil. Trans. R. Soc.* A **366** 2511–26

[26] Korn P 2019 A regularity-aware algorithm for variational data assimilation of an idealized coupled atmosphere—ocean model *J. Sci. Comput.* **79** 748–86

[27] Kurtz T G and Protter P 1991 Weak limit theorems for stochastic integrals and stochastic differential equations *Ann. Probab.* **19** 1035–70

[28] Lau K M and Chan P H 1985 The 40-50 day oscillation and the El Nino-Southern oscilation: a new perspective *Bull. Am. Meteorol. Soc.* **67** 533–4

[29] Lau K M and Shen S 1988 On the dynamics of intraseasonal oscillations and ENSO *J. Atmos. Sci.* **45** 1781–97

[30] Lorenz E N 1995 Climate is what you expect (in preparation) (available at: http://eaps4.mit.edu/research/Lorenz/Climate_expect.pdf)

[31] Matsuno T 1966 Quasi-geostrophic motions in the equatorial area *J. Meteorol. Soc. Japan* **43** 25–43

[32] Majda A J, Timofeyev I and Vanden Eijnden E 2001 A mathematical framework for stochastic climate models *Commun. Pure Appl. Math.* **54** 891–974

[33] Maz'ya V 2011 *Sobolev Spaces: With Applications to Elliptic Partial Differential Equations* (*Grundlehren der Mathematischen Wissenschaften* vol 341) (Springer) (https://doi.org/10.1007/978-3-642-15564-2)

[34] McKean H P Jr 1966 A class of Markov processes associated with nonlinear parabolic equations *Proc. Natl Acad. Sci. USA* **56** 1907

[35] Neelin J D, Battisti D S, Hirst A C, Jin F-F, Wakata Y, Yamagata T and Zebiak S E 1998 ENSO theory *J. Geophys. Res.* **103** 14261–90

[36] Palmer T 2019 Stochastic weather and climate models *Nat. Rev. Phys.* **1** 463–71

[37] Perez C L, Moore A M, Zavala-Garay J and Kleeman R 2005 A comparison of the influence of additive and multiplicative stochastic forcing on a coupled model of ENSO *J. Clim.* **18** 5066–85

[38] Rozovskii R L 1990 *Stochastic Evolution Systems* (Kluwer Academic Publishers)

[39] Röckner M, Schmuland B and Zhang X 2008 Yamada–Watanabe theorem for stochastic evolution equations in infinite dimensions *Condens. Matter Phys.* **11** 247–59

[40] Sznitman A-S 1991 Topics in propagation of chaos *Ecole d'Ete de Probabilites de Saint-Flour XIX—1989* (*Lecture Notes in Mathematics* vol 1464) (Springer) pp 165–251

[41] Vlasov A A 1968 The vibrational properties of an electron gas *Sov. Phys.-Usp.* **10** 721

[42] Zebiak S 1982 A simple atmospheric model of relevance to El Nino *J. Atmos. Sci.* **39** 2017–27

[43] Zebiak S 1986 Atmospheric convergence feedback in a simple model for El Nino *Mon. Weather Rev.* **114** 1263–71

[44] Zebiak S and Cane M 1987 A model El Nino-southern oscillation *Mon. Weather Rev.* **115** 2262–78