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A FEASIBLE CENTRAL LIMIT THEOREM FOR REALISED COVARIATION OF SPDES IN THE CONTEXT OF FUNCTIONAL DATA

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This article establishes an asymptotic theory for volatility estimation in an infinite-dimensional setting. We consider mild solutions of semilinear stochastic partial differential equations and derive a stable central limit theorem for the *semigroup-adjusted realised covariation* ($SARCV$), which is a consistent estimator of the integrated volatility and a generalisation of the realised quadratic covariation to Hilbert spaces. Moreover, we introduce *semigroup-adjusted multipower variations* ($SAMPV$) and establish their weak law of large numbers; using $SAMPV$, we construct a consistent estimator of the asymptotic covariance of the mixed-Gaussian limiting process appearing in the central limit theorem for the SARCV, resulting in a feasible asymptotic theory. Finally, we outline how our results can be applied even if observations are only available on a discrete space-time grid.

1. Introduction. Estimation of volatility is of great importance for capturing the second-order structure of a random dynamical system. In this work, we develop a feasible asymptotic distribution theory for the estimation of the integrated volatility operator $\int_0^t \Sigma_s ds := \int_0^t \sigma_s \sigma_s^* ds$ corresponding to a stochastic partial differential equation (SPDE) in a separable Hilbert space H of the form

$$dY_t = (\mathcal{A}Y_t + \alpha_t)dt + \sigma_t dW_t, \quad t \in [0,T], \tag{1}$$

based on discrete observations of its mild solution within a finite time-interval $[0,T]$ for $T>0$. Here $\mathcal{A}$ is the generator of a strongly continuous semigroup $\mathcal{S} := (\mathcal{S}(t))_{t\geq 0}$ on H, W is a cylindrical Wiener process, α and σ are the drift- and volatility processes, respectively (see Section 3 below for a detailed specification). Such SPDEs constitute a well-established framework for describing spatio-temporal dynamics with applications in, e.g., finance, physics, biology, meteorology and mechanics (cf. the textbooks [40], [63], [56] or [57]). In the context of infill-asymptotics and in the presence of time-discrete observations

$$Y_0, Y_{\Delta_n}, ..., Y_{\lfloor T/\Delta_n \rfloor}, \quad \Delta_n := \frac{1}{n}$$

of a realisation of a solution to (1), the role of integrated volatility is similar to the one of the covariance operator in the analysis of i.i.d. functional data. This becomes particularly evident if σ is independent of W. In this case integrated volatility is the conditional covariance of the driving noise, that is,

$$\int_0^t \sigma_s dW_s \big| \sigma \sim \mathcal{N}\left(0, \int_0^t \Sigma_s ds\right), \quad t \geq 0.$$

Hence, a feasible estimation theory for integrated volatility in this setting could allow standard functional data analysis methods to be applied to the analysis of observations of solutions to SPDEs.

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Our theory is based on the *semigroup-adjusted realised covariation* ($SARCV$), given for $n \in \mathbb{N}$ by

$$SARCV_t^n := \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \tilde{\Delta}_i^n Y^{\otimes 2} := \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \left(Y_{i\Delta_n} - \mathcal{S}(\Delta) Y_{(i-1)\Delta_n}\right)^{\otimes 2}, \tag{2}$$

which was shown to be a consistent estimator of the integrated volatility $\int_0^t \Sigma_s ds$ in [19]. Here $h^{\otimes 2} = \langle h, \cdot \rangle h$ denotes the usual tensor product. In this paper, we consider the more involved task of proving, under suitable regularity conditions, the functional central limit theorem

$$\Delta_n^{-\frac{1}{2}} \left(SARCV_t^n - \int_0^t \Sigma_s ds \right) \overset{\mathcal{L}-s}{\Longrightarrow} \mathcal{N}(0, \Gamma_t),$$

where $\overset{\mathcal{L}-s}{\Longrightarrow}$ stands for the stable convergence in law as a process in the Skorokhod space $\mathcal{D}([0,T], \mathcal{H})$. $\mathcal{N}(0, \Gamma_t)$ is an infinite-dimensional continuous mixed Gaussian process[1] with values in $\mathcal{H}$, the space of Hilbert-Schmidt operators on H, and with a conditional covariance operator Γ_t, called the *asymptotic variance*. The above central limit theorem is not feasible, as the asymptotic variance is a priori unknown, so we also derive a consistent estimator for Γ. As this can be done conveniently by appealing to laws of large numbers for certain adjusted power and bipower variations, we also provide consistency results for general *semigroup-adjusted realised multipower variations* ($SAMPV$) given by

$$SAMPV_t^n(m_1, ..., m_k) := \sum_{i=1}^{\lfloor t/\Delta_n \rfloor - k + 1} \bigotimes_{j=1}^{k} \tilde{\Delta}_{i+j-1}^n Y^{\otimes m_j}. \tag{3}$$

We refer to the preliminaries below for the general tensor power notation.

Compared with the finite-dimensional theory, the semigroup adjustment in the realised covariation and the multipower variations might seem unusual. Nevertheless, the results presented here should be understood as a direct generalisation of the theory for multivariate semimartingales to the setting of semilinear SPDEs as in (1). This is because the semigroup adjustment just becomes relevant if $(Y_t)_{t\in[0,T]}$ is not a semimartingale, which is a purely infinite-dimensional issue. In fact, if H is finite-dimensional, $(Y_t)_{t\in[0,T]}$ is automatically a semimartingale and dropping the semigroup adjustment in (2) still yields a consistent estimator, namely, the quadratic covariation

$$RV_t^n = \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} (Y_{i\Delta_n} - Y_{(i-1)\Delta_n})^{\otimes 2}. \tag{4}$$

One can equivalently think of choosing the semigroup to equal the identity operator on H (i.e. $\mathcal{S} \equiv I$) for the sake of the limit theorems and move the part of the (in this case) strong solution belonging to the original generator $\mathcal{A}$ in equation (1) into the drift α.

For over two decades, there have been many contributions to the asymptotic theory for stochastic volatility estimation in a finite-dimensional set-up. These include the articles [11, 12, 13, 6] and [50], amongst many others, and the textbooks [52] and [1], focusing on the semimartingale set-up. Moreover, recently, attention has also turned towards finite-dimensional volatility estimation in the context when the observed process is not necessarily a semimartingale, see e.g. [37], [8], [9], [36], [35], [30], [29], [66] and [45, 61].

[1] Recall that a centred Hilbert space-valued random variable X is mixed Gaussian with random covariance $C : H \to H$ if conditional on C the random variable $\langle X, h \rangle$ a one-dimensional centred Gaussian distributed random variable with variance $\langle Ch, h \rangle$ for all $h \in H$.

There are two recent strands of research that are related to the infinite-dimensional case: during the last decade, some effort went into the generalisation of ARCH and GARCH models for functional data, appearing at a possibly high frequent rate in [49], [7], [26], [65] and [55]. At the same time, a lot of recent research has been devoted to the intricate problem of estimating volatility based on observations of finite-dimensional realisations of second-order stochastic partial differential equations (cf. [24], [27], [28], [23], [47], [4], [33], [32], [3], [60] to mention some). We refer to [31] for a survey. In that sense, volatility estimation has been approached either discretely in time or discretely in space. So in contrast to the high research activity in both of these areas, to the best of the authors' knowledge, there appear to be no results at the intersection that allow making inference on a coherent and potentially smooth spatio-temporal volatility structure as we do here. Such results, however, may be desirable in many situations. We discuss some applications and relevant types of data in the following subsection.

The presentation of our results is divided into six sections, where after a short consideration of data and some brief preliminaries following this introduction, we outline the setting for the guiding example of term structure models in Section 2 which makes the otherwise rather abstract operator-theoretic notation more concrete. We present a detailed discussion on limit theorems and applications of the $SARCV$ in Section 3, where we also include a short section on the estimation of conditional covariances in Subsection 3.2 and establish the corresponding feasible limit theory (accounting for the unknown random covariance structure in the basic central limit theorem for this estimator) in Subsection 3.3. A discussion about the convergence behaviour of the naïve quadratic variation is added in Subsection 3.4. Afterwards, we outline, how the limit theory can be applied in the case of discrete observations in time and space in Section 3.5. Section 4 addresses the laws of large numbers for the general semigroup-adjusted multipower variations $SAMPV(m_1, ..., m_k)$. Section 5 outlines the proofs of the limit theorems, which are given in full length in an Appendix. We summarise and further discuss the results in the concluding Section 6.

1.1. *Considerations on data.* As the $SARCV$ and the $SAMPV$ take into account the Hilbert space-valued data $(Y_{i\Delta_n}, i = 1, ..., \lfloor t/\Delta_n \rfloor)$, the theory presented here is part of the realm of *functional data analysis*. Functional data, which are usually sampled discretely, are often smoothed in order to obtain an element in some suitable function space. In our case, this means that practically every datum $Y_{i\Delta_n}$ should be considered as a smoothed version of discretely sampled data. Assuming that data are of high resolution in the spatial dimension as well, one can obtain fully feasible consistency results and central limit theorems for the integrated volatility operators from our results (see Section 3.5 for how this can be done for a regular sampling grid). This means, however, that (at least locally when estimating functionals of the integrated volatility) we need to have dense samples in both space and time.

Taking into account the effort that went into the development of volatility estimation in the case of sampling the solution of an SPDE at a fixed finite number of points in space and a high frequent rate in time, it might be worth underlining the following: the wording "high frequent" can be misleading, as this is primarily a matter of scale.

For instance, in financial forward and futures markets, where one wants to capture price variations for contracts with times-to-maturity of more than a year, intra-daily patterns of variation might, for some purposes, not be as insightful as e.g., intra-monthly ones. Another example is meteorological data, where in several regions we find a considerable number of weather stations measuring for instance wind, temperature or rainfall at fixed time intervals such as every hour. This leads to a reasonable volume of spatio-temporal data for a week or a month rather than a day. Moreover, reducing volatility estimation on techniques that allow making inference based on fixed multivariate samples of the SPDE might make it hard to capture spatial features like slope and curvature induced by the dynamics of neighbouring stations via the asymptotic analysis. Dynamics that are dependent on this kind of *derivative information* are of course not just relevant to meteorological applications but are for instance considered important to describe the dynamics of term structure models in finance (c.f. [34]). Smooth features of the volatility operator can be conveniently accessed in the functional data framework we elaborate on here and derivative information are inherent in the estimator itself (due to the adjustment).

On the other hand, in contrast to possibly prevalent perception, there are intraday high-frequency financial data that should eventually be considered functional. One example can be found in the modern structure of intraday energy markets. In the European intraday energy markets, participants can continuously trade contracts for energy delivery each day (from late afternoon til midnight) for all 96 quarter-hours of the day ahead. Interpreting this as a discretisation of the curve of all potential forward contracts of the next day, this can, due to no-arbitrage arguments, be considered as a semimartingale in a Hilbert space of functions. We underline, that our results are new also in the semimartingale case $\mathcal{S} = I$, leading to an infinite-dimensional theory for realised covariation of H-valued semimartingales. Arguably, in that way, it becomes possible to estimate components of the recently treated infinite-dimensional stochastic volatility models (c.f. [39, 38], [21], [17], [20]).

Preliminaries and notation. Throughout this work, H, is a separable Hilbert space. The corresponding inner product and norm are denoted by $\langle\cdot,\cdot\rangle_H$ and $\|\cdot\|_H$ and the identity operator on H by I_H, where we will drop the H-dependence most of the time and simply write $\langle\cdot,\cdot\rangle$, $\|\cdot\|$ and I. If G is another separable Hilbert space, $h \in H$ and $g \in G$, we write $L(G,H)$ for the space of bounded linear operators from G to H and $L(H) := L(H,H)$. We write $\|\cdot\|_{\text{op}}$ for the operator norm on these spaces. $L_{\text{HS}}(G,H)$ denotes the Hilbert space of Hilbert-Schmidt operators from G into H, that is $B \in L(G,H)$ such that

$$\|B\|^2_{L_{\text{HS}}(U,H)} := \sum_{n=1}^{\infty} \|Be_n\|^2 < \infty,$$

for an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of G. If $G = H$, we write $\mathcal{H} := L_{\text{HS}}(H,H)$. The operator $h \otimes g := \langle h,\cdot\rangle g$ is a Hilbert-Schmidt and even nuclear operator from H to G. Recall that B is nuclear, if $\sum_{n=1}^{\infty} \|Be_n\| < \infty$ for some orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of G. Moreover, we shortly write $h^{\otimes p} = h \otimes (h \otimes (\cdots \otimes (h \otimes h))$ and $\bigotimes_{j=1}^k h_j := h_1 \otimes ... \otimes h_k := h_1 \otimes (... \otimes (h_{k-1} \otimes h_k))$. We write recursively $\mathcal{H}^2 = \mathcal{H} = L_{\text{HS}}(H,H)$ and $\mathcal{H}^m = L_{\text{HS}}(H,\mathcal{H}^{m-1})$, for $m > 2$. Thus, $\mathcal{H}^m$ is the space of operators spanned by the orthonormal basis $(e_{j_1} \otimes \cdots \otimes e_{j_m})_{j_1,\dots,j_m\in\mathbb{N}}$, for an orthonormal basis $(e_j)_{j\in\mathbb{N}}$ of H with respect to the Hilbert-Schmidt norm. As $\mathcal{H}^m$ is isometrically isomorphic to the space $L_{\text{HS}}(\mathcal{H}^p,\mathcal{H}^q)$ if $p+q = m$ and $p,q \geq 2$ (and $L_{\text{HS}}(H,\mathcal{H}^q)$ or $L_{\text{HS}}(\mathcal{H}^p,H)$ if p or q is equal to 1), we will alternate between the notations throughout the paper. For instance, if m is even, $\mathcal{H}^m$ can be identified with the space $L_{\text{HS}}(\mathcal{H}^{\frac{m}{2}},\mathcal{H}^{\frac{m}{2}})$, which is why we can speak without loss of generality of symmetric operators on these spaces. Recall moreover that

$$\Sigma_t := \sigma_t\sigma_t^* \quad \forall t \in [0,T], \tag{5}$$

where σ is the stochastically integrable Hilbert-Schmidt operator-valued volatility process (c.f. Section 3). We will also need the notation $\Sigma_s^{\mathcal{S}_n} := \mathcal{S}(i\Delta_n - s)\Sigma_s\mathcal{S}(i\Delta_n - s)^*$ for $s \in ((i-1)\Delta_n, i\Delta_n]$. We also need different concepts of convergence of stochastic processes. Recall that a sequence of random variables $(X_n)_{n\in\mathbb{N}}$ defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ and with values in a Polish space E converges stably in law to a random variable X defined on an extension $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ of $(\Omega,\mathcal{F},\mathbb{P})$ with values in E, if for all bounded continuous $f:E\to\mathbb{R}$ and all bounded random variables Y on $(\Omega,\mathcal{F})$ we have $\mathbb{E}[Yf(X_n)]\to\tilde{\mathbb{E}}[Yf(X)]$ as $n\to\infty$, where $\tilde{\mathbb{E}}$ denotes the expectation with respect to $\tilde{\mathbb{P}}$. If, for a Hilbert space-valued process X^n, we have that it converges stably in law as a process in the Skorokhod space $\mathcal{D}([0,T];H)$, we write $X^n \stackrel{\mathcal{L}-s}{\Longrightarrow} X$. Here and throughout we always assume the space $\mathcal{D}([0,T];H)$ to be endowed with the classical Skorokhod topology, making it a Polish space (c.f. for instance chapter VI in [25]). Moreover, by $X^n \stackrel{u.c.p.}{\longrightarrow} X$ we mean convergence uniformly on compacts in probability, i.e. for all $\epsilon>0$ it is $\mathbb{P}[\sup_{t\in[0,T]}\|X^n(t)-X(t)\|>\epsilon]\to 0$ for $T>0$.

2. A motivating example: term structure models. In this section, we discuss the example of term structure models from mathematical finance arising in bond and energy markets. Term structure models, which can conveniently be expressed in form of stochastic partial differential equations, relate the time to maturity of financial contracts to their empirical and theoretical characteristics. For an introduction to the SPDE approach to modelling forward curve evolutions we refer to [43] in the case of instantaneous forward rates in bond markets and to [18] in the case of instantaneous forward prices in energy and commodity markets.

Forward curves, respectively forward prices, are usually considered to take their values in some suitable Hilbert space of functions. Besides the space of square-integrable functions $L^2(0,1)$, reproducing kernel Hilbert spaces (RKHS) and in particular Sobolev spaces such as

$$H^1(0,1):=\{h:[0,1]\to\mathbb{R}: h \text{ is absolutely continuous and } h'\in L^2(0,1)\}$$

equipped with the norm $\|h\|:=h(0)^2+\int_0^1(h'(x))^2dx$ are a reasonable choice for a state space of instantaneous forward curves. The compact interval $[0,1]$ contains all observable times to maturity (normalised by the maximal time to maturity observable). The arbitrage-free dynamics of forward curves can then be expressed in terms of the Heath-Jarrow-Morton-Musiela equation

$$df_t=(\partial_x f_t+\alpha(\sigma_s))\,ds+\sigma_s dW_s,$$

where σ is a general Hilbert-Schmidt operator valued process from a noise space U into $H=H^1(0,1)$ and $\alpha: L_{\text{HS}}(U,H)\to H$ is a continuous mapping (c.f. [43, Section 4.3]) for forward rates and vanishes entirely for commodity and energy price curves (c.f. e.g. [15]). In the space $L^2(0,1)$ of square-integrable functions, ∂_x is defined on its domain $D(\partial_x)=\{h\in H^1(0,1): h(1)=0\}$ and according to [41, Section 2.11] generates the nilpotent semigroup of left shifts in $L^2(0,1)$ given by

$$\mathcal{S}(t)h(x):=\begin{cases} h(x+t), & x+t\leq 1,\\ 0, & x+t>1.\end{cases} \tag{6}$$

In the Sobolev space, the differential operator ∂_x can be defined on its domain $D(\partial_x)=\{h\in H^1(0,1): h'\in H^1(0,1)\}$ and combining Corollary 5.1.1 in [43] and [41, Section 2.3] it is then the generator of the strongly continuous semigroup of left shifts on $H^1(0,1)$ given by

$$\mathcal{S}(t)h(x):=\begin{cases} h(x+t), & x+t\leq 1,\\ h(1), & x+t>1.\end{cases} \tag{7}$$

We may choose the noise space to be $U=L^2(0,1)$, such that we can interpret σ_s as a Hilbert-Schmidt operator from $L^2(0,1)$ into itself or that it maps into $H^1(0,1)\hookrightarrow L^2(0,1)$ and is

Hilbert-Schmidt with respect to the norm on $H^1(0,1)$ if $H = H^1(0,1)$. As such, it is given as a kernel operator

$$\sigma_s f(x) = \int_0^1 q_s(x,y) f(y) dy, \quad \forall s \geq 0, x \in [0,1].$$

In the case that $H = H^1(0,1)$ we alternatively could have chosen $U = H^1(0,1)$, as by Theorem 9 in [22] we have that in an RKHS on $[0,1]$ with kernel k, every continuous linear operator L is given by a kernel operator with kernel $l(x,y) = \langle k(x,\cdot), L^* k(\cdot,y)\rangle$ in the sense that

$$Lf(x) = \langle f, l(\cdot,x)\rangle, \quad \forall x \in [0,1].$$

We will come back to the estimation of integrated volatility in this setting for $H = H^1(0,1)$ in Section 3.5.

3. Limit theorems for the $SARCV$. Throughout this work we fix $(Y_t)_{t\in[0,T]}$ for $T > 0$ to be the mild solution of the SPDE (1), i.e. Y is a continuous adapted stochastic process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ with right-continuous filtration $(\mathcal{F}_t)_{t\in[0,T]}$ taking values in the separable Hilbert space H and is given by the stochastic Volterra process

$$Y_t = \mathcal{S}(t)Y_0 + \int_0^t \mathcal{S}(t-s)\alpha_s ds + \int_0^t \mathcal{S}(t-s)\sigma_s dW_s, \quad t \in [0,T]. \tag{8}$$

Here, $\mathcal{S} := (\mathcal{S}(t))_{t\geq 0}$ is a strongly continuous semigroup on H generated by $\mathcal{A}$ and W is a cylindrical Wiener process potentially on another separable Hilbert space U (with covariance operator I_U). Moreover, α is an almost surely Bochner integrable adapted stochastic process with values in H and σ is a Hilbert-Schmidt operator-valued process that is stochastically integrable with respect to W, i.e. for $\Omega_T := [0,T] \times \Omega$,

$$\sigma \in \left\{ \Phi : \Omega_T \to L_{\text{HS}}(U,H) : \Phi \text{ predictable and } \mathbb{P}\left[\int_0^T \|\Phi(s)\|^2_{L_{\text{HS}}(U,H)} ds < \infty\right] = 1 \right\}$$

(c.f. for instance Chapter 2.5 in [56] for the definition of the stochastic integral in this context). Both coefficients α and σ can in principle be state (or even path) dependent, provided that there is a mild solution of the form (8) to the equation. We refer to $(Y_t)_{t\in[0,T]}$ as a mild Itô process.

We present first our result on the asymptotic behaviour of the semigroup-adjusted realised covariation ($SARCV$), as it is the most important example of the (semigroup-adjusted) power variations. The law of large numbers for general multipower variations is postponed to the next section.

3.1. *Infeasible central limit theorems for the $SARCV$.* As it was shown in [19], the law of large numbers needs no further assumption on Y[2] :

[2] There are two minor differences with respect to the limit theory established in [19]: First, the driver W was assumed to have a covariance that is of trace class. However, considering the stochastic integral of a Hilbert-Schmidt operator-valued process with respect to a cylindrical Wiener noise or the stochastic integral of a process with values in $L_{HS}(Q^{1/2}U, H)$ with respect to the corresponding trace class (Q-)Wiener process in U, does not make a difference. The stochastic integral can (on an extension of the probability space) in both cases be translated into one or the other, due to the martingale representation theorems (c.f. Section 2.2.5 in [57]). Second, the drift was assumed to be almost surely square-integrable. Here, in this paper, we do not aim to derive a rate of convergence via the laws of large numbers and are in that regard able to drop these conditions.

THEOREM 3.1. *For a mild Itô process Y of the form* (8)*, we have*

$$SARCV^n \overset{u.c.p.}{\longrightarrow} \left(\int_0^t \Sigma_s ds\right)_{t\in[0,T]}.$$

The derivation of a corresponding central limit theorem, that is, the asymptotic normality of

$$\tilde{X}_t^n := SARCV_t^n - \int_0^t \Sigma_s ds := \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} (\tilde{\Delta}_i^n Y)^{\otimes 2} - \int_0^t \Sigma_s ds, \tag{9}$$

is more involved. First of all, already in finite dimensions some further conditions have to be imposed, which is why we give an analogue of the fairly mild Assumption 5.4.1(i) from [52]:

ASSUMPTION 1. The coefficients α and σ satisfy the following local integrability condition:

$$\mathbb{P}\left(\int_0^T \|\alpha_s\|^2 + \|\sigma_s\|^4_{L_{\text{HS}}(U,H)} ds < \infty\right) = 1.$$

The law of large numbers, Theorem 3.1, is very general, as there are no additional assumptions imposed on Y. However, the subtle difference to the convergence of realised variation in the finite-dimensional case is hidden in the rate of convergence. Even if Assumption 1 holds, the speed of convergence may become arbitrarily slow and might not be of magnitude $\mathcal{O}_p(\sqrt{\Delta_n})$ anymore, where $\mathcal{O}_p$ denotes *boundedness in probability* (c.f. Example 2 below). The latter is however an important condition to obtain a general infinite-dimensional central limit theorem with respect to some uniform operator topology such as the one induced by the Hilbert-Schmidt norm. In order to overcome this issue, we impose further assumptions which increase the regularity of the sample paths of the process or consider limit theorems for the mild solution process evaluated at functionals h that induce some regularity of the respective finite-dimensional process $\langle Y_t, h\rangle$[3].

To this purpose, we introduce the notion of Favard spaces. Here, for $\gamma \in (0,1)$ the γ-Favard space $F_\gamma^{\mathcal{S}}$ is defined by

$$F_\gamma^{\mathcal{S}} = F_\gamma^{\mathcal{S}}(H) := \left\{ h \in H : \|h\|_{F_\gamma^{\mathcal{S}}(N)} := \sup_{t\in[0,N]} \left\| t^{-\gamma} (I - \mathcal{S}(t))\, h \right\| < \infty, \forall N > 0 \right\}.$$

As $D(\mathcal{A}) \subset F_\gamma^{\mathcal{S}}$, these spaces always form dense subsets of H and become Banach spaces when equipped with the norm $\sup_{N\geq 0} \|\cdot\|_{F_\gamma^{\mathcal{S}}(N)}$ as long as the semigroup has a negative growth bound (c.f. [41], Chapter II.5). An example of practical importance for a subset of a $1/2$-Favard space are the evaluation functionals in a Sobolev space (this is outlined further in Section 3.5).

For functionals in the $\frac{1}{2}$-Favard space, we have the following central limit theorem in the weak operator topology:

[3]One might hope to find a uniform rate c_n such that $c_n^{-1}(SARCV_t^n - \int_0^t \Sigma_s ds)$ converges in distribution to a nontrivial law with respect some operator-topology. This is not possible in the general context we are examining: Example 2 describes a case, in which for certain irregular functionals $\sqrt{n}\langle (SARCV_t^n - \int_0^t \Sigma_s ds)h, g\rangle$ diverges. On the other hand, for another choice of functionals ($h, g \in D(\mathcal{A})$ for instance) we obtain convergence in distribution to a centered Gaussian law.

THEOREM 3.2. *Define the covariance operator process* Γ_t *for* $t \in [0,T]$ *on* $\mathcal{H}$ *by*

$$\Gamma_t B := \int_0^t \Sigma_s (B + B^*) \Sigma_s ds, \quad B \in \mathcal{H}.$$

Let $B \in \mathcal{H}$ *be an operator with a finite-dimensional range of the form* $B = \sum_{l=1}^K \mu_l h_l \otimes g_l$ *for* $h_l, g_l \in F_{\frac{1}{2}}^{\mathcal{S}^*}$, $\mu_l \in \mathbb{R}$ *for* $l = 1, ..., K$, $K \in \mathbb{N}$ *and let Assumption 1 hold. Then*

$$\left(\Delta_n^{-\frac{1}{2}} \langle \tilde{X}_t^n, B \rangle_{\mathcal{H}} \right)_{t \in [0,T]} \overset{\mathcal{L}-s}{\Longrightarrow} \left(\mathcal{N}(0, \langle \Gamma_t B, B \rangle) \right)_{t \in [0,T]},$$

where the limiting process on the right is, conditionally on $\mathcal{F}$*, a continuous centered Gaussian process with independent increments defined on a very good filtered extension* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t, \tilde{\mathbb{P}})$ *of* $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$.

For the notion of a very good filtered extension we refer to [52, Sec. 2.4.1]. Let us now give two examples of operators B that can be chosen in Theorem 3.2 to make inference on term structure models.

EXAMPLE 1. We consider examples of practical importance: local averages and evaluation functionals.

(a) (Local averages) Consider the case that $H = L^2(0,1)$ and $\mathcal{S}$ is the nilpotent shift semigroup defined in (6). We have for $t \in [0,1]$ that $\mathcal{S}^*(t) f(x) = \mathbb{I}_{[t,1]}(x) f(x-t)$. Then it holds, for $0 < b \leq 1$ and $t < b$, that

$$\|(\mathcal{S}(t)^* - I)\mathbb{I}_{[0,b]}\|_{L^2(0,1)}^2 = (\min(b+t,1) - b) + t,$$

which shows that $\mathbb{I}_{[0,b]} \in F_{\frac{1}{2}}^{\mathcal{S}^*}$ but $\mathbb{I}_{[0,b]} \notin F_{\gamma}^{\mathcal{S}^*}$ for any $\gamma > 1/2$. Since Favard-spaces are vector spaces, this yields in particular, that by virtue of Theorem 3.2 we can analyse one-dimensional (or multivariate) stochastic processes that arise as local averages over certain areas of a mild solution. That is, we can readily analyse time series $\bar{y}_{i\Delta_n}^{a,b}$, $i = 0, ..., \lfloor T/\Delta_n \rfloor$ where

$$\bar{y}_{i\Delta_n}^{a,b} := \frac{1}{b-a} \int_a^b Y_{i\Delta_n}(x) dx = \frac{1}{b-a} \langle Y_{i\Delta_n}, \mathbb{I}_{a,b} \rangle_{L^2(0,1)}.$$

For forward curves in term structure models this kind of sampling structure appears naturally as differences of yield curve values or (log-)bond prices which can be observed in the market, since for a zero coupon bond price at time t with time to maturity $x + t$ we have

$$P_t(x) = e^{-\int_0^x f_t(y) dy}.$$

In energy markets we also observe prices as weighted averages of instantaneous forward prices in the form of energy-swap contracts guaranteeing delivery of energy over a certain time (c.f. [16]). A practically relevant class of operators are, hence, weighted sums of indicator functionals of the form

$$\sum_{i,j=1}^d w_{i,j} \mathbb{I}_{[a_i,b_i]} \otimes \mathbb{I}_{[a_j,b_j]},$$

for some intervals $[a_i, b_i] \subset [0,1]$ and $w_{i,j} \in \mathbb{R}$ for $i, j = 1, ..., d$.

(b) (Evaluation functionals) For $H = H^1(0,1)$ we can define evaluation functionals δ_x by $\delta_x f = f(x)$ for all $x \in [0,1]$. These functionals satisfy $\delta_x \in F_{\frac{1}{2}}^{\mathcal{S}^*}$, while $\delta_x \notin F_{\gamma}^{\mathcal{S}^*}$ for any $\gamma > 1/2$ if $x \in [0,1)$. This is shown in Lemma 3.13 below where statistical estimation within this framework is elaborated in a fully discrete setting. We can, hence, analyse one-dimensional (or multivariate) stochastic processes that arise as evaluations of mild solutions of first-order stochastic partial differential equations at a finite number of points. A practically relevant class of operators are, thus, weighted sums of evaluation functionals of the form

$$B = \sum_{i,j=1}^{d} w_{i,j} \delta_{x_i} \otimes \delta_{x_j},$$

for some elements $x_i \subset [0,1]$ and $w_{i,j} \in \mathbb{R}$ for $i,j = 1, ..., d$.

In order to derive a stable central limit theorem for the $SARCV$ with respect to the Hilbert-Schmidt norm, we need to impose regularity assumptions on the volatility process itself, namely:

ASSUMPTION 2. One of the two following conditions holds:

(i)

$$\int_0^T \sup_{t\in[0,T]} \mathbb{E}[\|t^{-\frac{1}{2}}(I - \mathcal{S}(t))\sigma_s\|_{\text{op}}^2] ds < \infty;$$

(ii)

$$\mathbb{P}\left[\int_0^T \sup_{t\in[0,T]} \|t^{-\frac{1}{2}}(I - \mathcal{S}(t))\sigma_s\|_{\text{op}}^2 ds < \infty\right] = 1.$$

REMARK 1. Observe that if the semigroup has negative growth bound and, thus, $F_{\frac{1}{2}}^{\mathcal{S}}$ is a Banach space, Assumption 2(i) and (ii) can be rewritten as

(i) $\sigma \in L^2\left([0,T], F_{\frac{1}{2}}^{\mathcal{S}}\left(L^2(\Omega, L(U,H))\right)\right)$

(ii) $\mathbb{P}\left[\sigma \in L^2\left([0,T], L\left(U, F_{\frac{1}{2}}^{\mathcal{S}}(H)\right)\right)\right] = 1.$

Now we state the associated central limit theorem.

THEOREM 3.3. *Let Γ be as in Theorem 3.2. Under Assumptions 1 and 2 we have that*

$$(\Delta_n^{-\frac{1}{2}} \tilde{X}_t^n)_{t\in[0,T]} \overset{\mathcal{L}-s}{\Longrightarrow} (\mathcal{N}(0,\Gamma_t))_{t\in[0,T]}, \tag{10}$$

where the limiting process on the right is, conditionally on $\mathcal{F}$, a continuous centered $\mathcal{H}$-valued Gaussian process with independent increments defined on a very good filtered extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t, \tilde{\mathbb{P}})$ of $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$.

Assumption 2 is a sharp regularity criterion for the validity of the central limit theorem in the Hilbert-Schmidt norm:

EXAMPLE 2. Assumption 2 is sharp in the sense that for all $\mathfrak{H} < \frac{1}{2}$ we can always find a deterministic and constant volatility σ, such that

$$\sup_{t\in[0,T]} \|t^{-\mathfrak{H}}(I-\mathcal{S}(t))\sigma\|_{L_{\text{HS}}(U,H)} < \infty, \tag{11}$$

but convergence in distribution of $\sqrt{n}\tilde{X}_t^n$ cannot take place, even with respect to the weak operator topology. Such a specification can be done for instance in the following way: Take $H = L^2[0,2]$, $(\mathcal{S}(t))_{t\geq 0}$ the nilpotent semigroup of left-shifts, such that for $x\in[0,2], t\geq 0$ it is $\mathcal{S}(t)f(x) = \mathbb{I}_{[0,2]}(x+t)f(x+t)$ and $\sigma = e\otimes X$, where $e\in H$ such that $\|e\| = 1$ and X is an appropriately chosen path of a rough fractional Brownian motion. That is, $X(x) = B_x^{\mathfrak{H}}(\omega)$ for a fractional Brownian motion $(B_x^{\mathfrak{H}})_{x\geq 0}$ with Hurst parameter $\mathfrak{H} < \frac{1}{2}$, defined on another probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ and $\omega\in\bar{\Omega}$ is such that $B_x^{\mathfrak{H}}$ is $\mathfrak{H}$-Hölder continuous and guarantees divergence of $\sqrt{n}\tilde{X}_t^n$. Clearly, $B^{\mathfrak{H}}(\omega)$ is globally $\mathfrak{H}$-Hölder continuous on $[0,2]$ and we can find a $C>0$ such that

$$\left\|\frac{(I-\mathcal{S}(t))}{t^{\mathfrak{H}}}\sigma\right\|_{L_{\text{HS}}(U,H)}^2 = \int_0^{2-t}\left(\frac{B_x^{\mathfrak{H}}(\omega)-B_{x+t}^{\mathfrak{H}}(\omega)}{t^{\mathfrak{H}}}\right)^2 dx + \int_{2-t}^2\left(\frac{B_x^{\mathfrak{H}}(\omega)}{t^{\mathfrak{H}}}\right)^2 dx \leq C$$

Hence, we have that (11) holds. However, it is intuitively clear, that the lower $\mathfrak{H}$ is chosen, the worse the impact on the regularity of Y is, which eventually leads to divergence of $\sqrt{n}\tilde{X}_t^n$ for the rough case $\mathfrak{H} < \frac{1}{2}$. We give a detailed verification of this counterexample as well as how to choose the appropriate ω in the Appendix.

In order to account for such irregularities, one often scales the increments in a particular way and still obtains a feasible limit theory, such as was done for second-order stochastic partial differential equations in [24] or [27] and for Brownian semistationary processes in [37], [8], [9], [36], [35] and [45, 61]. However, by the law of large numbers, Theorem 3.1 we deduce that these rescaling arguments would lead to inconsistent estimators.

To get an intuition about the regularity that is induced by Assumption 2, observe the following

REMARK 2. Assumption 2(i) (and 2(ii)) increases the regularity of Y in space and time: In fact, suppose that the volatility has bounded second moment, that is, $\sup_{s\in[0,T]}\mathbb{E}\left[\|\sigma_s\|_{L_{\text{HS}}(U,H)}^2\right] < \infty$. The assumption then says that the stochastic convolution is weakly mean-square $\frac{1}{2}$-regular in time, as for each $h\in H$ and $0\leq u<t\leq T$

$$\begin{aligned}
&\mathbb{E}\left[\left(\langle\int_0^t \mathcal{S}(t-s)\sigma_s dW_s - \int_0^u \mathcal{S}(u-s)\sigma_s dW_s, h\rangle\right)^2\right]^{\frac{1}{2}} \\
&\quad\leq \left(\int_0^u \mathbb{E}\left[\|((\mathcal{S}(t-u)-I)\mathcal{S}(u-s)\sigma_s)^* h\|^2\right] ds\right)^{\frac{1}{2}} \\
&\qquad + \left(\int_u^t \mathbb{E}\left[\|(\mathcal{S}(t-s)\sigma_s)^* h\|^2\right] ds\right)^{\frac{1}{2}} \\
&\quad= \mathcal{O}\left((t-u)^{\frac{1}{2}}\right). \qquad (12)
\end{aligned}$$

If we are in a reproducing kernel Hilbert space (i.e. a Hilbert space of functions, say over an interval in $\mathbb{R}$ such that the evaluation functionals δ_x are continuous) and the semigroup is the shift semigroup, it is easy to see that the assumption also gives mean-square $\frac{1}{2}$-regularity in space: To see this, we write δ_x for the evaluation functionals in H and observe that

$$\mathbb{E}\left[\left|\int_0^t \mathcal{S}(t-s)\sigma_s dW_s(x) - \int_0^t \mathcal{S}(t-s)\sigma_s dW_s(y)\right|^2\right]^{\frac{1}{2}}$$

$$= \mathbb{E}\left[\left|\delta_0\left(\int_0^t \mathcal{S}(t-s)\mathcal{S}(y)(\mathcal{S}(x-y)-I)\sigma_s dW_s\right)\right|^2\right]^{\frac{1}{2}}$$

$$\leq \|\delta_0\| \sup_{t\in[0,T]} \|\mathcal{S}(t)\|_{\text{op}} \left(\int_0^t \mathbb{E}\left[\|\mathcal{S}(x-y)-I)\sigma_s\|_{\text{op}}^2\right] ds\right)^{\frac{1}{2}}$$

$$= \mathcal{O}(|x-y|^{\frac{1}{2}}), \tag{13}$$

by Itô's formula for $x > y$. Combining (12) and (13) we find that the random field $(t,x) \mapsto \int_0^t \mathcal{S}(t-s)\sigma_s dW_s(x)$ has mean-square regularity $\frac{1}{2}$ in space and in time.

REMARK 3 (What if the semigroup adjustment is infeasible?). The semigroup adjustment can readily be implemented in situations in which the semigroup is known and has a simple form (e.g. a simple left-shift as in term structure models). However, it should be noted that the adjustment might be hard or even impossible to implement in some cases. For instance, a commonly encountered situation is $\mathcal{A} = \kappa\mathcal{A}'$ for some known generator $\mathcal{A}'$ of a strongly continuous semigroup $(\mathcal{T}(t))_{t\geq 0}$ in H and an unknown parameter κ. In this case, we have $\mathcal{S}(t) = \mathcal{T}(\kappa t)$ and without further knowledge of the parameter κ, SARCV is an infeasible estimator.

It is, hence, important to characterise situations, in which the semigroup adjustment is superfluous and we can use the simpler infinite-dimensional realised variation (4). We give weak regularity conditions on the volatility guaranteeing consistency and asymptotic normality of RV_t^n in section 3.4. A simple, yet very relevant situation is when the volatility has a finite second moment and is contained in the domain of the generator $\mathcal{A}$ of the semigroup. Assuming the drift to be zero for convenience, it is well known that in this case the stochastic convolution (8) is a strong solution to the SPDE

$$dY_t = \mathcal{A}Y_t dt + \sigma_t dW_t, \quad Y_0 = 0, t \in [0,T],$$

(which is especially fulfilled if $\mathcal{A}$ is continuous), c.f. [57, Theorem 3.2]. This yields that Y is of the form

$$Y_t = \int_0^t \mathcal{A}Y_s ds + \int_0^t \sigma_s dW_s,$$

such that we can reinterpret Y to be a mild Itô process of the form (8) with the semigroup to be the identity and $\alpha_t = \mathcal{A}Y_t$ for the sake of the limit theory. In that way, Assumption 2 is trivially fulfilled and the realised covariation RV_t^n (c.f. (4)) is consistent and asymptotically mixed normal.

At the same time, the adjustment with the initial semigroup (generated by $\mathcal{A}$) also leads to a consistent estimator, since the semigroup is Lipschitz-continuous on the range of the volatility due to the mean value theorem. Thus, $SARCV^n$ converges in probability to the same limit and has the same asymptotic normal distribution as RV^n. However, the assumption that the volatility is in the domain of the generator $\mathcal{A}$ or the existence of a strong solution is oftentimes too strong and we give some weaker regularity conditions in Section 3.4 enabling us to use RV_t^n even in some situations in which Y does not have the pleasant semimartingale structure of a strong solution. Yet, in some important cases also these conditions might be too strong and the asymptotic equality of the semigroup-adjusted and the nonadjusted variation is not in general fulfilled (c.f. Section 3.4).

REMARK 4 (Which CLT to use in practise?). Both results Theorem 3.2 and 3.3 are central limit theorems for the same process. While Theorem 3.3 yields a more general convergence, it comes along with the additional regularity Assumption 2, while Theorem 3.2 does not impose further assumptions on the mild Itô process Y itself, but rather on the functionals under which we observe it.

Hence, we might use Theorem 3.2 in situations in which regularity assumptions on the volatility are not reasonable or cannot be guaranteed to hold and we are interested in testing hypotheses or finding confidence intervals of sufficiently regular functionals of the integrated volatility (in terms of the assumption of the theorem). Two important classes of such functionals (or even linear combinations of these) are presented in Example 1. In term structure models, for instance, we might want to quantify the estimation error of the volatility corresponding to a particular economic parameter. For instance, it is usually important to consider the spread between two forward contracts with maturities far from each other. We are then interested in confidence intervals for the volatility of the process $\langle \delta_x - \delta_y, f_t \rangle)_{t\in[0,T]}$ for the long maturity x and the short term maturity y where δ_x and δ_y are evaluation functionals $\delta_x f = f(x)$ in the Sobolev space $H^1(0,1)$ which is defined in Section 2. In this case, we have to characterise the asymptotic distribution of $\int_0^T \langle \Sigma_s(\delta_x - \delta_y), (\delta_x - \delta_y)\rangle ds$. It turns out, that the evaluation functionals δ_x and δ_y are sharply in the space $F_{\frac{1}{2}}^{\mathcal{S}^*}$ for the shift semigroup $\mathcal{S}$ defined in Section 2, such that we can use Theorem 3.2 with the choice $B = (\delta_x - \delta_y)^{\otimes 2}$ (c.f. Lemma 3.13 below).

On the other hand, if regularity Assumption 2 is reasonable to assume, Theorem 3.3 makes Theorem 3.2 obsolete. Infinite-dimensional central limit theorems as Theorem 3.3 can be used to design hypothesis tests based on nonlinear functionals of integrated volatility via an infinite-dimensional Delta method (c.f. [68, Section 3.9]), or to make inference on the eigencomponents of integrated volatility in the same way infinite-dimensional limit theorems guarantee the asymptotic normality of empirical eigenfunctions for covariance operators (c.f. [54]) and we could also test for functionals that are not in the $1/2$ Favard-space of the dual of the semigroup. The latter is for instance the case for indicator functionals (hence, local averages) in $L^2(0,1)$ and the heat semigroup (c.f. Section 3.5.1 below), for which the Favard spaces are sharply embedded into Hölder spaces of continuous functions (c.f. [41, Proposition 5.33]).

3.2. *Estimation of conditional covariance.* As argued in the introduction, estimating integrated volatility corresponds to the estimation of the conditional covariance of the noise process if we assume that the volatility and the Wiener process are independent. As opposed to the semimartingale case, however, it is not the conditional covariance of the increments or adjusted increments of a mild solution of an SPDE. The latter can, nevertheless be estimated within our framework as well and might be used for inference on the dynamics.

As a motivation, we show in the next example how we can build time-series models from HJMM-term structure dynamics.

EXAMPLE 3 (HJMM-time series model). Let us come back to the term structure model described in Section 2. Assume that the drift and volatility processes are independent of the cylindrical Wiener process and stationary. We want to build a functional quarterly time-series $(F_i)_{i\in\mathbb{N}}$ for the forward curve process, that describes the dynamics of the arbitrage-free HJMM-dynamics well and might for instance be used in forecasting. Measuring time in years, it is then

$$F_i := Y_{\frac{i}{4}} = \mathcal{S}\left(\frac{1}{4}\right) Y_{\frac{i-1}{4}} + \int_{\frac{i-1}{4}}^{\frac{i}{4}} \mathcal{S}\left(\frac{i}{4} - s\right) \alpha_s ds + \int_{\frac{i-1}{4}}^{\frac{i}{4}} \mathcal{S}\left(\frac{i}{4} - s\right) \sigma_s dW_s$$
$$= \mathcal{S}\left(\frac{1}{4}\right) Y_{\frac{i-1}{4}} + \mu_i + \epsilon_i,$$

where

$$\mu_i := \int_{\frac{i-1}{4}}^{\frac{i}{4}} \mathcal{S}\left(\frac{i}{4} - s\right) \alpha_s ds, \quad \epsilon_i := \int_{\frac{i-1}{4}}^{\frac{i}{4}} \mathcal{S}\left(\frac{i}{4} - s\right) \sigma_s dW_s.$$

Defining $\Sigma_i^* := \int_0^{\frac{1}{4}} \mathcal{S}(\frac{1}{4} - s)\Sigma_{s+\frac{(i-1)}{4}} \mathcal{S}(\frac{1}{4} - s)^* ds$, we obtain a stationary time-series of covariance operators, such that

$$\epsilon_i | \sigma \sim N(0, \Sigma_i^*), \quad i \in \mathbb{N},$$

forms a weak white noise sequence.

Assuming the time-series μ_i to be deterministic and constant and potentially violating the no-arbitrage setting, we can proceed in a straightforward manner: If μ is deterministic and constant, estimation of mean μ and covariance $C = \mathbb{E}[\Sigma_i^*]$ can be based on their empirical counterparts via the adjusted increments $(Y_{\frac{i}{4}} - \mathcal{S}(\frac{1}{4})Y_{\frac{i-1}{4}})$. We might then conduct a dimension reduction of the model by functional principal component analysis.

The conditional heteroscedasticity of the F_i would necessitate a sharper analysis of the time series of conditional covariances $(\Sigma_i^*)_{i\in\mathbb{N}}$. We might assume that it follows a particular functional time-series model and treat it as observed rather than latent in the spirit of [6]. In the latter case, this is justified by the observation that in the case of continuous semimartingales integrated volatility is the same as the conditional covariance of the increments of the process and is observable under continuous observations. In our case integrated volatility is observable as well by virtue of Theorem 3.1 but does not correspond to the conditional covariance of adjusted increments anymore. Fortunately, adjusting our estimator appropriately makes observation of the conditional covariance possible as well. Even better, we can estimate it without imposing the regularity Assumption 2. This result can be found in Corollary 3.4 below.

Let us come back to the general setting. For $0 \le U \le T$, define

$$\int_U^T \Sigma_s^T ds := \int_U^T \mathcal{S}(T-s)\Sigma_s \mathcal{S}(T-s)^* ds. \tag{14}$$

In the case that the drift and the volatility are independent of the driving Wiener process this is the conditional covariance of the adjusted increments. I.e. we have

$$(Y_T - \mathcal{S}(T-U)Y_U)|\alpha, \sigma \sim \mathcal{N}\left(\int_U^T \mathcal{S}(T-s)\alpha_s ds, \int_U^T \Sigma_s^T ds\right).$$

In that regard, it is helpful to exploit that the process

$$Y_t^T := \mathcal{S}(T)Y_0 + \int_0^t \mathcal{S}(T-s)\alpha_s ds + \int_0^t \mathcal{S}(T-s)\sigma_s dW_s$$
$$= \tilde{Y}_0 + \int_0^t \tilde{\alpha}_s ds + \int_0^t \tilde{\sigma}_s dW_s, \quad t \in [0, T],$$

is a semimartingale on H, where $\tilde{Y}_0 := \mathcal{S}(T)Y_0$, $\tilde{\alpha}_t = \mathcal{S}(T-t)\alpha_t$ and $\tilde{\sigma}_t = \mathcal{S}(T-t)\sigma_t$. Hence, the associated nonadjusted realised covariation is a consistent and asymptotically normal estimator of $\int_0^T \Sigma_s^T ds$. Luckily, in the presence of the functional data $(Y_{i\Delta_n}, i = 1,...,\lfloor T/\Delta_n \rfloor)$, we can reconstruct the quadratic variation corresponding to Y^T by

$$Y^T_{i\Delta_n} - Y^T_{(i-1)\Delta_n} = \mathcal{S}(T-i\Delta_n)Y_{i\Delta_n} - \mathcal{S}(T-(i-1)\Delta_n)Y_{(i-1)\Delta_n}.$$

This yields the following limit theorems as a corollary of Theorem 3.3 and Remark 3, which do not need Assumption 2:

COROLLARY 3.4. *We have*

$$\sum_{i=1}^{\lfloor T/\Delta_n \rfloor} \left(\mathcal{S}(T-i\Delta_n)Y_{i\Delta_n} - \mathcal{S}(T-(i-1)\Delta_n)Y_{(i-1)\Delta_n}\right)^{\otimes 2} \xrightarrow{u.c.p.} \int_0^T \Sigma_s^T ds,$$

and, if Assumption 1 holds, we also have

$$\Delta_n^{-\frac{1}{2}} \left(\sum_{i=1}^{\lfloor T/\Delta_n \rfloor} \left(\mathcal{S}(T-i\Delta_n)Y_{i\Delta_n} - \mathcal{S}(T-(i-1)\Delta_n)Y_{(i-1)\Delta_n}\right)^{\otimes 2} - \int_0^T \Sigma_s^T ds \right)$$

$$\stackrel{\mathcal{L}-s}{\Longrightarrow} \mathcal{N}(0, \int_0^T \mathcal{S}(T-s)\Sigma_s\mathcal{S}(T-s)^*(\cdot + \cdot^*)\mathcal{S}(T-s)\Sigma_s\mathcal{S}(T-s)^* ds).$$

In particular, we obtain that

$$\sum_{i=\lfloor U/\Delta_n \rfloor+1}^{\lfloor T/\Delta_n \rfloor} \left(\mathcal{S}(T-i\Delta_n)Y_{i\Delta_n} - \mathcal{S}(T-(i-1)\Delta_n)Y_{(i-1)\Delta_n}\right)^{\otimes 2} \xrightarrow{u.c.p.} \int_U^T \Sigma_s^T ds,$$

and under Assumption 1 that

$$\Delta_n^{-\frac{1}{2}} \left(\sum_{i=\lfloor U/\Delta_n \rfloor+1}^{\lfloor T/\Delta_n \rfloor} \left(\mathcal{S}(T-i\Delta_n)Y_{i\Delta_n} - \mathcal{S}(T-(i-1)\Delta_n)Y_{(i-1)\Delta_n}\right)^{\otimes 2} - \int_U^T \Sigma_s^T ds \right)$$

$$\stackrel{\mathcal{L}-s}{\Longrightarrow} \mathcal{N}(0, \int_U^T \mathcal{S}(T-s)\Sigma_s\mathcal{S}(T-s)^*(\cdot + \cdot^*)\mathcal{S}(T-s)\Sigma_s\mathcal{S}(T-s)^* ds).$$

REMARK 5 (Inadequacy of the conditional covariance for dimension reduction). It should be noted that (conditional) covariances may not be a suitable tool for dimension reduction in situations where the stochastic dynamics imposed by the SPDE should be conserved, unlike in the case of i.i.d. functional data. This can be of great importance, as SPDE dynamics often encode important physical or economic principles (such as the absence of arbitrage opportunities in term structure models).

In the energy market, for instance, there is evidence that energy spot prices are not following semimartingale-dynamics (c.f. [14]). Energy spot prices as observed in the market are averages of the lower end of the forward price curve (see e.g. [16]) and are, thus, bounded linear functionals of these in the Hilbert-space $L^2([0,1])$. This implies in particular, that energy forward curves cannot follow a strong solution to the Heath-Jarrow-Morton-Musiela equation in $L^2(0,1)$ (c.f. section 2). Corollary 1 in [42] shows that this excludes the existence of a finite-dimensional submanifold of $L^2(0,1)$ on which the solution to the Heath-Jarrow-Morton-Musiela equation is viable. Hence, given that observed energy spot prices do indeed not follow semimartingale-dynamics, the projection onto a finite-dimensional linear subspace, which is usually done via a functional principal component technique based on the covariance, violates the principle of the absence of arbitrage in the market.

In contrast, the stochastic noise process and, hence, integrated volatility can conveniently be replaced by an approximated and potentially low-dimensional version without harming the stochastic dynamics imposed by the SPDE.

We next outline how to transform Theorems 3.2 and 3.3 (as well as Corollary 3.4) into feasible results.

3.3. *Feasible central limit theorems for the $SARCV$.* The central limit Theorems 3.2 and 3.3 (and Corollary 3.4) are infeasible in practice, as we do not know the asymptotic variance operator Γ a priori. A consistent estimator of this random operator is given by the difference of the corresponding (semigroup-adjusted) fourth power- and the second bipower variation, and therefore it will be possible to derive feasible versions of Theorems 3.2 and 3.3. For that, we introduce $\hat{\Gamma}^n$ given by

$$\hat{\Gamma}_t^n := \Delta_n^{-1}\big(SAMPV_t^n(4) - SAMPV_t^n(2,2)\big). \tag{15}$$

It can be seen by the following laws of large numbers in Theorems 4.1 and 4.2 that this defines a consistent estimator of Γ. I.e., we have in $\mathcal{H}^4$

$$\hat{\Gamma}^n \xrightarrow{u.c.p.} \Gamma \quad \text{as } n\to\infty \tag{16}$$

under the following Assumption:

ASSUMPTION 3. α is locally bounded and σ is a càdlàg process w.r.t. $\|\cdot\|_{L_{\text{HS}}(U,H)}$.

This assumption corresponds to Assumption (H) in [52, p.238]. Due to the next result, the estimator $\hat{\Gamma}^n$ behaves well in the sense that it remains in the space of covariance operators:

LEMMA 3.5. *$\hat{\Gamma}_t^n$ is a symmetric and positive semidefinite nuclear (and therefore Hilbert-Schmidt) operator.*

PROOF. That it is a symmetric nuclear operator follows immediately, since it is the difference of two symmetric nuclear operators. Notice that for any real vector $(x_1,...,x_N)$ for some $N\in\mathbb{N}$ we have

$$0\leq \sum_{i=1}^{N-1}(x_{i+1}-x_i)^2 = \sum_{i=1}^{N-1}x_{i+1}^2+\sum_{i=1}^{N-1}x_i^2 - 2\sum_{i=1}^{N-1}x_{i+1}x_i \leq 2\left[\sum_{i=1}^{N}x_i^2-\sum_{i=1}^{N-1}x_{i+1}x_i\right].$$

Using this elementary inequality we obtain positive semidefiniteness, since for each $B\in\mathcal{H}$

$$\begin{aligned}&\left\langle \Delta_n\hat{\Gamma}_t^n B,B\right\rangle_{\mathcal{H}}\\ &= \sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\left\langle(\tilde{\Delta}_i^n Y)^{\otimes 2},B\right\rangle_{\mathcal{H}}^2 - \sum_{i=1}^{\lfloor t/\Delta_n\rfloor-1}\left\langle(\tilde{\Delta}_i^n Y)^{\otimes 2},B\right\rangle_{\mathcal{H}}\left\langle(\tilde{\Delta}_{i+1}^n Y)^{\otimes 2},B\right\rangle_{\mathcal{H}}.\end{aligned}$$

Hence, $\hat{\Gamma}^n$ is positive semidefinite. □

The following two results are direct corollaries of the central limit theorems 3.2 and 3.3 and the fact that two sequences of random variables defined on the same probability space with values in a Polish space, where one converges stably in law and the other converges in probability, converge jointly stably in law (c.f. [46, Thm. 3.18 (b)]). We now give the feasible version of the central limit theorem 3.2, which can be used to find confidence intervals (e.g. for evaluations in a reproducing kernel Hilbert space setting as in Subsection 3.5):

COROLLARY 3.6. *Let Assumption 3 hold and $B \in \mathcal{H}$ be an operator with a finite-dimensional range of the form $B = \sum_{l=1}^{K} \mu_l h_l \otimes g_l$ for $h_l, g_l \in F_{1/2}^{\mathcal{S}^*}$, $l = 1, ..., K$, $K \in \mathbb{N}$. Then*

$$\frac{\Delta_n^{-\frac{1}{2}} \langle \tilde{X}_t^n, B \rangle_{\mathcal{H}}}{\sqrt{\langle \hat{\Gamma}_t B, B \rangle_{\mathcal{H}}}} \xrightarrow{d} \mathcal{N}(0,1),$$

conditional on the set $\{\langle \Gamma_t B, B \rangle_{\mathcal{H}} > 0\} \subseteq \Omega$.

We also obtain a "feasible" version of Theorem 3.3:

COROLLARY 3.7. *Under Assumptions 2 and 3, we obtain*

$$\left(\Delta_n^{-\frac{1}{2}} \tilde{X}_t^n, \hat{\Gamma}_t^n\right)_{t \in [0,T]} \overset{\mathcal{L}-s}{\Longrightarrow} \left(\mathcal{N}\left(0, \Gamma_t\right), \Gamma_t\right)_{t \in [0,T]}, \tag{17}$$

where we consider the processes in the space $\mathcal{H} \times \mathcal{H}^4$, equipped with the metric

$$d((B_1, \Psi_1), (B_2, \Psi_2)) := \|B_1 - B_2\|_{\mathcal{H}} + \|\Psi_1 - \Psi_2\|_{\mathcal{H}^4}.$$

3.4. *Is the semigroup adjustment necessary?.* Certainly, in many situations, it would be convenient to use the realised quadratic variation instead of the semigroup-adjusted variation. We shall show below when this is possible but start here with an example where the realised covariation diverges.

EXAMPLE 4. Assume that for an element $e \in H$ such that $\|e\| = 1$ and an H-valued random variable X the volatility takes the simple form

$$\sigma_s = e \otimes \mathcal{S}(s) X.$$

Moreover, we assume that there is no drift and $Y(0) = 0$ and let X (and hence σ_s) be independent of the driving cylindrical Wiener process W (i.e., no so-called leverage effect). The process $\beta_t := \langle e, W_t \rangle$ is well-defined and a one-dimensional standard Brownian motion. We obtain

$$Y_t := \int_0^t \mathcal{S}(t-s) \sigma_s dW_s = \beta_t \mathcal{S}(t) X \quad \forall t \in [0,T].$$

This simple form can be exploited in order to derive counterexamples for the validity of the law of large numbers and the central limit theorem for the quadratic variation. For that, we introduce two cases:

(i) (Counterexample for the law of large numbers) $H = L^2[0,2]$, $X(x) := B_x^{\mathfrak{H}}$, where $B^{\mathfrak{H}}$ is a fractional Brownian motion with Hurst parameter $\mathfrak{H} = \frac{1}{4}$ and $(\mathcal{S}(t))_{t \geq 0}$ is the (nilpotent) left-shift semigroup given by

$$\mathcal{S}(t) f(x) := f(x+t) \mathbb{I}_{[0,2]}(x+t) \quad t \geq 0, x \in [0,2].$$

(ii) (Counterexample for the central limit theorem) $H = L^2(\mathbb{R})$, $X(x) := \mathbb{I}_{[0,1]}(x)$, and $(\mathcal{S}(t))_{t\geq 0}$ is the left-shift semigroup given by

$$\mathcal{S}(t)f(x) := f(x+t) \quad x,t \geq 0.$$

Observe that in this case Assumptions 1 and 2 are satisfied, such that the central limit theorem 3.3 holds.

We start with the first case and make the following technical observation:

$$\begin{aligned}\left\|\sum_{i=1}^n ((\mathcal{S}(\Delta_n)-I)Y_{(i-1)\Delta_n})^{\otimes 2}\right\|_{\mathcal{H}}^2 &= \sum_{i,j=1}^n \langle ((\mathcal{S}(\Delta_n)-I)Y_{(i-1)\Delta_n})^{\otimes 2}, (\mathcal{S}(\Delta_n)-I)Y_{(j-1)\Delta_n})^{\otimes 2}\rangle_{\mathcal{H}} \\ &= \sum_{i,j=1}^n \langle ((\mathcal{S}(\Delta_n)-I)Y_{(i-1)\Delta_n}), ((\mathcal{S}(\Delta_n)-I)Y_{(j-1)\Delta_n})\rangle^2 \\ &\geq \sum_{i=1}^n \|(\mathcal{S}(\Delta_n)-I)Y_{(i-1)\Delta_n})\|^4.\end{aligned}$$

Assume now that the realised variation RV_t^n converges in probability to the integrated volatility. One can show, that $(RV_t^n - \int_0^t \Sigma_s ds - \sum_{i=1}^n ((\mathcal{S}(\Delta_n)-I)Y_{(i-1)\Delta_n})^{\otimes 2})$ and therefore $\sum_{i=1}^n \|(\mathcal{S}(\Delta_n)-I)Y_{(i-1)\Delta_n})\|^4$ converges in probability to 0 and that $\sum_{i=1}^n \|(\mathcal{S}(\Delta_n)-I)Y_{(i-1)\Delta_n})\|^4$ is uniformly integrable. This is a technical exercise, which can be found in the Appendix. Thus, in the first case, we must necessarily have by Jensen's inequality

$$\begin{aligned}0 = \lim_{n\to\infty} \sum_{i=1}^n \mathbb{E}\left[\|(\mathcal{S}(\Delta_n)-I)Y_{(i-1)\Delta_n})\|^4\right] &\geq \lim_{n\to\infty} \sum_{i=1}^n \mathbb{E}\left[\|(\mathcal{S}(\Delta_n)-I)Y_{(i-1)\Delta_n})\|^2\right]^2 \\ &= \lim_{n\to\infty} \Delta_n^{2+4\mathfrak{H}} \sum_{i=1}^n (i-1)^2 > 0,\end{aligned}$$

which is a contradiction.

Assume now that the realised variation $\sqrt{n}(RV_t^n - \int_0^t \Sigma_s ds)$ converges in distribution to a normal distribution. We now turn to the second example (ii). In this case, both $\sqrt{n}(RV_t^n - \int_0^t \Sigma_s ds)$ and $\sqrt{n}(SARCV_t^n - \int_0^t \Sigma_s ds)$ are uniformly integrable, such that their convergence in distribution implies convergence of their means. This is again a technical exercise and the details can be found in the Appendix. We observe that

$$\mathbb{E}\left[RV_t^n - \int_0^t \Sigma_s ds\right] = \mathbb{E}\left[SARCV_t^n - \int_0^t \Sigma_s ds\right] + \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}\left[[(\mathcal{S}(\Delta_n)-I)Y_{(i-1)\Delta_n}]^{\otimes 2}\right].$$

Normalising by $\sqrt{n}$ we find that the first summand converges to 0, due to the uniform integrability and the central limit theorem 3.3 (i.e. convergence in distribution to a centred random variable). With the notation $\Delta_i \mathcal{S} = \mathcal{S}(i\Delta_n) - \mathcal{S}((i-1)\Delta_n)$ we find, since $\mathbb{E}\left[[(\mathcal{S}(\Delta_n)-I)Y_{(i-1)\Delta_n}]^{\otimes 2}\right] = \int_0^{(i-1)\Delta_n} (\Delta_i \mathcal{S}\mathbb{I}_{[0,1]})^{\otimes 2} ds$ that

$$\begin{aligned}\left\|\mathbb{E}\left[\sum_{i=1}^n [(\mathcal{S}(\Delta_n)-I)Y_{(i-1)\Delta_n}]^{\otimes 2}\right]\right\|_{\mathcal{H}}^2 &= \sum_{i,j=1}^n (i-1)(j-1)\Delta_n^2 \langle \Delta_i \mathcal{S}\mathbb{I}_{[0,1]}, \Delta_j \mathcal{S}\mathbb{I}_{[0,1]}\rangle^2 \\ &\geq \Delta_n^2 \sum_{i=1}^n (i-1)^2 \|\Delta_i \mathcal{S}\mathbb{I}_{[0,1]}\|^4\end{aligned}$$

$$=\Delta_n^2 \sum_{i=1}^{n} (i-1)^2 2\Delta_n^2.$$

After normalisation by $n = (\sqrt{n})^2$ the expression converges to a positive constant, which verifies that the second case (ii) provides a counterexample for the central limit theorem.

We can, however, impose assumptions on the regularity of the semigroup on the range of the volatility, such that we again obtain a law of large numbers and a central limit theorem for the realised variations. The assumption for the law of large numbers is

ASSUMPTION 4. Let almost surely

$$\lim_{t\to 0} \int_0^T \|t^{-\frac{1}{2}}(I-\mathcal{S}(t))\sigma_s\|_{L_{\text{HS}}(U,H)}^2 ds = 0.$$

REMARK 6. Assumption 4 looks similar to Assumption 2. However, in contrast to the weaker Assumption 2, Assumption 4 excludes some elementary shapes for the volatility such as the one of Example 4, for which it is simple to see that $\|(I-\mathcal{S}(t))\sigma\|_{L_{\text{HS}}(U,H)} = 2t$.

Analogously, we obtain a central limit theorem under the following assumption.

ASSUMPTION 5. Let almost surely

$$\lim_{t\to 0} \int_0^T \|t^{-\frac{3}{4}}(I-\mathcal{S}(t))\sigma_s\|_{L_{\text{HS}}(U,H)}^2 ds = 0.$$

We have the following results.

THEOREM 3.8. (i) *(Law of large numbers) If Assumption 4 is valid, we have*

$$RV_t^n \overset{u.c.p.}{\longrightarrow} \int_0^t \Sigma_s ds. \tag{18}$$

(ii) *(Central limit theorem) If Assumptions 1 and 5 are valid, we have*

$$\Delta_n^{-\frac{1}{2}} \left(RV_t^n - \int_0^t \Sigma_s ds \right) \overset{\mathcal{L}-s}{\Longrightarrow} \mathcal{N}(0, \Gamma_t). \tag{19}$$

We also have a central limit theorem in the weak operator topology as well as a law of large numbers with mild conditions on the functionals:

THEOREM 3.9. (i) *(Law of large numbers) If $B \in \mathcal{H}$ is of the form $B = \sum_{l=1}^K \mu_l h_l \otimes g_l$ for $h_l, g_l \in F_{1/2}^{\mathcal{S}^*}$ for $l = 1, ..., K,\ K \in \mathbb{N}$, we have*

$$\langle RV_t^n, B\rangle_{\mathcal{H}} \overset{u.c.p.}{\longrightarrow} \int_0^t \langle \Sigma_s, B\rangle_{\mathcal{H}} ds. \tag{20}$$

(ii) *(Central limit theorem) If $B \in \mathcal{H}$ is of the form $B = \sum_{l=1}^K \mu_l h_l \otimes g_l$ for $h_l, g_l \in F_{3/4}^{\mathcal{S}^*}$ for $l = 1, ..., K,\ K \in \mathbb{N}$ and Assumption 1 holds, we have*

$$\langle \Delta_n^{-\frac{1}{2}} \left(RV_t^n - \int_0^t \Sigma_s ds \right), B\rangle_{\mathcal{H}} \overset{\mathcal{L}-s}{\Longrightarrow} \mathcal{N}(0, \langle \Gamma_t B, B\rangle_{\mathcal{H}}). \tag{21}$$

3.5. *Discrete samples in space and time.* We discuss in this subsection the case when we have observations which are discrete in space and time. Discretisation in space yields many nontrivial challenges (e.g. owing to asynchronicity or noise). Here we want to outline how our results can be used immediately for estimation of the second-order structure of a continuous mild Itô process and therefore we assume throughout this subsection that we have observations of Y on a discrete regular space-time grid. That is, we observe

$$Y_{i\Delta_n}(j\Delta_n) := Y_{t_i}(x_j), i, j = 1, ..., n, \tag{22}$$

where for notational reasons we fix $T = 1$. We assume that H is the Sobolev space

$$H^1(0,1) := \{h : [0,1] \to \mathbb{R} : h \text{ is absolutely continuous and } h' \in L^2([0,1])\},$$

equipped with the norm $\|h\| := h(0)^2 + \int_0^1 (h'(x))^2 dx$. This is a reproducing kernel Hilbert space in which the corresponding reproducing kernel is $k(x,y) := 1 + \min(x,y)$, c.f. [22]. We write $\delta_x = k(x,\cdot)$ for both the representer of the evaluation functionals and the evaluation functionals $\delta_x f = f(x)$ in H.

Define the operator $\Pi_n : H \to H$ as the orthogonal projection onto

$$H_n := span(\delta_{j\Delta_n}, j = 1, ..., n).$$

Then, for any $h \in H$, $\Pi_n h$ can readily be recovered from the finite number of evaluations $h(j\Delta_n)$, $j = 1, ..., n$. Indeed, as $\langle \delta_{j\Delta_n}, \Pi_n h\rangle = \langle \delta_{j\Delta_n}, h\rangle = h(j\Delta_n)$, $\Pi_n h$ is the unique element in $span(\delta_{j\Delta_n}, j = 1, ..., n)$ that interpolates the points $h(j\Delta_n), j = 1, ..., n$. Thus, it is of the form

$$\Pi_n Y_{i\Delta_n} = \sum_{j=1}^n \alpha_{j,i} k(j\Delta_n, \cdot), \tag{23}$$

where $(\alpha_{1,i}, ..., \alpha_{n,i})^\perp = (\mathbb{K}_n)^{-1}(Y_{i\Delta_n}(\Delta_n), ..., Y_{i\Delta_n}(1))^\perp$ and $\mathbb{K}_n$ denotes the positive definite matrix $\mathbb{K}_n = (k(j_1\Delta_n, j_2\Delta_n))_{j_1,j_2=1,...,n}$. Observe that in this particular case, the kernel matrix has a very simple form as $k(j_1\Delta_n, j_2\Delta_n) = 1 + \Delta_n \min(j_1, j_2)$ and its inverse is given by the symmetric tridiagonal matrix $\mathbb{K}_n^{-1}$ which has entries

$$(\mathbb{K}_n^{-1})_{i,j} = \begin{cases} -n & |i-j| = 1 \\ 2n & i = j \notin \{1, n\} \\ n & i = j = n \\ 2 + \frac{n^2-2}{n+1} & i = j = 1 \\ 0 & |i-j| > 1. \end{cases}$$

This method yields the interpolating element in H that is minimal with respect to the norm in H (c.f. [22, Theorem 58]) and is a very natural choice of reconstructing a curve from discrete data. The projections are also suitable for asymptotic theory due to the subsequent lemma.

LEMMA 3.10. *The projections Π_n converge strongly to the identity on $H = H^1(0,1)$.*

PROOF. According to [22, Theorem 3] $K_0 := span(\delta_x, x \in [0,1])$ is dense in $H^1(0,1)$. For an arbitrary element $h = \sum_{i=1}^d \lambda_i \delta_{x_i} \in K_0$ let $\hat{h}_n = \sum_{i=1}^d \lambda_i \delta_{\hat{x}_i^n}$, where $\hat{x}_i^n \in \{j\Delta_n, j = 1, ..., n\}$ which is closest to x_i. We then have $\|\delta_{x_i} - \delta_{\hat{x}_i}\| \leq |x_i - \hat{x}_i| \leq \Delta_n$ for all $i = 1, ..., d$ and, thus, $\|h - \hat{h}_n\| \leq \Delta_n \sum_{j=1}^d |\lambda_j|$. Now let $h \in H$ and $\epsilon > 0$. We can choose a $g \in span(\delta_x, x \in [0,1])$ such that $\|h - g\| \leq \frac{\epsilon}{2}$ and for g we can find an $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$ there is an $h_n \in span(\delta_{j\Delta_n}, j = 1, ..., n)$ such that $\|g - h_n\| \leq \epsilon/2$. Thus, since Π_n is an orthogonal projection, for all $n \geq n_0$ we have

$$\|(I - P_n)h\| \leq \|h - h_n\| \leq \|h - g\| + \|g - h_n\| \leq \epsilon.$$

□

Let us now derive asymptotic results in the fully discrete setting (22). We outline the situation here in two cases, which are of practical importance and well-suited for these observations. In the first case, we have a continuous Itô semimartingale in H. This covers suitable frameworks for intraday energy markets, as mentioned in the introductory section. In the second case, $\mathcal{S}$ is the semigroup of left shifts, which for instance corresponds to the framework of Heath-Jarrow-Morton term structure models, c.f. [43], for interest rates and for energy forward markets, c.f. [18]. For a different sampling scheme we will also include a short discussion on the stochastic heat equation in a separate subsection afterwards.

(a) (Semimartingale case) The semigroup is equal to the identity (or can be interpreted as such in the case of a strong solution as in Remark 3). That is, we observe a continuous Itô semimartingale

$$Y_t = Y_0 + \int_0^t \alpha_s ds + \int_0^t \sigma_s dW_s.$$

In that case, we define the operator

$$\hat{\Sigma}_t^n = \Pi_n RV_t^n \Pi_n = \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} (\Pi_n \Delta_i^n Y)^{\otimes 2}.$$

The latter is feasible, as we can derive the values $\Delta_i^n Y(j\Delta_n) = Y_{i\Delta_n}(j\Delta_n) - Y_{(i-1)\Delta_n}(j\Delta_n)$ from data and, hence, can derive $\Pi_n \Delta_i^n Y$ by (23).

(b) (Shift case) $\mathcal{S}$ is the semigroup of left shifts, given by

$$\mathcal{S}(t)h(x) := \begin{cases} h(x+t), & x+t \leq 1, \\ h(1), & x+t > 1, \end{cases}$$

which forms a the strongly continuous semigroup on $H^1(0,1)$. In that case, we define the operator

$$\hat{\Sigma}_t^n = \Pi_n SARCV_t^n \Pi_n = \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} (\Pi_n \tilde{\Delta}_i^n Y)^{\otimes 2}.$$

The latter is feasible, as we can derive the values $\tilde{\Delta}_i^n Y(j\Delta_n) = Y_{i\Delta_n}(j\Delta_n) - Y_{(i-1)\Delta_n}((j+1)\Delta_n)$ for $j = 1, ..., n-1$ and $\Delta_i^n Y(1) = 0$ (by the definition of the semigroup) from data and, hence, can derive $\Pi_n \tilde{\Delta}_i^n Y$ by (23) also in this case.

The proof of the next result makes use of Theorem 3.1.

LEMMA 3.11. *In both cases (a) and (b), we have*

$$\hat{\Sigma}_t^n \xrightarrow{u.c.p.} \int_0^t \Sigma_s ds,$$

with respect to the Hilbert-Schmidt norm on $\mathcal{H} = L_{HS}(H^1(0,1))$.

PROOF. Let A_n denote either RV_t^n in case (a) or $SARCV_t^n$ in case (b). Then it is

$$\|\Pi_n A_n \Pi_n - \Pi_n \int_0^t \Sigma_s ds \Pi_n\|_{\mathcal{H}} \leq \|A_n - \int_0^t \Sigma_s ds\|_{\mathcal{H}},$$

which converges to 0 uniformly on compacts in probability in both cases by Theorem 3.1. Moreover, $\Pi_n\Sigma_s\Pi_n$ converges to Σ_s with respect to the nuclear (and hence the Hilbert-Schmidt) norm for all $s\in[0,1]$, which follows by Lemma 3.10 and combining Proposition 4 and Lemma 5 in [59]. The u.c.p. convergence follows by dominated convergence as

$$\sup_{t\in[0,1]}\left\|\int_0^t \Pi_n\Sigma_s\Pi_n-\Sigma_s ds\right\|_{\mathcal{H}}\leq \int_0^1 \|\Pi_n\Sigma_s\Pi_n-\Sigma_s\|_{\mathcal{H}}\,ds.$$

□

Due to the semimartingale property of the processes $(Y_t(x))_{t\in[0,T]}$ in case (a), both by the finite-dimensional limit theory outlined in [52] or by appealing to Theorem 3.2 we have the following result.

COROLLARY 3.12. *In case (a), for $x\in[0,1]$, we have*

$$\sqrt{n}\left(\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\left(Y_{i\Delta_n}(x)-Y_{(i-1)\Delta_n}(x)\right)^2-\int_0^t\langle\sigma_s,\delta_x\rangle^2 ds\right)\xrightarrow{\mathcal{L}-s}\mathcal{N}(0,\langle\Gamma_t\delta_x^{\otimes 2},\delta_x^{\otimes 2}\rangle_{\mathcal{H}}).$$

A feasible version, conditional on the set $\{\langle\Gamma_t\delta_x^{\otimes 2},\delta_x^{\otimes 2}\rangle_{\mathcal{H}}>0\}\subseteq\Omega$, is given by

$$\begin{aligned}&\left(\sum_{i=1}^{n}\left(Y_{i\Delta_n}(x)-Y_{(i-1)\Delta_n}(x)\right)^4\right.\\&\left.-\sum_{i=1}^{n-1}\left(Y_{i\Delta_n}(x)-Y_{(i-1)\Delta_n}(x)\right)^2\left(Y_{(i+1)\Delta_n}(x)-Y_{(i)\Delta_n}(x)\right)^2\right)^{-\frac{1}{2}}\\&\times\left(\sum_{i=1}^{n}\left(Y_{i\Delta_n}(x)-Y_{(i-1)\Delta_n}(x)\right)^2-\int_0^t\langle\sigma_s,\delta_x\rangle^2 ds\right)\\&\xrightarrow{d}\mathcal{N}(0,1).\end{aligned}$$

It's notable that the central limit theorem can be recovered in case (b) as well, due to the following observation: In the case that $H=H^1(0,1)$, the representations δ_x of evaluation functionals are in the $\frac{1}{2}$-Favard spaces of the shift semigroup and its dual. Namely, we have

LEMMA 3.13. *Let $H=H^1(0,1)$ and $\mathcal{S}$ be the left shift semigroup. Then the representations δ_x, for any $0\leq x\leq 1$, of the evaluation functionals are elements in the Favard class $F_{1/2}^{\mathcal{S}}$ and $F_{1/2}^{\mathcal{S}^*}$, but for $x\in(0,1]$ not in the γ-Favard spaces $F_\gamma^{\mathcal{S}}$ and for $x\in[0,1)$ $F_\gamma^{\mathcal{S}^*}$ with respect to the shift semigroup for $\gamma>\frac{1}{2}$.*

Let us assume for the moment we are in case (b) for the process

$$Y_t(x)=Y_0(x+t)+\int_0^t\alpha_s(x+t-s)ds\int_0^t\langle\sigma_s,\delta_{x+t-s}\rangle dW_s.$$

This leads to the following useful limit theorem, which enables us to find confidence bounds for the process $(\int_0^t\langle\sigma_s,\delta_x\rangle^2 ds)_{t\in[0,T]}$ based on observations $(Y_{i\Delta_n}(x),Y_{i\Delta_n}(x+\Delta_n)),i=1,...,n$ in case (b):

COROLLARY 3.14. *In case (b), we have, for $x \in [0,1]$, due to the central limit Theorem 3.2 (respectively, Theorem 3.6)*

$$\sqrt{n}\left(\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\left(Y_{i\Delta_n}(x)-Y_{(i-1)\Delta_n}(x+\Delta_n)\right)^2-\int_0^t\langle\sigma_s,\delta_x\rangle^2ds\right)\xrightarrow{\mathcal{L}-s}\mathcal{N}(0,\langle\Gamma_t\delta_x^{\otimes 2},\delta_x^{\otimes 2}\rangle).$$

A feasible version, conditional on the set $\{\langle\Gamma_t\delta_x^{\otimes 2},\delta_x^{\otimes 2}\rangle_{\mathcal{H}}>0\}\subseteq\Omega$, is given by

$$\left(\sum_{i=1}^{n}\left(Y_{i\Delta_n}(x)-Y_{(i-1)\Delta_n}(x+\Delta_n)\right)^4\right.$$
$$\left.-\sum_{i=1}^{n-1}\left(Y_{i\Delta_n}(x)-Y_{(i-1)\Delta_n}(x+\Delta_n)\right)^2\left(Y_{(i+1)\Delta_n}(x)-Y_{i\Delta_n}(x+\Delta_n)\right)^2\right)^{-\frac{1}{2}}$$
$$\times\left(\sum_{i=1}^{n}\left(Y_{i\Delta_n}(x)-Y_{(i-1)\Delta_n}(x+\Delta_n)\right)^2-\int_0^t\langle\sigma_s,\delta_x\rangle^2ds\right)$$
$$\xrightarrow{d}\mathcal{N}(0,1).$$

We remark, that even for case (b), Lemma 3.13 also guarantees that Theorem 3.9(i) applies. Hence, it holds that

$$\sum_{i=1}^{n}\left(Y_{i\Delta_n}(x)-Y_{(i-1)\Delta_n}(x)\right)^2\xrightarrow{u.c.p.}\int_0^t\langle\sigma_s,\delta_x\rangle^2ds.$$

Therefore, we just need observations $Y_{i\Delta_n}(x), i=1,...,n$ to estimate the quadratic variation of the one-dimensional processes $(Y_t(x))_{t\in[0,T]}$ consistently.

3.5.1. *A note on the stochastic heat equation.* As already mentioned in Remark 3, the semigroup adjustment can be easily implemented in cases in which we know the semigroup and it has a simple form, which is not always the case. A prototypical example is the stochastic heat equation with an unknown diffusivity $\kappa>0$ taking the form

$$dY_t=\kappa\partial_{xx}Y_tdt+Q^{\frac{1}{2}}dW_t.$$

Here we assume that $\int_0^t Q^{\frac{1}{2}}dW_s$ is formally a Q-Wiener process taking values in $H=L^2[0,1]$ with an unknown nuclear covariance operator Q. The differential operator ∂_{xx} on the domain $D(\partial_{xx})=\{h\in L^2[0,1]:\|f'\|+\|f''\|<\infty, f(0)=f(1)=0\}$ generates an analytic semigroup on H given by

$$\mathcal{S}(t)f=\sum_{j=1}^{\infty}e^{t\lambda_j}\langle e_j,f\rangle e_j,$$

where $\lambda_j=\pi^2j^2\kappa$ and $e_j(x):=\sqrt{2}\sin(\pi jx)$ (see for instance Example B.12 in [63]). In this situation, the regularity of the dynamics is very often expressed in terms of Sobolev spaces, which can be formally defined as

$$\dot{H}^r:=D\left(\partial_{xx}^{\frac{r}{2}}\right)=\left\{h\in H:\|h\|_{\dot{H}^r}^2:=\sum_{j=1}^{\infty}\lambda_j^r\langle e_j,h\rangle^2<\infty\right\}. \tag{24}$$

Equipped with the norm $\|\cdot\|_{\dot{H}^r}=\|(-\mathcal{A})^{\frac{r}{2}}\cdot\|$, these are separable Hilbert spaces. Now, if W is a cylindrical Wiener process on $L^2(0,1)$ and

$$Q^{\frac{1}{2}}\in L_{\text{HS}}\left(L^2(0,1),\dot{H}^r\right), \tag{25}$$

it follows by Theorem 6.13 in Section 2.6 of [62]

$$\begin{aligned}\sup_{t\in[0,T]} t^{-\frac{r}{2}}\|(\mathcal{S}(t)-I)Q^{\frac{1}{2}}\|_{L_{\text{HS}}(U,H)}=&\sup_{t\in[0,T]} t^{-\frac{r}{2}}\|A^{-\frac{r}{2}}(\mathcal{S}(t)-I)A^{\frac{r}{2}}Q^{\frac{1}{2}}\|_{L_{\text{HS}}(U,H)}\\ \le& C\|A^{\frac{r}{2}}Q^{\frac{1}{2}}\|_{L_{\text{HS}}(U,H)}\\ =&C\|Q^{\frac{r}{2}}\|_{L_{\text{HS}}(U,\dot{H}^r)}<\infty.\end{aligned} \tag{26}$$

This yields

LEMMA 3.15. *If in (25) we have*

(a) $r=1$, *then Assumption 2 holds and the semigroup-adjusted realised covariation satisfies the infinite-dimensional central limit theorem 3.3;*
(b) $r>1$, *then Assumption 4 holds and the realised variation satisfies the infinite-dimensional law of large numbers Theorem 3.8(i);*
(b) $r>\frac{3}{2}$, *then Assumption 5 holds and the realised variation satisfies the infinite-dimensional central limit theorem 3.8(ii).*

As we do not necessarily know κ, it might not be possible to implement the semigroup adjustment. Even if we knew κ, on the basis of discrete observations we would need to approximate the semigroup appropriately to implement the adjustment such as it is done in [48]. In this regard, cases (b) and (c) of the previous theorem are particularly appealing, as they hold for the realised variation, which does not take into account an adjustment by the semigroup. Still, also the latter has to be approximated by discrete data. Here we assume that we sample data from the mild solution to the stochastic heat equation as local averages, that is, we have

$$\bar{y}_{i,j}^{n,m}:=\frac{1}{\Delta_m}\int_{(j-1)\Delta_m}^{j\Delta_m} Y_{i\Delta_n}(x)dx,\quad i=0,...,n, j=1,...,m.$$

Observe that in this case, we can have a different spatial and temporal resolution. Let Π_m denote the projection onto the subspace of $L^2[0,1]$ spanned by the orthonormal vectors $\Delta_m\mathbb{I}_{[(j-1)\Delta_m,j\Delta_m]}$. Then we can recover $\Pi_m\Delta_i^m Y$ from data as this is simply corresponding to the piecewise constant function given by

$$\Pi_m\Delta_i^m Y=\sum_{i=1}^m(\bar{y}_{i,j}^{n,m}-\bar{y}_{i-1,j}^{n,m})\mathbb{I}_{[(j-1)\Delta_m,j\Delta_m]}.$$

We can, thus, readily derive the estimator

$$\hat{\Sigma}_t^{n,m}:=\Pi_m RV_t^n\Pi_m=\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}(\Pi_m\Delta_i^n Y)^{\otimes 2},$$

from data as well. For a sufficiently regular Q, we then obtain an infinite-dimensional law of large numbers:

LEMMA 3.16. *Assume (25) holds with $r>1$. Then $\hat{\Sigma}_t^{n,m}$ is a consistent estimator, that is, with respect to the Hilbert-Schmidt norm it is as $n,m\to\infty$*

$$\hat{\Sigma}_t^{n,m}\stackrel{u.c.p.}{\longrightarrow}tQ.$$

PROOF. We have

$$\|\hat{\Sigma}_t^{n,m} - \Pi_m tQ\Pi_m\| \leq \|RV_t^n - tQ\|,$$

which converges to 0 by Lemma 3.15 (b) as $n \to \infty$. As $\Pi_m \to I$ strongly in $L^2(0,1)$ as $m \to \infty$ we also have that $\|\Pi_m(tQ)\Pi_m - tQ\|_{L_{\text{HS}}(L^1(0,1))}$ converges to 0 as $m \to \infty$ by combining Proposition 4 and Lemma 5 in [59]. □

We may also derive a central limit theorem for the one-dimensional observations.

LEMMA 3.17. *Assume that* (25) *holds with* $r > 3/2$ *and that* $m = m_n$ *with* $\lim_{n\to\infty} n\Delta_{m_n} = 0$. *Then for all* $h \in H$ *it is*

$$\sqrt{n}\langle\left(\hat{\Sigma}_t^{n,m} - tQ\right)h, h\rangle \overset{\mathcal{L}-s}{\Longrightarrow} \mathcal{N}\left(0, 2t\langle Qh, h\rangle^2\right).$$

PROOF. We decompose

$$\begin{aligned}&\sqrt{n}\left(\hat{\Sigma}_t^{n,m} - tQ\right)\\ =&\sqrt{n}\left(RV_t^n - tQ\right) + \left(\sqrt{n}\left(\hat{\Sigma}_t^{n,m} - \Pi_m tQ\Pi_m\right) - \sqrt{n}\left(RV_t^n - tQ\right)\right) + \sqrt{n}\left(\Pi_m tQ\Pi_m - tQ\right).\end{aligned}$$

The first term converges stably in law to the limiting Gaussian process as specified in the assertion as $n \to \infty$. It, thus, remains to show that the other two terms converge to 0 uniformly on compacts in probability.

For the second summand we denote $h_m = \Pi_m h$ and find that

$$\begin{aligned}&\sqrt{n}\left|\langle\left(\hat{\Sigma}_t^{n,m} - \Pi_m tQ\Pi_m\right)h, h\rangle - \sqrt{n}\langle\left(RV_t^n - tQ\right)h, h\rangle\right|\\ \leq&\sqrt{n}\,\|RV_t^n - tQ\|\,\|h_m - h\|(\|h_m\| + \|h\|).\end{aligned}$$

As the first factor is bounded in probability and $h_m \to h$ as $m \to \infty$, this converges to 0. For the second summand we have that $\sqrt{n}(\Pi_m Q\Pi_m - Q)$ we find

$$\langle\Pi_m Q\Pi_m h - Qh, h\rangle \leq \|(I - \Pi_m)Qh\| + \|(I - \Pi_m)Qh_m\| = (1)_m + (2)_m.$$

For the first summand we can argue that as Q maps into

$$\dot{H}^{\frac{3}{2}} \subset \dot{H}^1 \subset H^1(0,1)$$

by Lemma 3.1 in [67], we have that for any $h \in H$ that $\partial_x Qh = (Qh)' \in L^2(0,1)$ and $(Qh_m)' \in L^2(0,1)$ as well. Hence, for $Qh_m^*(\cdot) := \sum_{i=1}^m Qh(i\Delta_m)\mathbb{I}_{[(i-1)\Delta_m, i\Delta_m]}(\cdot) \in span(\mathbb{I}_{[(i-1)\Delta_m, i\Delta_m]} : i = 1, ..., m)$ we have

$$(1)_m^2 \leq \|Qh - Qh_m^*\|^2 = \left\|\sum_{i=1}^m\left(\int_x^{i\Delta_m}(Qh)'(y)dy\right)\mathbb{I}_{[(i-1)\Delta_m, i\Delta_m]}(x)\right\|^2 \leq \Delta_m\|(Qh)'\|^2$$

and in the same way and using Lemma 3.1 in [67]

$$(2)_m^2 \leq \Delta_m\|(Qh_m)'\|^2 = \Delta_m\|\partial_{xx}^{\frac{1}{2}} Qh_m\|^2 \leq \Delta_m\|Q\|^2_{L_{\text{HS}}(L^2(0,1),\dot{H}^1)}\|h_m\|^2.$$

Summing up, we get

$$\begin{aligned}\sqrt{n}\langle\Pi_m Q\Pi_m h - Qh, h\rangle \leq&\|(I - \Pi_m)Qh\| + \|(I - \Pi_m)Qh_m\|\\ =&\sqrt{n}\sqrt{\Delta_m}\left(\|(Qh)'\| + \|Q\|_{L_{\text{HS}}(L^2(0,1),\dot{H}^1)}\|h_m\|\right).\end{aligned}$$

This converges to 0 as $\sqrt{n\Delta_m} \to 0$ as $n \to \infty$ by assumption. □

Analytic semigroups such as the heat semigroups can impose regularity on the sample paths of Y and potentially allow to weaken the conditions of Lemma 3.15, which may not be sharp in this setting. We postpone a thorough analysis of these conditions in case of analytic semigroups to future research.

4. A law of large numbers for multipower variations. We still have to verify the consistency (16) of the estimator for the asymptotic variance Γ_t. Rather than proving only this specific result, we provide general laws of large numbers for power and multipower variations in this section.

For a positive symmetric trace-class operator Σ, we define the operator $\rho_\Sigma(m)$, as the mth *tensor moment* of an H-valued random variable $U \sim \mathcal{N}(0,\Sigma)$, i.e.,

$$\rho_\Sigma(m) = \mathbb{E}[U^{\otimes m}]. \tag{27}$$

This operator can be characterised by the identity

$$\langle \rho_{\Sigma_s}(m), h_1 \otimes ... \otimes h_m \rangle_{\mathcal{H}^m} = \sum_{p \in \mathcal{P}(m)} \prod_{(x,y)\in p} \langle \Sigma h_x, h_y \rangle, \tag{28}$$

for any collection $h_1, ..., h_m \in H$, where the sum is taken over all pairings over $(1,...,m)$, i.e., all ways to disjointly decompose $(1,...,m)$ into pairs. We denote the set of all these pairings by $\mathcal{P}(m)$, which is then given as

$$\mathcal{P}(m) = \Big\{ p \subset \{1,...,m\}^2 : \#p = \frac{m}{2} \text{ and if } (x,y),(x',y') \in p, \\ \text{then } x,y,x',y' \text{ are pairwise unequal and } x<y, x'<y' \Big\}.$$

In particular, $\rho_\Sigma(m) = 0$, if m is odd. In the case of power variations, we need

ASSUMPTION 6 (m). For a natural number $m \in \mathbb{N}$ we have

$$\mathbb{P}\left[\int_0^T \|\alpha_s\|^{\frac{2m}{2+m}} ds + \int_0^T \|\sigma_s\|^m_{L_{\text{HS}}(U,H)} ds < \infty \right] = 1. \tag{29}$$

Observe that the assumption above corresponds to Condition 3.4.6 in the finite-dimensional law of large numbers Theorem 3.4.1 in [52]. We now state a law of large numbers for semigroup-adjusted power variations:

THEOREM 4.1. *Let $m \geq 2$ be a natural number and Assumption 6(m) be valid. Then*

$$\Delta_n^{1-\frac{m}{2}} SAMPV^n(m) \xrightarrow{u.c.p.} \left(\int_0^t \rho_{\Sigma_s}(m) ds \right)_{t\in[0,T]},$$

with respect to the Hilbert-Schmidt norm on $\mathcal{H}^m$.

Let us study some examples:

EXAMPLE 5. If $m = 2$, there is just one way to decompose $\{1,2\}$ into pairs, i.e., $\mathcal{P}(2)$ consists of the pair $\{(1,2)\}$ only. Therefore $\rho_{\Sigma_s}(2) = \Sigma_s$, and in particular, the law of large numbers reads in this case

$$SARCV_t^n(2) \xrightarrow{u.c.p.} \int_0^t \Sigma_s ds,$$

which corresponds to the law of large numbers Theorem 3.1.

EXAMPLE 6. If $m = 4$, then we find that $\mathcal{P}(4)$ consists of the pairs $\{(1,2),(3,4)\}$, $\{(1,3),(2,4)\}$ and $\{(1,4),(2,3)\}$. Hence, it follows,

$$\begin{aligned}
&\langle \rho_{\Sigma_s}(4), h_1 \otimes ... \otimes h_4 \rangle_{\mathcal{H}^4} \\
&\quad = \langle \Sigma_s h_1, h_2 \rangle \langle \Sigma_s h_3, h_4 \rangle + \langle \Sigma_s h_1, h_3 \rangle \langle \Sigma_s h_2, h_4 \rangle + \langle \Sigma_s h_1, h_4 \rangle \langle \Sigma_s h_2, h_3 \rangle \\
&\quad = \langle \Sigma_s^{\otimes 2} + \Sigma_s(\cdot + \cdot^*)\Sigma_s, h_1 \otimes h_2 \otimes h_3 \otimes h_4 \rangle.
\end{aligned}$$

This yields $\rho_{\Sigma_s}(4) = \Sigma_s(\cdot + \cdot^*)\Sigma_s + \Sigma_s^{\otimes 2}$.

For a positive symmetric trace class operator $\Sigma : H \to H$, define for $m, m_1, ..., m_k \in \mathbb{N}$ such that $m = m_1 + ... + m_k$

$$\rho_\Sigma^{\otimes k}(m_1, ..., m_k) := \bigotimes_{j=1}^k \rho_\Sigma(m_j),$$

which is an operator in $\mathcal{H}^m$, such that for any collection $(h_{j,l}) \subset H$, $j = 1, ..., k$ and $l = 1, ..., m_j$ we have

$$\langle \rho_\Sigma^{\otimes k}(m_1, ..., m_k), \bigotimes_{l=1}^{m_1} h_{1,l} \otimes ... \otimes \bigotimes_{l=1}^{m_k} h_{k,l} \rangle_{\mathcal{H}^m} = \prod_{j=1}^k \sum_{p \in \mathcal{P}(m_l)} \prod_{(x,y) \in p} \langle \Sigma_s h_{x,j}, h_{y,j} \rangle.$$

We have the following law of large numbers for multipower variations:

THEOREM 4.2. *Let Assumption 3 hold and $m, m_1, m_2, \ldots, m_k$ be natural numbers such that $m_1 + ... + m_k = m \geq 2$. Then*

$$\Delta_n^{1-\frac{m}{2}} SAMPV^n(m_1, ..., m_k) \xrightarrow{u.c.p.} \left(\int_0^t \rho_{\Sigma_s}^{\otimes k}(m_1, ..., m_k) ds \right)_{t \in [0,T]}. \tag{30}$$

Let us consider the important example of bipower variation:

EXAMPLE 7 (Bipower variation). Let $m_1 = m_2 = k = 2$, i.e., $m = 4$, and define the bipower variation

$$SAMPV_t^n(2,2) = \sum_{i=1}^{\lfloor t/\Delta_n \rfloor - 1} \tilde{\Delta}_i^n Y^{\otimes 2} \otimes \tilde{\Delta}_{i+1}^n Y^{\otimes 2}. \tag{31}$$

Observe that $\rho_{\Sigma_s}^{\otimes 2}(2,2) = \rho_{\Sigma_s} \otimes \rho_{\Sigma_s} = \Sigma_s^{\otimes 2}$ by Example 5.

5. Outline of the Proofs. We will now provide an outline of the proofs of the main results (i.e. Theorems 3.1, 3.2, 3.3, 4.1 and 4.2). The remaining results Theorem 3.8, Theorem 3.9, Lemma 3.11 and Lemma 3.13 as well as Examples 2 and 4 are consequences of these limit theorems. The detailed proofs are relegated to the Appendix.

Throughout this section, we let p_N be the projection onto $v^N := \overline{lin(\{e_j : j \geq N\})}$, for some orthonormal basis $(e_j)_{j \in \mathbb{N}}$ that is contained in $D(\mathcal{A}^*)$, and P_N^m denote the projection onto $\overline{lin(\{\bigotimes_{l=1}^m e_{k_l} : k_l \geq N\})}$ (where m is variable, corresponding to the particular case). In the special case $m = 2$ we write $P_N^2 =: P_N$.

First, it is important to note that we can appeal to localised versions of the assumptions of Theorems 3.1, 3.2, 3.3, 4.1 and 4.2. This is a common procedure that follows the arguments presented in Section 4.4.1 in [52], which enables us to prove all theorems stated in this work under such localised assumptions. The localised assumptions essentially impose boundedness instead of almost sure finiteness, in order to ensure the existence of all necessary moments.

The first important observation is the following: By the localisation procedure, we can assume there is a constant A, such that

$$\int_0^T \|\alpha_s\|^{\frac{m}{2}} + \|\sigma_s\|^m_{L_{\text{HS}}(U,H)} ds < A. \tag{32}$$

In this case, the $SAMPV$, when projected onto functionals of the form $\bigotimes_{l=1}^m e_{j_l}$, for an orthonormal basis $(e_j)_{j\in\mathbb{N}}$ which is contained in $D(\mathcal{A}^*)$, $m \in \mathbb{N}$ and $j_1, ..., j_m \in \mathbb{N}$, corresponds asymptotically to the tensor multipower variations of the semimartingale

$$S_t := \int_0^t \alpha_s ds + \int_0^t \sigma_s dW_s.$$

We find that

$$\begin{aligned}\langle SAMPV_t^n(m_1,...,m_k), \bigotimes_{l=1}^m e_{j_l}\rangle_{\mathcal{H}^m}\\ = \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \langle \bigotimes_{j=1}^k \Delta_{i+j-1}^n S^{\otimes m_j}, \bigotimes_{l=1}^m e_{j_l}\rangle_{\mathcal{H}^m} + \mathcal{O}_p(\Delta_n^{\frac{m}{2}}).\end{aligned} \tag{33}$$

As the left-hand side of (33) corresponds to a multivariate continuous semimartingale, the limit theorems from [52] are readily available.

Now we come to the second important observation: For that, define the two sequences

$$a_N(z) := \sup_{n\in\mathbb{N}} \mathbb{E}\left[\int_0^T \|p_N \alpha_s^{\mathcal{S}_n}\|_{\mathcal{H}}^z ds\right], \quad b_N(z) := \sup_{n\in\mathbb{N}} \mathbb{E}\left[\int_0^T \|p_N \sigma_s^{\mathcal{S}_n}\|^z_{L_{\text{HS}}(U,H)} ds\right], \tag{34}$$

for $z \leq m$, $\sigma_s^{\mathcal{S}_n} = \mathcal{S}(i\Delta_n - s)\sigma_s$ and $\alpha_s^{\mathcal{S}_n} = \mathcal{S}(i\Delta_n - s)\alpha_s$ with $s \in ((i-1)\Delta_n, i\Delta_n]$. Observe that

$$\Sigma_s^{\mathcal{S}_n} = \sigma_s^{\mathcal{S}_n}(\sigma_s^{\mathcal{S}_n})^*.$$

Under (32) both $a_N(z)$ for $z \leq m/2$ and $b_N(z)$ for $z \leq m$ converge to 0 as $N \to \infty$ for $z \leq m$, respectively $z \leq \frac{m}{2}$. Moreover, we can find for all $m \in \mathbb{N}$ a universal constant $C > 0$ possibly depending on m, such that

$$\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}\left[\|p_N \tilde{\Delta}_i^n Y\|^m\right] \leq C\Delta_n^{\frac{m}{2}-1}(a_N(\frac{m}{2}) + b_N(m)) = o(\Delta_n^{\frac{m}{2}-1}). \tag{35}$$

We notice that the Hilbert-Schmidt structure of the volatility is crucial to establish that $b_N(z)$ converges to 0.

The proofs for limit theorems in this work follow a similar pattern. For the laws of large numbers:

(LLNa) Show that $(\Delta_n^{1-\frac{m}{2}}(I - P_N^m)(SAMPV_t^n - \int_0^t \rho^{\otimes k}(m_1, ..., m_k)ds))_{t\in[0,T]}$ converges for all $N \in \mathbb{N}$ to 0 as $n \to \infty$, due to the available limit theory for finite-dimensional semimartingales.

(LLNb) Show that $(\Delta_n^{1-\frac{m}{2}} P_N^m SAMPV_t^n)_{t\in[0,T]}$ converges to 0 uniformly in n and t as $N \to \infty$. Standard arguments then imply that

$$\left(\Delta_n^{1-\frac{m}{2}}\left(SAMPV_t^n - \int_0^t \rho^{\otimes k}(m_1, ..., m_k)ds\right)\right)_{t\in[0,T]} \xrightarrow{u.c.p.} 0 \quad \text{as } n \to \infty.$$

For the central limit theorems for the $SARCV$ we have

(CLTa) Show that

$$(36)\qquad (\tilde{Z}^{n,2}_t)_{t\in[0,T]} := \left(\Delta_n^{-\frac{1}{2}}\left(\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\tilde{\Delta}_i^n Y^{\otimes 2} - \int_{(i-1)\Delta_n}^{i\Delta_n}\Sigma_s^{\mathcal{S}_n}ds\right)\right)_{t\in[0,T]}$$

for $n\in\mathbb{N}$, which is a sequence of sums of martingale differences, is tight in $\mathcal{D}([0,T],\mathcal{H})$ provided that the (localised) Assumption 1 holds.

(CLTb) Prove that under (localised) Assumption 1 the finite-dimensional distributions $((I-P_N)\tilde{Z}^{n,2}_t)_{t\in[0,T]}$ converge to an asymptotically conditional Gaussian process with the covariance $(I-P_N)\Gamma_t(I-P_N)$ by virtue of (32) and the finite-dimensional limit Theorem 5.4.2 in [52].

(CLTc) In order to prove Theorem 3.3, we appeal to Assumption 2 to show that

$$\Delta_n^{-\frac{1}{2}}\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\int_{(i-1)\Delta_n}^{i\Delta_n}\left(\Sigma_s^{\mathcal{S}_n}-\Sigma_s\right)ds \xrightarrow{u.c.p.} 0,$$

and for Theorem 3.2 to the fact that the operator B has its finite-dimensional range in the $1/2$-Favard class of the dual semigroup in order to show that

$$\Delta_n^{-\frac{1}{2}}\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\int_{(i-1)\Delta_n}^{i\Delta_n}\langle(\Sigma_s^{\mathcal{S}_n}-\Sigma_s),B\rangle_{\mathcal{H}}ds \xrightarrow{u.c.p.} 0.$$

5.1. *Comments on the proof of the laws of large numbers.* The imposed conditions on the law of large numbers Theorems 4.1 and 4.2 state that the finite-dimensional multipower variations $\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}((I-P_N^m)\bigotimes_{j=1}^k\Delta_{i+j-1}^n S^{\otimes m_j})$ fulfil the required conditions of the corresponding laws of large numbers. In the case of power variations, that is under the localised version of Assumption 6, Theorem 3.4.1 in [52] is applicable. For the multipower variations with the localised version of Assumption 3, Theorem 8.4.1 in [52] applies and yields (LLNa).

Now, observe that the triangle inequality yields

$$\left\|P_N^m\left(SAMPV_t^n(m_1,...,m_k)-\int_0^t\rho_{\Sigma_s}^{\otimes k}(m_1,...,m_k)ds\right)\right\|_{\mathcal{H}^m}$$
$$\leq \|P_N^m SAMPV_t^n(m_1,...,m_k)\|_{\mathcal{H}^m}+\left\|P_N^m\int_0^t\rho_{\Sigma_s}^{\otimes k}(m_1,...,m_k)ds\right\|_{\mathcal{H}^m}.$$

For a given $\epsilon>0$, after appealing to the inequalities of Markov and Hölder together with (35), one finds that

$$\sup_{n\in\mathbb{N}}\mathbb{P}\left[\sup_{t\leq T}\Delta_n^{1-\frac{m}{2}}\|P_N^m SAMPV_t^n(m_1,...,m_k)\|_{\mathcal{H}^m}>\epsilon\right]\to 0 \quad \text{as } N\to\infty.$$

Moreover, straightforward calculations lead to

$$\|\rho_{p_N\Sigma_s p_N}(m)\|_{\mathcal{H}^m}^2\leq|\mathcal{P}(m)|^2(\sum_{j\geq N}\|\Sigma_s^{\frac{1}{2}}e_j\|^2)^m,$$

which converges to 0 as $N\to\infty$, since $\Sigma_s^{\frac{1}{2}}$ is a Hilbert-Schmidt operator. Through Markov's inequality, one finds

$$\mathbb{P}\left[\sup_{t\leq T}\left\|P_N^m\int_0^t\rho_{\Sigma_s}^{\otimes k}(m_1,...,m_k)ds\right\|_{\mathcal{H}^m}>\epsilon\right]\leq\frac{|\mathcal{P}(m)|}{\epsilon}\int_0^T\mathbb{E}\left[\left(\sum_{j\geq N}\|\Sigma_s^{\frac{1}{2}}e_j\|^2\right)^{\frac{m}{2}}\right]ds.$$

Dominated convergence implies that this converges to 0 as $N\to\infty$, which shows (LLNb).

5.2. *Comments on the proofs of the central limit theorem.* In order to show tightness for the sequence $\tilde{Z}^{n,2}$ we appeal to a criterion from [53, p.35]:

THEOREM 5.1. *Let H be a separable Hilbert space. The family of laws $(\mathbb{P}_{\psi^n})_{n\in\mathbb{N}}$ of a family of random variables $(\psi^n)_{n\in\mathbb{N}}$ in $\mathcal{D}([0,T],H)$ is tight if the following two conditions hold:*

(i) $(\mathbb{P}_{\psi^n_t})_{n\in\mathbb{N}}$ *is tight for each $t\in[0,T]$ and*
(ii) *(Aldous' condition) For all $\epsilon,\eta>0$ there is an $\delta>0$ and $n_0\in\mathbb{N}$ such that for all sequences of stopping times $(\tau_n)_{n\in\mathbb{N}}$ with $\tau_n\leq T-\delta$ we have*

$$\sup_{n\geq n_0}\sup_{\theta\leq\delta}\mathbb{P}[\|\psi^n_{\tau_n}-\psi^n_{\tau_n+\theta}\|_H>\eta]\leq\epsilon. \tag{37}$$

After some tedious estimations, one can verify Aldous' condition for $\tilde{Z}^{n,2}$ under the localised versions of Assumptions 1. Then it remains to show the spatial tightness, that is tightness of $(\tilde{Z}^{n,2}_t)_{n\in\mathbb{N}}$ as random sequences in $\mathcal{H}$ for each $t\in[0,T]$. In order to do this, we argue under condition (32) that, without loss of generality, we can assume $\alpha\equiv 0$. Moreover, we will appeal to the following criterion, which is based on the equi-small tails-characterisation of compact sets in Hilbert spaces and is well known (c.f. Lemma 1.8.1 in [68]):

LEMMA 5.2. *Let $(Y_n)_{n\in\mathbb{N}}$ be a sequence of random variables on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ with values in a separable Hilbert space H and having finite second moments. If for some orthonormal basis $(e_n)_{n\in\mathbb{N}}$ we have*

$$\lim_{N\to\infty}\sup_{n\in\mathbb{N}}\sum_{k\geq N}\mathbb{E}\left[\langle Y_n,e_k\rangle^2\right]=0, \tag{38}$$

then the sequence $(Y_n)_{n\in\mathbb{N}}$ is tight.

To show the spatial tightness of $\tilde{Z}^{n,2}$, we observe that

$$\sum_{m,k\geq N}\langle\tilde{Z}^{2,n}_t,e_k\otimes e_m\rangle^2_{\mathcal{H}}=\|\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\tilde{Z}^N_n(i)\|^2_{\mathcal{H}},$$

where

$$\tilde{Z}^N_n(i):=\Delta_n^{-\frac{1}{2}}\left((p_N\tilde{\Delta}^n_iY)^{\otimes 2}-\int_{t_{i-1}}^{t_i}p_N\mathcal{S}(t_i-s)\Sigma_s\mathcal{S}(t_i-s)^*p_Nds\right).$$

Next note that $t\mapsto\psi_t=\int_{(i-1)\Delta_n}^t p_N\mathcal{S}(i\Delta_n-s)\sigma_sdW_s$ is a martingale for $t\in[(i-1)\Delta_n,i\Delta_n]$. From [63, Theorem 8.2, p. 109] we then deduce that the process $(\zeta_t)_{t\geq0}$ given by $\zeta_t=(\psi_t)^{\otimes 2}-\langle\langle\psi\rangle\rangle_t$, where $\langle\langle\psi\rangle\rangle_t=\int_{(i-1)\Delta_n}^t p_N\mathcal{S}(t_i-s)\Sigma_s\mathcal{S}(t_i-s)^*p_Nds$, is a martingale w.r.t. $(\mathcal{F}_t)_{t\geq0}$. Therefore $\mathbb{E}[\tilde{Z}^N_n(i)|\mathcal{F}_{t_{i-1}}]=0$ and $\mathbb{E}[\langle\tilde{Z}^N_n(i),\tilde{Z}^N_n(j)\rangle_{\mathcal{H}}]=0$, which yields $\mathbb{E}[\|\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\tilde{Z}^N_n(i)\|^2_{\mathcal{H}}]=\sum_{i=1}^{\lfloor T/\Delta_n\rfloor}\mathbb{E}[\|\tilde{Z}^N_n(i)\|^2_{\mathcal{H}}]$. Moreover, it holds

$$\mathbb{E}[\|\tilde{Z}^N_n(i)\|^2_{\mathcal{H}}]\leq 4\Delta_n\int_{(i-1)\Delta_n}^{i\Delta_n}\mathbb{E}[\|p_N\sigma_s^{\mathcal{S}_n}\|^4_{L_{\text{HS}}(U,H)}]ds,$$

such that we ultimately obtain

$$\sum_{k,l\geq N}\mathbb{E}\left[\langle\tilde{Z}^{n,2}_t,e_k\otimes e_l\rangle^2\right]\leq 4\sup_{n\in\mathbb{N}}\int_0^T\mathbb{E}\left[\|p_N\sigma_s^{\mathcal{S}_n}\|^4_{L_{\text{HS}}(U,H)}\right]ds,$$

which converges to 0 due to (34). Lemma 5.2 yields the claim in (CLTa), i.e., we have shown the following intermediate result:

THEOREM 5.3. *Let Assumption 1 hold. Then the sequence of processes* $(\tilde{Z}_t^{n,2})_{t\in[0,T]}$ *is tight in* $\mathcal{D}([0,T],\mathcal{H})$.

We now outline the proof of the stable convergence in law as a process of the finite-dimensional distributions $(\langle \tilde{Z}_t^{n,2}, e_k \otimes e_l\rangle)_{k,l=1,\dots,d}$. Due to (33) and after some technical calculations, these finite-dimensional distributions can be asymptotically identified with the ones of the quadratic variation of the associated multivariate semimartingale, i.e., the stable limit of $(\langle \tilde{Z}_t^n e_k, e_l\rangle)_{k,l=1,\dots,d}$ is the same as the one of

$$\left(\Delta_n^{-\frac{1}{2}}\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\left(\langle \Delta_i^n S, e_k\rangle\langle \Delta_i^n S, e_l\rangle - \int_{(i-1)\Delta_n}^{i\Delta_n}\langle \Sigma_s e_k, e_l\rangle ds\right)\right)_{k,l=1,\dots,d}.$$

The latter is a component of the difference between realised quadratic covariation and the quadratic covariation of the d-dimensional continuous local martingale $S_t^d = (\langle S_t, e_1\rangle, \dots, \langle S_t, e_d\rangle)$. Therefore, $(\langle \tilde{Z}_t^n e_k, e_l\rangle)_{k,l=1,\dots,d}$ converges by Theorem 5.4.2 from [52] stably as a process to a continuous (conditional on $\mathcal{F}$) mixed normal distribution which can be realised on a very good filtered extension as

$$N_{k,l} = \frac{1}{\sqrt{2}}\sum_{c,b=1}^{d}\int_0^t \hat{\sigma}_{kl,bc}(s) + \hat{\sigma}_{lk,bc}(s) dB_s^{cb}.$$

Here, $\hat{\sigma}(s)$ is $d^2\times d^2$-matrix, being the square-root of the matrix $\hat{c}(s)$ with entries $\hat{c}_{kl,k'l'}(s) = \langle \Sigma_s e_k, e_{k'}\rangle\langle \Sigma_s e_l, e_{l'}\rangle$. Furthermore, B is a matrix of independent Brownian motions. As now all finite-dimensional distributions converge stably and the sequence of measures is tight, we obtain by a modification of Proposition 3.9 in [46] that the convergence is indeed stable in the Skorokhod space. One can then show that the asymptotic normal distribution has covariance Γ_t. This gives (CLTb) and thus an auxiliary central limit theorem, which does not rely on the spatial regularity condition in Assumption 2:

THEOREM 5.4. *Let Assumption 1 hold. We have that* $\tilde{Z}^{n,2} \overset{\mathcal{L}-s}{\Rightarrow} (\mathcal{N}(0,\Gamma_t))_{t\in[0,T]}$.

In order to prove Theorem 3.3 we have to show $\Delta_n^{-\frac{1}{2}}\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\int_{(i-1)\Delta_n}^{i\Delta_n}\Sigma_s^{\mathcal{S}_n} - \Sigma_s ds \overset{u.c.p.}{\longrightarrow} 0$. As $e_k \in D(\mathcal{A}^*)$ and due to the fact that $\|(\mathcal{S}(\Delta_n)^* - I)e_k\| = \|\int_0^{\Delta_n}\mathcal{S}(u)^*\mathcal{A}^* e_k du\| = \mathcal{O}(\Delta_n)$, it is relatively straightforward to show that for all $N\in\mathbb{N}$

$$(I-P_N)\Delta_n^{-\frac{1}{2}}\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\int_{(i-1)\Delta_n}^{i\Delta_n}\left(\Sigma_s^{\mathcal{S}_n} - \Sigma_s\right) ds \overset{u.c.p.}{\longrightarrow} 0. \tag{39}$$

Further, we find by the triangle, Bochner and Hölder inequalities

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\in[0,T]}\left\|P_N\Delta_n^{-\frac{1}{2}}\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\int_{(i-1)\Delta_n}^{i\Delta_n}\left(\Sigma_s^{\mathcal{S}_n} - \Sigma_s\right) ds\right\|_{\mathcal{H}}\right]\\ &\leq \left(\int_0^T \mathbb{E}\left[\left\|\Delta_n^{-\frac{1}{2}}(\mathcal{S}(\lfloor s/\Delta_n\rfloor\Delta_n - s) - I)\sigma_s\right\|_{\text{op}}^2\right] ds\right)^{\frac{1}{2}}\\ &\quad\times\left(\int_0^T \sqrt{2}\mathbb{E}\left[\|p_N\sigma_s\|_{L_{\text{HS}}(U,H)}^2 + \|p_N\mathcal{S}(\lfloor s/\Delta_n\rfloor\Delta_n - s)\sigma_s\|_{L_{\text{HS}}(U,H)}^2\right] ds\right)^{\frac{1}{2}}.\end{aligned}$$

The first factor is finite by Assumption (2)(i), whereas the second one converges to 0 as $N \to \infty$ by (34). By combining this with (39) the claim follows and Theorem 3.3 is proved.

In order to prove Theorem 3.2 we can argue similarly as for Theorem 3.3 that we just have to show $\Delta_n^{-\frac{1}{2}} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \int_{(i-1)\Delta_n}^{i\Delta_n} \langle \Sigma_s^{\mathcal{S}_n} - \Sigma_s, B \rangle_{\mathcal{H}} ds \overset{u.c.p.}{\longrightarrow} 0$. We can argue componentwise, which is why we assume without loss of generality that $B = h \otimes g$ and split the expression into two integral terms:

$$\begin{aligned}
&\Delta_n^{-\frac{1}{2}} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \int_{(i-1)\Delta_n}^{i\Delta_n} \langle (\Sigma_s^{\mathcal{S}_n} - \Sigma_s) h, g \rangle ds \\
&\quad = \Delta_n^{-\frac{1}{2}} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \int_{(i-1)\Delta_n}^{i\Delta_n} \langle ((\mathcal{S}(i\Delta_n - s) - I)\Sigma_s \mathcal{S}(i\Delta_n - s)^*) h, g \rangle ds \\
&\quad\quad + \Delta_n^{-\frac{1}{2}} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \int_{(i-1)\Delta_n}^{i\Delta_n} \langle (\Sigma_s (\mathcal{S}(i\Delta_n - s) - I)^*) h, g \rangle ds \\
&\quad = (1)_n + (2)_n.
\end{aligned}$$

We can show the convergence for $(1)_n$ only, as the convergence for $(2)_n$ is analogous. It holds that

$$\begin{aligned}
(1)_n =& \Delta_n^{-\frac{1}{2}} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \int_{(i-1)\Delta_n}^{i\Delta_n} \langle (I - p_N)(\Sigma_s \mathcal{S}(i\Delta_n - s)^*) h, (\mathcal{S}(i\Delta_n - s) - I)^* g \rangle ds \\
&+ \Delta_n^{-\frac{1}{2}} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \int_{(i-1)\Delta_n}^{i\Delta_n} \langle p_N (\Sigma_s \mathcal{S}(i\Delta_n - s)^*) h, (\mathcal{S}(i\Delta_n - s) - I)^* g \rangle ds \\
=& (1.1)_{n,N} + (1.2)_{n,N}.
\end{aligned}$$

Using that $(\mathcal{S}(i\Delta_n - s) - I) e_j = \int_s^{i\Delta_n} \mathcal{S}(u - s) \mathcal{A} e_j ds$ and that the projection $(I - P_N)$ has the form $(I - P_N) = \sum_{j=1}^{N-1} \langle \cdot, e_j \rangle e_j$, we can show that

$$\sup_{t \in [0,T]} |(1.1)_{n,N}| \leq \Delta_n^{\frac{1}{2}} \sum_{j=1}^{N-1} \int_0^T \|\Sigma_s\|_{\text{op}} ds \|h\| \|g\| \sup_{t \in [0,T]} \|\mathcal{S}(t)\|_{\text{op}}^2, \tag{40}$$

which converges to 0 as $n \to \infty$. In particular, $\sup_{t \in [0,T]} |(1.1)_{n,N}| \overset{u.c.p.}{\to} 0$ as $n \to \infty$. From the assumption that $g \in F_{1/2}^{\mathcal{S}^*}$ we can derive a finite constant

$$K := \sup_{t \in [0,T]} \|\mathcal{S}(t)\|_{\text{op}} \left(\int_0^T \mathbb{E}\left[\|\sigma_s^*\|_{\text{op}}^2 \right] \|h\|^2 ds \right)^{\frac{1}{2}} \sup_{t \leq T} \|t^{-\frac{1}{2}} (\mathcal{S}(t) - I)^* g\| < \infty$$

such that

$$\mathbb{E}\left[\sup_{t \in [0,T]} |(1.2)_{n,N}| \right] \leq K \left(\int_0^T \mathbb{E}\left[\|p_N \sigma_s\|_{\text{op}}^2 \right] ds \right)^{\frac{1}{2}},$$

which converges to 0 as $N \to \infty$ by (34). Thus, combining this uniform convergence result with (40) and the analogous argumentation for $(2)_n$ yields the assertion and, thus, (CLTc).

6. Conclusion. In this article, we introduced feasible central limit theorems for the semigroup-adjusted realised covariations and, thus, provided a basis for functional data analysis of mild solutions to a large number of semilinear stochastic partial differential equations. We also addressed the issue of how this can be translated into a fully discrete setting, whereby we assumed a regular spatio-temporal sampling grid. In general, finding closed forms for the semigroup-adjusted multipower variations is a task that must be addressed for each semigroup (or equivalently each infinitesimal generator), each sampling grid and any precise application separately. Certainly, the Hilbert space approach is well suited to account for potentially any sampling grid.

To gain an overview of the infinite-dimensional limit theory introduced for both $SARCV^n$ and RV^n in this article, it might be helpful to give a systematic summary. For the sake of presentation, it is tedious and eventually not very instructive to repeat all assumptions in full technical detail so instead we make a distinction on the basis of the magnitude of $p_n := \int_0^T \|(\mathcal{S}(\Delta_n) - I)\sigma_s\|_{L_{\text{HS}}(U,H)} ds$ in terms of Δ_n and assume the volatility σ of a mild Itô process of the form (8) to be deterministic. In this regard we can distinguish four cases:

(i) If $p_n = o(\Delta_n^{\frac{3}{4}}),$ then $SARCV^n$ and RV^n satisfy LLN and CLT .

(ii) If $p_n = o(\Delta_n^{\frac{1}{2}}),$ then RV^n satisfies LLN, $SARCV^n$satisfies LLN and CLT.

(iii) If $p_n = \mathcal{O}(\Delta_n^{\frac{1}{2}}),$ then $SARCV^n$ satisfies LLN and CLT.

(iv) In general $SARCV^n$ satisfies LLN,

where satisfying LLN (law of large numbers) means convergence to the integrated volatility in probability and satisfying CLT (central limit theorem) means asymptotic normality of the normalised estimator. Observe that Example 2 in Section 3 yields that we cannot reduce the regularity in (iii), if we want to guarantee the validity of a general central limit theorem for $SARCV^n$. Example 4 shows that RV^n does not have to satisfy a central limit theorem if $p_n = o(\Delta_n^{1/2})$ is not valid and underlines the necessity of the adjustment by the semigroup. If even $p_n = o(\Delta_n^{1/4})$ does not hold, then RV^n does not even have to satisfy the LLN.

Moreover, it is likely that in many realistic scenarios, the distribution underlying the data and the sampling itself yield some additional challenges, which can be approached in our setting. Let us comment on some of these points:

Functional sampling: In infinite dimensions, we witness sampling schemes that have no counterpart in finite dimensions. For instance, data could be sampled as averages (or in general smooth functionals) of the process of interest over certain time periods in the future or within a demarcated area. This is for instance the case for energy swap prices or meteorological forecasting data. Our framework yields an ideal basis to derive inferential statistical tools in these situations.

Jumps: Many processes are not considered to be continuous in time. In fact, many financial time series show jumps and spikes on a regular basis, which is, in particular, the case in energy markets, a potential application of our theory. This suggests the inclusion of a pure-jump component to our framework, such as in the framework of [44]. However, as in finite dimensions, jumps will considerably complicate expressions, applications and proofs and, thus, more effort has to go into the task of making inference on non-continuous behaviour in infinite-dimensional models. Arguably, the structure of our proof, which appeals to tightness and already existing limit theorems from finite dimensions, yields a promising approach.

Asynchronous sampling: It could very well be, that we sample at high frequency in time, but sparsely and irregularly in space. Ignoring this (for instance by naïve rearrangement to refresh times) can have unpleasant consequences such as the Epps effect, c.f. [1, Sec. 9.2.1].

Again, energy intraday market prices, in which all available maturities are unlikely traded at the exact same time instances, can be prone to this. Infinite-dimensionality and the potentially necessary adjustment by the semigroup might make it harder than in the finite-dimensional case to deal with this issue, as in addition to asynchronous sampling, one has to deal with the problem of smoothing the adequately aggregated data in space.

Noise: The task of accounting for noise in the samples, often called *market microstructure noise* in financial applications has received much attention by the research community (c.f., for example, [69], [10] or [51]), as noise lets the quadratic variation severely overshoot the integrated volatility in the presence of data sampled at very high frequency. In combination with the problem of smoothing (and asynchronous sampling) this appears to be a delicate question in infinite-dimensional applications. However, both finite-dimensional high-frequency statistics and functional data analysis have several tools available to deal with noise and it is intriguing to find out how they can be exploited to overcome this problem in the future.

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REFERENCES

[1] AÏT-SAHALIA, Y. and JACOD, J. (2014). *High-Frequency Financial Econometrics*. Princeton University Press, Princeton, New Jersey.

[2] ALIPRANTIS, C. D. and BORDER, K. C. (2006). *Infinite dimensional analysis*, Third ed. Springer, Berlin.

[3] ALTMEYER, R., BRETSCHNEIDER, T., JANÁK, J. and REISS, M. (2022). Parameter Estimation in an SPDE Model for Cell Repolarization. *SIAM/ASA J. Uncertain. Quantif.* **10** 179-199.

[4] ALTMEYER, R. and REISS, M. (2021). Nonparametric estimation for linear SPDEs from local measurements. *Ann. Appl. Probab.* **31** 1–38.

[5] AMBLARD, P., COEURJOLLY, J., LAVANCIER, F. and PHILIPPE, A. (2013). Basic properties of the Multivariate Fractional Brownian Motion. *Sémin. Congr.* **28** 65-87.

[6] ANDERSEN, T. G., BOLLERSLEV, T., DIEBOLD, F. X. and LABYS, P. (2003). Modeling and Forecasting Realized Volatility. *Econometrica* **71** 579–625.

[7] AUE, A., HORVÁTH, L. and F. PELLATT, D. (2017). Functional generalized autoregressive conditional heteroskedasticity. *J. Time Ser. Anal.* **38** 3–21.

[8] BARNDORFF-NIELSEN, O. E., CORCUERA, J. M. and PODOLSKIJ, M. (2011). Multipower variation for Brownian semistationary processes. *Bernoulli* **17** 1159–1194.

[9] BARNDORFF-NIELSEN, O. E., CORCUERA, J. M. and PODOLSKIJ, M. (2013). Limit theorems for functionals of higher order differences of Brownian semistationary processes. In *Prokhorov and Contemporary Probability* (A. E. SHIRYAEV, S. R. S. VARADHAN and E. PRESMAN, eds.). *Springer Proceedings in Mathematics and Statistics* **33** 69–96.

[10] BARNDORFF-NIELSEN, O. E., HANSEN, P. R., LUNDE, A. and SHEPHARD, N. (2008). Designing realized kernels to measure the ex post variation of equity prices in the presence of noise. *Econometrica* **76** 1481–1536.

[11] BARNDORFF-NIELSEN, O. E. and SHEPHARD, N. (2002). Econometric Analysis of Realized Volatility and Its Use in Estimating Stochastic Volatility Models. *J. R. Statist. Soc. B* **64** 253–280.

[12] BARNDORFF-NIELSEN, O. E. and SHEPHARD, N. (2003). Realized power variation and stochastic volatility models. *Bernoulli* **9** 243–265.

[13] BARNDORFF-NIELSEN, O. E. and SHEPHARD, N. (2004). Econometric Analysis of Realized Covariation: High Frequency Based Covariance, Regression, and Correlation in Financial Economics. *Econometrica* **72** 885–925.

[14] BENNEDSEN, M. (2017). A rough multi-factor model of electricity spot prices. *Energy Econ.* **63** 301–313.

[15] BENTH, F. E. and KRÜHNER, P. (2014). Approximation of forward curve models in commodity markets with arbitrage-free finite-dimensional models. *Finance Stoch.* **22** 327—366.

[16] BENTH, F. E., ŠALTYTĖ BENTH, J. and KOEKEBAKKER, S. (2008). *Stochastic Modeling of Electricity and Related Markets*. World Scientific.

[17] BENTH, F. E. and HARANG, F. A. (2021). Infinite dimensional pathwise Volterra processes driven by Gaussian Noise–Probabilistic Properties and Applications. *Electron. J. Probab.* **26**.

[18] BENTH, F. E. and KRÜHNER, P. (2014). Representation of infinite-dimensional forward price models in commodity markets. *Commun. Math. Statist.* **2** 47–106.

[19] BENTH, F. E., SCHROERS, D. and VERAART, A. E. D. (2022). A weak law of large numbers for realised covariation in a Hilbert space setting. *Stoch. Proc. Applic.* **145** 241-268.

[20] BENTH, F. E. and SGARRA, C. (2021). A Barndorff-Nielsen and Shephard model with leverage in Hilbert space for commodity forward markets. *Available at SSRN 3835053*.

[21] BENTH, F. E. and SIMONSEN, I. C. (2018). The Heston stochastic volatility model in Hilbert space. *Stoch. Analysis Applic.* **36** 733–750.

[22] BERLINET, A. and THOMAS-AGNAN, C. (2004). *Reproducing Kernel Hilbert Spaces in Probability and Statistics*. Springer Science & Business Media.

[23] BIBINGER, M. and TRABS, M. (2019). On Central Limit Theorems for Power Variations of the Solution to the Stochastic Heat Equation. In *Stochastic Models, Statistics and Their Applications* (A. STELAND, E. RAFAJŁOWICZ and O. OKHRIN, eds.) 69–84. Springer International Publishing, Cham.

[24] BIBINGER, M. and TRABS, M. (2020). Volatility estimation for stochastic PDEs using high-frequency observations. *Stoch. Proc. Applic.* **130** 3005 - 3052.

[25] BILLINGSLEY, P. (1999). *Convergence of Probability Measures*, 2 ed. John Wiley and Sons, New York.

[26] CEROVECKI, C., FRANCQ, C., HÖRMANN, S. and ZAKOIAN, J.-M. (2019). Functional GARCH models: the quasi-likelihood approach and its applications. *J. Econometrics* **209** 353–375.

[27] CHONG, C. (2020). High-frequency analysis of parabolic stochastic PDEs. *Ann. Statist.* **48** 1143–1167.

[28] CHONG, C. and DALANG, R. C. (2023). Power variations in fractional Sobolev spaces for a class of parabolic stochastic PDEs. *Bernoulli* **29** 1792–1820.

[29] CHONG, C., DELERUE, T. and LI, G. (2022). When frictions are fractional: Rough noise in high-frequency data. *E-print arXiv:2106.16149*.

[30] CHONG, C., DELERUE, T. and MIES, F. (2022). Rate-optimal estimation of mixed semimartingales. *E-print arXiv:2207.10464*.

[31] CIALENCO, I. (2018). Statistical Inference for SPDEs: an overview. *Statist. Inf. Stoch. Proc.* **20** 309–329.

[32] CIALENCO, I. and HUANG, Y. (2020). A note on parameter estimation for discretely sampled SPDEs. *Stoch. Dynamics* **20**.

[33] CIALENCO, I. and KIM, H. J. (2022). Parameter estimation for discretely sampled stochastic heat equation driven by space-only noise. *Stoch. Proc. Applic.* **143** 1–30.

[34] CONT, R. (2005). Modeling Term Structure Dynamics: An Infinite Dimensional Approach. *Int. J. Theor. Appl. Finance* **08** 357-380.

[35] CORCUERA, J. M., NUALART, D. and PODOLSKIJ, M. (2014). Asymptotics of weighted random sums. *Commun. Appl. Indust. Math.* **6** e-486, 11.

[36] CORCUERA, J. M., HEDEVANG, E., PAKKANEN, M. S. and PODOLSKIJ, M. (2013). Asymptotic theory for Brownian semi-stationary processes with application to turbulence. *Stoch. Proc. Applic.* **123** 2552–2574.

[37] CORCUERA, J. M., NUALART, D. and WOERNER, J. H. C. (2006). Power variation of some integral fractional processes. *Bernoulli* **12** 713–735.

[38] COX, S., KARBACH, S. and KHEDHER, A. (2021). An infinite-dimensional affine stochastic volatility model. *Math. Finance*.

[39] COX, S., KARBACH, S. and KHEDHER, A. (2022). Affine pure-jump processes on positive Hilbert-Schmidt operators. *Stoch. Proc. Applic.* **151** 191–229.

[40] DA PRATO, G. and ZABCZYK, J. (2014). *Stochastic Equations in Infinite Dimensions*, second ed. *Encyclopedia of Mathematics and its Applications* **152**. Cambridge University Press, Cambridge.

[41] ENGEL, K.-J. and NAGEL, R. (1999). *One-Parameter Semigroups for Linear Evolution Equations* **194**. Springer Science & Business Media.

[42] FILIPOVIĆ, D. (2000). Invariant manifolds for weak solutions to stochastic equations. *Probab. Theory Related Fields* **118** 323–341.

[43] FILIPOVIĆ, D. (2001). *Consistency Problems for HJM Interest Rate Models*. *Lecture Notes in Mathematics* **1760**. Springer, Berlin.

[44] FILIPOVIĆ, D., TAPPE, S. and TEICHMANN, J. (2010). Jump-diffusions in Hilbert spaces: existence, stability and numerics. *Stochastics* **82** 475-520.

[45] GRANELLI, A. and VERAART, A. E. D. (2019). A central limit theorem for the realised covariation of a bivariate Brownian semistationary process. *Bernoulli* **25** 2245–2278.

[46] HÄUSLER, E. and LUSCHGY, H. (2015). *Stable Convergence and Stable Limit Theorems*. *Probability Theory and Stochastic Modelling* **74**. Springer.

[47] HILDEBRANDT, F. and TRABS, M. (2021). Parameter estimation for SPDEs based on discrete observations in time and space. *Electron. J. Stat.* **15** 2716 – 2776.

[48] HILDEBRANDT, F. and TRABS, M. (2023). Nonparametric calibration for stochastic reaction-diffusion equations based on discrete observations. *Stochastic Processes and their Applications* **162** 171–217.

[49] HÖRMANN, S., HORVÁTH, L. and REEDER, R. (2013). A functional version of the ARCH model. *Econometric Theory* **29** 267–288.

[50] JACOD, J. (2008). Asymptotic properties of realized power variations and related functionals of semimartingales. *Stoch. Proc. Applic.* **118** 517–559.

[51] JACOD, J., LI, Y., MYKLAND, P., PODOLSKIJ, M. and VETTER, M. (2009). Microstructure noise in the continuous case: the pre-averaging approach. *Stoch. Proc. Applic.* **119** 2249–2276.

[52] JACOD, J. and PROTTER, P. (2012). *Discretization of Processes*. *Stochastic Modelling and Applied Probability* **67**. Springer, Heidelberg.

[53] JOFFE, A. and METIVIER, M. (1986). Weak convergence of sequences of semimartingales with applications to multitype branching processes. *Adv. Appl. Prob.* **18** 20–65.

[54] KOKOSZKA, P. and REIMHERR, M. (2013). Asymptotic normality of the principal components of functional time series. *Stoch. Proc. Applic.* **123** 1546-1562.

[55] KÜHNERT, S. (2020). Functional ARCH and GARCH models: A Yule-Walker approach. *Electron. J. Stat.* **14** 4321–4360.

[56] LIU, W. and RÖCKNER, M. (2015). *Stochastic Partial Differential Equations: An Introduction*. *Universitext*. Springer International Publishing.

[57] MANDREKAR, V. and GAWARECKI, L. (2011). *Stochastic Differential Equations in Infinite Dimensions*. *Probability and Its Applications*. Springer, Berlin, Heidelberg.

[58] MARINELLI, C. and RÖCKNER, M. (2016). On the maximal inequalities of Burkholder, Davis and Gundy. *Expo. Math.* **34** 1–26.

[59] MASAROTTO, V., PANARETOS, V. M. and ZEMEL, Y. (2019). Procrustes Metrics on Covariance Operators and Optimal Transportation of Gaussian Processes. *Sankhya A* **81** 172–213.

[60] PASEMANN, G., FLEMMING, S., ALONSO, S., BETA, C. and STANNAT, W. (2021). Diffusivity Estimation for Activator–Inhibitor Models: Theory and Application to Intracellular Dynamics of the Actin Cytoskeleton. *J Nonlinear Sci* **31**.

[61] PASSEGGERI, R. and VERAART, A. E. D. (2019). Limit theorems for multivariate Brownian semistationary processes and feasible results. *Adv. Appl. Probab.* **51** 667–716.

[62] PAZY, A. (1983). *Semigroups of linear operators and applications to partial differential equations*. *Appl. Math. Sci.* **44**. Springer-Verlag.

[63] PESZAT, S. and ZABCZYK, J. (2007). *Stochastic Partial Differential Equations with Lévy Noise*. *Encyclopedia of Mathematics and its Applications* **113**. Cambridge University Press, Cambridge.

[64] RUDIN, W. (1976). *Principles of Mathematical Analysis*, 3 ed. McGraw-Hill, New York.

[65] SUN, H. and YU, B. (2020). Volatility asymmetry in functional threshold GARCH model. *J. Time Ser. Anal.* **41** 95–109.

[66] SZYMANSKI, G. (2022). Optimal estimation of the rough Hurst parameter in additive noise. *E-print arXiv:2205.13035*.

[67] THOMÉE, V. (2007). *Galerkin finite element methods for parabolic problems*. Springer Science & Business Media.

[68] VAN DER VAART, A. W. and WELLNER, J. (1996). *Weak Convergence and Empirical Processes: with Applications to Statistics*. Springer Science & Business Media, New York.

[69] ZHANG, L., MYKLAND, P. and AÏT-SAHALIA, Y. (2005). A tale of two time scales: Determining integrated volatility with noisy high-frequency data. *J. Amer. Statist. Assoc.* **100** 1394–1411.

APPENDIX A: NOTATION

For convenience, we list some of the frequently used notation throughout this appendix:

- $L_{\text{HS}}(U,H)$: the space of Hilbert-Schmidt operators from U to H.
- $\mathcal{H}$: equals $L_{\text{HS}}(H,H)$, when H is the initial Hilbert space in which the mild Itô process Y takes its values.
- $\mathcal{H}^m$: for $m \geq 3$ this is the space of operators spanned by the orthonormal basis $(e_{j_1} \otimes \cdots \otimes e_{j_m})_{j_1,\dots,j_m \in \mathbb{N}}$ for an orthonormal basis $(e_j)_{j\in\mathbb{N}}$ of H with respect to the Hilbert-Schmidt norm.
- $\sigma_s^{\mathcal{S}_n} = \mathcal{S}(i\Delta_n - s)\sigma_s$ for $s \in ((i-1)\Delta_n, i\Delta_n]$.
- $\alpha_s^{\mathcal{S}_n} = \mathcal{S}(i\Delta_n - s)\alpha_s$ for $s \in ((i-1)\Delta_n, i\Delta_n]$.
- $\Sigma_s^{\mathcal{S}_n} := \sigma_s^{\mathcal{S}_n}(\sigma_s^{\mathcal{S}_n})^*$.
- p_N is the projection onto $v^N := \overline{lin(\{e_j : j \geq N\})}$, for some orthonormal basis $(e_j)_{j\in\mathbb{N}}$.
- P_N^m is the projection onto $\overline{lin(\{\bigotimes_{l=1}^m e_{k_l} : k_l \geq N\})}$.
- $P_N = P_N^2$ for the special case $m = 2$.

We start by giving several auxiliary results that are needed to prove the limit theorems.

APPENDIX B: TECHNICAL TOOLS

B.1. Localisation. For both the laws of large numbers Theorems 4.1 and 4.2 and the central limit Theorems 3.3, 3.2 we can work under the following stronger assumptions:

ASSUMPTION 7. There is a constant $A > 0$ such that almost surely

$$\int_0^T \|\alpha_s\|^2 + \|\sigma_s\|^4_{L_{\text{HS}}(U,H)} ds \leq A.$$

ASSUMPTION 8 (m). There is a constant $A > 0$ such that almost surely

$$\int_0^T \|\alpha_s\|^{\frac{2m}{2+m}} + \|\sigma_s\|^m_{L_{\text{HS}}(U,H)} ds \leq A.$$

ASSUMPTION 9. Assumption 3 holds and there is an $A > 0$ such that almost surely

$$\|\alpha_s\| + \|\sigma_s\|_{L_{\text{HS}}(U,H)} \leq A, \quad s \in [0,T].$$

We have then the following simplifying localisation result:

THEOREM B.1 (Localisation). *The following relaxations can be made for the limit theorems in this work:*

(a) *(localisation for CLT for functionals of $SARCV$ and RV) If the central limit Theorems 3.2 or 3.9(ii) hold under Assumption 7, then they also hold under Assumption 1.*
(b) *(localisation for LLN for power variations) If the law of large numbers Theorem 4.1 holds under Assumption 8(m), then it also holds under Assumption 6(m). Moreover, the laws of large numbers Theorems 3.8(i) or 3.9(i) hold, if they hold under the additional Assumption 8(2).*
(c) *(localisation for LLN for multipower variations)If the law of large numbers, Theorem 4.2, holds under Assumption 9, then it also holds under Assumption 3.*
(d) *(localisation for CLT for $SARCV$ and RV) If the central limit Theorem 3.3 holds under Assumptions 7 and 2(i) (respectively Theorem 3.8(ii) holds under Assumptions 7 and 5), then it also holds under Assumptions 1 together with 2(i) or (ii) (respectively 1 and 5).*

PROOF. Without loss of generality we may exchange the volatility process $(\sigma_t)_{t\in[0,T]}$ by a left continuous version $(\sigma_{t-})_{t\in[0,T]}$. This is because we assumed the filtration to be right-continuous and the stochastic integrals $\int_a^b H_s dW_s$ and $\int_a^b H_{s-} dW_s$ coincide for any predictable càdlàg process $(H_t)_{t\in[0,T]}$. In particular, in the case of a predictable càdlàg volatility process, we can assume it to be locally bounded since any left-continuous process is locally bounded.

Now, the same localisation procedure as for finite-dimensional semimartingales described in Section 4.4.1 in [52] holds: We define under each of the assumptions a different sequence $(\tau_N(i))_{N\in\mathbb{N}}$, $i=1,...,4$, of stopping times and the corresponding stopped process $Y_t(N,i) := Y_{t\wedge\tau_N(i)}$. Observe that on $\{t<\tau_N(i)\}$ we have $Y_t(N,i) = Y_t$: Set

$$X_t(1) := \int_0^t \|\alpha_s\|^2 + \|\sigma_s\|^4_{L_{\text{HS}}(U,H)} ds,$$

$$X_t(2) := \int_0^t \|\alpha_s\|^{\frac{2m}{2+m}} + \|\sigma_s\|^m_{L_{\text{HS}}(U,H)} ds,$$

$$X_t(3) := \|\alpha_t\| + \|\sigma_t\|_{L_{\text{HS}}(U,H)},$$

$$X_t(4) := \int_0^t \|\alpha_s\|^2 + \|\sigma_s\|^4_{L_{\text{HS}}(U,H)} + \sup_{r\in[0,t]} \|t^{-\frac{1}{2}}(\mathcal{S}(t)-I)\sigma_s\|^2_{\text{op}} ds.$$

We define for $i=1,...,4$

$$\tau_N(i) := \inf\{t\in[0,T] : X_t(i)\geq N\}, N\in\mathbb{N}.$$

Then Assumption 1 assures that $\tau_N(1)\uparrow T$, Assumption 6 assures that $\tau_N(2)\uparrow T$, Assumption 3 assures that $\tau_N(3)\uparrow T$, Assumption 1 together with 2(i) or (ii) ensures $\tau_N(4)\uparrow T$. Moreover, observe that $Y_t(N,1)$ satisfies Assumption 7, $Y_t(N,2)$ satisfies Assumption 8(m), $Y_t(N,3)$ satisfies Assumption 9, $Y_t(N,4)$ satisfies Assumption 7 and 2(i).

The localisation works now for all cases analogously: Observe, that as convergence in probability is a special case of stable convergence in law (when the underlying probability space for the limiting distribution coincides with the one on which the sequence of probability laws is defined), we just have to consider the stable convergence in law.

For any mild Itô process X, i.e. a process of the same form as Y as defined in (8), with values in H we write $U_n(X)$ either for the process of normalised multipower variations (for the central limit theorems) or the supremum over the unnormalised multipower variations (and as a special case, power variations). I.e., for proving (a) it is $U_n(X)_t = \Delta_n^{-1/2}\langle (SARCV_t^n - \int_0^t \Sigma_s ds), B\rangle$ (resp. $U_n(X)_t = \Delta_n^{-1/2}\langle (RV_t^n - \int_0^t \Sigma_s ds), B\rangle$) for $B=\sum_{l=1}^K \mu_l h_l\otimes g_l \in \mathcal{H}$ for $h_i, g_l \in F_{3/4}^{\mathcal{S}^*}$ and $U(X)$ is the asymptotic distribution as in Theorem 3.2, for (d) it is $U_n(X)_t = \Delta_n^{-1/2}(SARCV_t^n - \int_0^t \Sigma_s ds)$ (resp. $U_n(X)_t = \Delta_n^{-1/2}(RV_t^n - \int_0^t \Sigma_s ds)$) and $U(X)$ is the asymptotic distributions as described in Theorem 3.3 and (b) and (c) it is $U(X)=0$ and

$$U_n(X)_t = \sup_{s\in[0,t]} \|\Delta_n^{1-\frac{m}{2}} \sum_{i=1}^{\lfloor s/\Delta_n\rfloor -k+1} \bigotimes_{j=1}^k \tilde{\Delta}_{i+j-1}^n X^{\otimes m_j} - \int_0^t \rho_{\Sigma_s}^{\otimes k}(m_1,...,m_k) ds\|_{\mathcal{H}^m}.$$

Moreover, for the non-adjusted limit theorems, we exchange $\tilde{\Delta}$ by Δ in the above expressions.

In order to prove the claim, it is enough to show $U_n(Y)\to U(Y)$ stably in law as a process under the respective localised assumptions. I.e., if $\tilde{\Omega} = \Omega\times\Omega'$, $\tilde{\mathcal{F}} = \mathcal{F}\otimes\mathcal{F}'$ and $\tilde{\mathbb{P}} = \mathbb{P}[d\omega]\mathbb{Q}[\omega, d\omega']$ is the extension on which $U(X)$ can be realised, we want to show that for all bounded, continuous functions $g:\mathcal{D}([0,T],\mathcal{H})\to\mathbb{R}$ and all bounded $\mathcal{F}$-measurable real-valued random variables Z we have

$$\lim_{n\to\infty}\mathbb{E}\left[Zg(U_n(Y))\right]=\tilde{\mathbb{E}}\left[Zg(U(Y))\right], \tag{41}$$

where $\tilde{\mathbb{E}}$ is the expectation with respect to $\tilde{\mathbb{P}}$. If we write

$$\mathbb{Q}_Y(g)(\omega):=\int_{\Omega'}g(U(Y))(\omega,\omega')\mathbb{Q}(\omega,d\omega'),$$

we can express (41) as

$$\lim_{n\to\infty}\mathbb{E}\left[Zg(U_n(Y))\right]=\mathbb{E}\left[Z\mathbb{Q}_Y(g)\right].$$

Now assume that the convergence results hold for $Y(N,i)$, for which the localised assumptions are valid. We can therefore deduce that

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}\left[Zg(U_n(Y))\mathbb{I}_{t\leq\tau_N(i)}\right]&=\lim_{n\to\infty}\mathbb{E}\left[Zg(U_n(Y(N,i)))\right]\\&=\mathbb{E}\left[Z\mathbb{Q}_{Y(N,i)}(g)\right]=\mathbb{E}\left[Z\mathbb{Q}_Y(g)\mathbb{I}_{t\leq\tau_N(i)}\right]\end{aligned}$$

holds for all $N\in\mathbb{N}$. This implies (41) as boundedness of Z and g yield that

$$\sup_{n\in\mathbb{N}}\mathbb{E}\left[Z(g(U_n(Y))-\mathbb{Q}_Y(g))\mathbb{I}_{t>\tau_N}\right]\to 0\quad\text{ as } N\to\infty.$$

This proves the claim. □

In many occasions, we just need to impose a localised version of the condition

$$\mathbb{P}\left[\int_0^T\|\alpha_s\|^{\frac{m}{2}}+\|\sigma_s\|^m_{L_{\text{HS}}(U,H)}ds<\infty\right]=1. \tag{42}$$

In that regard, we introduce the auxiliary assumption:

ASSUMPTION 10 ((m)). There is a constant $A>0$ such that

$$\int_0^T\|\alpha_s\|^{\frac{m}{2}}+\|\sigma_s\|^m_{L_{\text{HS}}(U,H)}ds\leq A.$$

Based upon choosing the right m, this is satisfied under any (!) of the assumptions above. In particular Assumption 7 coincides with Assumption 10(4) while Assumption 8(2) coincides with Assumption 10(2). Assumption 8(m) is stricter than Assumption 10(m) if $m>2$.

B.2. Elimination of the semigroup on finite-dimensional projections. We will see that if we apply certain functionals, the semigroup-adjusted increments are essentially increments of finite-dimensional semimartingales. First, we recall the Burkholder-Davis-Gundy inequality (in what follows called the BDG inequality), c.f. [58].

THEOREM B.2. *For an H-valued local martingale $(M_t)_{t\in[0,T]}$ we have for all real numbers $m\geq 1$ and all $t\leq T$*

$$\mathbb{E}\left[\sup_{s\leq t}\|M_s\|^m\right]\leq C_m\mathbb{E}\left[[M,M]_t^{\frac{m}{2}}\right], \tag{43}$$

where the constant $C_m>0$ is just depending on m and $[M,M]$ is the scalar quadratic variation of M.

For our purposes, the most important application is given in the following example:

EXAMPLE 8. Let $M_t := \int_0^t \sigma_s dW_s$ be a stochastic integral. Then the BDG inequality (43) reads

$$\mathbb{E}\left[\sup_{s\le t}\|\int_0^t \sigma_s dW_s\|^m\right] \le C_m \mathbb{E}\left[\left(\int_0^t \mathrm{Tr}(\Sigma_s)ds\right)^{\frac{m}{2}}\right]$$
$$= C_m \mathbb{E}\left[\left(\int_0^t \|\sigma_s\|^2_{L_{\mathrm{HS}}(U,H)} ds\right)^{\frac{m}{2}}\right],$$

for the constant C_m just depending on m and for all $t \le T$.

Observe that there always exists an orthonormal basis $(e_i)_{i\in\mathbb{N}}$ of H that is contained in the domain $D(\mathcal{A}^*)$ of the generator $\mathcal{A}^*$ of the adjoint semigroup $(\mathcal{S}(t)^*)_{t\ge 0}$. The latter is a semigroup since H is a Hilbert space, see [41, p.44]. Besides tightness, the most important argument to be able to appeal to the finite-dimensional limit theory of semimartingales is the following one:

LEMMA B.3. *Suppose that Assumption 10(m) holds for $m \ge 2$. We define the H-valued semimartingale S by*

$$S_t := \int_0^t \alpha_s ds + \int_0^t \sigma_s dW_s,$$

and fix a basis $(e_j)_{j\in\mathbb{N}} \subset D(\mathcal{A}^*)$ of H. Let furthermore $\Delta_i^n S := S_{i\Delta_n} - S_{(i-1)\Delta_n}$ denote the non-adjusted increments of the semimartingale S. Then for $j_1,...,j_m \in \mathbb{N}$

$$\langle SAMPV_t^n(m_1,...,m_k), \bigotimes_{l=1}^m e_{j_l}\rangle_{\mathcal{H}^m}$$
$$= \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \langle \bigotimes_{j=1}^k \Delta_{i+j-1}^n S^{\otimes m_j}, \bigotimes_{l=1}^m e_{j_l}\rangle_{\mathcal{H}^m} + \mathcal{O}_p(\Delta_n^{\frac{m}{2}}). \tag{44}$$

PROOF. First, fix $i, j \in \mathbb{N}$. By the stochastic Fubini theorem (and since $Y_0 = 0$) we have

$$\int_0^t \int_0^u \langle \alpha_s, \mathcal{A}^*\mathcal{S}(u-s)^* e_j\rangle ds du$$
$$+ \int_0^t \int_0^u \langle \sigma_s, \mathcal{A}^*\mathcal{S}(u-s)^* e_j\rangle dW_s du + \langle S_t, e_j\rangle$$
$$= \int_0^t \int_0^u \frac{d}{du}\langle \alpha_s, \mathcal{S}(u-s)^* e_j\rangle ds du$$
$$+ \int_0^t \int_0^u \frac{d}{du}\langle \sigma_s, \mathcal{S}(u-s)^* e_j\rangle dW_s du + \langle S_t, e_j\rangle$$
$$= \int_0^t \int_s^t \frac{d}{du}\langle \alpha_s, \mathcal{S}(u-s)^* e_j\rangle du ds + \int_0^t \langle \alpha_s, e_j\rangle ds$$
$$+ \int_0^t \int_s^t \frac{d}{du}\langle \sigma_s, \mathcal{S}(u-s)^* e_j\rangle du dW_s + \int_0^t \langle \sigma_s, e_j\rangle dW_s$$

$$= \int_0^t \langle \alpha_s, \mathcal{S}(t-s)^* e_j \rangle ds + \int_0^t \langle \sigma_s, \mathcal{S}(t-s)^* e_j \rangle dW_s$$
$$= \langle Y_t, e_j \rangle.$$

Now define

$$\Delta_n^t a^j := \int_t^{t+\Delta_n} \int_t^u \frac{d}{du} \langle \alpha_s, \mathcal{S}(u-s)^* e_j \rangle ds du$$
$$+ \int_t^{t+\Delta_n} \int_t^u \langle \sigma_s, \mathcal{A}^* \mathcal{S}(u-s)^* e_j \rangle dW_s du. \tag{45}$$

Again, by using the stochastic Fubini theorem we obtain

$$\begin{aligned}
&\langle Y_{t+\Delta_n}, e_j \rangle - (\langle S_{t+\Delta_n}, e_j \rangle - \langle S_t, e_j \rangle) + \Delta_n^t a^j \\
&= \int_t^{t+\Delta_n} \int_0^u \frac{d}{du} \langle \alpha_s, \mathcal{S}(u-s)^* e_j \rangle ds du \\
&\quad - \int_t^{t+\Delta_n} \int_t^u \frac{d}{du} \langle \alpha_s, \mathcal{S}(u-s)^* e_j \rangle ds du \\
&\quad + \int_t^{t+\Delta_n} \int_0^u \frac{d}{du} \langle \sigma_s, \mathcal{S}(u-s)^* e_j \rangle dW_s du \\
&\quad - \int_t^{t+\Delta_n} \int_t^u \frac{d}{du} \langle \sigma_s, \mathcal{S}(u-s)^* e_j \rangle dW_s du + \langle Y_t, e_j \rangle \\
&= \int_t^{t+\Delta_n} \int_0^t \frac{d}{du} \langle \alpha_s, \mathcal{S}(u-s)^* e_j \rangle ds du \\
&\quad + \int_t^{t+\Delta_n} \int_0^t \frac{d}{du} \langle \sigma_s, \mathcal{S}(u-s)^* e_j \rangle dW_s du + \langle Y_t, e_j \rangle \\
&= \int_0^t \int_t^{t+\Delta_n} \frac{d}{du} \langle \alpha_s, \mathcal{S}(u-s)^* e_j \rangle du ds \\
&\quad + \int_0^t \int_t^{t+\Delta_n} \frac{d}{du} \langle \sigma_s, \mathcal{S}(u-s)^* e_j \rangle du dW_s + \langle Y_t, e_j \rangle \\
&= \int_0^t \int_s^{t+\Delta_n} \frac{d}{du} \langle \alpha_s, \mathcal{S}(u-s)^* e_j \rangle du ds \\
&\quad + \int_0^t \int_s^{t+\Delta_n} \frac{d}{du} \langle \sigma_s, \mathcal{S}(u-s)^* e_j \rangle du dW_s + \langle S_t, e_j \rangle \\
&= \int_0^t \langle \alpha_s, \mathcal{S}(t+\Delta_n-s)^* e_j \rangle ds + \int_0^t \langle \sigma_s, \mathcal{S}(t+\Delta_n-s)^* e_j \rangle dW_s \\
&= \langle \mathcal{S}(\Delta_n) Y_t, e_j \rangle.
\end{aligned}$$

Thus,

$$\langle \tilde{\Delta}_i^n Y, e_j \rangle = \langle \Delta_i^n S, e_j \rangle + \Delta_n^{(i-1)\Delta_n} a^j.$$

Now let $i_1, ..., i_m \in \mathbb{N}$. We get

$$\prod_{k=1}^m \langle \tilde{\Delta}_{i_k}^n Y, e_{j_k} \rangle = \prod_{k=1}^m \left(\langle \Delta_{i_k}^n S, e_{j_k} \rangle + \Delta_n^{(i_k-1)\Delta_n} a^{j_k} \right)$$

$$= \prod_{k=1}^{m} \langle \Delta_{i_k}^n S, e_{j_k} \rangle + \sum_{y=1,x=2}^{m} b_{xy}(i_1, ..., i_m, j_1, ..., j_m),$$

where each summand b_{xy} is of the form

$$b_{xy}(i_1, ..., i_m, j_1, ..., j_m) = \prod_{r=1}^{p} \langle \Delta_{i_{k_r}}^n S, e_{j_{k_r}} \rangle \prod_{s=1}^{q} \Delta_n^{(i_{l_s}-1)\Delta_n} a^{j_{l_s}},$$

for $p \leq m-1$, $q \leq m$, $p+q=m$ and

$$\{i_{k_1}, ..., i_{k_p}, i_{l_1}, ..., i_{l_q}\} = \{i_1, ..., i_m\}, \quad \{j_{k_1}, ..., j_{k_p}, j_{l_1}, ..., j_{l_q}\} = \{j_1, ..., j_m\}.$$

Then by the generalised Hölder inequality

$$\begin{aligned} &\mathbb{E}[|b_{xy}(i_1, ..., i_m, j_1, ..., j_m)|] \\ &\leq \mathbb{E}[|\langle \Delta_{i_{k_1}}^n S, e_{j_{k_1}} \rangle|^m]^{\frac{1}{m}} \times \cdots \times \mathbb{E}[|\langle \Delta_{i_{k_p}}^n S, e_{j_{k_p}} \rangle|^m]^{\frac{1}{m}} \\ &\quad \times \mathbb{E}[|\Delta_n^{(i_{l_1}-1)\Delta_n} a^{j_{l_1}}|^m]^{\frac{1}{m}} \times \cdots \times \mathbb{E}[|\Delta_n^{(i_{l_q}-1)\Delta_n} a^{j_{l_q}}|^m]^{\frac{1}{m}}. \end{aligned} \tag{46}$$

This means that in order to estimate $\mathbb{E}[|b_{xy}(i_1, ..., i_m, j_1, ..., j_m)|]$ we have to find bounds on $\mathbb{E}[|\langle \Delta_i^n S, e_j \rangle|^m]^{\frac{1}{m}}$ and $\mathbb{E}[|\Delta_n^{(i-1)\Delta_n} a^j|^m]^{\frac{1}{m}}$ for $i, j \in \mathbb{N}$. We start with the latter term.

For any $t \leq u \leq T$, $j \in \mathbb{N}$, we have for the quadratic variation

$$\begin{aligned} \left[\int_0^{\cdot} \mathbb{I}_{[t,u]} \langle \sigma_s, \mathcal{S}(u-s)^* \mathcal{A}^* e_j \rangle dW_s \right]_T &= \int_u^t \langle \mathcal{S}(u-s) \Sigma_s \mathcal{S}(t-s)^* \mathcal{A}^* e_j, \mathcal{A}^* e_j \rangle ds \\ &= \int_t^u \|\sigma_s^* \mathcal{S}(u-s)^* \mathcal{A}^* e_j\|^2 ds. \end{aligned}$$

Hence, by the BDG inequality (43) we obtain, for $t \leq u \leq t + \Delta_n$,

$$\begin{aligned} &\mathbb{E}\left[\left|\int_t^u \langle \alpha_s, \mathcal{A}^* \mathcal{S}(t-s)^* e_j \rangle ds + \int_t^u \langle \sigma_s, \mathcal{A}^* \mathcal{S}(u-s)^* e_j \rangle dW_s \right|^m\right]^{\frac{1}{m}} \\ \leq &\mathbb{E}\left[\left|\int_t^u \langle \alpha_s, \mathcal{A}^* \mathcal{S}(t-s)^* e_j \rangle ds \right|^m\right]^{\frac{1}{m}} + \mathbb{E}\left[\left|\int_t^u \langle \sigma_s, \mathcal{A}^* \mathcal{S}(u-s)^* e_j \rangle dW_s \right|^m\right]^{\frac{1}{m}} \\ \leq &\mathbb{E}\left[\left|\int_t^u \langle \alpha_s, \mathcal{A}^* \mathcal{S}(t-s)^* e_j \rangle ds \right|^m\right]^{\frac{1}{m}} + C_m^{\frac{1}{m}} \mathbb{E}\left[\left|\int_t^u \|\sigma_s^* \mathcal{S}(u-s)^* \mathcal{A}^* e_j\|^2 ds \right|^{\frac{m}{2}}\right]^{\frac{1}{m}} \\ \leq &\Delta_n^{\frac{m-2}{m}} \mathbb{E}\left[\left(\int_t^{t+\Delta_n} \|\alpha_s\|^{\frac{m}{2}} ds\right)^2\right]^{\frac{1}{m}} \sup_{t \in [0,T]} \|\mathcal{S}(t)\|_{\text{op}} \|\mathcal{A}^* e_j\| \\ &+ C_m^{\frac{1}{m}} \Delta_n^{\frac{m-2}{2m}} \left(\int_t^{t+\Delta_n} \mathbb{E}\left[\|\sigma_s^*\|_{\text{op}}^m\right] ds\right)^{\frac{1}{m}} \sup_{t \in [0,T]} \|\mathcal{S}(t)\|_{\text{op}} \|\mathcal{A}^* e_j\|. \end{aligned}$$

Therefore, as $m \geq 2$

$$\begin{aligned} &\mathbb{E}[|\Delta_n^t a^j|^m]^{\frac{1}{m}} \\ \leq &\Delta_n \left(\Delta_n^{\frac{m-2}{m}} \mathbb{E}\left[\left(\int_t^{t+\Delta_n} |\|\alpha_s\|^{\frac{m}{2}} ds\right)^2\right]^{\frac{1}{m}} + C_m^{\frac{1}{m}} \Delta_n^{\frac{m-2}{2m}} \left(\int_t^{t+\Delta_n} \mathbb{E}\left[\|\sigma_s^*\|_{\text{op}}^m\right] ds\right)^{\frac{1}{m}}\right) \end{aligned}$$

$$\times \sup_{t\in[0,T]} \|\mathcal{S}(t)\|_{\text{op}}\|\mathcal{A}^*e_j\|$$

$$\leq \Delta_n^{\frac{3m-2}{2m}} \left(\mathbb{E}\left[\left(\int_t^{t+\Delta_n} \||\alpha_s\|^{\frac{m}{2}} ds \right)^2 \right]^{\frac{1}{m}} + C_m^{\frac{1}{m}} \left(\int_t^{t+\Delta_n} \mathbb{E}\left[\|\sigma_s^*\|_{\text{op}}^m \right] ds \right)^{\frac{1}{m}} \right)$$

$$\times \sup_{t\in[0,T]} \|\mathcal{S}(t)\|_{\text{op}}\|\mathcal{A}^*e_j\|. \tag{47}$$

Now for the martingale differences $\mathbb{E}[|\langle \Delta_i^n S, e_j\rangle|^m]^{\frac{1}{m}}$ we can estimate by virtue of the BDG inequality (43):

$$\mathbb{E}[|\langle \Delta_i^n S, e_j\rangle|^m]^{\frac{1}{m}}$$

$$\leq \mathbb{E}\left[\| \int_{(i-1)\Delta_n}^{i\Delta_n} \alpha_s ds\|^m \right]^{\frac{1}{m}} + C_m^{\frac{1}{m}} \mathbb{E}\left[(\int_{(i-1)\Delta_n}^{i\Delta_n} \langle \Sigma_s e_j, e_j\rangle ds)^{\frac{m}{2}} \right]^{\frac{1}{m}}$$

$$\leq \left(\Delta_n^{\frac{m-2}{m}} \mathbb{E}\left[\left(\int_{(i-1)\Delta_n}^{i\Delta_n} \|\alpha_s\|^{\frac{m}{2}} ds \right)^2 \right]^{\frac{1}{m}} + \Delta_n^{\frac{m-2}{2m}} C_m^{\frac{1}{m}} \left(\int_{(i-1)\Delta_n}^{i\Delta_n} \mathbb{E}[\|\sigma_s\|_{\text{op}}^m] ds \right)^{\frac{1}{m}} \right)$$

$$\leq \Delta_n^{\frac{m-2}{2m}} \left(\mathbb{E}\left[\left(\int_{(i-1)\Delta_n}^{i\Delta_n} \|\alpha_s\|^{\frac{m}{2}} ds \right)^2 \right]^{\frac{1}{m}} + C_m^{\frac{1}{m}} \left(\int_{(i-1)\Delta_n}^{i\Delta_n} \mathbb{E}[\|\sigma_s\|_{\text{op}}^m] ds \right)^{\frac{1}{m}} \right). \tag{48}$$

Combining (46), (47) and (48) yields, as $q \geq 1$

$$\mathbb{E}[|b_{xy}(i_1, ..., i_m, j_1, ..., j_m)|]$$

$$\leq \Delta_n^{p\frac{m-2}{2m}} \prod_{l=1}^{p} \left(\mathbb{E}\left[\left(\int_{(i_{k_l}-1)\Delta_n}^{i_{k_l}\Delta_n} \|\alpha_s\|^{\frac{m}{2}} ds \right)^2 \right]^{\frac{1}{m}} + C_m^{\frac{1}{m}} \left(\int_{(i-1)\Delta_n}^{i\Delta_n} \mathbb{E}[\|\sigma_s\|_{\text{op}}^m] ds \right)^{\frac{1}{m}} \right)^p$$

$$\times \Delta_n^{q\frac{3m-2}{2m}} \prod_{l=1}^{q} \left(\mathbb{E}\left[\left(\int_{(i_{k_l}-1)\Delta_n}^{i_{k_l}\Delta_n} \|\alpha_s\|^{\frac{m}{2}} ds \right)^2 \right]^{\frac{1}{m}} \right.$$

$$\left. + C_m^{\frac{1}{m}} \left(\int_{(i_{k_l}-1)\Delta_n}^{i_{k_l}} \mathbb{E}\left[\|\sigma_s^*\|_{\text{op}}^m \right] ds \right)^{\frac{1}{m}} \right)^q$$

$$\times \sup_{t\in[0,T]} \|\mathcal{S}(t)\|_{\text{op}}^q \|\mathcal{A}^*e_j\|^q$$

$$\leq \Delta_n^{\frac{m}{2}} \prod_{l=1}^{m} \left(\mathbb{E}\left[\left(\int_{(i_l-1)\Delta_n}^{i_l\Delta_n} \|\alpha_s\|^{\frac{m}{2}} ds \right)^2 \right]^{\frac{1}{m}} + C_m^{\frac{1}{m}} \left(\int_{(i_l-1)\Delta_n}^{i_l\Delta_n} \mathbb{E}\left[\|\sigma_s\|_{\text{op}}^m \right] ds \right)^{\frac{1}{m}} \right)$$

$$\times \left(\sup_{t\in[0,T]q,j=1,...,m} \|\mathcal{S}(t)\|_{\text{op}}^q \|\mathcal{A}^*e_j\|^q \right).$$

Now we use introduce the notation $b_{x,y}(i_1, ..., i_m, j_1, ..., j_m) =: b_{x,y}(i)^{m_1,...,m_k}$ in the case of $i_1, ..., i_{m_1} = i, i_{m_1+1}, ..., i_{m_1+m_2} = i+1$ ... $i_{m_{k-2}+1}, ..., i_{m_{k-2}+m_{k-1}} = i+k-1$ (to help with the intuition, this is just a formal way to specify that the first m_1 components are equal and then the next m_2 are again equal and so on). Then

$$\begin{aligned}&\langle SAMPV_t^n(m_1,...,m_k),\bigotimes_{j=1}^m e_j\rangle\\&=\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\langle\bigotimes_{j=1}^k\tilde{\Delta}_{i+j-1}S,\bigotimes_{j=1}^m e_j\rangle+\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\sum_{x=2,y=1}^m b_{x,y}(i)^{m_1,...,m_k}.\end{aligned}$$

In order to prove the assertion, we just have to show that the latter summand is $\mathcal{O}(\Delta_n^{\frac{m}{2}})$. Therefore, the generalised Hölder inequality and the elementary inequality $(a+b)^m\leq 2^m(a^m+b^m)$ for positive real numbers $a,b\in\mathbb{R}_+$ yield

$$\begin{aligned}&\sum_{i=1}^{\lfloor t/\Delta_n\rfloor-k+1}\sum_{y=1,x=2}^m\mathbb{E}[|b_{xy}(i)^{m_1,...,m_k}|]\\&\leq m(m-1)\Delta_n^{\frac{m}{2}}\left(\sup_{t\in[0,T],q,j=1,...,m}\|\mathcal{S}(t)\|_{\text{op}}^q\|\mathcal{A}^*e_j\|^q\right)\\&\quad\times\sum_{i=1}^{\lfloor t/\Delta_n\rfloor-k+1}\prod_{j=1}^k\left(\mathbb{E}\left[\left(\int_{(i+j-2)\Delta_n}^{(i+j-1)\Delta_n}|\|\alpha_s\|^{\frac{m}{2}}ds\right)^2\right]^{\frac{1}{m}}\right.\\&\qquad\left.+C_m^{\frac{1}{m}}\left(\int_{(i+j-2)\Delta_n}^{(i+j-1)\Delta_n}\mathbb{E}\left[\|\sigma_s\|_{\text{op}}^m\right]ds\right)^{\frac{1}{m}}\right)^{m_j}\\&\leq m(m-1)\Delta_n^{\frac{m}{2}}\left(\sup_{t\in[0,T],q,j=1,...,m}\|\mathcal{S}(t)\|_{\text{op}}^q\|\mathcal{A}^*e_j\|^q\right)\\&\quad\times\sum_{i=1}^{\lfloor t/\Delta_n\rfloor-k+1}\prod_{j=1}^k 2^m\left(\mathbb{E}\left[\left(\int_{(i+j-2)\Delta_n}^{(i+j-1)\Delta_n}|\|\alpha_s\|^{\frac{m}{2}}ds\right)^2\right]\right.\\&\qquad\left.+C_m\int_{(i+j-2)\Delta_n}^{(i+j-1)\Delta_n}\mathbb{E}\left[\|\sigma_s\|_{\text{op}}^m\right]ds\right)^{\frac{m_j}{m}}.\end{aligned}$$

As $\mathbb{E}\left[\left(\int_{(i+j-2)\Delta_n}^{(i+j-1)\Delta_n}\|\alpha_s\|^{\frac{m}{2}}ds\right)^2\right]\leq\mathbb{E}\left[\int_{(i+j-2)\Delta_n}^{(i+j-1)\Delta_n}\|\alpha_s\|^{\frac{m}{2}}ds\right]$ for Δ_n small enough, we obtain

$$\begin{aligned}&\sum_{i=1}^{\lfloor t/\Delta_n\rfloor-k+1}\sum_{y=1,x=2}^m\mathbb{E}\left[|b_{xy}(i)^{m_1,...,m_k}|\right]\\&\leq m(m-1)\Delta_n^{\frac{m}{2}}\left(\sup_{t\in[0,T],q,j=1,...,m}\|\mathcal{S}(t)\|_{\text{op}}^q\|\mathcal{A}^*e_j\|^q\right)\\&\quad\times\sum_{i=1}^{\lfloor t/\Delta_n\rfloor-k+1}\prod_{j=1}^k 2^m\mathbb{E}\left[\int_{(i+j-2)\Delta_n}^{(i+j-1)\Delta_n}\|\alpha_s\|^{\frac{m}{2}}ds+C_m\|\sigma_s\|_{\text{op}}^m ds\right]^{\frac{m_j}{m}}\end{aligned}$$

$$\leq m(m-1)\Delta_n^{\frac{m}{2}}\left(\sup_{t\in[0,T],q,j=1,\dots,m}\|\mathcal{S}(t)\|_{\text{op}}^q\|\mathcal{A}^*e_j\|^q\right)$$

$$\times\, k!2^m\mathbb{E}\left[\int_0^T\|\alpha_s\|^{\frac{m}{2}}ds+C_m\|\sigma_s\|_{\text{op}}^m ds\right]$$

$$\leq m(m-1)\Delta_n^{\frac{m}{2}}\left(\sup_{t\in[0,T],q,j=1,\dots,m}\|\mathcal{S}(t)\|_{\text{op}}^q\|\mathcal{A}^*e_j\|^q\right)k!A.$$

This proves the assertion □

B.2.1. *Uniform convergence of the finite-dimensional projections.* We introduce the notation

$$\sigma_s^{\mathcal{S}_n}=\mathcal{S}(i\Delta_n-s)\sigma_s,$$

and

$$\alpha_s^{\mathcal{S}_n}=\mathcal{S}(i\Delta_n-s)\alpha_s,$$

for $s\in((i-1)\Delta_n,i\Delta_n]$, such that $\sigma_s^{\mathcal{S}_n}(\sigma_s^{\mathcal{S}_n})^*=\Sigma_s^{\mathcal{S}_n}$. We often need the following technical lemma:

LEMMA B.4. *Suppose that for $m\in\mathbb{N}$ we have*

$$\int_0^T\mathbb{E}\left[\|\alpha_s\|^{\frac{m}{2}}+\|\sigma_s\|_{L_{HS}(U,H)}^m\right]ds<\infty,$$

which holds in particular under Assumption 10(m). Let $(e_j)_{j\in\mathbb{N}}$ be an orthonormal basis of H and p_N be the projection onto $v^N:=\overline{span\{e_j:j\geq N\}}$. Then for all natural numbers $p\leq m$, we have

$$\lim_{N\to\infty}\sup_{n\in\mathbb{N}}\mathbb{E}\left[\int_0^T\|p_N\sigma_s^{\mathcal{S}_n}\|_{L_{HS}(U,H)}^p ds\right]=0.$$

Moreover, for all $q\leq\frac{m}{2}$, we have

$$\lim_{N\to\infty}\sup_{n\in\mathbb{N}}\mathbb{E}\left[\int_0^T\|p_N\alpha_s^{\mathcal{S}_n}\|^q ds\right]=0.$$

PROOF. It is enough to prove the assertion for the first limit as the second limit can be treated as a special case, if we replace $\alpha^{\mathcal{S}_n}$ by the Hilbert-Schmidt operator $e\otimes\alpha$ and using that

$$\|p_N\alpha_s^{\mathcal{S}_n}\|=\|e\otimes p_N\alpha_s^{\mathcal{S}_n}\|,$$

where e is an arbitrary element in U with $\|e\|=1$. No observe that the Bochner integrability of $\|p_N\sigma_s^{\mathcal{S}_n}\|_{L_{\text{HS}}(U,H)}^p$ is guaranteed by assumption and

$$\mathbb{E}\left[\int_0^{\lfloor T/\Delta_n\rfloor\Delta_n}\|p_N\sigma_s^{\mathcal{S}_n}\|_{L_{\text{HS}}(U,H)}^p ds\right]\leq\sum_{i=1}^{\lfloor T/\Delta_n\rfloor}\mathbb{E}\left[\int_{(i-1)\Delta_n}^{i\Delta_n}\sup_{r\in[0,T]}\|p_N\mathcal{S}(r)\sigma_s\|_{L_{\text{HS}}(U,H)}^p ds\right]$$

$$=\int_0^T\mathbb{E}\left[\sup_{r\in[0,T]}\|p_N\mathcal{S}(r)\sigma_s\|_{L_{\text{HS}}(U,H)}^p\right]ds. \tag{49}$$

Now observe that, for $s, r \in [0,T]$ fixed, we have almost surely

$$\|p_N\mathcal{S}(r)\sigma_s\|^2_{L_{\text{HS}}(U,H)} = \|\sigma_s^*\mathcal{S}(r)^*p_N\|^2_{L_{\text{HS}}(H,U)} = \sum_{k=N}^{\infty}\|\sigma_s^*\mathcal{S}(r)^*e_k\|^2 \to 0, \quad \text{as } N\to\infty,$$

since $\sigma_t^*\mathcal{S}(s)^*$ is almost surely a Hilbert-Schmidt operator. Moreover, the function $f_s(r) = \|p_N\mathcal{S}(r)\sigma_s\|_{L_{\text{HS}}(U,H)}$ is continuous in r, as for

$$\begin{aligned}|f_s(r_1) - f_s(r_2)| &\le \|p_N(\mathcal{S}(r_1) - \mathcal{S}(r_2))\sigma_s\|_{L_{\text{HS}}(U,H)} \\ &\le \sup_{r\in[0,T]}\|\mathcal{S}(r)\|_{\text{op}} \sup_{r\le|r_1-r_2|}\|(I-\mathcal{S}(r))\,\sigma_s\|_{L_{\text{HS}}(U,H)}.\end{aligned}$$

The latter converges to 0, as $r_1 \to r_2$, by Proposition 5.1 in [19]. As we also have

$$\|p_N\mathcal{S}(s)\sigma_t\|_{L_{\text{HS}}(U,H)} \ge \|p_{N+1}p_N\mathcal{S}(s)\sigma_t\|_{L_{\text{HS}}(U,H)} = \|p_{N+1}\mathcal{S}(s)\sigma_t\|_{L_{\text{HS}}(U,H)},$$

we find by virtue of Dini's theorem (c.f. Theorem 7.13 in [64]) that almost surely

$$\lim_{N\to\infty}\sup_{t\in[0,T]}\|p_N\mathcal{S}(r)\sigma_s\|_{L_{\text{HS}}(U,H)} = 0.$$

By the dominated convergence theorem, this immediately yields

$$\mathbb{E}\left[\int_0^{\lfloor T/\Delta_n\rfloor\Delta_n}\|p_N\sigma_s^{\mathcal{S}_n}\|^p_{L_{\text{HS}}(U,H)}ds\right] \le \mathbb{E}\left[\int_0^T\sup_{r\in[0,T]}\|p_N\mathcal{S}(r)\sigma_s\|^p_{L_{\text{HS}}(U,H)}ds\right] \to 0,$$

as $N\to\infty$. This proves the claim. □

B.3. Various estimates for increments. In this subsection, we will use the notation

$$\tilde{\Delta}_i^n A := \int_{(i-1)\Delta_n}^{i\Delta_n}\mathcal{S}(i\Delta_n - s)\alpha_s ds, \tag{50}$$

$$\tilde{\Delta}_i^n M := \int_{(i-1)\Delta_n}^{i\Delta_n}\mathcal{S}(i\Delta_n - s)\sigma_s dW_s. \tag{51}$$

We will make use of the following Lemma:

LEMMA B.5. *Suppose that Assumption 10(m) holds. Let $(e_j)_{j\in\mathbb{N}}$ be an orthonormal basis of H and p_N be the projection onto $v^N := \overline{\text{span}\{e_j : j\ge N\}}$. Moreover, let*

$$a_N(z) := \sup_{n\in\mathbb{N}}\mathbb{E}\left[\int_0^T\|p_N\alpha_s^{\mathcal{S}_n}\|^z ds\right],$$

and

$$b_N(z) := \sup_{n\in\mathbb{N}}\mathbb{E}\left[\int_0^T\|p_N\sigma_s^{\mathcal{S}_n}\|^z_{L_{HS}(U,H)}ds\right],$$

which both converge to 0 as $N\to\infty$ for $z\le m$, respectively $z\le\frac{m}{2}$ by Lemma B.4. Then we can find for all $m\in\mathbb{N}$ a universal constant $C = C(m) > 0$, such that

$$\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\mathbb{E}\left[\|p_N\tilde{\Delta}_i^n A\|^m\right] \le C\Delta_n^{m-1}a_N(m), \tag{52}$$

$$\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\mathbb{E}\left[\|p_N\tilde{\Delta}_i^n M\|^m\right] \le C\Delta_n^{\frac{m}{2}-1}b_N(m), \tag{53}$$

and

$$\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}\left[\|p_N \tilde{\Delta}_i^n Y\|^m\right] \leq C(\Delta_n^{m-1} a_N(m) + \Delta_n^{\frac{m}{2}-1} b_N(m)) = o(\Delta_n^{\frac{m}{2}-1}). \tag{54}$$

PROOF. Throughout this proof, we treat C as a generic constant that is chosen appropriately large in each step. The majorisation in (52) is an immediate implication of Bochner's inequality and the boundedness of α. For (53), we get by the BDG inequality (43) that

$$\begin{aligned}
&\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}\left[\|p_N \int_{(i-1)\Delta_n}^{i\Delta_n} \mathcal{S}(i\Delta_n - s)\sigma_s dW_s\|^m\right] \\
&\leq \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} C_m \mathbb{E}\left[\left(\int_{(i-1)\Delta_n}^{i\Delta_n} \|p_N \mathcal{S}(i\Delta_n - s)\sigma_s\|^2_{L_{\text{HS}}(U,H)} ds\right)^{\frac{m}{2}}\right] \\
&\leq \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} C_m \Delta_n^{\frac{m}{2}-1} \int_{(i-1)\Delta_n}^{i\Delta_n} \mathbb{E}\left[\|p_N \mathcal{S}(i\Delta_n - s)\sigma_s\|^m_{L_{\text{HS}}(U,H)}\right] ds \\
&= C_m \Delta_n^{\frac{m}{2}-1} b_N(m).
\end{aligned}$$

Moreover, inequality (54) holds as

$$\begin{aligned}
\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}\left[\|p_N \tilde{\Delta}_i^n Y\|^m\right] &\leq 2^{m-1} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}\left[\|p_N \int_{(i-1)\Delta_n}^{i\Delta_n} \mathcal{S}(i\Delta_n - s)\alpha_s ds\|^m\right] \\
&\quad + 2^{m-1} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}\left[\|p_N \int_{(i-1)\Delta_n}^{i\Delta_n} \mathcal{S}(i\Delta_n - s)\sigma_s dW_s\|^m\right] \\
&\leq 2^{m-1} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \Delta_n^{\frac{m}{2}(\frac{m}{2}-1)} \mathbb{E}\left[\left(\int_{(i-1)\Delta_n}^{i\Delta_n} \|p_N \alpha_s^{\mathcal{S}_n}\|^{\frac{m}{2}} ds\right)^{\frac{m}{2}}\right] \\
&\quad + 2^{m-1} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}\left[\left(\int_{(i-1)\Delta_n}^{i\Delta_n} \|p_N \sigma_s^{\mathcal{S}_n}\|^2_{L_{\text{HS}}(U,H)} ds\right)^{\frac{m}{2}}\right] \\
&\leq 2^{m-1} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \Delta_n^{\frac{m}{2}(\frac{m}{2}-1)} \mathbb{E}\left[\int_{(i-1)\Delta_n}^{i\Delta_n} \|p_N \alpha_s^{\mathcal{S}_n}\|^{\frac{m}{2}}\right] ds A^{\frac{m}{2}-1} \\
&\quad + 2^{m-1} \Delta_n^{\frac{m}{2}-1} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \mathbb{E}\left[\int_{(i-1)\Delta_n}^{i\Delta_n} \|p_N \sigma_s^{\mathcal{S}_n}\|^m_{L_{\text{HS}}(U,H)} ds\right] \\
&\leq \Delta_n^{\frac{m}{2}-1} 2^{m-1} \left(a_N(\frac{m}{2}) A^{\frac{m}{2}-1} + b_N(m)\right).
\end{aligned}$$

□

APPENDIX C: PROOF OF THE LAWS OF LARGE NUMBERS

PROOF OF THEOREMS 4.1 AND 4.2. Let $(e_j)_{j\in\mathbb{N}}$ be an orthonormal basis of H, such that $e_j \in \mathcal{D}(A^*)$ for all $j \in \mathbb{N}$. Recall that P_N^d is the projection onto $\mathcal{V}_N^d$, with

$$\mathcal{V}_N^d := \overline{span\{e_{j_1} \otimes \cdots \otimes e_{j_d} : j_i \geq N, i = 1, \ldots, d\}}.$$

We write $P_N^d = p_N$ if $d=1$. Then we can identify $(I-P_N^m)SAMPV_t^n(m_1,...,m_k)$ with the "matrix"

$$(\langle SAMPV_t^n(m_1,...,m_k), e_{j_1}\otimes ... \otimes e_{j_d}\rangle_{\mathcal{H}^m})_{(j_1,...,j_d)\in\{1,...,N\}^d}.$$

To obtain the asymptotic behaviour of this "matrix", and so the convergence

$$(I-P_N^m)\Delta_n^{1-\frac{m}{2}} SAMPV_t^n(m_1,...,m_k) \stackrel{u.c.p.}{\longrightarrow} (I-P_N^m)\int_0^t \rho_{\Sigma_s}^{\otimes k}(m_1,...,m_k)ds,$$

it is enough to check the convergence of the $\langle SAMPV_t^n(m_1,...,m_k), e_{j_1}\otimes ... \otimes e_{j_m}\rangle_{\mathcal{H}^m}$, for each $(j_1,...,j_m)\in\{1,...,N\}^m$ separately.

Fix therefore some $(e_{p_{i,j}})_{i=1,...,k,j=1,...,m_k}$ with $p_{i,j}\in\mathbb{N}$ and $e_{p_{i,j}}\in\{e_1,...,e_N\}$. Using Lemma B.3 and its notation, we find that

$$\Delta_n^{1-\frac{m}{2}}\langle SAMPV_t^n(m_1,...,m_k), \bigotimes_{i=1}^k\bigotimes_{j=1}^{m_j} e_{p_{i,j}}\rangle_{\mathcal{H}^m}$$

has the same asymptotic behaviour as $\Delta_n^{1-\frac{m}{2}}\sum_{i=1}^{\lfloor t/\Delta_n\rfloor -k+1}\prod_{l=1}^k\prod_{j=1}^{m_l}\langle \Delta_{i+l-1}^n S, e_{p_{l,j}}\rangle$, where we recall that

$$\Delta_{i+l-1}^n S = \int_{(i+j-2)\Delta_n}^{(i+j-1)\Delta_n}\alpha_s ds + \int_{(i+j-2)\Delta_n}^{(i+j-1)\Delta_n}\sigma_s dW_s.$$

This is however a component of the multipower variation of the multivariate semimartingale $(\langle S_t, e_1\rangle,...,\langle S_t, e_N\rangle)_{t\in]0,T]}$. Thus, in the case of power variations (i.e., under Assumption 8), Theorem 3.4.1 in [52] applies, while in the case of multipower variations (i.e., under Assumption 9), Theorem 8.4.1 in [52] applies. Hence, this yields

$$\Delta_n^{1-\frac{m}{2}}\langle SAMPV_t^n(m_1,...,m_k), \bigotimes_{i=1}^k\bigotimes_{j=1}^{m_j} e_{p_{i,j}}\rangle_{\mathcal{H}^m} \stackrel{u.c.p.}{\longrightarrow} \langle \int_0^t \rho_{\Sigma_s}^{\otimes k}(m_1,...,m_k)ds, \bigotimes_{i=1}^k\bigotimes_{j=1}^{m_j} e_{p_{i,j}}\rangle_{\mathcal{H}^m},$$

i.e.,

$$(I-P_N^m)\Delta_n^{1-\frac{m}{2}} SAMPV_t^n(m_1,...,m_k) \stackrel{u.c.p.}{\longrightarrow} (I-P_N^m)\int_0^t \rho_{\Sigma_s}^{\otimes k}(m_1,...,m_k)ds.$$

This establishes the result for finite-dimensional projections of the multipower variation.

The triangle inequality yields

$$\begin{aligned}&\|(SAMPV_t^n(m_1,...,m_k) - \int_0^t \rho_{\Sigma_s}^{\otimes k}(m_1,...,m_k)ds)\|_{\mathcal{H}^m}\\ &\quad\leq \|(I-P_N^m)(SAMPV_t^n(m_1,...,m_k) - \int_0^t \rho_{\Sigma_s}^{\otimes k}(m_1,...,m_k)ds)\|_{\mathcal{H}^m}\\ &\qquad + \|P_N^m SAMPV_t^n(m_1,...,m_k)\|_{\mathcal{H}^m} + \|P_N^m\int_0^t \rho_{\Sigma_s}^{\otimes k}(m_1,...,m_k)ds\|_{\mathcal{H}^m}.\end{aligned}$$

We next show the uniform convergence to zero of the latter two terms. The Markov and generalised Hölder inequality as well as (54) yield for a given $\epsilon>0$,

$$\mathbb{P}\left[\sup_{t\leq T}\Delta_n^{1-\frac{m}{2}}\|P_N^m SAMPV_t^n(m_1,...,m_k)\|_{\mathcal{H}^m} > \epsilon\right]$$

$$
\begin{aligned}
&\le \frac{1}{\epsilon}\Delta_n^{1-\frac{m}{2}}\mathbb{E}\left[\sup_{t\le T}\left\|\sum_{i=1}^{\lfloor t/\Delta_n\rfloor-k+1}\bigotimes_{j=1}^{k}(p_N\tilde{\Delta}_{i+j-1}^n Y)^{\otimes m_j}\right\|_{\mathcal{H}^m}\right]\\
&\le \frac{1}{\epsilon}\Delta_n^{1-\frac{m}{2}}\sum_{i=1}^{\lfloor T/\Delta_n\rfloor-k+1}\mathbb{E}\left[\prod_{j=1}^{k}\left\|p_N\tilde{\Delta}_{i+j-1}^n Y\right\|^{m_j}\right]\\
&\le \frac{1}{\epsilon}\Delta_n^{1-\frac{m}{2}}\sum_{i=1}^{\lfloor T/\Delta_n\rfloor-k+1}\prod_{j=1}^{k}\mathbb{E}\left[\left\|p_N\tilde{\Delta}_{i+j-1}^n Y\right\|^{m}\right]^{\frac{m_j}{m}}\\
&\le \frac{1}{\epsilon}\Delta_n^{1-\frac{m}{2}}\prod_{j=1}^{k}\left(\sum_{i=1}^{\lfloor T/\Delta_n\rfloor-k+1}\mathbb{E}\left[\left\|p_N\tilde{\Delta}_{i+j-1}^n Y\right\|^{m}\right]\right)^{\frac{m_j}{m}}\\
&\le \frac{1}{\epsilon}\Delta_n^{1-\frac{m}{2}}\prod_{j=1}^{k}\left(C\Delta_n^{\frac{m}{2}-1}(a_N(\frac{m}{2})+b_N(m))\right)^{\frac{m_j}{m}}.
\end{aligned}
$$

This converges to 0 as $N\to\infty$ uniformly in n. Now, notice that by definition we have

$$
\langle \rho_{p_N\Sigma_s p_N}, e_{j_1}\otimes ... \otimes e_{j_m}\rangle = \langle \rho_{\Sigma_s}, e_{j_1}\otimes ... \otimes e_{j_m}\rangle \delta_{j_1,...,j_n\ge N}.
$$

Therefore

$$
\begin{aligned}
\|\rho_{p_N\Sigma_s p_N}(m)\|_{\mathcal{H}^m}^2 &= \sum_{j_1,...,j_m\ge N}\langle \rho_{\Sigma_s}(m), e_{j_1}\otimes ... \otimes e_{j_m}\rangle^2\\
&= \sum_{j_1,...,j_m\ge N}(\sum_{p\in\mathcal{P}(m)}\prod_{(k,l)\in p}\langle \Sigma_s e_{j_l}, e_{j_k}\rangle)^2\\
&\le |\mathcal{P}(m)|\sum_{p\in\mathcal{P}(m)}\sum_{j_1,...,j_m\ge N}\prod_{(k,l)\in p}\langle \Sigma_s e_{j_l}, e_{j_k}\rangle^2\\
&\le |\mathcal{P}(m)|\sum_{p\in\mathcal{P}(m)}\sum_{j_1,...,j_m\ge N}\prod_{(k,l)\in p}\|\Sigma_s^{\frac{1}{2}}e_{j_l}\|^2\|\Sigma_s^{\frac{1}{2}}e_{j_k}\|^2.\\
&= |\mathcal{P}(m)|^2(\sum_{j\ge N}\|\Sigma_s^{\frac{1}{2}}e_j\|^2)^m,
\end{aligned}
$$

which converges to 0 almost surely as $N\to\infty$, as $\Sigma_s^{\frac{1}{2}}$ is a Hilbert-Schmidt operator. Hence, by the definition of $\rho_{\Sigma_s}^{\otimes k}(m_1,...,m_k)$, it holds,

$$
\begin{aligned}
&\mathbb{P}\left[\sup_{t\le T}\left\|P_N^m\int_0^t \rho_{\Sigma_s}^{\otimes k}(m_1,...,m_k)ds\right\|_{\mathcal{H}^m}>\epsilon\right]\\
&\quad\le \frac{1}{\epsilon}\int_0^T\mathbb{E}\left[\left\|P_N^m\rho_{\Sigma_s}^{\otimes k}(m_1,...,m_k)\right\|_{\mathcal{H}^m}\right]ds\\
&\quad= \frac{1}{\epsilon}\int_0^T\mathbb{E}\left[\left(\sum_{j_1,...,j_m\in\mathbb{N}}\langle \rho_{p_N\Sigma_s p_N}^{\otimes k}(m_1,...,m_k), e_1\otimes ... \otimes e_m\rangle^2\right)^{\frac{1}{2}}\right]ds
\end{aligned}
$$

$$= \frac{1}{\epsilon}\int_0^T \mathbb{E}\left[\prod_{l=1}^k \left(\sum_{j_1,...,j_{m_l}\in\mathbb{N}} \langle \rho_{p_N\Sigma_s p_N}(m_l), e_1\otimes ...\otimes e_{m_l}\rangle^2\right)^{\frac{1}{2}}\right] ds$$
$$= \frac{1}{\epsilon}\int_0^T \mathbb{E}\left[\prod_{l=1}^k \|\rho_{p_N\Sigma_s p_N}(m_l)\|_{\mathcal{H}^m}\right] ds$$
$$\leq \frac{|\mathcal{P}(m)|}{\epsilon}\int_0^T \mathbb{E}\left[\left(\sum_{j\geq N} \|\Sigma_s^{\frac{1}{2}} e_j\|^2\right)^{\frac{m}{2}}\right] ds.$$

This converges to 0 by the Dominated Convergence Theorem.

Summing up, we can, for each $\delta > 0$, find an $N \in \mathbb{N}$ independent of n, such that

$$\lim_{n\to\infty} \mathbb{P}\left[\sup_{t\leq T}\left(\|(SAMPV_t^n(m_1,...,m_k) - \int_0^t \rho_{\Sigma_s}^{\otimes k}(m_1,...,m_k)ds)\|_{\mathcal{H}^m}\right) > \epsilon\right]$$
$$\leq \lim_{n\to\infty} \mathbb{P}\left[\sup_{t\leq T}\left(\|(I-P_N^m)(SAMPV_t^n(m_1,...,m_k) - \int_0^t \rho_{\Sigma_s}^{\otimes k}(m_1,...,m_k)ds)\|_{\mathcal{H}^m}\right) > \epsilon\right]$$
$$+ \limsup_{n\to\infty}\left(\mathbb{P}\left[\sup_{t\leq T}(\|P_N^m SAMPV_t^n(m_1,...,m_k)\|_{\mathcal{H}^m}) > \epsilon\right]\right.$$
$$\left. + \mathbb{P}\left[\sup_{t\leq T}\left(\|P_N^m \int_0^t \rho_{\Sigma_s}^{\otimes k}(m_1,...,m_k)ds\|_{\mathcal{H}^m}\right) > \epsilon\right]\right)$$
$$\leq \lim_{n\to\infty} \mathbb{P}\left[\sup_{t\leq T}\left(\|(I-P_N^m)(SAMPV_t^n(m) - \int_0^t \rho_{\Sigma_s}^{\otimes k}(m_1,...,m_k)ds)\|_{\mathcal{H}^m}\right) > \epsilon\right] + 2\delta$$
$$= 2\delta.$$

This holds for all $\delta > 0$, and hence the assertion follows. □

APPENDIX D: PROOF OF THE CENTRAL LIMIT THEOREMS

We prove the central limit theorems by proving the tightness of the laws of the processes in the Skorokhod space $\mathcal{D}([0,T],H)$ first and then make use of the available finite-dimensional asymptotic limit theory in order to show convergence of the corresponding finite-dimensional distributions.

D.1. A short primer on tightness. Recall that a sequence of measures $(\mu_n)_{n\in\mathbb{N}}$ is tight on a Polish space B equipped with its Borel σ-algebra $(B,\mathcal{B}(B))$, if for each $\epsilon > 0$ there is a compact set $K_\epsilon \subset B$ such that $\sup_{n\in\mathbb{N}} \mu_n(B \setminus K_\epsilon) < \epsilon$. We will say that a sequence $(X_n)_{n\in\mathbb{N}}$ of Borel-measurable random variables in B (e.g. stochastic processes) is tight if the underlying sequence of laws $(\mu_{X_n})_{n\in\mathbb{N}}$ is tight.

D.1.1. *Tightness of random elements in the Skorokhod space $\mathcal{D}([0,T],H)$.* For the convenience of the reader, we repeat the following tightness criterion from [53, p.35].

THEOREM D.1. *Let H be a separable Hilbert space. The family of laws $(\mathbb{P}_{\psi^n})_{n\in\mathbb{N}}$ of a sequence of random variables $(\psi^n)_{n\in\mathbb{N}}$ in $\mathcal{D}([0,T],H)$ is tight if the following two conditions hold:*

(i) $(\mathbb{P}_{\psi_t^n})_{n\in\mathbb{N}}$ *is tight for each* $t \in [0,T]$ *and*
(ii) *(Aldous' condition) For all* $\epsilon, \eta > 0$ *there is an* $\delta > 0$ *and* $n_0 \in \mathbb{N}$ *such that for all sequences of stopping times* $(\tau_n)_{n\in\mathbb{N}}$ *with* $\tau_n \leq T - \delta$ *we have*

$$\sup_{n\geq n_0} \sup_{\theta\leq\delta} \mathbb{P}\left[\|\psi_{\tau_n}^n - \psi_{\tau_n+\theta}^n\|_H > \eta\right] \leq \epsilon. \tag{55}$$

Regarding point (i) above, to show tightness in the space $\mathcal{D}([0,T],H)$ it is necessary to find criteria for the tightness in the Hilbert space itself. This can be approached by an *equi-small tails*-argument and is well known (c.f. Lemma 1.8.1 in [68]):

THEOREM D.2. *Let* $(Y_n)_{n\in\mathbb{N}}$ *be a sequence of random variables on a probability space* $(\Omega,\mathcal{F},\mathbb{P})$ *with values in a separable Hilbert space* H*, such that for all* $\delta > 0$

$$\lim_{N\to\infty} \sup_{n\in\mathbb{N}} \mathbb{P}\left[\sum_{k\geq N} \langle Y_n, e_k\rangle^2 > \delta\right] = 0, \tag{56}$$

for some orthonormal basis $(e_n)_{n\in\mathbb{N}}$*. Then the sequence* $(\mathbb{P}_{Y_n})_{n\in\mathbb{N}}$ *is tight in* H.

PROOF. Fix some $\epsilon > 0$. By assumption we can define two increasing sequences of natural numbers $(N_k^\epsilon)_{k\in\mathbb{N}}$ and $(l_k)_{k\in\mathbb{N}}$, such that $N_1^\epsilon = 1$ and

$$\sup_{n\in\mathbb{N}} \mathbb{P}\left[\sum_{l\geq N_k^\epsilon} \langle Y_n, e_l\rangle^2 > \frac{1}{l_k}\right] \leq \epsilon \frac{1}{l_k^2 \sum_{j=1}^\infty \frac{1}{l_j^2}}. \tag{57}$$

Further, we introduce

$$A_k^\epsilon := \left\{ h \in H : \sum_{l\geq N_k^\epsilon} \langle h, e_l\rangle^2 \leq \frac{1}{l_k} \right\}. \tag{58}$$

We prove now that $K_\epsilon = \bigcap_{k\in\mathbb{N}} A_k^\epsilon$ is compact. It is obviously closed and bounded Then we have as $k \to \infty$

$$\sup_{h\in K_\epsilon} \sum_{l\geq N_k^\epsilon} \langle h, e_l\rangle^2 \leq \frac{1}{l_k} \to 0.$$

Hence, the set K_ϵ is totally bounded and by the Hausdorff theorem (c.f. Theorem 3.28 in [2]) compact.

It is now left to show that $1 - \mathbb{P}_{Y_n}[K_\epsilon] < \epsilon$. But, by Markov's inequality and the choice of N_k^ϵ we have

$$1 - \mathbb{P}_{Y_n}[K_\epsilon] \leq \sum_{k=1}^\infty \mathbb{P}_{Y_n}[(A_k^\epsilon)^c] \leq \epsilon,$$

which proves the claim. □

By Markov's inequality, we have the following Corollary to Theorem D.2,

COROLLARY D.3. *Let* $(Y_n)_{n\in\mathbb{N}}$ *be a sequence of random variables on a probability space* $(\Omega,\mathcal{F},\mathbb{P})$ *with values in a separable Hilbert space* H *and having finite second moments. If for some orthonormal basis* $(e_n)_{n\in\mathbb{N}}$ *we have*

$$\lim_{N\to\infty} \sup_{n\in\mathbb{N}} \sum_{k\geq N} \mathbb{E}\left[\langle Y_n, e_k\rangle^2\right] = 0, \tag{59}$$

then the sequence $(\mathbb{P}_{Y_n})_{n\in\mathbb{N}}$ *is tight in* H.

D.1.2. *Tightness and stable convergence.* Let E be a Polish space, $\mathcal{E} := \mathcal{B}(E)$ its Borel σ-algebra and $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Recall that a map $K : \Omega \times \mathcal{E} \to [0,1]$ is called a Markov kernel from $(\Omega, \mathcal{F})$ to $(E, \mathcal{E})$, if for all $\omega \in \Omega$ the map $K(\omega, \cdot)$ is a Borel probability measure on E and for all $A \in \mathcal{E}$ the map $K(\cdot, A)$ is an $\mathcal{F}$-measurable random variable.

Let $(Y_n)_{n\in\mathbb{N}}$ be a sequence of random variables with values in the Skorokhod space $\mathcal{D}([0,T], H)$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and Y a random variable with values in $\mathcal{D}([0,T], H)$ defined on an extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ of $(\Omega, \mathcal{F}, \mathbb{P})$. Observe, that we can specify a Markov kernel K by the conditional distribution

$$K(\omega, A) = \tilde{\mathbb{P}}[Y \in A|\mathcal{F}].$$

One can then see that stable convergence of the sequence $(Y_n)_{n\in\mathbb{N}}$ to Y can be written as

$$\mathbb{E}[Zf(Y_n)] \to \tilde{\mathbb{E}}[Zf(Y)] = \int_\Omega Z(\omega) \int_{\mathcal{D}([0,T],H)} f(x)K(\omega, dx)\mathbb{P}[d\omega], \quad \text{as } n \to \infty \tag{60}$$

for all bounded continuous functions $f : \mathcal{D}([0,T], H) \to \mathbb{R}$ and all bounded random variables Z on $(\Omega, \mathcal{F})$. In that way we can identify stable convergence of a sequence of random variables as convergence towards a Markov kernel in the sense of (60). We will use this in the proof of the next theorem, which can be found in [46, Proposition 3.9] for continuous processes. Here we extend the proof for processes with values in the Skorokhod space.

THEOREM D.4. *Let $(Y_n)_{n\in\mathbb{N}}$ be a sequence of random variables with values in the Skorokhod space $\mathcal{D}([0,T], H)$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and Y a random variable with values in $\mathcal{D}([0,T], H)$ defined on an extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ of $(\Omega, \mathcal{F}, \mathbb{P})$. If $(Y_n)_{n\in\mathbb{N}}$ is tight and $(Y_n(t_1), ..., Y_n(t_d)) \to (Y(t_1), ..., Y(t_d))$ stably for each finite collection $t_1, ..., t_d \in [0,T]$, $d \in \mathbb{N}$, then $Y_n \to Y$ stably as $n \to \infty$.*

PROOF. Assume $(Y_n)_{n\in\mathbb{N}}$ is tight and $(Y_n(t_1), ..., Y_n(t_d)) \to (Y(t_1), ..., Y(t_d))$ stably for each finite collection $t_1, ..., t_d \in [0,T]$ and $(Y_n)_{n\in\mathbb{N}}$ does not converge stably to Y. Then we can find a subsequence $(n_k)_{k\in\mathbb{N}}$, $\epsilon > 0$, a bounded real-valued random variable Z on $(\Omega, \mathcal{F}, \mathbb{P})$ and $h \in C_b(\mathcal{D}([0,T], H))$, such that

$$|\mathbb{E}[Zh(Y_{n_k})] - \tilde{\mathbb{E}}[Zh(Y)]| \geq \epsilon. \tag{61}$$

Equivalently, this means for the kernel

$$K(\omega, A) := \tilde{\mathbb{P}}[Y \in A|\mathcal{F}](\omega)$$

that

$$|\mathbb{E}[Zh(Y_{n_k})] - \int_\Omega Z(\omega) \int_{\mathcal{D}([0,T],H)} f(x)K(\omega, dx)\mathbb{P}[d\omega]| \geq \epsilon. \tag{62}$$

By the tightness of $(Y_n)_{n\in\mathbb{N}}$, we can appeal to Theorem 3.4(a) in [46] and obtain a subsequence $(n_{k_l})_{l\in\mathbb{N}}$ of $(n_k)_{k\in\mathbb{N}}$, such that $Y_{n_{k_l}} \to L$ stably for some Markov kernel $L : \Omega \times \mathcal{B}(\mathcal{D}([0,T], H)) \to [0,1]$ as $n_{k_l} \to \infty$. Then we have for all $F \in \mathcal{F}$ with $\mathbb{P}(F) > 0$ that

$$\mathbb{P}^F \circ Y_{n_{k_l}}^{-1} \xrightarrow{d} \mathbb{P}^F[L] := \int_\Omega L(\omega, \cdot)\mathbb{P}^F[d\omega]$$

weakly by Theorem 3.2 (iv) in [46], where $\mathbb{P}^F[A] := \frac{\mathbb{P}[A\cap F]}{\mathbb{P}[F]}$ is the conditional probability. According to [25, p.138-139] there is then for each $F \in \mathcal{F}$ with $\mathbb{P}(F) > 0$ a dense set $T_{\mathbb{P}^F[L]} \subset [0,T]$ (depending on the limiting distribution), that contains 0 and T and

$$\mathbb{P}^F \circ Y_{n_{k_l}}^{-1} \circ (\pi_{t_1,...,t_d}) \xrightarrow{d} \mathbb{P}^F[L] \circ (\pi_{t_1,...,t_d}) \tag{63}$$

whenever $t_1,...,t_d \in T_{\mathbb{P}^F[L]}$ for d arbitrary. Here, $\pi_{t_1,...,t_d}(f) = (f(t_1),...,f(t_d))$ denotes the finite-dimensional projections. By Theorem 12.5 in [25], the sets

$$\pi^{-1}_{t_1,...,t_d}(A), \quad A \in \mathcal{B}(H^d), t_1,...,t_d \in T_{\mathbb{P}^F[L]}$$

generate $\mathcal{B}(\mathcal{D}([0,T],H))$, where $H^d = H \times ... \times H$ is equipped with the product topology, and, in particular, two measures $\mathbb{Q}_1$ and $\mathbb{Q}_2$ coincide on $\mathcal{B}(\mathcal{D}([0,T],H))$, if $\mathbb{Q}_1 \circ \pi^{-1}_{t_1,...,t_d} = \mathbb{Q}_2 \circ \pi^{-1}_{t_1,...,t_d}$ for all $t_1,...,t_d \in T_{\mathbb{P}^F[L]}$, $d \in \mathbb{N}$. By assumption we have that $(Y_n(t_1),...,Y_n(t_d)) \to (Y(t_1),...,Y(t_d))$ stably, which is equivalent to

$$\mathbb{P}^F \circ Y_n^{-1} \circ \pi_{t_1,...,t_d} \xrightarrow{d} \tilde{\mathbb{P}}^F \circ Y^{-1} \circ \pi_{t_1,...,t_d}$$

for all $t_1,...,t_d \in [0,T]$, $d \in \mathbb{N}$ and all $F \in \mathcal{F}$ with $\mathbb{P}(F) > 0$ by [46, Theorem 3.4(iv)]. This together with (63) yields

$$\tilde{\mathbb{P}}^F \circ Y^{-1} \circ \pi_{t_1,...,t_d} = \mathbb{P}^F[L] \circ (\pi_{t_1,...,t_d})$$

for all $t_1,...,t_d \in T_{\mathbb{P}^F[L]}$, $d \in \mathbb{N}$ and, hence,

$$\mathbb{P}^F[K] = \int_\Omega K(\omega,\cdot)\mathbb{P}^F[d\omega] = \tilde{\mathbb{P}}^F \circ Y^{-1} = \mathbb{P}^F[L]$$

for all $F \in \mathcal{F}$ with $\mathbb{P}(F) > 0$. However, this shows that $K = L$, which is a contradiction, as by construction of L it is

$$\mathbb{E}[Zf(Y_{n_{k_l}})] \to \int_\Omega Z(\omega) \int_{\mathcal{D}([0,T],H)} f(x)L(\omega,dx)\mathbb{P}[d\omega], \quad \text{as } l \to \infty.$$

□

D.2. Tightness results for the central limit theorems. In this section, we are going to prove in several steps the following theorem. Recall the notation $\Sigma_s^{\mathcal{S}_n} := \mathcal{S}(i\Delta_n - s)\Sigma_s\mathcal{S}(i\Delta_n - s)^*$ for $s \in [(i-1)\Delta_n, i\Delta_n)$, that we will use extensively here.

THEOREM D.5. *Let Assumption 7 hold. Then the sequence of processes*

$$(\tilde{Z}_t^{n,2})_{t\in[0,T]} := \left(\Delta_n^{-\frac{1}{2}}\left(\sum_{i=1}^{\lfloor t/\Delta_n\rfloor} \tilde{\Delta}_i^n Y^{\otimes 2} - \int_{(i-1)\Delta_n}^{i\Delta_n} \Sigma_s^{\mathcal{S}_n} ds\right)\right)_{t\in[0,T]}, \quad n \in \mathbb{N}$$

is tight in $\mathcal{D}([0,T],\mathcal{H})$.

Despite the rather extensive notation, it is relatively straightforward to show that $(\tilde{Z}^{n,2})_{t\in[0,T]}$ satisfies Aldous' condition.

THEOREM D.6. *(Temporal tightness) Let* $(\mathbb{P}_n)_{n\in\mathbb{N}}$ *be given by* $\mathbb{P}_n = \mathbb{P}_{(\tilde{Z}_t^{n,2})_{t\in[0,T]}}$ *and Assumption 7 hold. Then* $(\mathbb{P}_n)_{n\in\mathbb{N}}$ *satisfies Aldous' condition.*

PROOF. The Markov inequality yields

$$\begin{aligned}
&\mathbb{P}\left[\left\|\tilde{Z}^{n,2}_{\tau_n} - Z^{n,2}_{\tau_n+\theta}\right\|_{\mathcal{H}} > \eta\right] \\
\leq &\frac{1}{\eta}\mathbb{E}\left[\left\|\tilde{Z}^{n,2}_{\tau_n} - \tilde{Z}^{n,2}_{\tau_n+\theta}\right\|_{\mathcal{H}}\right] \\
\leq &\frac{1}{\eta}\left(\Delta_n^{-\frac{1}{2}}\mathbb{E}\left[\left\|\sum_{i=\lfloor \tau_n/\Delta_n\rfloor}^{\lfloor(\tau_n+\theta)/\Delta_n\rfloor} \tilde{\Delta}_i^n Y^{\otimes 2}\right\|_{\mathcal{H}} + \Delta_n^{-\frac{1}{2}}\left\|\int_{\tau_n}^{\tau_n+\theta} \Sigma_s ds\right\|_{\mathcal{H}}\right]\right).
\end{aligned} \tag{64}$$

Now, set $\theta < \delta < \Delta_n$. We can estimate further

$$\begin{aligned}
&\mathbb{E}\left[\left\|\sum_{i=\lfloor \tau_n/\Delta_n\rfloor}^{\lfloor(\tau_n+\theta)/\Delta_n\rfloor} \tilde{\Delta}_i^n Y^{\otimes 2}\right\|_{\mathcal{H}} + \left\|\int_{\tau_n}^{\tau_n+\theta} \Sigma_s ds\right\|_{\mathcal{H}}\right] \\
&\leq \left(\mathbb{E}\left[\left\|\tilde{\Delta}^n_{\lfloor \tau_n+\theta/\Delta_n\rfloor} Y^{\otimes 2}\right\|_{\mathcal{H}} + \left\|\tilde{\Delta}^n_{\lfloor \tau_n/\Delta_n\rfloor} Y^{\otimes 2}\right\|_{\mathcal{H}}\right]\right) \\
&\qquad + \mathbb{E}[\int_{\tau_n}^{\tau_n+\theta} \|\Sigma_s\|_{\mathcal{H}}\, ds] \\
&= \left(\mathbb{E}\left[\left\|\tilde{\Delta}^n_{\lfloor \tau_n+\theta/\Delta_n\rfloor} Y\right\|^2 + \left\|\tilde{\Delta}^n_{\lfloor \tau_n/\Delta_n\rfloor} Y\right\|^2\right]\right) + \mathbb{E}[\int_{\tau_n}^{\tau_n+\theta} \|\Sigma_s\|_{\mathcal{H}} ds] \\
&= (1)_n + (2)_n.
\end{aligned}$$

We obtain

$$\begin{aligned}
&\Delta_n^{-\frac{1}{2}}\mathbb{E}\left[\left\|\tilde{\Delta}^n_{\lfloor \tau_n+\theta/\Delta_n\rfloor} Y\right\|^2\right] \\
&\leq \Delta_n^{-\frac{1}{2}} 2\left(\mathbb{E}\left[\left\|\int_{(\lfloor \tau_n+\theta/\Delta_n\rfloor)\Delta_n}^{(\lfloor \tau_n+\theta/\Delta_n\rfloor+1)\Delta_n} \alpha_s^{\mathcal{S}_n} ds\right\|^2\right]\right. \\
&\qquad \left. + \mathbb{E}\left[\left\|\int_{(\lfloor \tau_n+\theta/\Delta_n\rfloor)\Delta_n}^{(\lfloor \tau_n+\theta/\Delta_n\rfloor+1)\Delta_n} \sigma_s^{\mathcal{S}_n} dW_s\right\|^2\right]\right) \\
&\leq \Delta_n^{-\frac{1}{2}} 2\left(\mathbb{E}\left[\Delta_n^{-1}\int_{(\lfloor \tau_n+\theta/\Delta_n\rfloor)\Delta_n}^{(\lfloor \tau_n+\theta/\Delta_n\rfloor+1)\Delta_n} \left\|\alpha_s^{\mathcal{S}_n}\right\|^2 ds\right]\right. \\
&\qquad \left. + \mathbb{E}\left[\int_{(\lfloor \tau_n+\theta/\Delta_n\rfloor)\Delta_n}^{(\lfloor \tau_n+\theta/\Delta_n\rfloor+1)\Delta_n} \|\sigma_s\|^2_{L_{\text{HS}}(U,H)} ds\right]\right) \\
&\leq 2\Delta_n^{\frac{1}{2}} \sup_{r\in[0,T]} \|\mathcal{S}(r)\|_{\text{op}}^2 \int_0^T \mathbb{E}\left[\|\alpha_s\|^2\right] ds \\
&\qquad + 2\mathbb{E}\left[\left(\int_{(\lfloor \tau_n+\theta/\Delta_n\rfloor)\Delta_n}^{(\lfloor \tau_n+\theta/\Delta_n\rfloor+1)\Delta_n} \|\sigma_s\|^4_{L_{\text{HS}}(U,H)} ds\right)^{\frac{1}{2}}\right], \qquad (65)
\end{aligned}$$

which converges to 0 as $n \to \infty$ since the function $t \mapsto \int_0^t \|\sigma_s\|^4_{L_{\text{HS}}(U,H)} ds$ is uniformly continuous on $[0,T]$ and bounded.

Analogously we obtain

$$\Delta_n^{-\frac{1}{2}}\mathbb{E}\left[\|\tilde{\Delta}^n_{\lfloor \tau_n/\Delta_n\rfloor} Y^{\otimes 2}\|_{\mathcal{H}}\right] \to 0,$$

as $n \to \infty$. This yields $\lim_{n\to\infty} \Delta_n^{-\frac{1}{2}}(1)_n = 0$.

It remains to show $\lim_{n\to\infty} \Delta_n^{-\frac{1}{2}}(2)_n = 0$. Observe that since $s \mapsto \Sigma_s$ is bounded by assumption, the following convergence holds almost surely as $n \to \infty$:

$$\Delta_n^{-\frac{1}{2}}\int_{\tau_n}^{\tau_n+\theta} \|\Sigma_s\|_{\mathcal{H}} ds \leq \left(\int_{\tau_n}^{\tau_n+\theta} \|\Sigma_s\|^2_{\mathcal{H}} ds\right)^{\frac{1}{2}} \leq \sup_{t\in[0,T-\theta]} \left(\int_t^{t+\theta} \|\Sigma_s\|^2_{\mathcal{H}} ds\right)^{\frac{1}{2}}$$

$$\leq \sup_{t,s\in[0,T],t-s\leq\Delta_n} \left(\int_s^t \|\Sigma_s\|_{\mathcal{H}}^2 ds\right)^{\frac{1}{2}}$$

converges to zero, as $t \mapsto \int_0^t \|\Sigma_s\|_{\mathcal{H}}^2 ds$ is uniformly continuous on $[0,T]$ almost surely, as $\theta \leq \Delta_n$. Moreover, we have the integrable majorant

$$\int_{\tau_n}^{\tau_n+\theta} \|\Sigma_s\|_{\mathcal{H}} ds \leq \int_0^T \|\Sigma_s\|_{\mathcal{H}} ds,$$

such that we obtain by dominated convergence $\Delta_n^{-\frac{1}{2}}(2) \to 0$ as $n \to \infty$. Hence, we conclude the Aldous condition in Theorem D.1. □

In order to show tightness of the sequences $(\tilde{Z}^{n,2})_{n\in\mathbb{N}}$ in $\mathcal{D}([0,T],\mathcal{H})$ under the conditions of Theorem D.5, we have to verify the tightness of each $(\tilde{Z}_t^{n,2})_{n\in\mathbb{N}}$ in $\mathcal{H}$ for each $t \in [0,T]$ separately. This is what we do in the remainder of this subsection. We first argue that we can assume $\alpha \equiv 0$ in the proof of tightness of $\tilde{Z}_t^{n,2}$ in $\mathcal{H}$. Observe that with the notation of Section B.3, we have

$$\begin{aligned}
&\Delta_n^{-\frac{1}{2}}\left(SARCV_t^n - \int_0^t \Sigma_s^{\mathcal{S}_n} ds\right)\\
=&\Delta_n^{-\frac{1}{2}}\left(\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\left(\tilde{\Delta}_i^n A + \tilde{\Delta}_i^n M\right)^{\otimes 2} - \int_0^t \Sigma_s^{\mathcal{S}_n} ds\right)\\
=&\Delta_n^{-\frac{1}{2}}\left(\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\tilde{\Delta}_i^n A^{\otimes 2} + \tilde{\Delta}_i^n A \otimes \tilde{\Delta}_i^n M + \tilde{\Delta}_i^n M \otimes \tilde{\Delta}_i^n A\right)\\
&+\Delta_n^{-\frac{1}{2}}\left(\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\tilde{\Delta}_i^n M^{\otimes 2} - \int_0^t \Sigma_s^{\mathcal{S}_n} ds\right)\\
=:&(I)_t^n + (II)_t^n. \qquad (66)
\end{aligned}$$

We obtain:

THEOREM D.7 (Elimination of the drift). *Suppose that Assumption 10(2m) holds. In this case, the first summand in (66), that is $(I)_t^n$ is tight. In particular, in order to show Theorem D.8 we can assume $\alpha \equiv 0$.*

PROOF. We show first that

$$\lim_{N\to\infty}\sup_{n\in\mathbb{N}}\mathbb{E}\left[\sup_{t\in[0,T]}\left\|P_N^2(I)_t^n\right\|_{\mathcal{H}}\right] = 0. \qquad (67)$$

We can compute, using Hölder's inequality

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\left\|P_N^2(I)_t^n\right\|_{\mathcal{H}}\right]\\
\leq&\Delta_n^{-\frac{1}{2}}\sum_{i=1}^{\lfloor T/\Delta_n\rfloor}\mathbb{E}\left[\|P_N^2\tilde{\Delta}_i^n A\|^2 + 2\|P_N^2\tilde{\Delta}_i^n A\|\|P_N^2\tilde{\Delta}_i^n M\|\right]
\end{aligned}$$

$$
\begin{aligned}
=&\Delta_n^{-\frac{1}{2}} \sum_{i=1}^{\lfloor T/\Delta_n \rfloor} \mathbb{E}\left[\|P_N^2 \tilde{\Delta}_i^n A\|^2\right] \\
&+ 2\Delta_n^{-\frac{1}{2}} \left(\sum_{i=1}^{\lfloor T/\Delta_n \rfloor} \mathbb{E}\left[\|P_N^2 \tilde{\Delta}_i^n A\|^2\right]\right)^{\frac{1}{2}} \left(\sum_{i=1}^{\lfloor T/\Delta_n \rfloor} \mathbb{E}\left[\|P_N^2 \tilde{\Delta}_i^n M\|\right]^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

Now using (52) and (53) as well as the corresponding notation from Lemma B.5 we find

$$
\mathbb{E}\left[\sup_{t\in[0,T]} \left\|P_N^2 (I)_t^n\right\|_{\mathcal{H}}\right] \leq \Delta_n^{\frac{1}{2}} C\Delta_n a_N(2) + 2\left(Ca_N(2)\right)^{\frac{1}{2}} \left(Cb_N(2)\right)^{\frac{1}{2}}
$$

The latter converges to 0, as $N \to \infty$, uniformly in n. Thus (67) holds. A straightforward application of Markov's inequality and Theorem D.2 yield the assertion. □

In the subsections below we make the remaining steps in order to prove Theorem D.8, i.e., in view of Theorem D.1 and Theorem D.6 we have to prove that $(\tilde{Z}_t^{n,2})_{n\in\mathbb{N}}$ is tight in $\mathcal{H}$. In view of Theorem D.7 we further assume that $\alpha \equiv 0$ throughout these subsections.

D.2.1. *Spatial tightness for quadratic variation.* Recalling that Assumption 7 is satisfied under the assumptions of Theorem D.8, the tightness of the sequence of laws corresponding to $(\tilde{Z}^{n,2})_{n\in\mathbb{N}}$ is tight in $\mathcal{D}([0,T],\mathcal{H})$ by

THEOREM D.8. *Assume that*

$$
\int_0^T \mathbb{E}\left[\|\sigma_s^{\mathcal{S}_n}\|_{L_{HS}(U,H)}^4\right] ds < \infty,
$$

which is in particular the case, if Assumption 7 holds. We have

$$
\begin{aligned}
&\lim_{N\to\infty} \sup_{t\in[0,T]} \sup_{n\in\mathbb{N}} \sum_{m,k\geq N} \mathbb{E}\left[\langle \tilde{Z}_t^{2,n}, e_k \otimes e_m\rangle_{\mathcal{H}}^2\right] \\
&= \lim_{N\to\infty} \sup_{t\in[0,T]} \sup_{n\in\mathbb{N}} \mathbb{E}\left[\|p_N \tilde{Z}_t^{2,n}\|_{\mathcal{H}}^2\right] = 0, \qquad (68)
\end{aligned}
$$

and thus, the sequence $(\tilde{Z}_t^{n,2})_{n\in\mathbb{N}}$ is tight in $\mathcal{H}$. Hence, the sequence $(\tilde{Z}^{n,2})_{n\in\mathbb{N}}$ is tight in $\mathcal{D}([0,T],\mathcal{H})$.

PROOF. We define

$$
\begin{aligned}
\tilde{Z}_n^N(i) :=& \Delta_n^{-\frac{1}{2}} \left((p_N \tilde{\Delta}_i^n Y)^{\otimes 2} - \langle\langle p_N \tilde{\Delta}_i^n Y\rangle\rangle\right) \\
=& \Delta_n^{-\frac{1}{2}} \left((p_N \tilde{\Delta}_i^n Y)^{\otimes 2} - \int_{t_{i-1}}^{t_i} p_N \mathcal{S}(t_i - s)\Sigma_s \mathcal{S}(t_i - s)^* p_N ds\right).
\end{aligned}
$$

First we show that $\sup_{t\in[0,T]} \|\sum_{i=1}^{\lfloor t/\Delta_n\rfloor} \tilde{Z}_n^N(i)\|_{\text{HS}}$ has finite second moment. Note that, by the BDG inequality (43), we have

$$
\begin{aligned}
\mathbb{E}\left[\left\|p_N \tilde{\Delta}_i^n Y\right\|^4\right] \leq& \mathbb{E}\left[\left(\int_{(i-1)\Delta_n}^{i\Delta_n} \|\sigma_s^{\mathcal{S}_n}\|_{L_{\text{HS}}(U,H)}^2 ds\right)^2\right] \qquad (69) \\
\leq& \Delta_n \mathbb{E}\left[\int_{(i-1)\Delta_n}^{i\Delta_n} \|\sigma_s^{\mathcal{S}_n}\|_{L_{\text{HS}}(U,H)}^4 ds\right]. \qquad (70)
\end{aligned}
$$

Therefore, by the triangle and Cauchy-Schwarz inequalities, we have

$$
\begin{aligned}
&\mathbb{E}\left[\left\|\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\tilde{Z}_n^N(i)\right\|_{\mathcal{H}}^2\right]\\
&\quad\leq \mathbb{E}\left[\left(\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\left\|\tilde{Z}_n^N(i)\right\|_{\mathcal{H}}\right)^2\right]\\
&\quad\leq \mathbb{E}\left[\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\left\|\tilde{Z}_n^N(i)\right\|_{\mathcal{H}}^2\right]\lfloor t/\Delta_n\rfloor\\
&\quad\leq \Delta_n^{-1}\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\mathbb{E}\left[\left(\left\|p_N\tilde{\Delta}_i^n Y\right\|^2+\int_{(i-1)\Delta_n}^{i\Delta_n}\|\Sigma_s^{\mathcal{S}_n}\|_{L_{\text{HS}}(U,H)}\,ds\right)^2\right]\lfloor t/\Delta_n\rfloor\\
&\quad\leq \Delta_n^{-1}\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}2\left(\mathbb{E}\left[\left\|p_N\tilde{\Delta}_i^n Y\right\|^4\right]+\int_{(i-1)\Delta_n}^{i\Delta_n}\mathbb{E}\left[\|\sigma_s^{\mathcal{S}_n}\|_{L_{\text{HS}}(U,H)}^4\right]ds\right)\lfloor t/\Delta_n\rfloor\\
&\quad\leq \Delta_n^{-1}\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}4\Delta_n\int_{(i-1)\Delta_n}^{i\Delta_n}\mathbb{E}\left[\|\sigma_s^{\mathcal{S}_n}\|_{L_{\text{HS}}(U,H)}^4\right]ds\lfloor t/\Delta_n\rfloor\\
&=\int_0^{\lfloor t/\Delta_n\rfloor}\mathbb{E}\left[\|\sigma_s^{\mathcal{S}_n}\|_{L_{\text{HS}}(U,H)}^4\right]ds\lfloor t/\Delta_n\rfloor<\infty,
\end{aligned}
$$

where the finiteness is due to the assumption.

Now note that $t\mapsto\psi_t=\int_{(i-1)\Delta_n}^t p_N\mathcal{S}(t_i-s)\sigma_s dW_s$ is a martingale for $t\in[(i-1)\Delta_n,i\Delta_n]$. From [63, Theorem 8.2, p. 109] we deduce that the process $(\zeta_t)_{t\geq 0}$, with

$$\zeta_t=(\psi_t)^{\otimes 2}-\langle\langle\psi\rangle\rangle_t,$$

is a martingale with respect to $(\mathcal{F}_t)_{t\geq 0}$, and hence,

$$
\begin{aligned}
\mathbb{E}\left[(p_N\tilde{\Delta}_i^n Y)^{\otimes 2}|\mathcal{F}_{(i-1)\Delta_n}\right]&=\mathbb{E}\left[(\psi_{i\Delta_n})^{\otimes 2}|\mathcal{F}_{(i-1)\Delta_n}\right]\\
&=\mathbb{E}\left[\langle\langle\psi\rangle\rangle_{i\Delta_n}|\mathcal{F}_{(i-1)\Delta_n}\right]\\
&=\int_{(i-1)\Delta_n}^{i\Delta_n}\mathbb{E}\left[p_N\mathcal{S}(t_i-s)\Sigma_s\mathcal{S}(t_i-s)^*p_N|\mathcal{F}_{t_{i-1}}\right]ds.
\end{aligned}
$$

So, $\mathbb{E}\left[\tilde{Z}_n^N(i)\middle|\mathcal{F}_{t_{i-1}}\right]=0$. Moreover, for $j<i$, as each $\tilde{Z}_n^N(i)$ is $\mathcal{F}_{(i-1)\Delta_n}$ measurable and the conditional expectation commutes with bounded linear operators, we find by using the tower property of conditional expectation that

$$
\begin{aligned}
\mathbb{E}\left[\langle\tilde{Z}_n^N(i),\tilde{Z}_n^N(j)\rangle_{\mathcal{H}}\right]&=\mathbb{E}\left[\mathbb{E}\left[\langle\tilde{Z}_n^N(i),\tilde{Z}_n^N(j)\rangle_{\mathcal{H}}|\mathcal{F}_{(i-1)\Delta_n}\right]\right]\\
&=\mathbb{E}\left[\langle\mathbb{E}\left[\tilde{Z}_n^N(i)|\mathcal{F}_{(i-1)\Delta_n}\right],\tilde{Z}_n^N(j)\rangle_{\mathcal{H}}\right]=0.
\end{aligned}
$$

Thus, we obtain

$$\mathbb{E}\left[\left\|\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\tilde{Z}_n^N(i)\right\|_{\mathcal{H}}^2\right]\leq\sum_{i=1}^{\lfloor T/\Delta_n\rfloor}\mathbb{E}\left[\|\tilde{Z}_n^N(i)\|_{\mathcal{H}}^2\right].$$

Applying the triangle and Bochner inequalities, the basic inequality $(a+b)^2 \leq 2(a^2+b^2)$ and appealing to (69), we find

$$\begin{aligned}
&\mathbb{E}\left[\|\tilde{Z}_n^N(i)\|_{\mathcal{H}}^2\right] \\
&\leq 2\Delta_n^{-1}\mathbb{E}\left[\|(p_N\tilde{\Delta}_i^n Y)^{\otimes 2}\|_{\mathcal{H}}^2 + \left(\int_{(i-1)\Delta_n}^{i\Delta_n} \|p_N\mathcal{S}(i\Delta_n - s)\Sigma_s\mathcal{S}(i\Delta_n - s)^* p_N\|_{\mathcal{H}} ds\right)^2\right] \\
&\leq 4\int_{(i-1)\Delta_n}^{i\Delta_n} \mathbb{E}\left[\|p_N\sigma_s^{\mathcal{S}_n}\|_{L_{\text{HS}}(U,H)}^4\right] ds.
\end{aligned}$$

Summing up, we have

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left\|\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\tilde{Z}_n^N(i)\right\|_{\mathcal{H}}^2\right] \leq 4\sup_{n\in\mathbb{N}}\int_0^T \mathbb{E}\left[\|p_N\sigma_s^{\mathcal{S}_n}\|_{L_{\text{HS}}(U,H)}^4\right] ds,$$

which converges to 0 by Lemma B.4. Hence, as

$$\sup_{t\in[0,T]}\sup_{n\in\mathbb{N}}\sum_{m,k\geq N}\mathbb{E}\left[\langle\tilde{Z}_t^{2,n}, e_k\otimes e_m\rangle_{\mathcal{H}}^2\right] = \|\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\tilde{Z}_n^N(i)\|_{\mathcal{H}}^2,$$

the Theorem follows by Corollary D.3. □

D.3. Convergence of finite-dimensional distributions and remainders.

D.3.1. *Proof of the central limit theorem for realised covariation.* Before we can finally prove the central limit theorem for realised covariation, we need the following auxiliary Lemma:

LEMMA D.9. *Let $(e_j)_{j\in\mathbb{N}}$ be an orthonormal basis of H that is contained in $D(\mathcal{A}^*)$. Then for any $k,l\in\mathbb{N}$ we have*

$$\int_{(i-1)\Delta_n}^{i\Delta_n}\langle\Sigma_s^{\mathcal{S}_n}e_k, e_l\rangle ds = \int_{(i-1)\Delta_n}^{i\Delta_n}\langle\Sigma_s e_k, e_l\rangle ds + \psi_n^{i,k,l},$$

where $\psi_n^{i,k,l}$ is a sequence of random variables such that,

$$\mathbb{E}[|\psi_n^{i,k,l}|] \leq K\Delta_n^2,$$

for a constant $K = K_2(k,l) > 0$ independent of i.

PROOF. Since $\mathcal{S}(i\Delta_n - s)^* e_k = \int_s^{i\Delta_n}\mathcal{S}(u-s)^*\mathcal{A}^* e_k du + e_k$ for all $k\in\mathbb{N}$, we have that,

$$\begin{aligned}
&\int_{(i-1)\Delta_n}^{i\Delta_n}\langle\Sigma_s^{\mathcal{S}_n}e_k, e_l\rangle ds \\
&\qquad = \int_{(i-1)\Delta_n}^{i\Delta_n}\langle\Sigma_s\int_s^{i\Delta_n}\mathcal{S}(u-s)^*\mathcal{A}^* e_k du + \Sigma_s e_k, \int_s^{i\Delta_n}\mathcal{S}(u-s)^*\mathcal{A}^* e_l du + e_l\rangle ds \\
&\qquad = \int_{(i-1)\Delta_n}^{i\Delta_n}\langle\Sigma_s e_k, e_l\rangle + \langle\Sigma_s\int_s^{i\Delta_n}\mathcal{S}(u-s)^*\mathcal{A}^* e_k du, e_l\rangle
\end{aligned}$$

$$+ \langle \Sigma_s e_k, \int_s^{i\Delta_n} \mathcal{S}(u-s)^* \mathcal{A}^* e_l du\rangle$$
$$+ \langle \Sigma_s \int_s^{i\Delta_n} \mathcal{S}(u-s)^* \mathcal{A}^* e_k du, \int_s^{i\Delta_n} \mathcal{S}(u-s)^* \mathcal{A}^* e_l du\rangle ds.$$

It holds by Bochner's inequality

$$\mathbb{E}\left[\left|\langle \Sigma_s e_k, \int_s^{i\Delta_n} \mathcal{S}(u-s)^* \mathcal{A}^* e_l du\rangle\right|\right] \leq \int_s^{i\Delta_n} \mathbb{E}\left[\|\Sigma_s \mathcal{S}(u-s)^* \mathcal{A}^* e_l\|\right] du$$
$$\leq \Delta_n \sup_{t,s\in[0,T]} \mathbb{E}\left[\|\Sigma_s \mathcal{S}(t)^* \mathcal{A}^* e_l\|\right],$$

and analogous by swapping k and l

$$\mathbb{E}\left[\left|\langle \Sigma_s e_l, \int_s^{i\Delta_n} \mathcal{S}(u-s)^* \mathcal{A}^* e_k du\rangle\right|\right] \leq \int_s^{i\Delta_n} \mathbb{E}\left[\|\Sigma_s \mathcal{S}(u-s)^* \mathcal{A}^* e_k\|\right] du$$
$$\leq \Delta_n \sup_{t,s\in[0,T]} \mathbb{E}\left[\|\Sigma_s \mathcal{S}(t)^* \mathcal{A}^* e_k\|\right].$$

Moreover

$$\mathbb{E}\left[\left|\langle \Sigma_s \int_s^{i\Delta_n} \mathcal{S}(u-s)^* \mathcal{A}^* e_k du, \int_s^{i\Delta_n} \mathcal{S}(u-s)^* \mathcal{A}^* e_l du\rangle\right|\right]$$
$$= \int_s^{i\Delta_n}\int_s^{i\Delta_n} \mathbb{E}\left[|\langle \Sigma_s \mathcal{S}(u-s)^* \mathcal{A}^* e_k, \mathcal{S}(v-s)^* \mathcal{A}^* e_l\rangle|\right] du dv$$
$$\leq \Delta_n^2 \sup_{t,s\in[0,T],j=k,l} \mathbb{E}\left[\|\sigma_s \mathcal{S}(t)^* \mathcal{A}^* e_j\|^2\right].$$

By the choice $K = \sup_{t,s\in[0,T],j=k,l} \mathbb{E}\left[\|\sigma_s \mathcal{S}(t)^* \mathcal{A}^* e_j\|^2\right]$, the assertion follows. □

We can now prove an auxiliary central limit theorem, which essentially does not rely on the spatial regularity condition in Assumption 2.

THEOREM D.10. *Let Assumption 7 hold. We have that* $\tilde{Z}^{n,2} \stackrel{\mathcal{L}-s}{\Rightarrow} (\mathcal{N}(0,\Gamma_t))_{t\in[0,T]}$.

PROOF. Let $(e_i)_{i\in\mathbb{N}}$ be an orthonormal basis of H that is contained in the domain $D(\mathcal{A}^*)$ of the adjoint of the generator $\mathcal{A}$. Since tightness of $(\mathbb{P}_{\tilde{Z}^{n,2}})_{n\in\mathbb{N}}$ in the Skorokhod topology is guaranteed by Theorems D.6 and D.8 in combination with Theorem D.1, it is enough to show the stable convergence in law as a process of the finite-dimensional distributions $\tilde{Z}_t^{n,2}(d) := (\langle \tilde{Z}_t^{n,2}, e_k\otimes e_l\rangle)_{k,l=1,\dots,d}$.

Therefore, for $k,l = 1,\dots,d$ we find by Lemma B.3 and Lemma D.9 (and using the same notation) that

$$\langle \tilde{Z}_t^{n,2}, e_k\otimes e_l\rangle_{\mathcal{H}} = \Delta_n^{-\frac{1}{2}} \sum_{i=1}^{\lfloor t/\Delta_n\rfloor} \left(\langle \tilde{\Delta}_i^n Y, e_k\rangle\langle \tilde{\Delta}_i^n Y, e_l\rangle - \int_{(i-1)\Delta_n}^{i\Delta_n} \langle \Sigma_s^{\mathcal{S}_n} e_k, e_l\rangle ds\right)$$
$$= \Delta_n^{-\frac{1}{2}} \sum_{i=1}^{\lfloor t/\Delta_n\rfloor} \left(\langle \Delta_i^n S, e_k\rangle\langle \Delta_i^n S, e_l\rangle - \int_{(i-1)\Delta_n}^{i\Delta_n} \langle \Sigma_s e_k, e_l\rangle ds\right)$$
$$+ \Delta_n^{-\frac{1}{2}} \sum_{i=1}^{\lfloor t/\Delta_n\rfloor} \left(\zeta_n^{i,k,l} + \psi_n^{i,k,l}\right).$$

The second summand converges to 0 in probability uniformly on compacts, which is why we have that the stable limit of $\langle \tilde{Z}_t^{n,2} e_k, e_l\rangle$ is the same as the one of

$$\Delta_n^{-\frac{1}{2}} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \left(\langle \Delta_i^n S, e_k\rangle \langle \Delta_i^n S, e_l\rangle - \int_{(i-1)\Delta_n}^{i\Delta_n} \langle \Sigma_s e_k, e_l\rangle ds \right).$$

The latter is a component of the difference of realised quadratic covariation and the quadratic covariation of the d-dimensional continuous local martingale $S_t^d = (\langle S_t, e_1\rangle, ..., \langle S_t, e_d\rangle)$. Therefore, $\tilde{Z}^{n,2}(d) = (\langle \tilde{Z}^n, e_k, e_l\rangle)_{k,l=1,...,d}$ converges by Theorem 5.4.2 from [52] stably as a process to a process that is defined on a very good filtered extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t, \tilde{\mathbb{P}})$ of $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. This limiting process is conditionally on $\mathcal{F}$ a centered Gaussian which can be realised on the very good filtered extension as

$$N_{k,l} = \frac{1}{\sqrt{2}} \sum_{c,b=1}^{d} \int_0^t \left(\hat{\sigma}_{kl,bc}(s) + \hat{\sigma}_{lk,bc}(s)\right) dB_s^{cb}.$$

Here, $\hat{\sigma}(s)$ is a $d^2 \times d^2$-matrix, being the square-root of the matrix $\hat{c}(s)$ with entries $\hat{c}_{kl,k'l'}(s) = \langle \Sigma_s e_k, e_{k'}\rangle \langle \Sigma_s e_l, e_{l'}\rangle$. Furthermore, B is a matrix of independent Brownian motions. This corresponds to the covariance Γ_t, as by the Itô isometry we obtain

$$\begin{aligned}
\mathbb{E}[N_{k,l} N_{k',l'} | \mathcal{F}] &= \frac{1}{2} \sum_{c,b=1}^{d} \int_0^t \left(\hat{\sigma}_{kl,bc}(s) + \hat{\sigma}_{lk,bc}(s)\right) \left(\hat{\sigma}_{k'l',bc}(s) + \hat{\sigma}_{l'k',bc}(s)\right) ds \\
&= \frac{1}{2} \sum_{c,b=1}^{d} \int_0^t \left(\hat{\sigma}_{kl,bc}(s)\hat{\sigma}_{k'l',bc}(s) + \hat{\sigma}_{kl,bc}(s)\hat{\sigma}_{l'k',bc}(s)\right. \\
&\qquad \left. + \hat{\sigma}_{lk,bc}(s)\hat{\sigma}_{k'l',bc}(s) + \hat{\sigma}_{lk,bc}(s)\hat{\sigma}_{l'k',bc}(s)\right) ds \\
&= \frac{1}{2} \int_0^t \hat{c}_{kl,k'l'}(s) + \hat{c}_{kl,l'k'}(s) + \hat{c}_{lk,k'l'}(s) + \hat{c}_{lk,l'k'}(s) ds \\
&= \frac{1}{2} \int_0^t \langle \Sigma_s e_k, e_{k'}\rangle \langle \Sigma_s e_l, e_{l'}\rangle + \langle \Sigma_s e_k, e_{l'}\rangle \langle \Sigma_s e_l, e_{k'}\rangle \\
&\qquad + \langle \Sigma_s e_l, e_{k'}\rangle \langle \Sigma_s e_k, e_{l'}\rangle + \langle \Sigma_s e_l, e_{l'}\rangle \langle \Sigma_s e_k, e_{k'}\rangle ds \\
&= \int_0^t \langle \Sigma_s e_k, e_{k'}\rangle \langle \Sigma_s e_l, e_{l'}\rangle + \langle \Sigma_s e_k, e_{l'}\rangle \langle \Sigma_s e_l, e_{k'}\rangle ds \\
&= \int_0^t \left(\langle \Sigma_s (e_{k'} \otimes e_{l'}) \Sigma_s, e_k \otimes e_l\rangle_{\mathcal{H}} + \langle \Sigma_s (e_{l'} \otimes e_{k'}) \Sigma_s, e_k \otimes e_l\rangle_{\mathcal{H}}\right) ds \\
&= \langle \Gamma_t e_{k'} \otimes e_{l'}, e_k \otimes e_l\rangle_{\mathcal{H}}.
\end{aligned}$$

As now all finite-dimensional distributions converge stably and the sequence of measures is tight, we obtain by Theorem D.4 that the convergence is indeed stable to a process Z in the Skorokhod space, whose corresponding finite-dimensional components $Z(d) := (\langle Z, e_k, e_l\rangle)_{k,l=1,...,d}$ are conditionally on $\mathcal{F}$ a centred Gaussian process. It is itself conditionally centred Gaussian. Moreover, the process is continuous as well, since for Z in $\mathcal{D}([0,T], \mathcal{H})$ we have

$$\|Z_t - Z_s\|_{\mathcal{H}} \leq \|P_d Z_t\|_{\mathcal{H}} + \|Z_t(d) - Z_s(d)\|_{\mathbb{R}^{d\times d}} + \|P_d Z_s\|_{\mathcal{H}}.$$

The outer terms can be made arbitrarily small since by (68) it holds

$$\lim_{d\to\infty}\sup_{t\in[0,T]}\mathbb{E}\left[\|P_d\tilde{Z}_t^{2,n}\|_{\mathcal{H}}^2\right]=0.$$

The middle term converges for each fixed d to 0 as $|t-s|\to 0$, as P_dZ is continuous as an Itô integral. The proof is complete. □

Now we are in the position to prove the central limit theorems 3.3 and 3.2 for realised covariations.

PROOF OF THEOREM 3.3. It is clear that $\Delta_n^{-\frac{1}{2}}\int_{\lfloor t/\Delta_n\rfloor}^t\Sigma_s ds$ converges to 0 in the $u.c.p.$-sense, and since

$$\tilde{X}_t^n=\tilde{Z}_t^{n,2}+\Delta_n^{-\frac{1}{2}}\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\int_{(i-1)\Delta_n}^{i\Delta_n}\left(\Sigma_s^{\mathcal{S}_n}-\Sigma_s\right)ds+\Delta_n^{-\frac{1}{2}}\int_{\lfloor t/\Delta_n\rfloor}^t\Sigma_s ds,$$

we have to show

$$\Delta_n^{-\frac{1}{2}}\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\int_{(i-1)\Delta_n}^{i\Delta_n}\left(\Sigma_s^{\mathcal{S}_n}-\Sigma_s\right)ds\stackrel{u.c.p.}{\longrightarrow}0.$$

Recall that P_N denotes the projection onto $\overline{\{e_k\otimes e_l:k,l\geq N\}}$, where again $(e_j)_{j\in\mathbb{N}}$ is an orthonormal basis of H that is contained in $D(\mathcal{A}^*)$. Notice that for each $A\in\mathcal{H}$

$$\begin{aligned}\|(I-P_N)A\|_{\mathcal{H}}&=\|\sum_{k,l\leq N-1}\langle A,e_k\otimes e_l\rangle_{\mathcal{H}}e_k\otimes e_l\|_{\mathcal{H}}\\&\leq\sum_{k,l\leq N-1}|\langle A,e_k\otimes e_l\rangle_{\mathcal{H}}|\|e_k\otimes e_l\|_{\mathcal{H}}\\&=\sum_{k,l\leq N-1}|\langle A,e_k\otimes e_l\rangle_{\mathcal{H}}|\\&=\sum_{k,l\leq N-1}|\langle Ae_k,e_l\rangle|.\end{aligned}$$

Then, by Lemma D.9

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\in[0,T]}\left\|(I-P_N)\Delta_n^{-\frac{1}{2}}\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\int_{(i-1)\Delta_n}^{i\Delta_n}\left(\Sigma_s^{\mathcal{S}_n}-\Sigma_s\right)ds\right\|_{\mathcal{H}}\right]\\&\leq\mathbb{E}\left[\sup_{t\in[0,T]}\sum_{k,l=1}^{N-1}\left|\Delta_n^{-\frac{1}{2}}\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\int_{(i-1)\Delta_n}^{i\Delta_n}\langle\left(\Sigma_s^{\mathcal{S}_n}-\Sigma_s\right)e_k,e_l\rangle ds\right|\right]\\&=\mathbb{E}\left[\sup_{t\in[0,T]}\sum_{k,l=1}^{N-1}\left|\Delta_n^{-\frac{1}{2}}\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\psi_{i,k,l}\right|\right]\\&\leq\sum_{k,l=1}^{N-1}\Delta_n^{-\frac{1}{2}}\sum_{i=1}^{\lfloor T/\Delta_n\rfloor}K(k,l)\Delta_n^2\end{aligned}$$

$$\leq \left(\sum_{k,l=1}^{N-1} K(k,l)\right) T\Delta_n^{\frac{1}{2}}, \tag{71}$$

which converges to 0 as $n \to \infty$. Thus, for all $N \in \mathbb{N}$,

$$(I - P_N)\Delta_n^{-\frac{1}{2}} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \int_{(i-1)\Delta_n}^{i\Delta_n} \left(\Sigma_s^{\mathcal{S}_n} - \Sigma_s\right) ds \xrightarrow{u.c.p.} 0.$$

Further, we have, by using the triangle and Bochner inequalities,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} \left\| P_N \Delta_n^{-\frac{1}{2}} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \int_{(i-1)\Delta_n}^{i\Delta_n} \left(\Sigma_s^{\mathcal{S}_n} - \Sigma_s\right) ds \right\|_{\mathcal{H}}\right] \\
\leq &\Delta_n^{-\frac{1}{2}} \int_0^T \mathbb{E}\left[\left\| P_N \left(\Sigma_s^{\mathcal{S}_n} - \Sigma_s\right)\right\|_{\mathcal{H}}\right] ds \\
\leq &\Delta_n^{-\frac{1}{2}} \int_0^T \mathbb{E}\,[\| P_N (\mathcal{S}(\lfloor s/\Delta_n \rfloor \Delta_n - s) - I)\Sigma_s \mathcal{S}(\lfloor s/\Delta_n \rfloor \Delta_n - s)^* \|_{\mathcal{H}} \\
&\qquad + \| P_N \Sigma_s (\mathcal{S}(\lfloor s/\Delta_n \rfloor \Delta_n - s)^* - I)\|_{\mathcal{H}}]\, ds.
\end{aligned}$$

Now, estimating further using the Cauchy-Schwarz inequality along with the fact that $\|AB\|_{\mathcal{H}} \leq \|A\|_{\text{op}} \|B\|_{L_{\text{HS}}(U,H)}$ for any Hilbert Schmidt operator $B : H \to U$ and continuous linear operator $A : U \to H$,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} \left\| P_N \Delta_n^{-\frac{1}{2}} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \int_{(i-1)\Delta_n}^{i\Delta_n} \left(\Sigma_s^{\mathcal{S}_n} - \Sigma_s\right) ds \right\|_{\mathcal{H}}\right] \\
\leq &\Delta_n^{-\frac{1}{2}} \int_0^T \mathbb{E}\,[\| p_N (\mathcal{S}(\lfloor s/\Delta_n \rfloor \Delta_n - s) - I)\sigma_s \|_{\text{op}} \\
&\qquad \left(\| p_N \sigma_s \|_{L_{\text{HS}}(U,H)} + \| p_N \mathcal{S}(\lfloor s/\Delta_n \rfloor \Delta_n - s)\sigma_s \|_{L_{\text{HS}}(U,H)}\right)]\, ds \\
\leq &\left(\int_0^T \mathbb{E}\left[\|\Delta_n^{-\frac{1}{2}} (\mathcal{S}(\lfloor s/\Delta_n \rfloor \Delta_n - s) - I)\sigma_s\|_{\text{op}}^2\right] ds\right)^{\frac{1}{2}} \\
&\qquad \times \left(\int_0^T \sqrt{2}\mathbb{E}\left[\| p_N \sigma_s \|_{L_{\text{HS}}(U,H)}^2 + \| p_N \mathcal{S}(\lfloor s/\Delta_n \rfloor \Delta_n - s)\sigma_s \|_{L_{\text{HS}}(U,H)}^2\right] ds\right)^{\frac{1}{2}}.
\end{aligned}$$

The first factor is finite by Assumption 2, whereas the second one converges to 0 as $N \to \infty$ by Lemma B.4. Observe that we used that the Lemma holds also in the special case $\mathcal{S}(t) = I$ on H for all $t \geq 0$. Thus, we have

$$\lim_{N\to\infty} \sup_{n\in\mathbb{N}} \mathbb{E}\left[\sup_{t\in[0,T]} \left\| P_N \Delta_n^{-\frac{1}{2}} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \int_{(i-1)\Delta_n}^{i\Delta_n} \left(\Sigma_s^{\mathcal{S}_n} - \Sigma_s\right) ds \right\|_{\mathcal{H}}\right] = 0.$$

Therefore, we can find for each $\delta > 0$ an $N_\delta \in \mathbb{N}$ such that for all $N \geq N_\delta$

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left\| \Delta_n^{-\frac{1}{2}} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \int_{(i-1)\Delta_n}^{i\Delta_n} \left(\Sigma_s^{\mathcal{S}_n} - \Sigma_s\right) ds \right\|_{\mathcal{H}}\right]$$

$$
\begin{aligned}
\leq & \mathbb{E}\left[\sup_{t\in[0,T]}\left\|(I-P_N)\Delta_n^{-\frac{1}{2}}\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\int_{(i-1)\Delta_n}^{i\Delta_n}\left(\Sigma_s^{\mathcal{S}_n}-\Sigma_s\right)ds\right\|_{\mathcal{H}}\right] \\
&\qquad + \sup_{n\in\mathbb{N}}\mathbb{E}\left[\sup_{t\in[0,T]}\left\|P_N\Delta_n^{-\frac{1}{2}}\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\int_{(i-1)\Delta_n}^{i\Delta_n}\left(\Sigma_s^{\mathcal{S}_n}-\Sigma_s\right)ds\right\|_{\mathcal{H}}\right] \\
\leq & \mathbb{E}\left[\sup_{t\in[0,T]}\left\|(I-P_N)\Delta_n^{-\frac{1}{2}}\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\int_{(i-1)\Delta_n}^{i\Delta_n}\left(\Sigma_s^{\mathcal{S}_n}-\Sigma_s\right)ds\right\|_{\mathcal{H}}\right]+\delta \\
& \to \delta.
\end{aligned}
$$

As this holds for all $\delta>0$, we obtain that

$$
\mathbb{E}\left[\sup_{t\in[0,T]}\left\|\Delta_n^{-\frac{1}{2}}\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\int_{(i-1)\Delta_n}^{i\Delta_n}\left(\Sigma_s^{\mathcal{S}_n}-\Sigma_s\right)ds\right\|_{\mathcal{H}}\right]\to 0,
$$

as $n\to\infty$, and the assertion follows. □

PROOF OF THEOREM 3.2. For any $B\in\mathcal{H}$,

$$
\langle\tilde{X}_t^n,B\rangle_{\mathcal{H}}=\langle\tilde{Z}_t^{n,2},B\rangle_{\mathcal{H}}+\langle\Delta_n^{-\frac{1}{2}}\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\int_{(i-1)\Delta_n}^{i\Delta_n}\left(\Sigma_s^{\mathcal{S}_n}-\Sigma_s\right)ds,B\rangle_{\mathcal{H}}.
$$

Since the stable convergence with respect to the Hilbert-Schmidt norm as proven in Theorem D.10 implies the stable convergence in law with respect to the (analytically) weak topology, we only have to show

$$
\Delta_n^{-\frac{1}{2}}\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\int_{(i-1)\Delta_n}^{i\Delta_n}\langle\left(\Sigma_s^{\mathcal{S}_n}-\Sigma_s\right),B\rangle_{\mathcal{H}}ds\overset{u.c.p.}{\longrightarrow}0.
$$

We can argue componentwise, which is why we assume without loss of generality that $B=h\otimes g$ for $h,g\in F_{1/2}^{\mathcal{S}^*}$. We split into two terms as follows:

$$
\begin{aligned}
&\Delta_n^{-\frac{1}{2}}\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\int_{(i-1)\Delta_n}^{i\Delta_n}\langle(\Sigma_s^{\mathcal{S}_n}-\Sigma_s)h,g\rangle ds \\
&\quad=\Delta_n^{-\frac{1}{2}}\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\int_{(i-1)\Delta_n}^{i\Delta_n}\langle(((\mathcal{S}(i\Delta_n-s)-I)\Sigma_s\mathcal{S}(i\Delta_n-s)^*)h,g\rangle ds \\
&\qquad+\Delta_n^{-\frac{1}{2}}\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\int_{(i-1)\Delta_n}^{i\Delta_n}\langle(\Sigma_s(\mathcal{S}(i\Delta_n-s)-I)^*)h,g\rangle ds \\
&\quad=(1)_n+(2)_n.
\end{aligned}
$$

We only show the convergence for $(1)_n$ since the argument for $(2)_n$ is analogous. It holds

$$
(1)_n=\Delta_n^{-\frac{1}{2}}\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\int_{(i-1)\Delta_n}^{i\Delta_n}\langle(I-p_N)(\Sigma_s\mathcal{S}(i\Delta_n-s)^*)h,(\mathcal{S}(i\Delta_n-s)-I)^*g\rangle ds
$$

$$
\begin{aligned}
&+ \Delta_n^{-\frac{1}{2}} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \int_{(i-1)\Delta_n}^{i\Delta_n} \langle p_N(\Sigma_s \mathcal{S}(i\Delta_n - s)^*)h, (\mathcal{S}(i\Delta_n - s) - I)^* g\rangle ds \\
=&(1.1)_{n,N} + (1.2)_{n,N},
\end{aligned}
$$

where again we denoted by p_N the projection onto $\overline{\{e_j : j \geq N\}}$ for an orthonormal basis $(e_j)_{j\in\mathbb{N}}$ of H that is contained in $D(\mathcal{A})$. We have

$$
\mathcal{S}(t-s)e_i - e_i = \int_s^t \mathcal{S}(u-s)\mathcal{A}e_i du,
$$

and therefore it holds for the first summand that,

$$
\begin{aligned}
&\sup_{t\in[0,T]} |(1.1)_{n,N}| \\
&= \sup_{t\in[0,T]} |\Delta_n^{-\frac{1}{2}} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \int_{(i-1)\Delta_n}^{i\Delta_n} \langle (I-p_N)(\Sigma_s \mathcal{S}(i\Delta_n - s)^*)h, (\mathcal{S}(i\Delta_n - s) - I)^* g\rangle ds| \\
&= \sup_{t\in[0,T]} |\Delta_n^{-\frac{1}{2}} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \int_{(i-1)\Delta_n}^{i\Delta_n} \sum_{j=1}^{N-1} \langle (\Sigma_s \mathcal{S}(i\Delta_n - s)^*)h, e_j\rangle \langle e_j, (\mathcal{S}(i\Delta_n - s) - I)^* g\rangle ds| \\
&= \sup_{t\in[0,T]} |\sum_{j=1}^{N-1} \Delta_n^{-\frac{1}{2}} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \int_{(i-1)\Delta_n}^{i\Delta_n} \langle (\Sigma_s \mathcal{S}(i\Delta_n - s)^*)h, e_j\rangle \langle \int_s^{i\Delta_n} \mathcal{S}(u-s)\mathcal{A}e_j ds, g\rangle ds \\
&\leq \sum_{j=1}^{N-1} \Delta_n^{-\frac{1}{2}} \int_0^T \|\Sigma_s\|_{\text{op}} \|h\| \Delta_n \|\mathcal{A}e_j\| \|g\| ds \sup_{t\in[0,T]} \|\mathcal{S}(t)\|_{\text{op}}^2
\end{aligned}
$$

(72)

$$
\leq \Delta_n^{\frac{1}{2}} \sum_{j=1}^{N-1} \int_0^T \|\Sigma_s\|_{\text{op}} ds \|h\| \|g\| \sup_{t\in[0,T]} \|\mathcal{S}(t)\|_{\text{op}}^2.
$$

The last expression converges to 0 as $n \to \infty$ almost surely. In particular, we have as $n \to \infty$ that

$$
|(1.1)_{n,N}| \overset{u.c.p.}{\to} 0.
$$

It follows from the fact that $g \in F_{1/2}^{\mathcal{S}^*}$ that we can find a constant $K := \sup_{t\leq T} \|t^{-\frac{1}{2}}(\mathcal{S}(t) - I)^* g\| < \infty$ such that

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} |(1.2)_{n,N}|\right] \\
\leq& \Delta_n^{-\frac{1}{2}} \sum_{i=1}^{\lfloor T/\Delta_n \rfloor} \int_{(i-1)\Delta_n}^{i\Delta_n} \mathbb{E}\left[|\langle p_N(\Sigma_s \mathcal{S}(i\Delta_n - s)^*)h, (\mathcal{S}(i\Delta_n - s) - I)^* g\rangle|\right] ds \\
\leq& \Delta_n^{-\frac{1}{2}} \sup_{t\leq\Delta_n} \|(\mathcal{S}(t) - I)^* g\| \int_0^T \sup_{t\leq\Delta_n} \mathbb{E}\left[\|p_N \sigma_s\|_{\text{op}} \|\sigma_s^* \mathcal{S}(t)^*\|_{\text{op}} \|h\|\right] ds \\
\leq& K \left(\int_0^T \mathbb{E}\left[\|p_N \sigma_s\|_{\text{op}}^2\right] ds\right)^{\frac{1}{2}} \left(\int_0^T \sup_{t\leq\Delta_n} \mathbb{E}\left[\|\sigma_s^* \mathcal{S}(t)^*)\|_{\text{op}}^2\right] \|h\|^2 ds\right)^{\frac{1}{2}}
\end{aligned}
$$

$$\leq K \left(\int_0^T \mathbb{E}\left[\|p_N \sigma_s\|_{\text{op}}^2 \right] ds \right)^{\frac{1}{2}} \left(\int_0^T \mathbb{E}\left[\|\sigma_s^*\|_{\text{op}}^2 \right] \|h\|^2 ds \right)^{\frac{1}{2}} \sup_{t\in[0,T]} \|\mathcal{S}(t)\|_{\text{op}}. \tag{73}$$

As the first factor converges to 0 as $N \to \infty$ by Lemma B.4, we obtain the convergence of $\mathbb{E}[\sup_{t\in[0,T]} |(1.2)_{n,N}|]$ to 0 as $N \to \infty$ uniformly in n. Therefore, we can find for each $\delta > 0$ an $N \in \mathbb{N}$, such that by Markov's inequality

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{P}\left[\sup_{t\in[0,T]} |(1)_n| > \epsilon \right] \\
&\leq \lim_{n\to\infty} \mathbb{P}\left[\sup_{t\in[0,T]} |(1.1)_{n,N}| > \epsilon \right] + \sup_{n\in\mathbb{N}} \mathbb{P}\left[\sup_{t\in[0,T]} |(1.2)_{n,N}| > \epsilon \right] \\
&= 0 + \frac{1}{\epsilon} \sup_{n\in\mathbb{N}} \mathbb{E}\left[\sup_{t\in[0,T]} |(1.2)_{n,N}| \right] \leq \delta.
\end{aligned}$$

As this holds for all $\delta > 0$ we obtain $(1)_n \overset{u.c.p.}{\longrightarrow} 0$ as $n \to \infty$. The assertion for $(2)_n \overset{u.c.p.}{\longrightarrow} 0$ follows analogously. □

APPENDIX E: REMAINING PROOFS

We will now prove the remaining results, i.e. Theorem 3.8, Theorem 3.9, Lemma 3.11 and Lemma 3.13 as well as Examples 2 and 4. We start with the proof of Example 2.

PROOF OF EXAMPLE 2. We start with some general observations. If the central limit theorem should be valid, i.e., if $\Delta_n^{-\frac{1}{2}} \left(SARCV_t^n - \int_0^t \Sigma_s ds \right)$ should converge in distribution, we must necessarily have that it is tight. As the sum of two tight sequences is tight itself and since we have

$$\begin{aligned}
&\Delta_n^{-\frac{1}{2}} \left(SARCV_t^n - \int_0^t \Sigma_s ds \right) - \Delta_n^{-\frac{1}{2}} \left(SARCV_t^n - \int_0^t \Sigma_s^{\mathcal{S}_n} ds \right) \\
&= \left(\Delta_n^{-\frac{1}{2}} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \int_0^t \Sigma_s^{\mathcal{S}_n} - \Sigma_s ds \right),
\end{aligned}$$

we find that $\left(\Delta_n^{-\frac{1}{2}} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \int_0^t \Sigma_s^{\mathcal{S}_n} - \Sigma_s ds \right)$ must be tight, due to Theorem D.10. I.e. for all $\epsilon > 0$ there is a compact set $K_\epsilon \subset \mathcal{H}$ such that $\sup_{n\in\mathbb{N}} \mathbb{P}[X_n \notin K_\epsilon] < \epsilon$. All compact sets $K \subset \mathcal{H}$ are bounded and hence contained in a ball with radius large enough. Hence, from tightness we obtain that for all $\epsilon > 0$ there is an $M_\epsilon > 0$ such that

$$\sup_{n\in\mathbb{N}} \mathbb{P}\left[\left\| \left(\Delta_n^{-\frac{1}{2}} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \int_0^t \Sigma_s^{\mathcal{S}_n} - \Sigma_s ds \right) \right\|_{\mathcal{H}} > M_\epsilon \right] < \epsilon.$$

Thus, if for some specification of σ, there is an $\epsilon_0 > 0$ such that

$$\limsup_{M\uparrow\infty} \sup_{n\in\mathbb{N}} \mathbb{P}\left[\left\| \left(\Delta_n^{-\frac{1}{2}} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \int_0^t \Sigma_s^{\mathcal{S}_n} - \Sigma_s ds \right) \right\|_{\mathcal{H}} > M \right] \geq \epsilon_0 \tag{74}$$

we necessarily have that the central limit theorem cannot hold. For some $f_1 \in H$ let the volatility have the form

$$\sigma_s = e \otimes f_1,$$

where $e \in H$ is such that $\|e\| = 1$. Moreover, we let $f_2 \in H$ be such that

$$|\langle (\mathcal{S}(x) - I) f_1, f_2 \rangle| \leq C x^{\frac{1}{4}}, \tag{75}$$

for some constant $C > 0$, which does not depend on $x \geq 0$ and

$$\limsup_{n\to\infty} \left| \Delta_n^{-\frac{1}{2}} \langle f_1, f_2 \rangle \sum_{i=1}^{n} \int_{(i-1)\Delta_n}^{i\Delta_n} \langle (\mathcal{S}(i\Delta_n - s) - I) f_1, f_2 \rangle ds \right| = \infty. \tag{76}$$

Moreover, we have $\Sigma_s = f_1^{\otimes 2}$ and hence

$$\begin{aligned}
&\Delta_n^{-\frac{1}{2}} \langle \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \int_0^t \Sigma_s^{\mathcal{S}_n} - \Sigma_s ds, f_2^{\otimes 2} \rangle \\
=&\Delta_n^{-\frac{1}{2}} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \int_{(i-1)\Delta_n)}^{i\Delta_n} \langle \mathcal{S}(i\Delta_n - s) f_1, f_2 \rangle^2 - \langle f_1, f_2 \rangle^2 ds \\
=&\Delta_n^{-\frac{1}{2}} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \int_{(i-1)\Delta_n)}^{i\Delta_n} \langle (\mathcal{S}(i\Delta_n - s) - I) f_1, f_2 \rangle \langle (\mathcal{S}(i\Delta_n - s) + I) f_1, f_2 \rangle ds \\
=&\Delta_n^{-\frac{1}{2}} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \int_{(i-1)\Delta_n)}^{i\Delta_n} \langle (\mathcal{S}(i\Delta_n - s) - I) f_1, f_2 \rangle^2 ds \\
&+ 2\langle f_1, f_2 \rangle \Delta_n^{-\frac{1}{2}} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \int_{(i-1)\Delta_n)}^{i\Delta_n} \langle (\mathcal{S}(i\Delta_n - s) - I) f_1, f_2 \rangle ds.
\end{aligned}$$

Due to (75), it is simple to see that the first term converges to 0 as $n \to \infty$. Now we have that (74) holds, since

$$\begin{aligned}
&\|f_2\|^2 \limsup_{n\in\mathbb{N}} \left\| \left(\Delta_n^{-\frac{1}{2}} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \int_0^t \Sigma_s^{\mathcal{S}_n} - \Sigma_s ds \right) \right\|_{\mathcal{H}} \\
\geq &\limsup_{n\to\infty} \Delta_n^{-\frac{1}{2}} \left| \langle \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \int_{(i-1)\Delta_n)}^{i\Delta_n} (\mathcal{S}(i\Delta_n - s) - I) f_1 ds, f_2 \rangle \right| |\langle f_1, f_2 \rangle| = \infty.
\end{aligned}$$

In order to show that Example 2 is indeed a valid counterexample, it is, thus, left to show that for the choice $H = L^2[0,2]$, $(\mathcal{S}(t))_{t\geq 0}$ the nilpotent semigroup of left-shifts and f_1 a path of a fractional Brownian motion, we can find an $f_2 \in H$ such that (75) and (76) hold. We do this as follows: We define $(B_1(t), B_2(t))_{t\in\mathbb{R}}$ to be a multivariate fractional Brownian motion on some probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$, i.e. a bivariate Gaussian stochastic process with stationary increments such that the multivariate self-similarity

$$(B_1(\lambda t), B_2(\lambda t)) \sim (\lambda^{\mathfrak{H}} B_1(t), \lambda^{\epsilon} B_2(t)) \quad \forall \lambda > 0, t \in \mathbb{R}$$

holds for $0 < \mathfrak{H} < \frac{1}{2}$, $0 < \epsilon < \frac{1}{2} - \mathfrak{H}$ and $\max(\mathfrak{H}, \epsilon) > \frac{1}{4}$. Moreover, we assume that $\mathbb{E}[B_1(t)B_2(t)] =: \rho > 0$ for $t \in \mathbb{R}$. We also assume that $(B_1(t), B_2(t))_{t\in\mathbb{R}}$ is time-reversible, i.e. $(B_1(t), B_2(t)) = B_1(-t), B_2(-t))$ for all $t \in \mathbb{R}$. In that case, the covariance structure of this process is given by

$$\mathbb{E}[B_i(t)B_j(s)] = \frac{\rho_{i,j}}{2}(|s|^{\mathfrak{H}_{i,j}} + |t|^{\mathfrak{H}_{i,j}} - |t-s|^{\mathfrak{H}_{i,j}}),$$

where

$$\rho_{i,j} = \begin{cases} 1, & i = j, \\ \rho, & i \neq j, \end{cases}$$

and

$$\mathfrak{H}_{i,j} = \begin{cases} \mathfrak{H}, & i = j = 1, \\ \epsilon, & i = j = 2, \\ \mathfrak{H} + \epsilon, & i \neq j. \end{cases}$$

Observe that we can always find a $\rho > 0$ that guarantees the existence of such a process (c.f. Proposition 9 in [5]) We want to find an $\omega \in \bar{\Omega}$, such that with the choice

$$(f_1(x), f_2(x)) = (B_1(x)(\omega), B_2(x)(\omega)), \quad x \in [0,2], \tag{77}$$

we have (75) and (76). For that, observe that we have for $t < 2$, as the fractional Brownian motion is globally Hölder on the compact interval $[0,2]$ that for almost all $\omega \in \bar{\Omega}$ there is a $\bar{C}_\omega > 0$ such that

$$\|(\mathcal{S}(t) - I)B^{\mathfrak{H}}(\omega)\|^2 = \int_0^{2-t} (B^{\mathfrak{H}}_{t+x}(\omega) - B^{\mathfrak{H}}_x(\omega))^2 dx + \int_{2-t}^{2} (B^{\mathfrak{H}}_x(\omega))^2 dx \leq \bar{C}_\omega t^{2\mathfrak{H}}.$$

By analogous reasoning and since the adjoint semigroup $(\mathcal{S}(t)^*)_{t\geq 0}$ is given by the right-shift

$$\mathcal{S}(t)^* f(x) = f(x-t)\mathbb{I}_{[0,2]}(x-t),$$

we obtain for almost all $\omega \in \bar{\Omega}$ a $\bar{C}_\omega > 0$ such that

$$\begin{aligned} |\langle(\mathcal{S}(x) - I)f_1, f_2\rangle| \leq & \min(\|(\mathcal{S}(x) - I)B_1(\omega)\|\|B_2(\omega)\|, \|(\mathcal{S}(x)^* - I)B_2(\omega)\|\|B_1(\omega)\|) \\ \leq & \bar{C}_\omega x^{\frac{1}{4}}, \end{aligned}$$

and hence (75) holds. It is now enough to prove that (76) holds for all $\omega \in A$ which are in a set $A \in \bar{\mathcal{F}}$ such that $\bar{\mathbb{P}}[A] > 0$. For that, it is enough to prove that there is a $c > 0$ and an $N \in \mathbb{N}$, such that for all $n \geq N$ we have

$$\left| \mathbb{E}\left[\langle B_1, B_2\rangle \sum_{i=1}^{n} \int_{(i-1)\Delta_n}^{i\Delta_n} \Delta_n^{-(\mathfrak{H}+\epsilon)} \langle(\mathcal{S}(i\Delta_n - s) - I)B_1, B_2\rangle ds \right] \right| > c, \tag{78}$$

since in this case, for $n \geq N$ we have $\bar{\mathbb{P}}[A] > 0$ for the choice

$$A = \left\{ \left| \langle B_1, B_2\rangle \sum_{i=1}^{n} \int_{(i-1)\Delta_n}^{i\Delta_n} \Delta_n^{-\frac{1}{2}} \langle(\mathcal{S}(i\Delta_n - s) - I)B_1, B_2\rangle ds \right| > c \right\}.$$

If this would not be the case, there would be a subsequence $(n_k)_{k\in\mathbb{N}}$ such that on a full $\bar{\mathbb{P}}$-measure set $\bar{\Omega}_1$ we have

$$\left| \langle B_1, B_2\rangle \sum_{i=1}^{n_k} \int_{(i-1)\Delta_{n_k}}^{i\Delta_{n_k}} \Delta_{n_k}^{-\frac{1}{2}} \langle(\mathcal{S}(i\Delta_{n_k} - s) - I)B_1, B_2\rangle ds \right| \leq c.$$

Letting $\mathbb{E}_{\bar{\mathbb{P}}}$ denote the expectation with respect to the probability measure $\bar{\mathbb{P}}$, we would have

$$\left| \mathbb{E}_{\bar{\mathbb{P}}}\left[\langle B_1, B_2\rangle \sum_{i=1}^{n_k} \int_{(i-1)\Delta_{n_k}}^{i\Delta_{n_k}} \Delta_{n_k}^{-(\mathfrak{H}+\epsilon)} \langle(\mathcal{S}(i\Delta_{n_k} - s) - I)B_1, B_2\rangle ds \right] \right|$$

$$
\begin{aligned}
&= \left| \int_{\bar{\Omega}_1} \langle B_1, B_2 \rangle \sum_{i=1}^{n_k} \int_{(i-1)\Delta_{n_k}}^{i\Delta_{n_k}} \Delta_{n_k}^{-(\mathfrak{H}+\epsilon)} \langle (\mathcal{S}(i\Delta_{n_k} - s) - I) B_1, B_2 \rangle ds(\omega) \bar{\mathbb{P}}[d\omega] \right| \\
&\leq \int_{\bar{\Omega}_1} \left| \langle B_1, B_2 \rangle \sum_{i=1}^{n_k} \int_{(i-1)\Delta_{n_k}}^{i\Delta_{n_k}} \Delta_{n_k}^{-\frac{1}{2}} \langle (\mathcal{S}(i\Delta_{n_k} - s) - I) B_1, B_2 \rangle ds(\omega) \right| \bar{\mathbb{P}}[d\omega] \\
&\leq \int_{\bar{\Omega}_1} c \bar{\mathbb{P}}[d\omega] \\
&= c.
\end{aligned}
$$

This would contradict (78). That yields, in particular, that if (78) is valid we obtain that (76) holds for the choice $f_1 := B_1(\omega)$ and $f_2 := B_2(\omega)$ for any $\omega \in A$, as in this case

$$
\begin{aligned}
&\limsup_{n\to\infty} \left| \Delta_n^{-\frac{1}{2}} \langle f_1, f_2 \rangle \sum_{i=1}^{n} \int_{(i-1)\Delta_n}^{i\Delta_n} \langle (\mathcal{S}(i\Delta_n - s) - I) f_1, f_2 \rangle ds \right| \\
&= \limsup_{n\to\infty} \Delta_n^{-(\frac{1}{2} - (\mathfrak{H}+\epsilon))} \left| \Delta_n^{-(\mathfrak{H}+\epsilon)} \langle f_1, f_2 \rangle \sum_{i=1}^{n} \int_{(i-1)\Delta_n}^{i\Delta_n} \langle (\mathcal{S}(i\Delta_n - s) - I) f_1, f_2 \rangle ds \right| \\
&\geq \limsup_{n\to\infty} \Delta_n^{-(\frac{1}{2} - (\mathfrak{H}+\epsilon))} c \\
&= \infty.
\end{aligned}
$$

Let us now prove that (78) holds to complete the proof. By the Isserlis-Wick formula we obtain

$$
\begin{aligned}
&\mathbb{E}_{\bar{\mathbb{P}}} \left[\langle B_1, B_2 \rangle \sum_{i=1}^{n} \int_{(i-1)\Delta_n}^{i\Delta_n} \Delta_n^{-(\mathfrak{H}+\epsilon)} \langle (\mathcal{S}(i\Delta_n - s) - I) B_1, B_2 \rangle ds \right] \\
&= \sum_{i=1}^{n} \int_{(i-1)\Delta_n}^{i\Delta_n} \int_0^2 \int_{2-i\Delta_n+s}^{2} \Delta_n^{-(\mathfrak{H}+\epsilon)} \mathbb{E}_{\bar{\mathbb{P}}} [B_1(x)(B_1(y + i\Delta_n - s) \\
&\qquad\qquad - B_1(y)) B_2(y) B_2(x)] \, dydxds \\
&\quad + \sum_{i=1}^{n} \int_{(i-1)\Delta_n}^{i\Delta_n} \int_0^2 \int_0^{2-i\Delta_n+s} \Delta_n^{-(\mathfrak{H}+\epsilon)} \mathbb{E}_{\bar{\mathbb{P}}} [B_1(x)(B_1(y + i\Delta_n - s) - B_1(y))] \\
&\qquad\qquad \times \mathbb{E}_{\bar{\mathbb{P}}} [B_2(y) B_2(x)] \, dydxds \\
&\quad + \sum_{i=1}^{n} \int_{(i-1)\Delta_n}^{i\Delta_n} \int_0^2 \int_0^{2-i\Delta_n+s} \Delta_n^{-(\mathfrak{H}+\epsilon)} \mathbb{E}_{\bar{\mathbb{P}}} [B_2(x)(B_1(y + i\Delta_n - s) - B_1(y))] \\
&\qquad\qquad \times \mathbb{E} [B_2(y) B_1(x)] \, dydxds \\
&\quad + \sum_{i=1}^{n} \int_{(i-1)\Delta_n}^{i\Delta_n} \int_0^2 \int_0^{2-i\Delta_n+s} \Delta_n^{-(\mathfrak{H}+\epsilon)} \mathbb{E}_{\bar{\mathbb{P}}} [B_2(y)(B_1(y + i\Delta_n - s) - B_1(y))] \\
&\qquad\qquad \times \mathbb{E}_{\bar{\mathbb{P}}} [B_2(x) B_1(x)] \, dydxds \\
&=: (1)_t^n + (2)_t^n + (3)_t^n + (4)_t^n.
\end{aligned}
$$

Clearly, since

$$\sup_{r_{1,1},\dots,r_{1,d_1},r_{2,1},\dots,r_{2,d_2}\in[0,2]}\left|\mathbb{E}\left[\prod_{k=1}^{d_1}B_1(r_{i,1})\prod_{k=1}^{d_2}B_2(r_{2,k})\right]\right|<\infty, \tag{79}$$

the first term goes to 0 as $n\to\infty$, as

$$\begin{aligned}
&|(1)_t^n|\\
=&\left|\sum_{i=1}^n\int_{(i-1)\Delta_n}^{i\Delta_n}\int_0^2\int_{2-i\Delta_n+s}^2\Delta_n^{-(\mathfrak{H}+\epsilon)}\mathbb{E}_{\bar{\mathbb{P}}}\left[B_1(x)(B_1(y+i\Delta_n-s)\right.\right.\\
&\qquad\qquad\left.\left.-B_1(y))B_2(y)B_2(x)\right]dydxds\right|\\
\leq&\sum_{i=1}^n\int_{(i-1)\Delta_n}^{i\Delta_n}2(\Delta_n-s)\Delta_n^{-(\mathfrak{H}+\epsilon)}ds\\
&\qquad\qquad\times\sup_{r_{1,1},r_{1,2},r_{2,1},r_{2,2}}\left|\mathbb{E}_{\bar{\mathbb{P}}}\left[B_1(r_{1,1})B_1(r_{1,2})B_2(r_{2,1})B_2(r_{2,1})\right]\right|\\
\leq&2\Delta_n^{1-(\mathfrak{H}+\epsilon)}\sup_{r_{1,1},r_{1,2},r_{2,1},r_{2,2}}\left|\mathbb{E}_{\bar{\mathbb{P}}}\left[B_1(r_{1,1})B_1(r_{1,2})B_2(r_{2,1})B_2(r_{2,1})\right]\right|.
\end{aligned}$$

For the second term, observe that, by the mean value theorem, we have for all $x,y\in(0,2]$ such that $y,y-x,y+i\Delta_n-s-x\neq0$ for $s\in[(i-1)\Delta_n,i\Delta_n]$ that

$$\begin{aligned}
&\left||x-y|^{2\mathfrak{H}}+|y+i\Delta_n-s|^{2\mathfrak{H}}-|x-y-(i\Delta_n-s)|^{2\mathfrak{H}}-|y|^{2\mathfrak{H}}\right|\\
\leq&\max(y^{2\mathfrak{H}-1},|x-y|^{2\mathfrak{H}-1},|x-y-(i\Delta_n-s)|^{2\mathfrak{H}-1})(i\Delta_n-s)\\
\leq&(y^{2\mathfrak{H}-1}+|x-y|^{2\mathfrak{H}-1}+|x-y-(i\Delta_n-s)|^{2\mathfrak{H}-1})(i\Delta_n-s).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left|\int_0^{2-i\Delta_n+s}\mathbb{E}_{\bar{\mathbb{P}}}\left[B_2(y)B_2(x)\right]\right.\\
&\qquad\left.\times\left(|x-y|^{2\mathfrak{H}}+(y+i\Delta_n-s)^{2\mathfrak{H}}-|x-y-(\Delta_n-s)|^{2\mathfrak{H}}-y^{2\mathfrak{H}}\right)dy\right|\\
\leq&\Delta_n\int_0^{2-i\Delta_n+s}(y^{2\mathfrak{H}-1}+|x-y|^{2\mathfrak{H}-1}+|x-y-(i\Delta_n-s)|^{2\mathfrak{H}-1})dy\\
&\qquad\times\sup_{r,s\in[0,2]}\left|\mathbb{E}_{\bar{\mathbb{P}}}\left[B_2(r)B_2(s)\right]\right|.
\end{aligned}$$

It is

$$\begin{aligned}
\int_0^{2-i\Delta_n+s}|x-y|^{2\mathfrak{H}-1}dy\leq\int_0^2|x-y|^{2\mathfrak{H}-1}dy=&\int_0^x(x-y)^{2\mathfrak{H}-1}dy+\int_x^2(y-x)^{2\mathfrak{H}-1}dy\\
=&\frac{x^{2\mathfrak{H}}+(2-x)^{2\mathfrak{H}}}{2\mathfrak{H}}\\
\leq&\frac{2^{2\mathfrak{H}}}{\mathfrak{H}},
\end{aligned}$$

and

$$\int_0^{2-i\Delta_n+s}|x-y-i\Delta_n+s|^{2\mathfrak{H}-1}dy=\int_{i\Delta_n-s}^2|x-y|^{2\mathfrak{H}-1}dy\leq\int_0^2|x-y|^{2\mathfrak{H}-1}dy\leq\frac{2^{2\mathfrak{H}}}{\mathfrak{H}},$$

as well as

$$\int_0^{2-i\Delta_n+s} y^{2\mathfrak{H}-1}dy = \frac{(2-i\Delta_n+s)^{2\mathfrak{H}}}{\mathfrak{H}} \leq \frac{2^{2\mathfrak{H}}}{\mathfrak{H}}.$$

Hence,

$$\begin{aligned}&\Big|\int_0^{2-i\Delta_n+s} \mathbb{E}_{\bar{\mathbb{P}}}\left[B_2(y)B_2(x)\right]\left(|x-y|^{2\mathfrak{H}}+|y+i\Delta_n-s|^{2\mathfrak{H}}\right.\\&\qquad\qquad\left.-|x-y-(i\Delta_n-s)|^{2\mathfrak{H}}-|y|^{2\mathfrak{H}}\right)dy\Big|\\&\leq \Delta_n 3\frac{2^{2\mathfrak{H}}}{\mathfrak{H}}\sup_{r,s\in[0,2]}|\mathbb{E}_{\bar{\mathbb{P}}}\left[B_2(r)B_2(s)\right]|.\end{aligned}$$

This yields

$$\begin{aligned}&|(2)_t^n|\\&=\Big|\sum_{i=1}^n\int_{(i-1)\Delta_n}^{i\Delta_n}\int_0^2\int_0^{2-i\Delta_n+s}\Delta_n^{-(\mathfrak{H}+\epsilon)}\mathbb{E}_{\bar{\mathbb{P}}}\left[B_1(x)(B_1(y+i\Delta_n-s)-B_1(y))\right]\\&\qquad\qquad\times\mathbb{E}_{\bar{\mathbb{P}}}\left[B_2(y)B_2(x)\right]dydxds\Big|\\&=\Big|\sum_{i=1}^n\int_{(i-1)\Delta_n}^{i\Delta_n}\int_0^2\int_0^{2-i\Delta_n+s}\Delta_n^{-(\mathfrak{H}+\epsilon)}\frac{1}{2}\left(|x-y|^{2\mathfrak{H}}+|y+i\Delta_n-s|^{2\mathfrak{H}}\right.\\&\qquad\qquad\left.-|x-y-(i\Delta_n-s)|^{2\mathfrak{H}}-|y|^{2\mathfrak{H}}\right)\\&\qquad\qquad\times\mathbb{E}_{\bar{\mathbb{P}}}\left[B_2(y)B_2(x)\right]dydxds\Big|\\&\leq\sum_{i=1}^n\int_{(i-1)\Delta_n}^{i\Delta_n}\int_0^2\Delta_n^{1-(\mathfrak{H}+\epsilon)}3\frac{2^{2\mathfrak{H}}}{\mathfrak{H}}dxds\sup_{r,s\in[0,2]}|\mathbb{E}_{\bar{\mathbb{P}}}\left[B_2(r)B_2(s)\right]|\\&=\Delta_n^{1-(\mathfrak{H}+\epsilon)}6\frac{2^{2\mathfrak{H}}}{\mathfrak{H}}\sup_{r,s\in[0,2]}|\mathbb{E}\left[B_2(r)B_2(s)\right]|,\end{aligned}$$

which goes to 0 as $n\to\infty$. By analogous reasoning we obtain that the third summand $(3)_t^n$ goes to 0 as $n\to\infty$. We now come to the fourth term. For that, we find

$$\begin{aligned}&\sum_{i=1}^n\int_{(i-1)\Delta_n}^{i\Delta_n}\int_0^{2-i\Delta_n+s}\Delta_n^{-(\mathfrak{H}+\epsilon)}\mathbb{E}\left[B_2(y)(B_1(y+i\Delta_n-s)-B_1(y))\right]dyds\\&=\sum_{i=1}^n\int_{(i-1)\Delta_n}^{i\Delta_n}\Delta_n^{-(\mathfrak{H}+\epsilon)}\int_0^{2-(i\Delta_n-s)}\mathbb{E}\left[\left(B_1(x+i\Delta_n-s)-B_1(x)\right)B_2(x)\right]dxds\\&\quad-\sum_{i=1}^n\int_{(i-1)\Delta_n}^{i\Delta_n}\Delta_n^{-(\mathfrak{H}+\epsilon)}\int_{2-(i\Delta_n-s)}^2\mathbb{E}\left[B_1(x)B_2(x)\right]dxds\\&=\sum_{i=1}^n\int_{(i-1)\Delta_n}^{i\Delta_n}\frac{\rho}{2\Delta_n^{\mathfrak{H}+\epsilon}}\int_0^{2-(i\Delta_n-s)}(x+i\Delta_n-s)^{\mathfrak{H}+\epsilon}-x^{\mathfrak{H}+\epsilon}-(i\Delta_n-s)^{\mathfrak{H}+\epsilon}dxds\\&\quad-\sum_{i=1}^n\int_{(i-1)\Delta_n}^{i\Delta_n}\frac{\rho}{\Delta_n^{(\mathfrak{H}+\epsilon)}}\int_{2-(i\Delta_n-s)}^2 x^{\mathfrak{H}+\epsilon}dxds\end{aligned}$$

$$
\begin{aligned}
=&\frac{\rho}{2}\sum_{i=1}^{n}\int_{(i-1)\Delta_n}^{i\Delta_n}\int_0^{2-(i\Delta_n-s)}\frac{(x+i\Delta_n-s)^{\mathfrak{H}+\epsilon}-x^{\mathfrak{H}+\epsilon}}{\Delta_n^{\mathfrak{H}+\epsilon}}dxds\\
&-\frac{\rho}{2}\sum_{i=1}^{n}\int_{(i-1)\Delta_n}^{i\Delta_n}\left(\frac{(i\Delta_n-s)}{\Delta_n}\right)^{\mathfrak{H}+\epsilon}(2-(i\Delta_n-s))ds\\
&-\sum_{i=1}^{n}\int_{(i-1)\Delta_n}^{i\Delta_n}\frac{\rho}{\Delta_n^{(\mathfrak{H}+\epsilon)}}\int_{2-(i\Delta_n-s)}^{2}x^{\mathfrak{H}+\epsilon}dxds. \qquad (80)
\end{aligned}
$$

The first and the third summand converge to 0 as $n\to\infty$, as by the mean value theorem

$$
\begin{aligned}
&\left|\frac{\rho}{2}\sum_{i=1}^{n}\int_{(i-1)\Delta_n}^{i\Delta_n}\int_0^{2-(i\Delta_n-s)}\frac{(x+i\Delta_n-s)^{\mathfrak{H}+\epsilon}-x^{\mathfrak{H}+\epsilon}}{\Delta_n^{\mathfrak{H}+\epsilon}}dxds\right|\\
\leq&\frac{\rho}{2}\sum_{i=1}^{n}\int_{(i-1)\Delta_n}^{i\Delta_n}\int_0^{2-(i\Delta_n-s)}\Delta_n^{1-(\mathfrak{H}+\epsilon)}x^{\mathfrak{H}+\epsilon-1}dxds\\
\leq&\Delta_n^{1-(\mathfrak{H}+\epsilon)}\frac{\rho}{2}\int_0^2 x^{\mathfrak{H}+\epsilon-1}dx.
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\sum_{i=1}^{n}\int_{(i-1)\Delta_n}^{i\Delta_n}\frac{\rho}{\Delta_n^{(\mathfrak{H}+\epsilon)}}\int_{2-(i\Delta_n-s)}^{2}x^{\mathfrak{H}+\epsilon}dxds\right|\\
\leq&\rho\Delta_n^{1-(\mathfrak{H}+\epsilon)}2^{\mathfrak{H}+\epsilon}.
\end{aligned}
$$

Summing up, we obtain that for any $\eta>0$ there is an $N\in\mathbb{N}$, such that for all $n\geq N$ we have as $(1)_t^n$, $(2)_t^n$, $(3)_t^n$ and the first and third term in (80) go to 0 as $n\to\infty$

$$
\begin{aligned}
&\left|\mathbb{E}\left[\langle B_1,B_2\rangle\sum_{i=1}^{n}\int_{(i-1)\Delta_n}^{i\Delta_n}(i\Delta_n)^{-(\mathfrak{H}+\epsilon)}\langle(\mathcal{S}(i\Delta_n-s)-I)B_1,B_2\rangle ds\right]\right|\\
\geq&\left|\int_0^2\mathbb{E}\left[B_1(x)B_2(x)\right]dx\right|\frac{\rho}{2}\sum_{i=1}^{n}\int_{(i-1)\Delta_n}^{i\Delta_n}\left(\frac{(i\Delta_n-s)}{\Delta_n}\right)^{\mathfrak{H}+\epsilon}(2-(i\Delta_n-s))ds-\eta\\
\geq&\left|\int_0^2\mathbb{E}\left[B_1(x)B_2(x)\right]dx\right|\frac{\rho}{2\Delta_n^{\mathfrak{H}+\epsilon}}\sum_{i=1}^{n}\frac{\Delta_n^{\mathfrak{H}+\epsilon+1}}{\mathfrak{H}+\epsilon+1}-\eta\\
=&\left|\int_0^2\mathbb{E}\left[B_1(x)B_2(x)\right]dx\right|\frac{\rho}{2(\mathfrak{H}+\epsilon+1)}-\eta.
\end{aligned}
$$

As this holds for all $\eta>0$, we obtain (78) and hence the proof. □

We continue by proving the remaining assertions of Example 4

PROOF OF THE REMAINING ASSERTIONS OF EXAMPLE 4. In order to verify the validity of the counterexample 4 we still have to show that if $X=B^{\mathfrak{H}}$,

(i) $(RV_t^n-\int_0^t\Sigma_s ds-\sum_{i=1}^n((\mathcal{S}(\Delta_n)-I)Y_{(i-1)\Delta_n})^{\otimes 2})$ converges in probability to 0 and
(ii) $\sum_{i=1}^n\|(\mathcal{S}(\Delta_n)-I)Y_{(i-1)})\|^4$ is uniformly integrable.

Moreover, in the second case, in which $X=\mathbb{I}_{[0,1]}$ we must show that

(iii) $\Delta_n^{-\frac{1}{2}}(SARCV_t^n - \int_0^t \Sigma_s ds)$ is uniformly integrable and

(iv) $\Delta_n^{-\frac{1}{2}}(RV_t^n - \int_0^t \Sigma_s ds)$ is uniformly integrable.

Observe that

$$
\begin{aligned}
RV_t^n &= \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \Delta_i^n Y^{\otimes 2} \\
&= \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \tilde{\Delta}_i^n Y^{\otimes 2} + \tilde{\Delta}_i^n Y \otimes (\mathcal{S}(\Delta_n) - I) Y_{(i-1)\Delta_n} \\
&\qquad + (\mathcal{S}(\Delta_n) - I) Y_{(i-1)\Delta_n} \otimes \tilde{\Delta}_i^n Y + [(\mathcal{S}(\Delta_n) - I) Y_{(i-1)\Delta_n}]^{\otimes 2}.
\end{aligned}
$$

Thus, recalling that $\Sigma_s^{\mathcal{S}_n} = \mathcal{S}(i\Delta_n - s)\Sigma_s \mathcal{S}(i\Delta_n - s)^*$ for $s \in ((i-1)\Delta_n, i\Delta_n]$

$$
\begin{aligned}
&(RV_t^n - \int_0^t \Sigma_s ds) \\
=&(SARCV_t^n - \int_0^t \Sigma_s^{\mathcal{S}_n} ds) + \Delta_n^{-\frac{1}{2}} \int_0^t \Sigma_s^{\mathcal{S}_n} - \Sigma_s ds \\
&+ \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \tilde{\Delta}_i^n Y \otimes (\mathcal{S}(\Delta_n) - I) Y_{(i-1)\Delta_n} \\
&+ \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} (\mathcal{S}(\Delta_n) - I) Y_{(i-1)\Delta_n} \otimes \tilde{\Delta}_i^n Y \\
&+ \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} [(\mathcal{S}(\Delta_n) - I) Y_{(i-1)\Delta_n}]^{\otimes 2} \\
=&(1)_t^n + (2)_t^n + (3)_t^n + (4)_t^n + (5)_t^n.
\end{aligned}
$$

The law of large numbers 3.1 guarantees that $(1)_t^n + (2)_t^n$ converges to 0 in probability with respect to the Hilbert-Schmidt norm. Moreover, for $(3)_t^n$ (and analogously $(4)_t^n$) we find in the first case that, with the notation $\Delta_i \mathcal{S} = \mathcal{S}(i\Delta_n) - \mathcal{S}((i-1)\Delta_n)$,

$$
\begin{aligned}
&\mathbb{E}\left[\|(3)_t^n\|^2\right] \\
=&\sum_{i,j=1}^n \mathbb{E}\left[\langle \tilde{\Delta}_i^n Y \otimes (\mathcal{S}(\Delta_n) - I) Y_{(i-1)\Delta_n}, \tilde{\Delta}_j^n Y \otimes (\mathcal{S}(\Delta_n) - I) Y_{(j-1)\Delta_n} \rangle_{\mathcal{H}}\right] \\
=&\sum_{i,j=1}^n \mathbb{E}\left[(\beta_{i\Delta_n} - \beta_{(i-1)\Delta_n})\beta_{(i-1)\Delta_n}(\beta_{j\Delta_n} - \beta_{(j-1)\Delta_n})\beta_{(j-1)\Delta_n}\right] \\
&\qquad \times \mathbb{E}\left[\langle \mathcal{S}(i\Delta_n) B^{\mathfrak{H}} \otimes \Delta_i \mathcal{S} B^{\mathfrak{H}}, \mathcal{S}(j\Delta_n) B^{\mathfrak{H}} \otimes \Delta_j \mathcal{S} B^{\mathfrak{H}} \rangle_{\mathcal{H}}\right] \\
=&\sum_{i=1}^n (i-1)\Delta_n^2 \mathbb{E}\left[\|\mathcal{S}(i\Delta_n) B^{\mathfrak{H}} \otimes \Delta_i \mathcal{S} B^{\mathfrak{H}}\|_{\mathcal{H}}^2\right] \\
\leq&\sum_{i=1}^n (i-1)\Delta_n^2 \mathbb{E}\left[\|\mathcal{S}(i\Delta_n) B^{\mathfrak{H}}\|_{\mathcal{H}}^4\right]^{\frac{1}{2}} \mathbb{E}\left[\|\Delta_i \mathcal{S} B^{\mathfrak{H}}\|_{\mathcal{H}}^4\right]^{\frac{1}{2}}
\end{aligned}
$$

$$= \sup_{r\in[0,T]} \|\mathcal{S}(r)\|_{\text{op}}^4 \mathbb{E}\left[\|B^{\mathfrak{H}}\|_{\mathcal{H}}^4\right]^{\frac{1}{2}} \mathbb{E}\left[\|(\mathcal{S}(\Delta_n)-I)B^{\mathfrak{H}}\|_{\mathcal{H}}^4\right]^{\frac{1}{2}} \sum_{i=1}^{n}(i-1)\Delta_n^2.$$

This converges to 0, as $n\to\infty$ and thus $(3)_t^n \to 0$ (as well as $(4)_t^n \to 0$) as $n\to\infty$ in $L^2(\Omega)$. This shows the first point (i).

In order to prove (ii), observe that as by Jensen's inequality it holds (since the central eight's moment of a Gaussian random variable $Z \sim N(0,\rho^2)$ is $\mathbb{E}[Z^8] = 105\rho^8$)

$$\mathbb{E}\left[\|\Delta_j \mathcal{S} B^{\mathfrak{H}}\|_H^8\right] \leq \int_0^2 \mathbb{E}\left[\left(B^{\mathfrak{H}}_{x+i\Delta_n} - B^{\mathfrak{H}}_{x+(i-1)\Delta_n}\right)^8\right] dx = 210\Delta_n^{8H} = 210\Delta_n^2,$$

we have

$$\begin{aligned}
&\mathbb{E}\left[\left(\sum_{i=1}^{n}\|(\mathcal{S}(\Delta_n)-I)Y_{(i-1)\Delta_n}\|^4\right)^2\right] \\
&= \sum_{i,j=1}^{n} \mathbb{E}\left[\beta^4_{(i-1)\Delta_n}\beta^4_{(j-1)\Delta_n}\right]\mathbb{E}\left[\|\Delta_i\mathcal{S}B^{\mathfrak{H}}\|_H^4\|\Delta_j\mathcal{S}B^{\mathfrak{H}}\|_H^4\right] \\
&\leq \sum_{i,j=1}^{n} \mathbb{E}\left[\beta^8_{(i-1)\Delta_n}\right]^{\frac{1}{2}}\mathbb{E}\left[\beta^8_{(j-1)\Delta_n}\right]^{\frac{1}{2}}\mathbb{E}\left[\|\Delta_i\mathcal{S}B^{\mathfrak{H}}\|_H^8\right]^{\frac{1}{2}}\mathbb{E}\left[\|\Delta_j\mathcal{S}B^{\mathfrak{H}}\|_H^8\right]^{\frac{1}{2}} \\
&\leq 210\times 105\sum_{i,j=1}^{n}(i-1)^2(j-1)^2\Delta_n^4\Delta_n^2 \\
&\leq 210\times 105,
\end{aligned}$$

which yields the $L^2(\Omega)$-boundedness of $\sum_{i=1}^n \|(\mathcal{S}(\Delta_n)-I)Y_{(i-1)}\|^4$ and, in particular, point (ii).

In order to show uniform integrability for the points (iii) and (iv) in the second case, in which $X=\mathbb{I}_{[0,1]}$, it is enough to show (after normalisation by $\sqrt{n}$) that all summands $\sqrt{n}(1)_t^n - \sqrt{n}(5)_t^n$ are bounded in $L^2(\Omega)$ uniformly in $n\in\mathbb{N}$. The first summand $\sqrt{n}(1)_t^n$ is bounded in $L^2(\Omega)$, due to Theorem D.8.

For the second term we observe

$$\begin{aligned}
&\mathbb{E}\left[\left\|\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\int_{(i-1)\Delta_n}^{i\Delta_n}\left(\Sigma_s^{\mathcal{S}_n}-\Sigma_s\right)ds\right\|_{\mathcal{H}}^2\right]^{\frac{1}{2}} \\
&\leq \int_0^T \mathbb{E}\left[\|(\mathcal{S}(\lfloor s/\Delta_n\rfloor\Delta_n - s)-I)\Sigma_s\mathcal{S}(\lfloor s/\Delta_n\rfloor\Delta_n - s)^*\|_{\mathcal{H}}^2\right]^{\frac{1}{2}} \\
&\qquad + \mathbb{E}\left[\|\Sigma_s(\mathcal{S}(\lfloor s/\Delta_n\rfloor\Delta_n - s)^*-I)\|_{\mathcal{H}}^2\right]^{\frac{1}{2}} ds \\
&\leq 2\sup_{r\in[0,T]}\|\mathcal{S}(r)\|_{\text{op}}\int_0^T \sup_{x\in[0,\Delta_n]}\mathbb{E}\left[\left\|(\mathcal{S}(x)-I)\sigma_s\right\|_{\text{op}}^2\left\|\sigma_s\right\|_{\mathcal{H}}^2\right]^{\frac{1}{2}}.
\end{aligned}$$

It holds for $x\geq 0$, $\|\sigma_s\|_{L_{\text{HS}}(U,H)}=1$ and

$$\|(\mathcal{S}(x)-I)\sigma_s\|^2_{L_{\text{HS}}(U,H)} = \|(\mathcal{S}(x+s)-\mathcal{S}(s))\mathbb{I}_{[0,1]}\|_H^2 = 2x.$$

This yields

$$\begin{aligned}
&\Delta_n^{-\frac{1}{2}}\mathbb{E}\left[\left\|\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\int_{(i-1)\Delta_n}^{i\Delta_n}\left(\Sigma_s^{\mathcal{S}_n}-\Sigma_s\right)ds\right\|_{\mathcal{H}}^2\right]^{\frac{1}{2}}\\
\leq&\Delta_n^{-\frac{1}{2}}2\sup_{r\in[0,T]}\|\mathcal{S}(r)\|_{\text{op}}\int_0^T\sup_{x\in[0,\Delta_n]}\|(\mathcal{S}(x)-I)\sigma_s\|_{\text{op}}\|\sigma_s\|_{\mathcal{H}}ds\\
\leq&4\sup_{r\in[0,T]}\|\mathcal{S}(r)\|_{\text{op}}^2T.
\end{aligned}$$

Hence $\sqrt{n}(2)_t^n$ is bounded in $L^2(\Omega)$. We just proved point (iii).

Now we turn to the $L^2(\Omega)$-boundedness of $\sqrt{n}(3)_t^n$ ($\sqrt{n}(4)_t^n$ is analogous). We have

$$\begin{aligned}
&\langle(\mathcal{S}(i\Delta_n)-\mathcal{S}(\Delta_n(i-1)))\,\mathbb{I}_{[0,1]},(\mathcal{S}(i\Delta_n)-\mathcal{S}(\Delta_n(i-1)))\,\mathbb{I}_{[0,1]}\rangle\\
=&\int_{\mathbb{R}}\left(\mathbb{I}_{[-i\Delta_n,-(i-1)\Delta_n]}(y)-\mathbb{I}_{[1-i\Delta_n,1-(i-1)\Delta_n]}(y)\right)\\
&\qquad\times\left(\mathbb{I}_{[-j\Delta_n,-(j-1)\Delta_n]}(y)-\mathbb{I}_{[1-j\Delta_n,1-(j-1)\Delta_n]}(y)\right)dy\\
=&\delta_{i,j}2\Delta_n.
\end{aligned}$$

This yields

$$\begin{aligned}
&\langle(\mathcal{S}(\Delta_n)-I)\,Y_{(i-1)\Delta_n},(\mathcal{S}(\Delta_n)-I)\,Y_{(j-1)\Delta_n}\rangle\\
=&\beta_{(i-1)\Delta_n}\beta_{(j-1)\Delta_n}\langle(\mathcal{S}(\Delta_n i)-\mathcal{S}(\Delta_n(i-1)))\,\mathbb{I}_{[0,1]},(\mathcal{S}(\Delta_n j)-\mathcal{S}(\Delta_n(j-1)))\,\mathbb{I}_{[0,1]}\rangle\\
=&\delta_{i,j}\beta_{(i-1)\Delta_n}^2 2\Delta_n.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\|\Delta_n^{-\frac{1}{2}}(3)_t\|_{\mathcal{H}}^2\\
=&\Delta_n^{-1}\sum_{i,j=1}^n\langle\tilde{\Delta}_i^nY\otimes(\mathcal{S}(\Delta_n)-I)Y_{(i-1)\Delta_n},\tilde{\Delta}_j^nY\otimes(\mathcal{S}(\Delta_n)-I)Y_{(j-1)\Delta_n}\rangle_{\mathcal{H}}\\
=&\Delta_n^{-1}\sum_{i,j=1}^n\langle\tilde{\Delta}_i^nY,\tilde{\Delta}_j^nY\rangle\langle(\mathcal{S}(\Delta_n)-I)Y_{(i-1)\Delta_n},(\mathcal{S}(\Delta_n)-I)Y_{(j-1)\Delta_n}\rangle\\
=&2\sum_{i=1}^n\|\tilde{\Delta}_i^nY\|^2\beta_{(i-1)\Delta_n}^2.
\end{aligned}$$

This gives, by independence of $\beta_{(i-1)\Delta_n}$ and $\tilde{\Delta}_i^nY$,

$$\mathbb{E}\left[\|\Delta_n^{-\frac{1}{2}}(3)_t\|_{\mathcal{H}}^2\right]=2\Delta_n\sum_{i=1}^n(i-1)\int_{(i-1)\Delta_n}^{i\Delta_n}\mathbb{E}\left[\|\sigma_s^{\mathcal{S}_n}\|_{L_{\text{HS}}(U,H)}^2\right]ds\leq 2\sup_{r\in[0,T]}\|\mathcal{S}(r)\|_{\text{op}}^2.$$

Hence, the $L^2(\Omega)$-boundedness of $\sqrt{n}(3)_t^n$ (and $\sqrt{n}(4)_t^n$) follows. It remains to show the $L^2(\Omega)$-boundedness of $(5)_t^n$. We find

$$\mathbb{E}\left[\|(5)_t^n\|^2\right]=\mathbb{E}\left[\|\Delta_n^{-\frac{1}{2}}\sum_{i=1}^n[(\mathcal{S}(\Delta_n)-I)Y_{(i-1)}]^{\otimes 2}\|^2\right]$$

$$
\begin{aligned}
&= \mathbb{E}\left[\Delta_n^{-1} \sum_{i,j=1}^{n} \langle \mathcal{S}(\Delta_n) - I)Y_{(i-1)}, \mathcal{S}(\Delta_n) - I)Y_{(j-1)}\rangle^2\right] \\
&= \mathbb{E}\left[\Delta_n^{-1} \sum_{i=1}^{n} 2\Delta_n^2 \beta_{(i-1)\Delta_n}^4\right] \\
&\leq 6.
\end{aligned}
$$

This yields the $L^2(\Omega)$-boundedness of $(5)_t^n$ and, thus, we proved (iv). □

Now we give the proof of Theorem 3.8.

PROOF OF THEOREM 3.8. Recall that by Theorem B.1(b) and (d) we can assume that Assumption 8 holds. This yields for the proof of the law of the large numbers that by the dominated convergence theorem

$$
\lim_{t\to 0} \mathbb{E}\left[\int_0^T \|t^{-\frac{1}{2}}(I - \mathcal{S}(t))\sigma_s\|_{L_{\text{HS}(U,H)}}^2 ds\right] = 0,
$$

and for the proof of the central limit theorem

$$
\lim_{t\to 0} \mathbb{E}\left[\int_0^T \|t^{-\frac{3}{4}}(I - \mathcal{S}(t))\sigma_s\|_{L_{\text{HS}(U,H)}}^2 ds\right] = 0.
$$

Then observe that we have

$$
\begin{aligned}
RV_t^n &= \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \Delta_i^n Y^{\otimes 2} \\
&= \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \tilde{\Delta}_i^n Y^{\otimes 2} + \tilde{\Delta}_i^n Y \otimes (\mathcal{S}(\Delta_n) - I)Y_{(i-1)\Delta_n} \\
&\qquad + (\mathcal{S}(\Delta_n) - I)Y_{(i-1)\Delta_n} \otimes \tilde{\Delta}_i^n Y + [(\mathcal{S}(\Delta_n) - I)Y_{(i-1)\Delta_n}]^{\otimes 2} \\
&= (1)_t + (2)_t + (3)_t + (4)_t.
\end{aligned}
$$

We know that the first summand converges in u.c.p. to the integrated volatility $\int_0^t \Sigma_s ds$. Under this assumption, we obtain for the second and third summand

$$
\begin{aligned}
\frac{1}{2} \sup_{t\in[0,T]} \mathbb{E}\left[\|(2)_t + (3)_t\|_{\mathcal{H}}\right] &\leq \sum_{i=1}^{\lfloor T/\Delta_n \rfloor} \mathbb{E}\left[\|(\mathcal{S}(\Delta_n) - I)Y_{(i-1)\Delta_n}\|\|\tilde{\Delta}_i^n Y\|\right] \\
= \left(\sum_{i=1}^{\lfloor T/\Delta_n \rfloor} \mathbb{E}\left[\|\tilde{\Delta}_i^n Y\|^2\right]\right)^{\frac{1}{2}} & \left(\sum_{i=1}^{\lfloor T/\Delta_n \rfloor} \mathbb{E}\left[\|(\mathcal{S}(\Delta_n) - I)Y_{(i-1)\Delta_n}\|^2\right]\right)^{\frac{1}{2}} \\
\leq \sup_{r\in[0,T]} \|\mathcal{S}(r)\|_{\text{op}} & \left(\sum_{i=1}^{\lfloor T/\Delta_n \rfloor} \int_{(i-1)\Delta_n}^{i\Delta_n} \mathbb{E}\left[\|\sigma_s\|_{L_{\text{HS}(U,H)}}^2\right] ds\right)^{\frac{1}{2}} \\
\times & \left(\sum_{i=1}^{\lfloor T/\Delta_n \rfloor} \int_0^T \mathbb{E}\left[\|(\mathcal{S}(\Delta_n) - I)\sigma_s\|_{L_{\text{HS}(U,H)}}^2\right] ds\right)^{\frac{1}{2}} \\
= o(1), &
\end{aligned}
$$

where the last equality is by assumption. Moreover, the last summand fulfills immediately

$$\sup_{t\in[0,T]}(4)_t \leq \sup_{r\in[0,T]}\|\mathcal{S}(r)\|_{\text{op}}\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\int_0^T \mathbb{E}\left[\|(\mathcal{S}(\Delta_n)-I)\sigma_s\|_{L_{\text{HS}(U,H)}}^2\right]ds = o(1).$$

This proves the claim for the law of large numbers. The central limit theorem follows by analogous reasoning, after normalising by $\Delta_n^{-\frac{1}{2}}$. □

Now we give the proof of Theorem 3.9

PROOF OF THEOREM 3.9. We can argue componentwise, which is why we assume without loss of generality that $B = h\otimes g$ for $h,g\in F_{1/2}^{\mathcal{S}^*}$ for (i) or $h,g\in F_{1/4}^{\mathcal{S}^*}$ for (ii) respectively. Again, we appeal to the decomposition

$$\begin{aligned}\langle RV_t^n h,g\rangle &= \sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\langle \Delta_i^n Y^{\otimes 2}h,g\rangle\\ &= \sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\langle \tilde{\Delta}_i^n Y^{\otimes 2}h,g\rangle + \langle \tilde{\Delta}_i^n Y\otimes(\mathcal{S}(\Delta_n)-I)Y_{(i-1)\Delta_n}h,g\rangle\\ &\qquad + \langle (\mathcal{S}(\Delta_n)-I)Y_{(i-1)\Delta_n}\otimes\tilde{\Delta}_i^n Y h,g\rangle + \langle[(\mathcal{S}(\Delta_n)-I)Y_{(i-1)}]^{\otimes 2}h,g\rangle\\ &= (1)_n^t + (2)_n^t + (3)_n^t + (4)_n^t.\end{aligned}$$

The first summand converges to $\int_0^T\langle\Sigma_s h,g\rangle ds$, and after normalisation with $\Delta_n^{\frac{1}{2}}$ it is asymptotically centred normal with variance $\langle\Gamma_t h\otimes g, h\otimes g\rangle$. Therefore, it is for the law of large numbers enough to show

$$\left((2)_n^t + (3)_n^t + (4)_n^t\right)\xrightarrow{u.c.p.} 0 \quad \text{as } n\to\infty.$$

and for the central limit theorem to show

$$\Delta_n^{-\frac{1}{2}}\left((2)_n^t + (3)_n^t + (4)_n^t\right)\xrightarrow{u.c.p.} 0 \quad \text{as } n\to\infty.$$

For the third (and analogously for the second) summand we have

$$\begin{aligned}|(3)_n^t| &= \left|\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\langle(\mathcal{S}(\Delta_n)-I)Y_{(i-1)\Delta_n}\otimes\tilde{\Delta}_i^n Y h,g\rangle\right|\\ &\leq \left(\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\langle(\mathcal{S}(\Delta_n)-I)Y_{(i-1)\Delta_n},h\rangle^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\langle\tilde{\Delta}_i^n Y,g\rangle^2\right)^{\frac{1}{2}}\\ &= \left(\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\langle(\mathcal{S}(\Delta_n)-I)Y_{(i-1)\Delta_n},h\rangle^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\langle\tilde{\Delta}_i^n Y^{\otimes 2}g,g\rangle\right)^{\frac{1}{2}},\end{aligned}$$

and the second factor converges to an asymptotically normal law. For the fourth summand we have

$$|(4)_n^t| = \left|\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\langle(\mathcal{S}(\Delta_n)-I)Y_{(i-1)}^{\otimes 2}h,g\rangle\right|$$

$$\leq\left(\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\langle(\mathcal{S}(\Delta_n)-I)Y_{(i-1)},h\rangle^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\langle(\mathcal{S}(\Delta_n)-I)Y_{(i-1)},g\rangle^2\right)^{\frac{1}{2}},$$

and, thus, for the law of large numbers it is enough to show for all $h\in F_{\frac{1}{2}}^{\mathcal{S}^*}$,

$$\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\langle(\mathcal{S}(\Delta_n)-I)Y_{(i-1)},h\rangle^2\stackrel{u.c.p.}{\longrightarrow}0,\quad\text{as } n\to\infty \tag{81}$$

and for the central limit theorem for all $h\in F_{\frac{3}{4}}^{\mathcal{S}^*}$,

$$\Delta_n^{-\frac{1}{2}}\sum_{i=1}^{\lfloor t/\Delta_n\rfloor}\langle(\mathcal{S}(\Delta_n)-I)Y_{(i-1)},h\rangle^2\stackrel{u.c.p.}{\longrightarrow}0,\quad\text{as } n\to\infty. \tag{82}$$

Due to Theorem B.1(b) (respectively (a) for the central limit theorem) we can suppose that Assumption 8 (or Assumption 7 for the central limit theorem) is valid. In that case, we have for the proof of the law of the large numbers by the dominated convergence theorem

$$\lim_{t\to 0}\mathbb{E}\left[\int_0^T\|t^{-\frac{1}{2}}(I-\mathcal{S}(t))\|_{\mathcal{H}}^2ds\right]=0$$

and for the proof of the central limit theorem

$$\lim_{t\to 0}\mathbb{E}\left[\int_0^T\|t^{-\frac{3}{4}}(I-\mathcal{S}(t))\sigma_s\|_{\mathcal{H}}^2ds\right]=0.$$

Moreover, we have

$$\begin{aligned}&\sum_{i=1}^{\lfloor T/\Delta_n\rfloor}\mathbb{E}\left[\langle(\mathcal{S}(\Delta_n)-I)Y_{(i-1)},h\rangle^2\right]\\ =&\sum_{i=1}^{\lfloor T/\Delta_n\rfloor}\int_0^{(i-1)\Delta_n}\mathbb{E}\left[\langle(\mathcal{S}(i\Delta_n)-\mathcal{S}((i-1)\Delta_n))\Sigma_s^{\mathcal{S}_n}(\mathcal{S}(i\Delta_n)-\mathcal{S}((i-1)\Delta_n))^*h,h\rangle\right]ds\\ \leq& T\int_0^T\sup_{t\in[0,T]}\mathbb{E}\left[\Delta_n^{-1}\langle(\mathcal{S}(t+\Delta_n)-\mathcal{S}(t))\Sigma_s^{\mathcal{S}_n}(\mathcal{S}(t+\Delta_n)-\mathcal{S}(t))\Delta_n))^*h,h\rangle\right]ds.\end{aligned}$$

Henceforth, in order to show (81) and (82) it is enough to show that, for all $\gamma\in(0,1)$ and $h\in F_{\gamma}^{\mathcal{S}^*}$, we have, as $n\to\infty$,

$$\int_0^T\sup_{t\in[0,T]}\mathbb{E}\left[\Delta^{-2\gamma}\langle(\mathcal{S}(t+\Delta)-\mathcal{S}(t))\Sigma_s(\mathcal{S}(t+\Delta)-\mathcal{S}(t))^*h,h\rangle\right]ds\stackrel{u.c.p.}{\longrightarrow}0.$$

We note that

$$\begin{aligned}&\int_0^T\sup_{t\in[0,T]}\mathbb{E}\left[\Delta^{-2\gamma}\langle(\mathcal{S}(t+\Delta)-\mathcal{S}(t))\Sigma_s(\mathcal{S}(t+\Delta)-\mathcal{S}(t))^*h,h\rangle\right]ds\\ \leq&\int_0^T\sup_{t\in[0,T]}\mathbb{E}\left[\Delta^{-2\gamma}\langle(I-p_N)\Sigma_s(\mathcal{S}(t+\Delta)-\mathcal{S}(t))^*h,(\mathcal{S}(t+\Delta)-\mathcal{S}(t))^*h\rangle\right]ds\\ &+\int_0^T\sup_{t\in[0,T]}\mathbb{E}\left[\Delta^{-2\gamma}\langle p_N\Sigma_s(\mathcal{S}(t+\Delta_n)-\mathcal{S}(t))^*h,\Delta_n^{-\gamma}(\mathcal{S}(t+\Delta)-\mathcal{S}(t))^*h\rangle^2\right]ds\\ =&(1)_{n,N}+(2)_{n,N},\end{aligned}$$

where again we denoted by p_N the projection onto $v^N := \overline{\{e_j : j \geq N\}}$ for an orthonormal basis $(e_j)_{j\in\mathbb{N}}$ of H that is contained in $D(\mathcal{A})$. We have

$$\mathcal{S}(t+\Delta_n)e_i - \mathcal{S}(t)e_i = \int_t^{t+\Delta_n} \mathcal{S}(u)\mathcal{A}e_i du$$

and therefore we find for the first summand that

$$\begin{aligned}
&(1)_{n,N}\\
&\leq \sum_{j=1}^{N-1} \left| \int_0^T \sup_{t\in[0,T]} \mathbb{E}\left[\Delta^{-2\gamma} \langle \Sigma_s(\mathcal{S}(t+\Delta_n)-\mathcal{S}(t))^* h, e_j\rangle \right.\right.\\
&\qquad\qquad \left.\left. \times \langle e_j, \Delta_n^{-\gamma}(\mathcal{S}(t+\Delta_n)-\mathcal{S}(t))^* h\rangle\right] ds\right|\\
&= \sum_{j=1}^{N-1} \Delta^{-2\gamma} \left| \int_0^T \sup_{t\in[0,T]} \mathbb{E}\left[\langle \Sigma_s(\mathcal{S}(t+\Delta_n)-\mathcal{S}(t))^* h, e_j\rangle \langle \int_t^{t+\Delta_n} \mathcal{S}(u)\mathcal{A}e_j ds, h\rangle\right] ds\right|\\
&\leq \sum_{j=1}^{N-1} \Delta_n^{1-\gamma} \left(\int_0^T \|\sigma_s\|_{\text{op}}^2 ds\right)^{\frac{1}{2}} \|h\|^2 \|\mathcal{A}e_j\| \sup_{t\in[0,T]} \|\mathcal{S}(t)\|_{\text{op}}\\
&\qquad\qquad \times \sup_{t\in[0,T]} \|\Delta_n^{-\gamma}(\mathcal{S}(t+\Delta_n)-\mathcal{S}(t))^* h\|,
\end{aligned}$$

which converges to 0 as $n \to \infty$. Moreover, it follows that for the second summand we have

$$(2)_{n,N} \leq \Delta_n^{-2\gamma} \|(\mathcal{S}(\Delta)-I)^* h\|^2 \left(\int_0^T \sup_{t\in[0,T]} \mathbb{E}\left[\|p_N \mathcal{S}(t)\sigma_s\|_{\text{op}}^2\right] ds\right)^{\frac{1}{2}},$$

where the first factor is bounded by Assumption on h and $\int_0^T \mathbb{E}\left[\|p_N \mathcal{S}(t)\sigma_s\|_{\text{op}}^2\right] ds$ converges to 0 as $N \to \infty$, as it can be shown analogously to the proof of Lemma B.4. We obtain that $\mathbb{E}[\sup_{n\in\mathbb{N}} |(1.2)_{n,N}|]$ converges to 0 as $N \to \infty$. Therefore, we can find for each $\delta > 0$ an $N \in \mathbb{N}$, such that by Markov's inequality

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{P}\left[\int_0^T \sup_{t\in[0,T]} \mathbb{E}\left[\Delta^{-2\gamma} \langle (\mathcal{S}(t+\Delta)-\mathcal{S}(t))\Sigma_s(\mathcal{S}(t+\Delta)-\mathcal{S}(t))^* h, h\rangle\right] ds > \epsilon\right]\\
&\leq \lim_{n\to\infty} \mathbb{P}\left[|(1)_{n,N}| > \epsilon\right] + \sup_{n\in\mathbb{N}} \mathbb{P}\left[|(2)_{n,N}| > \epsilon\right]\\
&= 0 + \frac{1}{\epsilon} \sup_{n\in\mathbb{N}} \mathbb{E}\left[|(2)_{n,N}|\right] \leq \delta.
\end{aligned}$$

As this holds for all $\delta > 0$, we obtain the assertion. □

Next, we give the proof of Lemma 3.13.

PROOF OF LEMMA 3.13. We recall that $\delta_x(\cdot) = 1 + \min(x, \cdot)$. By the mean value theorem it holds for $x \in (0,1)$ and $t > 0$ small enough, such that $t < x < 1-t$

$$\|\mathcal{S}(t)\delta_x - \delta_x\|^2 = \int_0^1 \left(\mathbb{I}_{[0,x]}\left((y+t)\wedge 1\right) - \mathbb{I}_{[0,x]}(y)\right)^2 dy$$

$$
\begin{aligned}
&= \int_0^{1-t} \mathbb{I}_{[x-t,x]}(y)dy + \int_{1-t}^1 \mathbb{I}_{[0,x]}(y)dy \\
&= (x \wedge (1-t)) - (x-t) + x - (1-t) \\
&= x + 2t - 1 \\
&\in (t, 2t).
\end{aligned}
$$

If $x = 1$ and t small enough such that $t \leq x$, it is

$$
\begin{aligned}
\|\mathcal{S}(t)\delta_x - \delta_x\|^2 &= \int_0^1 \left(\mathbb{I}_{[0,1]}\left((y+t)\wedge 1\right) - \mathbb{I}_{[0,1]}(y)\right)^2 dy \\
&= \int_0^{1-t} \mathbb{I}_{[x-t,x]}(y)dy + t \\
&= 2t.
\end{aligned}
$$

This shows that $x \in (0,1]$ $\delta_x \in F^{\mathcal{S}}_{\frac{1}{2}}$ but $\delta_x \notin\in F^{\mathcal{S}}_{\gamma}$ for any $\gamma > \frac{1}{2}$. Moreover, $\delta_0 \in F^{\mathcal{S}}_{\gamma}$ for all $\gamma \in [0,1]$ holds as well.

We show that this holds for the adjoint semigroup $(\mathcal{S}(t))_{t\geq 0}$ as well: For this purpose, we first derive an explicit representation of the adjoint operator $\mathcal{S}(t)^*$. Let $g \in H$ be arbitrary. Then for $x < 1$, we have, as $\delta_x'(1) = 0$,

$$
\begin{aligned}
S^*(t)g(x) =& \langle \mathcal{S}(t)\delta_x, g\rangle \\
=& \delta_x(t)g(0) + \int_0^1 \delta_x'\left((y+t)\wedge 1\right) g'(y)dy \\
=& \delta_x(t)g(0) + \int_0^{1-t} \delta_x'\left(y+t\right) g'(y)dx + \int_{1-t}^1 \delta_x'\left(1\right) g'(y)dy \\
=& \left(\int_0^t \delta_x'(y)dy + \delta_x(0)\right) g(0) + \int_t^1 \delta_x'\left(y\right) g'(y-t)dx \\
=& \left(\int_0^t \mathbb{I}_{[0,x]}(y)dx + 1\right) g(0) + \int_t^1 \mathbb{I}_{[0,x]}\left(y\right) g'(y-t)dx \\
=& g(0) + \int_0^1 \mathbb{I}_{[0,x]}(y)\left(\mathbb{I}_{[0,t]}(y)g(0) + \mathbb{I}_{[t,1]}(y)g'(y-t)\right) dy.
\end{aligned}
$$

This yields $(\mathcal{S}(t)^*g)(0) = g(0)$ and, for all $0 < t < 1$ and $x \in [0,1)$,

$$
(\mathcal{S}(t)^*g)'(x) = \left(\mathbb{I}_{[0,t]}(x)g(0) + \mathbb{I}_{[t,1]}(x)g'(x-t)\right).
$$

In particular, $\mathcal{S}(t)^*\delta_x(0) = 1$ and, for all $0 < t < 1$ and $y \in [0,1)$,

$$
\begin{aligned}
(\mathcal{S}(t)^*\delta_x)'(y) &= \left(\mathbb{I}_{[0,t]}(y) + \mathbb{I}_{[t,1]}(y)\mathbb{I}_{[0,x]}(y-t)\right) \\
&= \left(\mathbb{I}_{[0,t]}(y) + \mathbb{I}_{[t,(x+t)\wedge 1]}(y)\right).
\end{aligned}
$$

Therefore, for t small enough such that $0 \leq x < 1-t$

$$
\|\mathcal{S}(t)^*\delta_x - \delta_x\|^2 = \int_0^1 \left(\mathbb{I}_{[0,t]}(y) + \mathbb{I}_{[t,x+t]}(y) - \mathbb{I}_{[0,x]}(y)\right)^2 dy = \int_0^1 \mathbb{I}_{[x,x+t]}(y)dy = t.
$$

This shows that all δ_x with $0 \leq x < 1$ are contained in the Favard space $F^{\mathcal{S}^*}_{\frac{1}{2}}$, but not in $F^{\mathcal{S}^*}_{\gamma}$ for $\gamma > \frac{1}{2}$. Moreover, $\delta_1 \in F^{\mathcal{S}^*}_{\gamma}$ for all $\gamma \in [0,1]$ holds as well. □