A mean field game model of firm–level innovation

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Abstract

Knowledge spillovers occur when a firm researches a new technology and that technology is adapted or adopted by another firm, resulting in a social value of the technology that is larger than the initially predicted private value. As a result, firms systematically under–invest in research compared with the socially optimal investment strategy. Understanding the level of under–investment, as well as policies to correct it, is an area of active economic research. In this paper, we develop a new model of spillovers, taking inspiration from the available microeconomic data. The model developed is a mean field game model, which allows for heterogeneity in the productivity of a firm and allows for a novel approach to describing sector–level spillovers. The model is constructed from a network of interacting firms, whose connections represent knowledge transfers. We prove existence and uniqueness of solutions to the model, and we conduct some initial simulations to understand how indirect spillovers contribute to the productivity of a sector.

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1 Introduction

When a business invests in research and development (R&D), such strategy only takes into account how a potential innovation may increase the investing company's private value. However, other businesses may utilise innovations made by the original investing company to increase their own profits. However, returns from investment in innovation does not only benefit the investing company but has overall social value. Indeed, When a business invests in research and development (R&D), its strategy only takes into account how a potential innovation may increase the investing company's private value. However, other businesses may utilise innovations made by the original investing company to increase their own profits. This is known in economic literature as the knowledge spillover effect. By only considering its private return, businesses systematically undervalue their own innovations and hence under–invest in R&D, compared with the socially optimal investment level. Chronic under-investment in innovation by companies is indeed a textbook fact(25). To counteract the under–investment, governments introduce R&D subsidy policies for certain sectors of the economy. In order to effectively allocate such subsidies, it is therefore important to understand the extent of under–investment and how it varies between sectors.

To understand the spillover effect we develop a mean field game (MFG) model of firms distributed heterogeneously between sectors and according to their productivity level, taking into account their microscopic behaviour. From a microeconomic perspective, the size of knowledge spillovers can be inferred from the network of patent citations.(23) When an industrial technology is developed, it often gets patented. As part of the patent any previous technology that has been used must be cited. This results in a network of patent citations, where each citation can be used as a proxy for a spillover from one technology to another, so spillover sizes can be evaluated.(17) In the model we develop, sectors are connected by a graph that is informed by and can be calibrated to the microeconomic network of patent citation data.

A first model of knowledge spillovers, by Cohen and Levinthal,(16) considered the stock of knowledge of a firm to depend on the amount of investment in R&D of that firm and the total amount of investment by all other firms, through a mean field–type interaction. Only an initial analysis of the model was conducted.(16) A later model(1) (see Section 13.2 of this reference) started from a macroscopic perspective. Hence only the aggregate knowledge of the entire economy was considered and spillovers were assumed to increase the aggregate uniformly. This did not explain how spillovers heterogeneously affect firms. Similar models were also used to study entrepreneurship and intellectual property rights.(2) There has been particularly extensive research of knowledge spillovers in cross–country models.

In such models, a country's own output is aggregated and the knowledge level increases at a rate that depends on the leading country's knowledge level. However, simplifying assumptions were made that may affect their accuracy, such as interactions taking place in a discrete time setting,(18) or interactions between firms being described only through the evolution of aggregate quantities.(3; 27) In the present paper, we use an MFG model to both increase the complexity of the description of firms, and to link firm–level evolution directly to microeconomic data for spillover sizes. There have been several other papers focussing on MFG–type models of knowledge spillovers.(11; 35) The Boltzmann model studied in these papers did not consider how innovation among firms evolves, nor did it incorporate the microeconomic data related to patent citations in its formulation. Therefore, the model studied in the present paper can give greater insight into the relationship between knowledge spillovers and firm–level dynamics. Mean field games have been successfully applied in other areas of economics, including modelling stochastic growth,(29) and in equilibrium dynamics of supply and demand.(22; 24; 39) More generally, the mathematical theory of active particle systems has been successfully applied to various areas of economics and finance(7; 8; 10; 30).

In the present paper, we consider an MFG approach with a consumer–level consumption model through a representative agent model (see Eq. (2)). In earlier work using an MFG approach,(29) consumption was considered as a reduction in capital stock, whereas here we consider consumption as the purchase of goods produced by firms. The focus of this earlier reference was on the behaviour of consumers as a result of a relative utility function. In comparison here we focus on the behaviour of firms, and only consider consumers as a method to endogenise the price formation of a product. In the literature, the link between consumers and producers under the MFG framework has been explored in various ways. For instance, a trading model in which agents control the frequency with which they trade commodities has been proposed.(22) In that paper, the focus was on a financial market with trading in commodities, while in the present paper we consider buying and selling of products with firms that are price–setters. Solar renewable energy certificate markets have been modelled in the MFG framework.(39) In that paper, a model for pricing under specific conditions described by those markets was developed but is not directly applicable to the situation being modelled in the present paper. Finally, a price model in which agents control the rate at which they trade their products has been developed.(24) In comparison, in the present work, we consider a situation where firms control the price they sell products at, and the rate of consumption is determined by the consumers' consumption optimisation problem.

In this paper, we analyse a stationary MFG model describing the spillover effect. The MFG model describes the long–term behaviour of firms with full anticipation of the future. MFGs were described mathematically by Lasry and Lions,(32; 33) and simultaneously by Huang, Caines and Malhamé(28) and they build on the work of Aumann and related authors on anonymous games.(5; 37) The MFG system we develop includes distribution dependence that enters into the drift term but not in the cost functional, and we are considering more than one population of agents. Therefore our MFG model can be classed as a multi–population MFG with a non–separable Hamiltonian. There has been some work in either multi–population MFGs(15) or MFG models with non–separable Hamiltonians(4; 20) or both.(12; 13; 22) The multi–population MFG we develop in this paper is based on random interactions between firms, occurring on a graph structure. On a theoretical level, such interactions generally result in graphon mean field games (GMFGs). The theory around GMFGs is an emerging field, with the linear–quadratic case recently being considered.(6) Due to a simplifying assumption that we make, we are able to model the interactions as a multi–population game, with the interactions between populations being described by a graph between the populations. The combination of considering a growth model based on knowledge spillovers, along with a multi-population MFG with graph-based interaction is a novel approach that we believe provides interesting contributions to the existing literature. The techniques we use to prove existence and uniqueness rely on the ability to write the solution of a stationary Fokker–Planck equation in the form of an exponential. This characteristic has previously been used(9) to prove existence and uniqueness in MFG and best reply strategy models in a slightly different framework.

The paper is organised as follows. In Section 2, we develop the spillover model by describing firm behaviour at a microscopic level and formally deriving the mean field limit. In section 3, we describe the MFG problem and prove existence of solutions to it. We also show uniqueness of such solutions holds, provided the coupling strength between sectors is small enough. In Section 4, we provide some deeper insights into the effects of the modelling parameters, through numerical simulations. The first simulations show how parameters describing effects unrelated to spillovers (for example the discount factor, the noise level and the labour efficiency) change the MFG model. Our second group of simulations demonstrate the effect of the spillover network on the model. The spillover network is a sector–level network that aggregates the patent citation network. We show that the effect of a spillover on any sector is a result of all paths to that sector in the associated network, and not just the immediate connections between sectors, which is contrary to the current economic state of the art. Finally, in Section 5, we briefly discuss future research prospects for the model, including how we intend to apply the model to economic questions relevant to R&D subsidy policy.

2 Model development

The description of the microscopic model and the derivation of the MFG system, developed in this section, is purposefully kept at a heuristic level because the focus of the paper is on the PDE system. The modelling framework is important, however getting too involved in the technicalities may take away from the broader picture. We do expand on the technical aspects of the microscopic framework when the ambiguity reduces the clarity of the exposition.

2.1 The microscopic model

2.1.1 Firms

Assume there are L sectors within the economy, and in sector ℓ there are N_ℓ firms. Each firm, uniquely denoted by (ℓ, i) with $1 \leq \ell \leq L$ and $1 \leq i \leq N_\ell$, has a complete probability space, $(\Omega, \mathcal{F}, \mathbb{P})_{\ell,i}$, associated with it. Each probability space has a complete filter $\left((\mathcal{F}_t)_{t\geq 0}\right)_{\ell,i}$, induced by the augmentation of the natural filtration generated by the one-dimensional Brownian motion B_i, the random variable $\xi_{\ell,i} : \Omega_i \to [0, \infty)$ and the random variables $s_{i,j}^{\ell,\ell'} : \Omega_i \to \mathbb{N}$ for all (j, ℓ').

We assume firm i in group ℓ has $s_{i,j}^{\ell,\ell'}$ links with firm j in group ℓ', furthermore we assume the probability distribution of $s_{i,j}^{\ell,\ell'}$ depends on ℓ and ℓ' only. In particular, we assume there exists a function $p : \{0, \dots, L\}^2 \times \mathbb{N} \to [0, 1]$ such that $\mathbb{P}\left(s_{i,j}^{\ell,\ell'} = s\right) = p(\ell, \ell', s)$. The i^{th} firm in sector ℓ has a productivity–related knowledge level $Z_{\ell,i} \in (0, \infty)$, which increases as a result of employing labour $h_{\ell,i}$ or due to knowledge spillovers from firms that they are linked with. The knowledge dynamics are also affected by noise with strength $\sigma \in (0, \infty)$. As a result, $Z_{\ell,i}$ evolves according to the following reflected SDE

$$dZ_{\ell,i}^{h}(t) = \left((h_{\ell,i}(t))^{\gamma} + \frac{1}{N} \sum_{\ell'=1}^{L} \sum_{j=1}^{N_{\ell'}} s_{i,j}^{\ell,\ell'} Z_{\ell',j}^{h}(t) \right) dt + \sigma dB_{\ell,i}(t) + d\varphi_{\ell,i}^{h}(t) \,, \tag{1a}$$

$$Z_{\ell,i}^{h_{\ell,i}}(0) = \xi_{\ell,i} \sim m_\ell^0 \,, \tag{1b}$$

where $h = \left((h_{\ell,i})_{i=1}^{N_\ell}\right)_{\ell=1}^{L}$, $N = \sum_{\ell=1}^{L} N_\ell$, $B_{\ell,i}$ is the independent Brownian motion that generates the natural filtration $\left((\mathcal{F}_t)_{t\geq 0}\right)_{\ell,i}$, and $\gamma \in (0, 1)$ represents the inefficiency in converting one unit of labour to one unit of knowledge. The function $\varphi_{\ell,i}^{h}$ is the unique Skorokhod map of the unrestricted SDE.(40) This paper is not concerned with the technical intricacies of reflected Brownian motion which have been explored in previous literature.(45; 31; 36; 42) In the initial condition (1b), $\xi_{\ell,i} \sim m_\ell^0$ means the law of $\xi_{\ell,i}$ is equal to m_ℓ^0, where m_ℓ^0 is a given initial distribution, which may be different for each sector. We assume that

$h_{\ell,i}$ is a stationary Markovian feedback control, in particular, $h_{\ell,i}(t) = g_{\ell,i}(Z(t))$, for some deterministic function $g : (0,\infty)^N \times [0,\infty)$.

2.1.2 Consumers

In order to highlight the effect of knowledge spillovers on firm dynamics, we take a very standard approach to consumption.(1) We assume a representative consumer has consumption preferences given by $u(C, y)$, where u is a strictly increasing, differentiable and strictly jointly concave function, y is a generic good representing consumption external to the model, and $C = \left[\frac{1}{N}\sum_{\ell'=1}^{L}\sum_{j=1}^{N_{\ell'}} c_{\ell',j}^{\alpha}\right]^{\frac{1}{\alpha}}$ is a consumption index of the N goods produced by the firms in the model, each firm producing exactly one good. The constant $\alpha \in (0,1)$ is related to the elasticity of substitution, which describes the extent to which goods are interchangeable. In this model we choose y to be the numéraire, which means that all prices are normalised by the price of the generic good, y, and the price of good y is taken as 1. We assume the representative household has a fixed budget Y, and therefore the consumer's optimal consumption is defined in the following problem:

$$\text{Maximise} \quad u(C, y)\ , \tag{2a}$$

$$\text{subject to} \quad y + \frac{1}{N}\sum_{\ell'=1}^{L}\sum_{j=1}^{N_{\ell'}} p_{\ell',j}c_{\ell',j} \le Y\ . \tag{2b}$$

Using first–order conditions, we get

$$\frac{p_{\ell,i}}{p_{\ell',j}} = \left(\frac{c_{\ell,i}}{c_{\ell',j}}\right)^{\alpha-1}\ , \tag{3}$$

for any $\ell, \ell' \in \{0,\dots,L\}$, any $i \in \{1,\dots,N_\ell\}$ and any $j \in \{1,\dots,N_{\ell'}\}$. If P denotes the ideal price index, i.e.

$$\frac{p_{\ell,i}}{P} = \left(\frac{c_{\ell,i}}{C}\right)^{\alpha-1}\ , \tag{4}$$

then, by substituting (4) into (3) and rearranging, we find

$$P = \left[\frac{1}{N}\sum_{\ell'=1}^{L}\sum_{j=1}^{N_{\ell'}} p_{\ell',j}^{\frac{\alpha}{\alpha-1}}\right]^{\frac{\alpha-1}{\alpha}}\ ,$$

$$PC = \frac{1}{N}\sum_{\ell'=1}^{L}\sum_{j=1}^{N_{\ell'}} p_{\ell',j}c_{\ell',j}\ .$$

To determine the relationship between C and y, the function $u\left(C,y\right)$ is maximised subject to $y+PC\leq Y$. Then,

$$\frac{\partial_y u}{\partial_C u}=\frac{1}{P}\,.$$

Now we assume, for simplicity's sake, that $u\left(C,y\right)=C^\alpha+y^\alpha$. So,

$$y=Y\left(1+P^{\frac{\alpha}{\alpha-1}}\right)^{-1}=g\left(P,Y\right)\,, \tag{5a}$$

$$C=\frac{Y-g\left(P,Y\right)}{P}=\frac{YP^{\frac{1}{\alpha-1}}}{\left(1+P^{\frac{\alpha}{\alpha-1}}\right)}\,. \tag{5b}$$

2.1.3 Firms revisited

Now, we assume that firm i faces a cost $\psi_{\ell,i}$ of producing a unit of good. As in many papers,(1; 3; 26) we assume that the knowledge level of firm i reduces the unit cost. So, there exists a fixed constant $\psi\in(0,\infty)$ such that

$$\psi^h_{\ell,i}=\frac{\psi}{Z^h_{\ell,i}}\,.$$

Then firms choose to sell at a price p_i to maximise profits $\pi^h_{\ell,i}$ given by

$$\pi^h_{\ell,i}=p_{\ell,i}c_{\ell,i}-\frac{\psi}{Z^h_{\ell,i}}c_{\ell,i}\,.$$

By substituting (4) and (5b) in the above equation, assuming P is fixed (which is true for any individual firm as $N\to\infty$), then

$$\pi^h_{\ell,i}=B\left(Z^h_{\ell,i}\right)^{\frac{\alpha}{1-\alpha}}\,, \tag{6a}$$

$$B=(1-\alpha)\left[1+\left(\frac{\alpha}{\psi}\right)^{\frac{\alpha}{1-\alpha}}\left(\frac{1}{N}\sum_{\ell'=1}^{L}\sum_{j=1}^{N_{\ell'}}\left(Z^h_{\ell',j}\right)^{\frac{\alpha}{1-\alpha}}\right)\right]^{-1}\left(\frac{\alpha}{\psi}\right)^{\frac{\alpha}{1-\alpha}}Y\,. \tag{6b}$$

With their profits, firms can choose to invest in research to increase their future returns. Then, the agents' profit functional is given by

$$J_{\ell,i}\left(h\right)=\mathbb{E}\left[\int_0^\infty\left(B\left(Z^h_{\ell,i}\right)^{\frac{\alpha}{1-\alpha}}-wh_{\ell,i}\left(t\right)\right)e^{-\rho t}\,dt\right]\,. \tag{7}$$

The wage, w, and the discount rate, ρ, are given constants.

2.2 Mean field limit

When there are large numbers of firms in each sector, the microscopic model developed in Section 2.1 can become intractable. Instead, we assume the number of firms in each sector, N_ℓ, goes to infinity while $\frac{N_\ell}{N} \to A_\ell$ for some $A_\ell \in (0,1)$, which represents the proportion of firms in sector ℓ. In order to derive the limiting mean field model, we first define the empirical distributions for each sector $\ell = 1, \ldots, L$ by $m_\ell^{N_\ell,h} = \frac{1}{N_\ell} \sum_{i=1}^{N_\ell} \delta_{Z_{\ell,i}^h}$, where $\delta_{Z_{\ell,i}^h}$ is a Dirac delta at the point $Z_{\ell,i}^h$. We also need to define, for fixed $\ell, \ell' \in \{0, \ldots, L\}$ and $s \in \mathbb{N}$, the following graph empirical distribution $m_{\ell,\ell',s}^{N_{\ell,\ell',s},h} = \frac{1}{N_{\ell,\ell',s}} \sum_{j: s_{i,j}^{\ell,\ell'} = s} \delta_{Z_{\ell',j}^h}$. In this case, the interaction term in the dynamics (1) can be written as:

$$\frac{1}{N} \sum_{\ell'=1}^{L} \sum_{j=1}^{N_{\ell'}} s_{i,j}^{\ell,\ell'} Z_{\ell',j}^h = \sum_{\ell'=1}^{L} \frac{N_{\ell'}}{N} \sum_{s=1}^{\infty} \frac{N_{\ell,\ell',s}}{N_{\ell'}} s \int_0^{\bar{z}} z \, dm_{\ell,\ell',s}^{N_{\ell,\ell' s},h} ,$$

for each $\ell \in \{0, \ldots, L\}$. We can then rewrite the dynamics (1) using $m_{\ell,\ell',s}^{N_{\ell,\ell',s},h}$ as

$$\begin{aligned} dZ_{\ell,i}^h(t) = &\left((h_{\ell,i}(t))^\gamma + \sum_{\ell'=1}^{L} \frac{N_{\ell'}}{N} \sum_{s=1}^{\infty} \frac{N_{\ell,\ell',s}}{N_{\ell'}} s \int_0^{\bar{z}} z \, dm_{\ell,\ell',s}^{N_{\ell,\ell' s},h} \right) dt \\ &+ \sigma dB_{\ell,i}(t) + d\varphi_{\ell,i}^h(t) , \\ \mathcal{L}\left(Z_{\ell,i}^h(0)\right) = &\, m_\ell^0 , \end{aligned}$$

Assuming $m_\ell^{N_\ell,h}$ has a limit, m_ℓ^h, as $N_\ell \to \infty$ then, using the assumption that the distribution of $s_{i.j}^{\ell,\ell'}$ depends on ℓ and ℓ' only, it can be seen heuristically that $m_{\ell,\ell',s}^{N_{\ell,\ell',s},h} \to m_\ell^h$ as $N_{\ell\to\infty}$ as well. Therefore, in the limiting model, a representative firm in sector ℓ evolves according to the SDE

$$\begin{aligned} dZ_\ell^{h,m^h}(t) = &\left((h_\ell(t))^\gamma + \sum_{\ell'=1}^{L} A_{\ell'} p(\ell,\ell') \int_0^{\bar{z}} z' \; dm_{\ell'}^h(z',t) \right) dt \\ &+ \sigma dB_\ell(t) + d\varphi_\ell^{h,m^h}(t) \end{aligned} \tag{8a}$$

$$\mathcal{L}\left(Z_\ell^{h,m}(0)\right) = m_\ell^0 , \tag{8b}$$

where, $p(\ell,\ell') = \sum_{s=0}^{\infty} s \, p(\ell,\ell',s)$, because $\frac{N_{\ell,\ell',s}}{N_{\ell'}} \to p(\ell,\ell',s)$ as $N_\ell \to \infty$ by the law of large numbers. The corresponding profit functional is

$$J_\ell(h;m) = \mathbb{E}\left[\int_0^\infty \left(B \left(Z_\ell^{h,m^h}(t) \right)^{\frac{\alpha}{1-\alpha}} - w h_\ell(t) \right) e^{-\rho t} \; dt \right] . \tag{9}$$

If all firms act in the same way as the representative firm, then the distribution of firms with respect to productivity level is given by a system of L Fokker–Planck equations

$$\partial_t m_\ell^h + \partial_z \left[\left((h_\ell)^\gamma + \sum_{\ell'=1}^{L} A_{\ell'} p\left(\ell, \ell'\right) \int_0^{\bar{z}} z' \, dm_{\ell'}^h\left(z', t\right) \right) m_\ell^h \right] - \frac{\sigma^2}{2} \partial_{zz}^2 m_\ell^h = 0\,, \tag{10a}$$

$$- \left((h_\ell)^\gamma + \sum_{\ell'=1}^{L} A_{\ell'} p\left(\ell, \ell'\right) \int_0^{\bar{z}} z' \, dm_{\ell'}^h\left(z', t\right) \right) m_\ell^h + \frac{\sigma^2}{2} \partial_z m_\ell^h \Bigg|_{z=0,\bar{z}} = 0\,, \tag{10b}$$

$$m_\ell^h\left(z, 0\right) = m_\ell^0\left(z\right)\,, \tag{10c}$$

where the boundary condition comes from the reflected stochastic process.(41; 46)

3 The MFG model

3.1 Problem formulation

The MFG problem is related to the search for Nash equilibria in the optimisation of the profit functional (9) while agents evolve according to the dynamics (8).

Definition 1 *The MFG problem is to find a pair* (h^*, m^*)*, where* $h^* = (h_\ell^*)_{\ell=1}^L$ *is a sequence of controls and* $m^* = (m_\ell^*)_{\ell=1}^L$ *is a sequence of probability distributions on* $[0, \bar{z}]$*, such that for any other sequence of controls* h *and every* ℓ

$$J_\ell\left(h_\ell^*, m^*\right) \geq J_\ell\left(h_\ell, m^*\right) \tag{11a}$$

$$\text{and } m_\ell^* = \mathcal{L}\left(Z_\ell^{h^*, m^*}\right)\,. \tag{11b}$$

Such a distribution is called an MFG equilibrium.

To find an MFG equilibrium we first describe the Hamilton–Jacobi–Bellman (HJB) PDE related to the optimisation part of the problem (11a). Then we couple the HJB PDE to the Fokker–Planck PDE (10) to solve the consistency part (11b). We start by defining L Hamiltonians $H_\ell : (0, \bar{z}) \times \left(H^1\left(0, \bar{z}\right)\right)^L \times \mathbb{R} \to \mathbb{R}$, for $\ell = 1, \dots L$ as

$$\begin{aligned} H_\ell\left(z, m, \lambda\right) = & \sup_{h \geq 0} \left(h^\gamma + \sum_{\ell'=1}^{L} A_{\ell'} p\left(\ell, \ell'\right) \int_0^{\bar{z}} z' m_{\ell'}\left(z'\right) \, dz' \right) \lambda + B z^{\frac{\alpha}{1-\alpha}} - wh \\ = & \left(1 - \gamma\right) \left(\frac{\gamma}{w}\right)^{\frac{\gamma}{1-\gamma}} \max\left(0, \lambda\right)^{\frac{1}{1-\gamma}} \\ & + \lambda \sum_{\ell'=1}^{L} A_{\ell'} p\left(\ell, \ell'\right) \int_0^{\bar{z}} z' m_{\ell'}\left(z'\right) \, dz' + B z^{\frac{\alpha}{1-\alpha}}\,, \end{aligned} \tag{12}$$

where $z \in (0, \bar{z})$ is productivity, $m = (m_\ell)_{\ell=1}^{L}$ is a distribution of firms in each sector and λ is an adjoint variable. The optimal control is given by $h_\ell^* = \left(\frac{\gamma}{w} \max(0, \lambda)\right)^{\frac{1}{1-\gamma}}$, for $\ell = 1, \ldots, L$. Then we define the running profit $V_\ell(z, t)$, for $\ell = 1, \ldots, L$, by

$$V_\ell(z,t) = \sup_{h_\ell} \mathbb{E}\Bigg[\int_t^\infty \Big(B\left(Z_\ell^h(s)\right)^{\frac{\alpha}{1-\alpha}} - w h_\ell\left(Z_\ell^h(s)\right)\Big) e^{-\rho(s-t)}\, ds \,\Bigg|\, Z_\ell^h(t) = z\Bigg], \tag{13}$$

where $Z_\ell^h(s)$ follows (8). If we let the equilibrium distribution be given by m_ℓ (for $\ell = 1, \ldots, L$), then the MFG PDE system is stationary and given by

$$\begin{cases} V_\ell \in H^1(0, \bar{z}) & \text{(14a)} \\ m_\ell \in H^1(0, \bar{z}) & \text{(14b)} \\ -\dfrac{\sigma^2}{2} V_\ell'' + \rho V_\ell - H_\ell(z, m, V_\ell') = 0 & \text{(14c)} \\ -\dfrac{\sigma^2}{2} m_\ell'' + \left(\partial_\lambda H_\ell(z, m, V_\ell')\, m_\ell\right)' = 0 & \text{(14d)} \\ V_\ell'|_{z=0,\bar{z}} = 0 & \text{(14e)} \\ -\dfrac{\sigma^2}{2} m_\ell' + \partial_\lambda H_\ell(z, m, 0)\, m_\ell \Big|_{z=0,\bar{z}} = 0 & \text{(14f)} \\ \displaystyle\int_0^{\bar{z}} m_\ell(z)\, dz = 1\,. & \text{(14g)} \end{cases}$$

It can be shown, using either the dynamic programming principle(44) or the stochastic maximum principle,(14) that $V_\ell(z)$, as defined by (13), satisfies the HJB equation (14c), (14e), provided smooth solutions to the HJB equation exist. The Fokker–Planck system (14d), (14f), (14g) comes from the distribution in the previous section (10) and the consistency condition (11b). The verification theorem, Corollary 2, proves that existence(and uniqueness) of MFG equilibria, as defined in Definition 1 is a corollary to existence (and uniqueness) of smooth solutions to (14).

3.2 Existence and uniqueness of solutions to the MFG

Definition 2 *A solution to the innovation MFG model (14) is defined to be a tuple $(m, V) = (m_1, \ldots, m_L, V_1, \ldots, V_L)$ such that $m_\ell : (0, \bar{z}) \to (0, \infty)$, $V_\ell : (0, \bar{z}) \to \mathbb{R}$ satisfy (14) in the weak sense for each $\ell = 1, \ldots, L$.*

Theorem 1 *There exists a solution $(m, V) \in \left[C^2(0, \bar{z}) \cap C^1[0, \bar{z}]\right]^{2L}$ to (14). Furthermore,* *if* $\sum_{\ell'=1}^{L} A_{\ell'} p(\ell,\ell')$ *is small enough for every* $\ell = 1,\dots L$, *then the solution is unique.*

Proof. As noted in the introduction, this proof is based on an earlier proof of existence and uniqueness.(9) The proof presented here has some technical differences. Hence it is reproduced in full. However, it follows a similar framework and so we do not claim the proof to be new. First, for $k \in [0,\infty)$ we introduce an auxiliary system of PDEs defined by

$$\begin{cases} V^k \in H^1(0,\bar{z}) & \text{(15a)} \\ -\dfrac{\sigma^2}{2}\left(V^k\right)'' + \rho V^k - H^k\left(z, \left(V^k\right)'\right) = 0 & \text{(15b)} \\ \left(V^k\right)'\Big|_{z=0,\bar{z}} = 0\,, & \text{(15c)} \end{cases}$$

and

$$\begin{cases} m^k \in H^1(0,\bar{z}) & \text{(16a)} \\ -\dfrac{\sigma^2}{2}\left(m^k\right)'' + \left(\left[\left(\dfrac{\gamma}{w}\max\left(0,\left(V^k\right)'\right)\right)^{\frac{\gamma}{1-\gamma}} + k\right] m^k\right)' = 0 & \text{(16b)} \\ -\dfrac{\sigma^2}{2}\left(m^k\right)' + k m^k\Big|_{z=0,\bar{z}} = 0 & \text{(16c)} \\ \displaystyle\int_0^{\bar{z}} m^k(z)\, dz = 1\,, & \text{(16d)} \end{cases}$$

where

$$H^k(z,\lambda) = (1-\gamma)\left(\frac{\gamma}{w}\right)^{\frac{\gamma}{1-\gamma}} (\max(0,\lambda))^{\frac{1}{1-\gamma}} + k\lambda + B z^{\frac{\alpha}{1-\alpha}}.$$

We use a modified version of upper and lower solutions(38) to prove existence and uniqueness of a weak solution V^k to (15) for any $k \in [0,\infty)$, and use elliptic regularity theory to show $V^k \in C^2(0,\bar{z}) \cap C^1[0,\bar{z}]$. Next we define $m^k = \frac{1}{\left\|\bar{m}^k\right\|_1}\bar{m}^k$, where

$$\bar{m}^k = e^{\frac{2}{\sigma^2}\left(kz + \int_0^z \left(\frac{\gamma}{w}\max\left(0,\left(V^k\right)'\right)\right)^{\frac{\gamma}{1-\gamma}} dy\right)},$$

and $\left\|\bar{m}^k\right\|_1 = \int_0^{\bar{z}} \bar{m}^k dz$, for $k \in [0,\infty)$. We prove that $m^k \in C^2(0,\bar{z}) \cap C^1[0,\bar{z}]$ and that m^k is the unique solution of (16). Finally, we define a map

$$\Phi : \begin{cases} [0,\infty)^L & \to & [0,\infty)^L\,, \\ (k_1,\dots,k_L) & \mapsto & \left(\displaystyle\sum_{\ell'=1}^{L} A_{\ell'} p(\ell,\ell') \int_0^{\bar{z}} z m^{k_{\ell'}}(z)\, dz\right)_{\ell=1}^{L} \end{cases},$$

and using the Brouwer fixed point theorem we prove there exists $\bar{k} \in [0,\infty)^L$ such that

$\Phi\left(\bar{k}\right) = \bar{k}$. We use the contraction mapping theorem to prove uniqueness under certain smallness assumptions for the data. Then it follows, by replacing $\bar{k}_\ell$ with $\Phi\left(\bar{k}_\ell\right)$ in (15) and (16), that $\left(m^{\bar{k}}, V^{\bar{k}}\right) = \left(m^{\bar{k}_1}, \ldots, m^{\bar{k}_L}, V^{\bar{k}_1}, \ldots, V^{\bar{k}_L}\right)$ is a (unique) solution to (14) with the required regularity. ■

Corollary 2 *Let (m, V) be a solution to (14). Then h^* defined coordinate–wise by*

$$h^*_\ell = \left(\frac{\gamma}{w} \max\left(0, V'_\ell\right)\right) ,$$

is an equilibrium control to the MFG problem defined in Definition 1, with $m^* = m$. Furthermore, if the solution to (14) is unique, then the MFG equilibrium is unique.

Proof. Let (m, V) be a solution to (14). Then, applying earlier results(21) (see in particular Theorem 9.1 of this reference) with adjustments to take into account the reflection at the boundary, we get

$$V_\ell(z) \leq J_\ell\left(h_\ell, m^*\right) ,$$

for every stationary Markovian feedback control. Note that the proof relies on the application of Dynkin's formula, which can be readily extended to the case of reflected Markov processes.(46) Furthermore, as shown in earlier results(21) (see in particular Theorem 9.1 of this reference), we have

$$h^*_\ell = \left(\frac{\gamma}{w} \max\left(0, V'_\ell\right)\right)^{\frac{\gamma}{1-\gamma}} ,$$

is an optimal control of $J_\ell\left(h_\ell, m^*\right)$. Finally, to show (h^*, m^*) is an MFG equilibrium, it remains to show that an agent evolving according to (8) with a control h^*_ℓ has a distribution equal to m^*_ℓ. However, this can immediately be seen by comparing the Fokker–Planck equation governing the dynamics (8), given by (10), with the equation describing the distribution m^*, given by (14d). Note that the fact that m^* is stationary, and the right hand side of (10) equals the left hand side of (14d), implies the dynamics of the agent are stationary in law, and hence $m^* = m$.

Now, assume the solution to (14) is unique, and take any MFG equilibrium (h^*, m^*). Then, each h^*_ℓ is an optimal control for the following problem

$$\begin{aligned} &\text{Maximise} && \mathbb{E}\left[\int_0^\infty \left(B\left(Z^{h_\ell}_\ell(t)\right)^{\frac{\alpha}{1-\alpha}} - w h_\ell(t)\right) e^{-\rho t}\, dt\right] , \\ &\text{Subject to} && dZ^{h_\ell}(t) = \left(\left(h_\ell(t)\right)^\gamma + k_\ell\right) dt + \sigma dB_\ell(t) + d\varphi^{h_\ell}_\ell(t) , \\ &&& \mathcal{L}\left(Z^{h_\ell}_\ell(0)\right) = m^0_\ell , \end{aligned}$$

where $k_\ell = \sum_{\ell'=1}^{L} A_{\ell'} p\left(\ell, \ell'\right) \int_0^{\bar{z}} z m^*_{\ell'}(z)\, dz$. However, we can then define the HJB equation related to that optimal control problem as (15b) with $k = k_\ell$, which has a unique C^2 solution. Then, by Dynkin's formula applied to $e^{-\rho t} V^{k_\ell}\left(Z_\ell^{h_\ell}(t)\right)$, and following earlier proofs,(21; 47)

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^\infty \left(B\left(Z_\ell^{h_\ell}(t)\right)^{\frac{\alpha}{1-\alpha}} - w h_\ell(t)\right) e^{-\rho t}\,dt \,\middle|\, Z_\ell^{h_\ell}(0) = z\right] \\
&= \lim_{t\to\infty} \mathbb{E}\left[e^{-\rho t} V^{k_\ell}\left(Z_\ell^{h_\ell}(t)\right) \middle|\, Z_\ell^{h_\ell}(0) = z\right] \\
&\quad + \mathbb{E}\left[\int_0^\infty \left(B\left(Z_\ell^{h_\ell}(t)\right)^{\frac{\alpha}{1-\alpha}} - w h_\ell(t)\right) e^{-\rho t}\,dt \,\middle|\, Z_\ell^{h_\ell}(0) = z\right] \\
&= V^{k_\ell}(z) \\
&\quad + \mathbb{E}\left[\int_0^\infty \left(\left((h_\ell(t))^\gamma + k_\ell\right)\left(V^{k_\ell}\right)'\left(Z_\ell^{h_\ell}(t)\right)\right) e^{-\rho t}\,dt \,\middle|\, Z_\ell^{h_\ell}(0) = z\right] \\
&\quad + \mathbb{E}\Bigg[\int_0^\infty \left(\frac{\sigma^2}{2}\left(V^{k_\ell}\right)''\left(Z_\ell^{h_\ell}(t)\right) - \rho V^{k_\ell}\left(Z_\ell^{h_\ell}(t)\right) - w h_\ell(t)\right) \\
&\qquad\qquad\qquad\qquad e^{-\rho t}\,dt \,\Bigg|\, Z_\ell^{h_\ell}(0) = z\Bigg] \\
&= V^{k_\ell}(z) \\
&\quad + \mathbb{E}\left[\int_0^\infty \left((h_\ell(t))^\gamma \left(V^{k_\ell}\right)'\left(Z_\ell^{h_\ell}(t)\right) - w h_\ell(t)\right) e^{-\rho t}\,dt \,\middle|\, Z_\ell^{h_\ell}(0) = z\right] \\
&\quad - \mathbb{E}\Bigg[\int_0^\infty \left((1-\gamma)\left(\frac{\gamma}{w}\right)^{\frac{\gamma}{1-\gamma}}\left(\max\left(0, \left(V^{k_\ell}\right)'\left(Z_\ell^{h_\ell}(t)\right)\right)\right)^{\frac{1}{1-\gamma}}\right) \\
&\qquad\qquad\qquad\qquad e^{-\rho t}\,dt \,\Bigg|\, Z_\ell^{h_\ell}(0) = z\Bigg] \\
&\leq V^{k_\ell}(z)\ ,
\end{aligned}
$$

with equality if and only if $h_\ell(t) = \left(\frac{\gamma}{w}\left(V^{k_\ell}\right)'\left(Z_\ell^{h_\ell}(t)\right)\right)^{\frac{\gamma}{1-\gamma}}$, so there is a unique maximising control, given in feedback form as $h_\ell^*(z) = \left(\frac{\gamma}{w}\left(V^{k_\ell}\right)'(z)\right)^{\frac{\gamma}{1-\gamma}}$. Then, the dynamics of an agent using the control h_ℓ^* are governed by

$$
\begin{aligned}
\partial_t m_\ell^{h_\ell^*} + \partial_z\left(\partial_\lambda H^{k_\ell}\left(z, \left(V^{k_\ell}\right)'\right) m_\ell^{h_\ell^*}\right) - \frac{\sigma^2}{2}\partial_{zz}^2 m_\ell^{h_\ell^*} &= 0\,, \\
-\frac{\sigma^2}{2}\partial_z m_\ell^{h_\ell^*} + \partial_\lambda H^{k_\ell}\left(z, \left(V^{k_\ell}\right)'\right) m_\ell^{h_\ell^*}\bigg|_{z=0,\bar{z}} &= 0\,.
\end{aligned}
$$

So, since $m_\ell^{h_\ell^*} = m_\ell^*$ for all $\ell = 1, \dots L$, then $k = \Phi(k)$ and $\partial_t m_\ell^{h_\ell^*} = 0$, therefore (m^*, V^*) must be the unique solution to (14), and finally $h_\ell^* = \left(\frac{\gamma}{w}(V_\ell)'(z)\right)^{\frac{\gamma}{1-\gamma}}$. As a result, any generic equilibrium (h^*, m^*) must equal the equilibrium found in the existence part of the proof, and hence the equilibrium is unique. ■

3.2.1 Solutions to the auxiliary HJB PDE

Theorem 3 *There exists a unique solution* $V^k \in C^{2,\tau}[0,\bar{z}]$ *to the auxiliary HJB PDE* (15) *for any* $k \in [0,\infty)$ *and some* $\tau \in (0,1)$*, where* $C^{2,\tau}[0,\bar{z}]$ *is the set of* C^2 *functions on* $[0,\bar{z}]$ *whose second derivative is Hölder continuous with exponent* τ*. Furthermore,* $0 \le V^k \le \frac{B}{\rho}\bar{z}^{\frac{\alpha}{1-\alpha}}$

Proof. The existence part of the proof uses the theory of upper and lower solutions(34) (see specifically Theorem 4.3 of this reference) along similar lines as earlier results(9) (see proof of Proposition 3.12 in this reference). This shows that a solution $V^k \in W^{1,p}(0,\bar{z})$ to the auxiliary HJB PDE exists, for some $p \ge 1$, provided the following hold true:

1. There exist constants $\underline{V} \le \bar{V}$ such that $\rho\underline{V} - B\bar{z}^{\frac{\alpha}{1-\alpha}} \le 0 \le \rho\bar{V} - B\bar{z}^{\frac{\alpha}{1-\alpha}}$, for every $z \in [0,\bar{z}]$.
2. There exist constants $a_k \in \mathbb{R}$ and $b_k > 0$ such that

$$\left|\rho u - (1-\gamma)\left(\frac{\gamma}{w}\right)^{\frac{\gamma}{1-\gamma}}(\max(0,\lambda))^{\frac{1}{1-\gamma}} - k\lambda - B\bar{z}^{\frac{\alpha}{1-\alpha}}\right| \le a_k + b_k|\lambda|^p ,$$

   for every $z \in (0,\bar{z})$, $u \in \left[\underline{V},\bar{V}\right]$ and every $\lambda \in \mathbb{R}$.

If these two properties hold, then $\underline{V} \le V^k \le \bar{V}$. The first assertion is true by taking $\underline{V} = 0$ and $\bar{V} = \frac{B}{\rho}\bar{z}^{\frac{\alpha}{1-\alpha}}$, which also gives the required bounds for V^k. The second assertion is true with $b_k = k + (1-\gamma)\left(\frac{\gamma}{w}\right)^{\frac{\gamma}{1-\gamma}}$, $a_k = B\bar{z}^{\frac{\alpha}{1-\alpha}} + b_k$, and $p = \frac{2}{1-\gamma}$, as then

$$\begin{aligned}&\left|\rho u - (1-\gamma)\left(\frac{\gamma}{w}\right)^{\frac{\gamma}{1-\gamma}}(\max(0,\lambda))^{\frac{1}{1-\gamma}} - k\lambda - B\bar{z}^{\frac{\alpha}{1-\alpha}}\right| \\ &\le \rho|u| + \left(k + (1-\gamma)\left(\frac{\gamma}{w}\right)^{\frac{\gamma}{1-\gamma}}\right)\max\left(1,|\lambda|^{\frac{1}{1-\gamma}}\right) + B\bar{z}^{\frac{\alpha}{1-\alpha}} \le a_k + b_k|\lambda|^p .\end{aligned}$$

Now, since $(0,\bar{z})$ is bounded and $p > 2$, $V^k \in H^1(0,\bar{z})$. To show $V^k \in C^{2,\tau}[0,\bar{z}]$, take any solution V^k to (15) and define

$$\begin{aligned}f = \frac{2}{\sigma^2}\Bigg(&\left(\frac{\sigma^2}{2} - \rho\right)V^k + k\left(V^k\right)' \\ &+ (1-\gamma)\left(\frac{\gamma}{w}\right)^{\frac{\gamma}{1-\gamma}}\left(\max\left(0,\left(V^k\right)'\right)\right)^{\frac{1}{1-\gamma}} + B\bar{z}^{\frac{\alpha}{1-\alpha}}\Bigg).\end{aligned}$$

Then V^k is a solution of $-u'' + u = f$, where $f \in L^2(0, \bar{z})$. So, from elliptic regularity(43) (see Proposition 7.2. p.404 of this reference), $V^k \in H^2(0, \bar{z})$. Therefore $\left(V^k\right)' \in H^1(0, \bar{z})$, and so $f \in H^1(0, \bar{z})$ because $\alpha > 0$. So, from elliptic regularity(43) (see Proposition 7.4. p.407 of this reference), $V^k \in H^3(0, \bar{z})$. Then, by the Sobolev inequality(19) (see Theorem 6 p.270 of this reference) $V^k \in C^{2,\tau}[0, \bar{z}]$.

To prove uniqueness we use the strong maximum principle and Hopf's lemma(19) (see Section 6.4.2. pp. 330–333 of this reference). Suppose, for some $k \in [0, \infty)$, there are two solutions $V_1, V_2 \in C^2(0, \bar{z}) \cap C^1[0, \bar{z}]$ to (15) and $V_1 \neq V_2$. If we define $u = V_1 - V_2$, then u must attain its maximum at some point $z^* \in [0, \bar{z}]$. Suppose at this point $u > 0$. Note that if this were not the case, we could consider the minimum, as either its maximum or its minimum must be non–zero. The argument for the minimum is the same as the one for the maximum, so it is omitted. First suppose $z^* \in (0, \bar{z})$. Since this is the maximal point, $u'(z^*) = 0$, so $V_1'(z^*) = V_2'(z^*)$. Hence, there exists an open, connected and bounded region U such that $U \subset (0, \bar{z})$, $z^* \in U$ and

$$-\frac{\sigma^2}{2}u'' = -\rho u + ku' + (1-\gamma)\left(\frac{\gamma}{w}\right)^{\frac{\gamma}{1-\gamma}}\left[\max\left(0, V_1'\right)^{\frac{1}{1-\gamma}} - \max\left(0, V_2'\right)^{\frac{1}{1-\gamma}}\right] \leq 0\,,$$

for every $z \in U$. So, by the strong maximum principle, u is constant in U. In particular, using (15b), $u(z^*) = 0$. But this is a contradiction. The only other case is $z^* \in \{0, \bar{z}\}$ and $u(z) < u(z^*)$ for every $z \in (0, \bar{z})$. Then, $\frac{\partial u}{\partial \nu}\big|_{z^*} > 0$ by Hopf's Lemma, but by (15c), $\frac{\partial u}{\partial \nu} = \frac{\partial V_1}{\partial \nu} - \frac{\partial V_2}{\partial \nu} = 0$. This again leads to a contradiction. Therefore $V_1 = V_2$ and solutions to (15) are unique for every $k \in [0, \infty)$. ■

Proposition 1 *Fix $k, k_1, k_2 \in [0, \infty)$. Then, the unique classical solution to the auxiliary HJB PDE (15), as found in Theorem 3, satisfies the following properties:*

1. *V^k is an increasing function on $[0, \bar{z}]$ i.e. $\left(V^k\right)' \geq 0$*
2. *$\left(V^k\right)' > 0$ for all $z \in (0, \bar{z})$*
3. *$\| \left(V^k\right)' \|_\infty = \sup_{z \in (0, \bar{z})} \left(V^k\right)'(z) \leq \left[\frac{B}{1-\gamma}\bar{z}^{\frac{\alpha}{1-\alpha}}\right]^{1-\gamma}\left(\frac{w}{\gamma}\right)^{\gamma}$*
4. *$\left\|V^{k_1} - V^{k_2}\right\|_\infty \leq \frac{1}{\rho}\left[\frac{B}{1-\gamma}\bar{z}^{\frac{\alpha}{1-\alpha}}\right]^{1-\gamma}\left(\frac{w}{\gamma}\right)^{\gamma}|k_1 - k_2|$*
5. *V^k is strictly increasing with respect to k*
6. *$\left\|\left(V^{k_1}\right)' - \left(V^{k_2}\right)'\right\|_\infty \leq \frac{4\bar{z}}{\sigma^2}\left[\frac{B}{1-\gamma}\bar{z}^{\frac{\alpha}{1-\alpha}}\right]^{1-\gamma}\left(\frac{w}{\gamma}\right)^{\gamma}|k_1 - k_2|$*
7. *$\left(V^k\right)''(0) > 0 > \left(V^k\right)''(\bar{z})$.*

Proof. Property (1): Suppose, for a contradiction, there exists $z \in [0, \bar{z}]$ such that $\left(V^k\right)'(z) < 0$. First, by the boundary condition (15c), $z \in (0, \bar{z})$. So, by the boundary conditions and continuity of $\left(V^k\right)'$, there exists $z_0, z_1 \in [0, \bar{z}]$ with $z_0 < z_1$, $\left(V^k\right)'(z_0) = \left(V^k\right)'(z_1) = 0$ and $\left(V^k\right)'(z) \leq 0$ for all $z \in (z_0, z_1)$. Suppose that $z_0, z_1 \in (0, \bar{z})$. Then $\left(V^k\right)''(z_0) \leq 0 \leq \left(V^k\right)''(z_1)$ by construction of z_0, z_1 and differentiability of $\left(V^k\right)'$. Furthermore, $V^k(z_0) > V^k(z_1)$ because $\left(V^k\right)' < 0$ in (z_0, z_1). So, using (15b)

$$\begin{aligned} 0 = &-\frac{\sigma^2}{2}\left(\left(V^k\right)''(z_1) - \left(V^k\right)''(z_0)\right) \\ &+ \rho\left(V^k(z_1) - V^k(z_0)\right) - B\left(z_1^{\frac{\alpha}{1-\alpha}} - z_0^{\frac{\alpha}{1-\alpha}}\right) < 0\,. \end{aligned}$$

This is a contradiction, so $z_0 = 0$ or $z_1 = \bar{z}$. Assume $z_0 = 0$, we will again prove a contradiction (the other two cases of $z_1 = \bar{z}$ and both $z_0 = 0, z_1 = \bar{z}$ follow along similar arguments so their proofs are omitted). Since $\left(V^k\right)'(0) = \left(V^k\right)'(z_1) = 0$ and $\left(V^k\right)'(z) < 0$ for all $z \in (0, z_1)$ then, by continuity of $\left(V^k\right)''$, we can find $\epsilon_1, \delta_1 \in \left(0, \frac{z_1}{2}\right)$ such that $\left(V^k\right)''(z) \leq 0$ for all $z \in (0, \epsilon_1]$ and $\left(V^k\right)''(z) \geq 0$ for all $z \in [z_1 - \delta_1, z_1)$. Furthermore, V^k is strictly decreasing on (z_0, z_1). So, using these two facts and continuity of $\left(V^k\right)'$ there exists $\delta \in (0, \delta_1]$ and $\epsilon \in (0, \epsilon_1]$ such that

1. $\left(V^k\right)'(\epsilon) = \left(V^k\right)'(z_1 - \delta) = \min\left(\left(V^k\right)'(\epsilon_1), \left(V^k\right)'(z_1 - \delta_1)\right)$
2. $V^k(\epsilon) > V^k(z_1 - \delta)$
3. $\left(V^k\right)''(\epsilon) \leq 0 \leq \left(V^k\right)''(z_1 - \delta)$.

Then,

$$\begin{aligned} 0 > &-\frac{\sigma^2}{2}\left(\left(V^k\right)''(z_1 - \delta) - \left(V^k\right)''(\epsilon)\right) \\ &+ \rho\left(V^k(z_1 - \delta) - V^k(\epsilon)\right) - B\left((z_1 - \delta)^{\frac{\alpha}{1-\alpha}} - \epsilon^{\frac{\alpha}{1-\alpha}}\right)\,, \end{aligned}$$

which contradicts the fact that V^k is a solution to (15). Therefore, $\left(V^k\right)' \geq 0$ in $[0, \bar{z}]$.

Property (2): From Property (1), we know $\left(V^k\right)' \geq 0$. Now suppose, for a contradiction, there exists $z^* \in (0, \bar{z})$ such that $\left(V^k\right)'(z^*) = 0$. Then $\left(V^k\right)''(z^*) = 0$, since it is a minimum of $\left(V^k\right)'$. So, by (15b), $V^k(z^*) = \frac{B}{\rho} z^{\frac{\alpha}{1-\alpha}}$ and, since $\left(V^k\right)'(z^*) < \frac{d}{dz}\left(\frac{B}{\rho} z^{\frac{\alpha}{1-\alpha}}\right)$, there exists $z_0, z_1 \in (0, \bar{z})$ with $z_0 < z^* < z_1$ such that

1. $\left(V^k\right)'(z_0) = \left(V^k\right)'(z_1)$
2. $V^k(z_0) > \frac{B}{\rho} z_0^{\frac{\alpha}{1-\alpha}}$ and $V^k(z_1) < \frac{B}{\rho} z_1^{\frac{\alpha}{1-\alpha}}$
3. $\left(V^k\right)''(z_0) \leq 0 \leq \left(V^k\right)''(z_1)$.

Then,

$$-\frac{\sigma^2}{2}\left(\left(V^k\right)''(z_1)-\left(V^k\right)''(z_0)\right)+\rho\left(V^k(z_1)-V^k(z_0)\right)-B\left(z_1^{\frac{\alpha}{1-\alpha}}-z_0^{\frac{\alpha}{1-\alpha}}\right)<0\,,$$

which is a contradiction of (15b). Therefore, $\left(V^k\right)'(z)>0$ for all $z\in(0,\bar{z})$.

Property (3): Since $\left(V^k\right)'\geq 0$, $\left(V^k\right)'(0)=\left(V^k\right)'(\bar{z})=0$ and $\left(V^k\right)'$ is continuous on $[0,\bar{z}]$, then $\left(V^k\right)'$ must have a maximum that it attains at some point $z^*\in(0,\bar{z})$. Furthermore, since $\left(V^k\right)'$ is continuously differentiable in $(0,\bar{z})$, then $\left(V^k\right)''(z^*)=0$. So, using the bound on V^k found in Theorem 3

$$\begin{aligned}0\leq\left(V^k\right)'(z)&\leq\left(V^k\right)'(z^*)\\&=\left[\frac{w^{\frac{\gamma}{1-\gamma}}}{(1-\gamma)\,\gamma^{\frac{\gamma}{1-\gamma}}}\left(\rho V^k(z^*)-k\left(V^k\right)'(z^*)-B\,(z^*)^{\frac{\alpha}{1-\alpha}}\right)\right]^{1-\gamma}\\&\leq\left[\frac{B}{(1-\gamma)}\bar{z}^{\frac{\alpha}{1-\alpha}}\right]^{1-\gamma}\left(\frac{w}{\gamma}\right)^{\gamma}.\end{aligned}$$

Property (4): Take $k_1,k_2\in[0,\infty)$ with $k_1<k_2$. First we show $V^{k_2}-V^{k_1}\geq 0$, then we show $V^{k_2}-V^{k_1}\leq\frac{\left\|\left(V^{k_1}\right)'\right\|_\infty}{\rho}(k_2-k_1)$, and we can conclude using Property (3). Let $u_1=V^{k_2}-V^{k_1}$ and assume, for a contradiction, there exists $z\in[0,\bar{z}]$ such that $u_1(z)<0$. Then, u_1 attains a minimum at $z^*\in[0,\bar{z}]$ and $u_1(z^*)<0$. First suppose $z^*\in(0,\bar{z})$, then $u_1'(z^*)=0$ and from (15b)

$$-\frac{\sigma^2}{2}u_1''(z^*)=-\rho u_1(z^*)+(k_2-k_1)\left(V^{k_1}\right)'(z^*)>0\,,$$

since $\left(V^{k_1}\right)'\geq 0$ and $u_1<0$. Then, by continuity of u_1'', there exists an open bounded, connected $U\subset[0,\bar{z}]$ such that $z^*\in U$ and $u_1''<0$ for all $z\in U$. So, by the strong maximum principle, u_1 is constant in U. In particular, $u_1''=0$, which contradicts $u_1''<0$ for all $z\in U$. So, $z^*\in\{0,\bar{z}\}$ and $u_1(z)<u_1(z^*)$ for all $z\in(0,\bar{z})$. However, from Hopf's lemma $u_1'(z^*)\neq 0$, which contradicts (15c). So, we conclude that $u_1\geq 0$. Now let $u_2=V^{k_2}-V^{k_1}-\epsilon$, with $\epsilon=\frac{\left\|\left(V^{k_1}\right)'\right\|_\infty}{\rho}(k_2-k_1)<\infty$. We assume, for a contradiction, there exists $z\in[0,\bar{z}]$ such that $u_2(z)>0$. Then u_2 attains a maximum at $z^*\in[0,\bar{z}]$ and $u_2(z^*)>0$. First suppose $z^*\in(0,\bar{z})$, then $u_2'(z^*)=0$ and from (15b)

$$-\frac{\sigma^2}{2}u_2''(z^*)=-\rho u_2(z^*)+(k_2-k_1)\left(V^{k_1}\right)'(z^*)-\rho\epsilon<0\,,$$

since $u_2>0$ and $\rho\epsilon\geq(k_1-k_2)\left(V^{k_1}\right)'(z^*)$. Then, by continuity of u_2'', there exists an open bounded, connected $U\subset(0,\bar{z})$ such that $z^*\in U$ and $u_2''>0$ for all $z\in U$. So, by the strong maximum principle, u_2 is constant in U. In particular, $u_2'' = 0$, which contradicts $u_2'' > 0$ for all $z \in U$. So, $z^* \in \{0, \bar{z}\}$ and $u_2(z) > u_2(z^*)$ for all $z \in (0, \bar{z})$. However, from Hopf's lemma $u_2'(z^*) \neq 0$, which contradicts (15c). So, we can conclude that $u_2 \leq 0$.

Property (5): The proof of Property (4) shows V^k is increasing with respect to k. Now suppose, for a contradiction, there exists $z^* \in [0, \bar{z}]$ such that $k_1 < k_2$ but $V^{k_1}(z^*) = V^{k_2}(z^*)$. First, assume $z^* \in (0, \bar{z})$ and define $u = V^{k_2} - V^{k_1}$. Then, $u(z^*) = 0$, $u'(z^*) = 0$ and $u''(z^*) \geq 0$, since z^* is a minimum of u. Furthermore, from Property (2), $\left(V^{k_1}\right)'(z^*) > 0$. Therefore, using (15b), we get the contradiction

$$0 = -\frac{\sigma^2}{2}u'' + (k_1 - k_2)\left(V^{k_1}\right)' < 0\,.$$

Hence, $z^* \in \{0, \bar{z}\}$, so $u'(z^*) = 0$,using (15c). But, $u'(z^*) \neq 0$ by Hopf's lemma, which is a contradiction. So, V^k is strictly increasing with respect to k.

Property (6): Fix $k_1, k_2 \in [0, \infty)$. Let $u = V^{k_1} - V^{k_2}$. Then, u satisfies

$$\begin{aligned}\frac{\sigma^2}{2}u'' =& \rho u - k_1 u' + (k_2 - k_1)\left(V^{k_2}\right)' \\ &- (1-\gamma)\left(\frac{\gamma}{w}\right)^{\frac{\gamma}{1-\gamma}}\left(\left(\left(V^{k_1}\right)'\right)^{\frac{1}{1-\gamma}} - \left(\left(V^{k_2}\right)'\right)^{\frac{1}{1-\gamma}}\right)\,.\end{aligned}$$

Suppose for $z \in (0, \bar{z})$, $u'(z) \geq 0$. Then, since $u'(0) = 0$, there exists $z_0 \in [0, z]$ such that $u'(y) \geq 0$ for all $y \in [z_0, z]$ and $u'(z_0) = 0$. Therefore

$$\begin{aligned}0 \leq u'(z) = \int_{z_0}^{z} u''(y)\ dy &\leq \frac{2\bar{z}}{\sigma^2}\left(\rho\left\|u\right\|_\infty + |k_2 - k_1|\left\|\left(V^{k_2}\right)'\right\|_\infty\right) \\ &\leq \frac{4\bar{z}}{\sigma^2}\left[\frac{B}{(1-\gamma)}\bar{z}^{\frac{\alpha}{1-\alpha}}\right]^{1-\gamma}\left(\frac{w}{\gamma}\right)^{\gamma}|k_2 - k_1|\,.\end{aligned}$$

We can similarly show that

$$u'(z) \geq -\frac{4\bar{z}}{\sigma^2}\left[\frac{B}{(1-\gamma)}\bar{z}^{\frac{\alpha}{1-\alpha}}\right]^{1-\gamma}\left(\frac{w}{\gamma}\right)^{\gamma}|k_2 - k_1|\,,$$

if $u'(z) \leq 0$. Hence, $\left(V^k\right)'$ is Lipschitz continuous with respect to k with the required constant.

Property (7): First, we will show $\left(V^k\right)''(0) > 0$, and then that $\left(V^k\right)''(\bar{z}) < 0$. Both steps use a similar method. Note that $\left(V^k\right)''(0) \geq 0$ and $\left(V^k\right)''(\bar{z}) \leq 0$, because $\left(V^k\right)'(0) = \left(V^k\right)'(\bar{z}) = 0$ and $\left(V^k\right)(z) > 0$ for all $z \in (0, \bar{z})$. So suppose, for a contradiction, that $\left(V^k\right)''(0) = 0$. Then, since $V^k \in C^{2,\tau}[0, \bar{z}]$, we can use continuity of V^k, $\left(V^k\right)'$, $\left(V^k\right)''$ and (15b), (15c) to show

$V^k(0) = 0$. We can also use continuity of $\left(V^k\right)''$ to show that for every $C > 0$ there exists $\epsilon_1 \in (0,1)$ such that $z \in (0, \epsilon_1) \implies \left(V^k\right)''(z) < \frac{C\sigma^2}{\rho}$. Therefore, for any $z \in (0, \epsilon_1)$

$$V^k(z) = \int_0^z \int_0^y \left(V^k\right)''(y')\ dy'\ dy - z\left(V^k\right)'(0) - V^k(0) < \frac{C\sigma^2}{2\rho}z^2\,.$$

Using (15b), and the fact that $\left(V^k\right)'(z) \geq 0$, we in fact find $\left(V^k\right)''(z) \leq \frac{2\rho}{\sigma^2}V(z) < Cz^2$, on $(0, \epsilon)$. Repeating this procedure as many times as necessary, we can show that for $C = \frac{B}{\rho}$ and $k > \frac{\alpha}{1-\alpha}$ there exists $\epsilon_2 \in (0,1)$ such that

$$V^k(z) < \frac{B}{\rho}z^k < \frac{B}{\rho}z^{\frac{\alpha}{\alpha-1}}\,, \quad \text{for every } z \in (0, \epsilon_2)\ .$$

However, since $\left(V^k\right)' > 0$ in $(0, \bar{z})$, there exists $\epsilon_3 > 0$ such that $\left(V^k\right)'$ increases on $(0, \epsilon_3)$. Therefore, $\left(V^k\right)'' \geq 0$ on $(0, \epsilon_2)$. So, taking $\epsilon = \min(\epsilon_2, \epsilon_3)$, and using (15b):

$$\frac{B}{\rho}z^{\frac{\alpha}{\alpha-1}} > V^k(z) > \frac{B}{\rho}z^{\frac{\alpha}{\alpha-1}}\,,$$

which is a contradiction. Hence, $\left(V^k\right)''(0) > 0$. Now let's turn to $\left(V^k\right)''(\bar{z})$. Suppose, for a contradiction, that $\left(V^k\right)''(\bar{z}) = 0$. Then, since $V^k \in C^{2,\tau}[0, \bar{z}]$, we can use continuity of $V^k, \left(V^k\right)', \left(V^k\right)''$ and (15b), (15c) to show $V^k(\bar{z}) = \frac{B}{\rho}\bar{z}^{\frac{\alpha}{1-\alpha}}$. We can also use continuity of $\left(V^k\right)''$ to show that for every $C > 0$ there exists $\epsilon_1 >\in (0,1)$ such that $z \in (\bar{z} - \epsilon_1, \bar{z}) \implies \left(V^k\right)''(z) > -C$. Therefore, for any $z \in (\bar{z} - \epsilon_1, \bar{z})$

$$\begin{aligned}\left(V^k\right)'(z) &< C(\bar{z} - z)\,,\\ V^k(z) &> B\bar{z}^{\frac{\alpha}{1-\alpha}} - \frac{C}{2}(\bar{z} - z)^2\,.\end{aligned}$$

Then, using (15b),

$$\begin{aligned}V''(z) > \frac{2}{\sigma^2}\Bigg(&\frac{B}{\rho}\left(\bar{z}^{\frac{\alpha}{1-\alpha}} - z^{\frac{\alpha}{1-\alpha}}\right) - \frac{\rho C}{2}(\bar{z} - z)^2\\ &\qquad - kC(\bar{z} - z)^2 - (1-\gamma)\left(\frac{\gamma}{w}\right)^{\frac{\gamma}{1-\gamma}}\left(\left(V^k\right)'\right)^{\frac{1}{-\gamma}}\Bigg)\\ > -\tilde{C}(\bar{z} - z)\ .&\end{aligned}$$

Furthermore, this is true for any $\tilde{C} > 0$ provided ϵ_1 is small enough. Repeating this argument as many times as necessary, we find that there exists $\epsilon \in (0,1)$ such that the following hold for any $z \in (\bar{z} - \epsilon, \bar{z})$:

$$\begin{aligned} \left(V^k\right)''(z) > -C\left(\bar{z} - z\right)^k > -C\left(\bar{z} - z\right)^{\frac{\alpha}{1-\alpha}}\ , \\ \left(V^k\right)' < C\left(\bar{z} - z\right)^{\frac{\alpha}{1-\alpha}}\ , \\ V^k > \frac{B}{\rho}\bar{z}^{\frac{\alpha}{1-\alpha}} - C\left(\bar{z} - z\right)^{\frac{\alpha}{1-\alpha}}\ , \\ C + \frac{C}{\rho}\left(\frac{\sigma^2}{2} + k + (1-\gamma)\left(\frac{\gamma}{w}\right)^{\frac{\gamma}{1-\gamma}}\right) \le \frac{B}{\rho}\,. \end{aligned}$$

Then, using (15b), we get

$$V^k > \frac{B}{\rho}\bar{z}^{\frac{\alpha}{1-\alpha}} - C\left(\bar{z} - z\right)^{\frac{\alpha}{1-\alpha}} \tag{17a}$$

$$V^k < \frac{C}{\rho}\left(\frac{\sigma^2}{2} + k + (1-\gamma)\left(\frac{\gamma}{w}\right)^{\frac{\gamma}{1-\gamma}}\right)\left(\bar{z} - z\right)^{\frac{\alpha}{1-\alpha}} + \frac{B}{\rho}z^{\frac{\alpha}{1-\alpha}}\,. \tag{17b}$$

Rearranging gives

$$\left(\bar{z}^{\frac{\alpha}{1-\alpha}} - z^{\frac{\alpha}{1-\alpha}}\right) < \left(\bar{z} - z\right)^{\frac{\alpha}{1-\alpha}}$$

If $\alpha = \frac{1}{2}$, this is immediately a contradiction. If $\alpha > \frac{1}{2}$, then $\left(\bar{z}^{\frac{\alpha}{1-\alpha}} - z^{\frac{\alpha}{1-\alpha}}\right) - \left(\bar{z} - z\right)^{\frac{\alpha}{1-\alpha}}$ is strictly decreasing for all $z \in \left(\frac{\bar{z}}{2}, \bar{z}\right)$, and therefore $\left(\bar{z}^{\frac{\alpha}{1-\alpha}} - z^{\frac{\alpha}{1-\alpha}}\right) - \left(\bar{z} - z\right)^{\frac{\alpha}{1-\alpha}} > 0$, which is a contradiction. Finally, if $\alpha < \frac{1}{2}$, we can take $k = 1$ and modify C appropriately in (17), and use concavity of $z^{\frac{\alpha}{1-\alpha}}$ to show $\frac{\alpha}{1-\alpha}\bar{z}^{\frac{\alpha}{1-\alpha}}\left(\bar{z} - z\right) < \frac{\alpha}{1-\alpha}\bar{z}^{\frac{\alpha}{1-\alpha}}\left(\bar{z} - z\right)$, which is a contradiction again. Therefore $\left(V^k\right)''(\bar{z}) < 0$. ■

3.2.2 The auxiliary Fokker–Planck equation

Definition 3 *Fix $k \in [0, \infty)$ and let $V^k \in C^2(0, \bar{z}) \cap C^1[0, \bar{z}]$ denote the unique solution to (15). Then, we define the function $m^k : (0, \bar{z}) \to (0, \infty)$ by*

$$\bar{m}^k = e^{\frac{2}{\sigma^2}\left(kz + \int_0^z \left(\frac{\gamma}{w}\left(V^k\right)'\right)^{\frac{\gamma}{1-\gamma}}\ dy\right)} \tag{18a}$$

$$\left\|\bar{m}^k\right\|_1 = \int_0^{\bar{z}} \bar{m}^k dz \tag{18b}$$

$$m^k = \frac{1}{\left\|\bar{m}^k\right\|_1}\bar{m}^k\,. \tag{18c}$$

Proposition 2 *For every $k \in [0, \infty)$, $m^k \in C^2(0, \bar{z}) \cap C^1[0, \bar{z}]$ where m^k is defined by (3).*

Proof. First, note that m^k is well defined because $\left(V^k\right)' \ge 0$ and $\left(V^k\right)'$ is uniformly bounded. Hence, there exists $C \in (1, \infty)$ such that $\bar{m}^k(z) \in [1, C]$ and $\left\|\bar{m}^k\right\|_1 \in [\bar{z}, C\bar{z}]$, so $m^k(z) \in$

$\left[\frac{1}{C\bar{z}}, \frac{C}{\bar{z}}\right]$. Furthermore, $m^k \in C\,[0,\bar{z}]$ because $V^k \in C^1\,[0,\bar{z}]$. Now, if $m^k \in C^2\,(0,\bar{z}) \cap C^1\,[0,\bar{z}]$, then its derivatives would be

$$\left(m^k\right)' = \frac{2}{\sigma^2}\left(k + \left(\frac{\gamma}{w}\left(V^k\right)'\right)^{\frac{\gamma}{1-\gamma}}\right) m^k \tag{19a}$$

$$\begin{aligned}\left(m^k\right)'' =& \frac{2}{\sigma^2}\left(k + \left(\frac{\gamma}{w}\left(V^k\right)'\right)^{\frac{\gamma}{1-\gamma}}\right)\left(m^k\right)' \\ &+ \frac{2\gamma^2}{\sigma^2 w\,(1-\gamma)}\left(\frac{\gamma}{w}\left(V^k\right)'\right)^{\frac{2\gamma-1}{1-\gamma}}\left(V^k\right)'' m^k\,.\end{aligned} \tag{19b}$$

But, since $V^k \in C^1\,[0,\bar{z}]$ and $m^k \in C\,[0,\bar{z}]$, then $\frac{2}{\sigma^2}\left(k + \left(\frac{\gamma}{w}\left(V^k\right)'\right)^{\frac{\gamma}{1-\gamma}}\right) m^k$ is well–defined and continuous for all $z \in [0,\bar{z}]$. Hence, $m^k \in C^1\,[0,\bar{z}]$. Then, $\left(m^k\right)'$, $\left(V^k\right)$ and $\left(V^k\right)''$ are continuous in $(0,\bar{z})$ and from Proposition 1 $\left(V^k\right)' > 0$ in $(0,\bar{z})$. Hence, $\left(m^k\right)''$ is well–defined in $(0,\bar{z})$, $\left(m^k\right)'' \in C\,(0,\bar{z})$ and $m^k \in C^2\,(0,\bar{z}) \cap C^1\,[0,\bar{z}]$. ■

Theorem 4 *There exists a unique solution $m^k \in C^2\,(0,\bar{z}) \cap C^1\,[0,\bar{z}]$ to the auxiliary Fokker–Planck PDE (16) for any $k \in [0,\infty)$.*

Proof. Take m^k defined in Definition 3. Then, $m^k \in C^2\,(0,\bar{z}) \cap C^1\,[0,\bar{z}]$ by Proposition 2. Furthermore, from (19), m^k satisfies (16b), (16c). Finally, by construction, m^k satisfies (16d). Therefore, a solution to the auxiliary Fokker–Planck equation (16) exists, it is given by m^k, and $m^k \in C^2\,(0,\bar{z}) \cap C^1\,[0,\bar{z}]$. To prove uniqueness we follow the same proof as in earlier works.(9) For brevity we only outline the argument here. First, with $\bar{m}^k$ defined as in (18), we can use regularity of $\bar{m}^k$ from Proposition 2 to show (16) is equivalent to

$$m^k, \frac{m^k}{\bar{m}^k} \in H^1\,(0,\bar{z}) \tag{20a}$$

$$\left(\bar{m}^k\left(\frac{m^k}{\bar{m}^k}\right)'\right)' = 0 \tag{20b}$$

$$\bar{m}^k\left(\frac{m^k}{\bar{m}^k}\right)'\Bigg|_{z=0,\bar{z}} = 0\,, \qquad \int_0^{\bar{z}} m^k\,dz = 1\,. \tag{20c}$$

Then, by multiplying (20b) by $\frac{m^k}{\bar{m}^k}$, integrating over $(0,\bar{z})$ and using integration by parts, the system (20) is equivalent to

$$m^k \in H^1(0, \bar{z}) \,, \tag{21a}$$

$$\text{there exists } Z > 0 \text{ such that } m^k = \frac{1}{Z}\bar{m}^k \,, \tag{21b}$$

$$\int_0^{\bar{z}} m^k \; dz = 1 \,. \tag{21c}$$

From the previous results in this section, we have shown there exists a unique solution to (21) given by m^k from Definition 3. Hence, existence and uniqueness of the auxiliary Fokker–Planck PDE follows from the equivalence between (16) and (21). ∎

3.2.3 The fixed point problem

Definition 4 *Fix $k = (k_\ell)_{\ell=1}^L \in [0, \infty)^L$. For $\ell = 1, \ldots, L$, let V^{k_ℓ} be the unique solution to the auxiliary HJB PDE (15) with constant k_ℓ, and let m^{k_ℓ} be the unique solution to the auxiliary Fokker–Planck PDE (16) with constant k_ℓ. Then we define the function $\Phi : [0, \infty)^L \to [0, \infty)^L$ by*

$$\Phi_\ell(k) = \sum_{\ell'=1}^{L} A_{\ell'} p(\ell, \ell') \int_0^{\bar{z}} z m^{k_{\ell'}}(z) \; dz \,, \quad \ell = 1, \ldots, L \,.$$

Proposition 3 *The function Φ defined in Definition 4 is bounded. Furthermore, defining P as the $L \times L$ matrix with entries $P_{\ell,\ell'} = p(\ell, \ell')$ and A as the column vector $(A_1, \ldots, A_L)^T$, then*

$$0 \leq \|\Phi(k)\|_1 \leq \bar{z} \|PA\|_1 \,,$$

where the 1–norm $\|\cdot\|_1$ is defined as $\|x\|_1 = \sum_{\ell=1}^L |x_\ell|$ for any $x \in$ mathbbR^L.

Remark 5 *Due to this proposition, we can define $\zeta = \bar{z} \|PA\|_1$ and consider only the restriction of Φ to $[0, \zeta]^L$, which we will still denote by Φ for convenience.*

Proof. Take $\ell = 1, \ldots, L$. Then $\Phi_\ell(k) = \sum_{\ell'=1}^L A_{\ell'} p(\ell, \ell') \int_0^{\bar{z}} z m^{k_{\ell'}}(z) \; dz \geq 0$, since $p(\ell, \ell') \geq 0$ and $m^{k_{\ell'}} \geq 0$. Similarly, since $m^{k_{\ell'}}$ is a probability distribution, $\Phi_\ell(k) \leq \bar{z} \sum_{\ell'=1}^L A_{\ell'} p(\ell, \ell') \int_0^{\bar{z}} m^{k_{\ell'}}(z) \; dz = \bar{z} \sum_{\ell'=1}^L A_{\ell'} p(\ell, \ell')$. Therefore,

$$0 \leq \sum_{\ell=1}^{L} \Phi_\ell(k) \leq \bar{z} \sum_{\ell=1}^{L} \sum_{\ell'=1}^{L} A_{\ell'} p(\ell, \ell') = \bar{z} \|PA\|_1 \,.$$

∎

Theorem 6 *The function $\Phi : [0,\zeta]^L \to [0,\zeta]^L$ defined in Definition 4 is Lipschitz in the 1–norm on*
mathbbR^L. The Lipschitz constant is given by $\underset{\sim}{C}\ \max_{\ell=1,\dots,L} A_\ell P_\ell$, where $P_\ell = \sum_{\ell'=1}^{L} p(\ell',\ell)$ and $\underset{\sim}{C}$ depends on $\|PA\|_1$, but not explicitly on P or A.

Proof. First, fix $k \in [0,\zeta]$. From Property (5) of Proposition 1, the continuity of V^k, $\left(V^k\right)'$, $\left(V^k\right)''$ with respect to z in $[0,\bar z]$, and equations (15b), (15c), we find

$$\left(V^k\right)''(0) = \frac{2\rho}{\sigma^2} V^k(0) \geq \frac{2\rho}{\sigma^2} V^0(0) = \left(V^0\right)''(0) > 0\,.$$

Similarly, $\left(V^k\right)''(\bar z) \leq \left(V^\zeta\right)''(\bar z) < 0$ with the first inequality an equality if and only if $k = 0$. Moreover, $\left(V^k\right)''$ is continuous with respect to k due to (15b) and continuity of $V^k, \left(V^k\right)'$ with respect to k, which was proven in Proposition 1. Therefore, there exists $\epsilon_1, \epsilon_2 \in (0,1)$ and $C_1, C_2 > 0$, independent of k, such that

$$\begin{aligned}
\left(V^k\right)'(z) &= \int_0^z \left(V^k\right)''(y)\ dy \geq \int_0^z \left(V^0\right)''(y)\ dy \geq C_1 z\,, && \text{if } z \in [0,\epsilon_1]\\
\left(V^k\right)'(z) &= -\int_z^{\bar z} \left(V^k\right)''(y)\ dy \geq -\int_z^{\bar z} \left(V^\zeta\right)''(y)\ dy \geq C_2 z\,, && \text{if } z \in [\bar z - \epsilon_2, \bar z]\ .
\end{aligned}$$

Furthermore, by continuity of $\left(V^k\right)'$ with respect to k and compactness of $[0,\zeta]$, there exists $C_3 > 0$ such that $\inf_{k\in[0,\zeta]} \left(V^k\right)'(z) \geq C_3$ if $z \in [\epsilon_1, \bar z - \epsilon_2]$. Note that C_j for $j = 1,2,3$ are all independent of $k \in [0,\zeta]$. Therefore, if $\gamma \leq \frac{1}{2}$, for any $k_1, k_2 \in [0,\zeta]$:

$$\begin{aligned}
&\int_0^{\bar z} \left[\min\left(\left(V^{k_1}\right)'(z), \left(V^{k_2}\right)'(z)\right)\right]^{\frac{2\gamma-1}{1-\gamma}}\ dz\\
&\qquad \leq \int_0^{\epsilon_1} \left(C_1 z\right)^{\frac{2\gamma-1}{1-\gamma}}\ dz + \int_{\epsilon_1}^{\bar z-\epsilon_2} C_3^{\frac{2\gamma-1}{1-\gamma}}\ dz + \int_{\bar z-\epsilon_2}^{\bar z} \left(C_2\left(\bar z - z\right)\right)^{\frac{2\gamma-1}{1-\gamma}}\ dz \qquad (22)\\
&\qquad \leq \frac{1-\gamma}{\gamma}\left(C_1^{\frac{2\gamma-1}{1-\gamma}} + C_2^{\frac{2\gamma-1}{1-\gamma}}\right) + C_3^{\frac{2\gamma-1}{1-\gamma}}\,\bar z\,,
\end{aligned}$$

while, using Proposition 1, if $\gamma \geq \frac{1}{2}$

$$\begin{aligned}
&\int_0^{\bar z} \left[\max\left(\left(V^{k_1}\right)'(z), \left(V^{k_2}\right)'(z)\right)\right]^{\frac{2\gamma-1}{1-\gamma}}\ dz\\
&\qquad\qquad \leq \bar z\left(\frac{B}{(1-\gamma)}\bar z^{\frac{\alpha}{1-\alpha}}\right)^{2\gamma-1}\left(\frac{w}{\gamma}\right)^{\frac{\gamma(2\gamma-1)}{1-\gamma}}\ . \qquad (23)
\end{aligned}$$

Now, with the definition of $\bar{m}^k$ in (18), for any $k_1, k_2 \in [0, \zeta]$ we have

$$\begin{aligned}\left|\bar{m}^{k_1} - \bar{m}^{k_2}\right| &= \Bigg| e^{\frac{2}{\sigma^2}\left(k_1 z + \int_0^z \left(\frac{\gamma}{w}\left(V^{k_1}\right)'(y)\right)^{\frac{\gamma}{1-\gamma}} \, dy\right)} \\ &\qquad - e^{\frac{2}{\sigma^2}\left(k_2 z + \int_0^z \left(\frac{\gamma}{w}\left(V^{k_2}\right)'(y)\right)^{\frac{\gamma}{1-\gamma}} \, dy\right)} \Bigg| \\ &= \left| \frac{2}{\sigma^2} \int_{k_2 z + \int_0^z \left(\frac{\gamma}{w}\left(V^{k_2}\right)'(y)\right)^{\frac{\gamma}{1-\gamma}} \, dy}^{k_1 z + \int_0^z \left(\frac{\gamma}{w}\left(V^{k_1}\right)'(y)\right)^{\frac{\gamma}{1-\gamma}} \, dy} e^{\frac{2}{\sigma^2} u} \, du \right| .\end{aligned}$$

Then, using the uniform bound on $\left(V^k\right)'(y)$ with respect to k given by Proposition 1, we get

$$\begin{aligned}\left|\bar{m}^{k_1} - \bar{m}^{k_2}\right| &\leq \frac{2\bar{C}_1}{\sigma^2} \Bigg| \left(k_1 - k_2\right) z \\ &\qquad + \int_0^z \left[\left(\frac{\gamma}{w}\left(V^{k_1}\right)'(y)\right)^{\frac{\gamma}{1-\gamma}} - \left(\frac{\gamma}{w}\left(V^{k_2}\right)'(y)\right)^{\frac{\gamma}{1-\gamma}} \right] \, dy \Bigg| \\ &\leq \frac{2\bar{C}_1}{\sigma^2} \left(\left|k_1 - k_2\right| z + \left(\frac{\gamma}{w}\right)^{\frac{\gamma}{1-\gamma}} \int_0^{\bar{z}} \int_{\left(V^{k_2}\right)'(y)}^{\left(V^{k_1}\right)'(y)} \frac{\gamma}{1-\gamma} u^{\frac{2\gamma-1}{1-\gamma}} \, du \, dy \right) ,\end{aligned}$$

where $\bar{C}_1 = e^{\frac{2\bar{z}}{\sigma^2}\left(\zeta + \left[\frac{\gamma B}{(1-\gamma)w}\bar{z}^{\frac{\alpha}{1-\alpha}}\right]^{\gamma}\right)}$. Then, using Proposition 1 and either (22) or (23), we get

$$\begin{aligned}\left|\bar{m}^{k_1} - \bar{m}^{k_2}\right| \leq& \frac{2\bar{C}_1}{\sigma^2} \Bigg(\left|k_1 - k_2\right| z + \left(\frac{\gamma}{w}\right)^{\frac{\gamma}{1-\gamma}} \frac{\gamma}{1-\gamma} \left\| \left(V^{k_1}\right)' - \left(V^{k_2}\right)' \right\|_\infty \\ &\qquad \int_0^z \max\left[\left(\left(V^{k_1}\right)'(y)\right)^{\frac{2\gamma-1}{1-\gamma}}, \left(\left(V^{k_2}\right)'(y)\right)^{\frac{2\gamma-1}{1-\gamma}} \right] dy \Bigg) \qquad (24) \\ \leq& \frac{2\bar{C}_1}{\sigma^2} \left(z + \bar{C}_2\right) \left|k_1 - k_2\right| ,\end{aligned}$$

where $\bar{C}_2 = \frac{4\bar{z}}{\sigma^2} \left(\frac{B}{(1-\gamma)}\bar{z}^{\frac{\alpha}{1-\alpha}}\right)^{1-\gamma} \left(\frac{\gamma}{w}\right)^{\frac{\gamma^2}{1-\gamma}} \left(C_1^{\frac{2\gamma-1}{1-\gamma}} + C_2^{\frac{2\gamma-1}{1-\gamma}} + \frac{\gamma}{1-\gamma} C_3^{\frac{2\gamma-1}{1-\gamma}} \bar{z}\right)$, if $\gamma < \frac{1}{2}$. While $\bar{C}_2 = \frac{\gamma}{1-\gamma} \frac{4\bar{z}^2}{\sigma^2} \left(\frac{wB}{\gamma(1-\gamma)}\bar{z}^{\frac{alpha}{1-\alpha}}\right)^{\gamma}$, if $\gamma \geq \frac{1}{2}$. Note that for any $k \in [0, \zeta]$, $\left\|\bar{m}^k\right\|_1$ satisfies

$$\left\|\bar{m}^k\right\|_1 = \int_0^{\bar{z}} e^{\frac{2}{\sigma^2}\left[kz + \int_0^z \left(\frac{\gamma}{w}\left(V^k\right)'(y)\right)^{\frac{\gamma}{1-\gamma}} \, dy\right]} \, dz \geq 1 , \qquad (25)$$

as $\left(V^k\right)' \geq 0$. So, for any $k_1, k_2 \in [0, \zeta]$, using (24) and (25), we have

$$\begin{aligned}\left|\int_0^{\bar{z}} z\left(m^{k_1}-m^{k_2}\right)\, dz\right| \leq & \frac{1}{\left\|\bar{m}^{k_1}\right\|_1}\left|\int_0^{\bar{z}} z\left(\bar{m}^{k_1}-\bar{m}^{k_2}\right)\, dz\right| \\ & +\int_0^{\bar{z}} z \frac{\bar{m}^{k_2}}{\left\|\bar{m}^{k_1}\right\|_1\left\|\bar{m}^{k_2}\right\|_1}\, dz\left|\,\left\|\bar{m}^{k_1}\right\|_1-\left\|\bar{m}^{k_2}\right\|_1\right| \\ \leq & 2\bar{z} \int_0^{\bar{z}}\left|\bar{m}^{k_1}-\bar{m}^{k_2}\right|\, dz \leq \frac{4\bar{C}_1\bar{z}}{\sigma^2} \int_0^{\bar{z}}\left(z+\bar{C}_2\right)\, dz\left|k_1-k_2\right| \\ = & \frac{2\bar{C}_1\bar{z}^2\left(\bar{z}+2\bar{C}_2\right)}{\sigma^2}\left|k_1-k_2\right| := \underline{C}\ \left|k_1-k_2\right| .\end{aligned} \tag{26}$$

Now take $k^{(1)}, k^{(2)} \in [0, \zeta]^L$. Define $P_\ell = \sum_{\ell'=1}^L p\left(\ell', \ell\right)$, then recalling the definition of Φ given in Definition 4 and using (26)

$$\begin{aligned}\left\|\Phi\left(k^{(1)}\right)-\Phi\left(k^{(2)}\right)\right\|_1 & =\sum_{\ell=1}^L \sum_{\ell'=1}^L A_{\ell'} p\left(\ell, \ell'\right)\left|\int_0^{\bar{z}} z\left(m^{k_{\ell'}^{(1)}}-m^{k_{\ell'}^{(2)}}\right)\, dz\right| \\ & \leq \underline{C}\ \sum_{\ell'=1}^L A_{\ell'} P_{\ell'}\left|k_{\ell'}^{(1)}-k_{\ell'}^{(2)}\right| \leq \underline{C}\ \max_{\ell=1,\ldots,L} A_\ell P_\ell\left\|k^{(1)}-k^{(2)}\right\|_1 ,\end{aligned} \tag{27}$$

which concludes the proof. ■

Theorem 7 *For any given data, there exists a solution to the innovation MFG (14). Furthermore, if $\|PA\|_1$ is fixed, this solution is unique provided $A_\ell P_\ell < \frac{1}{\underline{C}}$ for every $\ell = 1, \ldots, L$.*

Proof. From Proposition 3 and Theorem 6, the function $\Phi : [0, \zeta]^L \to [0, \zeta]^L$ is a continuous function from a convex compact subset of $\mathbb{R}^L$ to itself. Therefore, by Brouwerâs fixed point theorem, Φ has a fixed point. Furthermore, Theorem 6 shows that Φ is a Lipschitz function in $\|\cdot\|_1$. The Lipschitz constant is given by $\underline{C}\ \max_{\ell=1,\ldots,L} A_\ell P_\ell$, where $\underline{C}$ depends on $\|PA\|_1$ but not directly on P_ℓ or A_ℓ. Therefore, for fixed $\|PA\|_1$, Φ is a contraction map provided $A_\ell P_\ell < \frac{1}{\underline{C}}$ for every $\ell = 1, \ldots, L$, and in this case the fixed point is unique.

Theorems 3 and 4 proved existence and uniqueness of solutions to equations (15) and (16) respectively for any $k \in [0, \zeta]$. Now, if k^* is a fixed point of Φ then $(m^*, V^*) := \left(m^{k_\ell^*}, V^{k_\ell^*}\right)_{\ell=1}^L$ is a solution to (14), which can be seen by replacing k_ℓ^* with $\Phi_\ell(k^*)$ in (15), (16) for every $\ell = 1, \ldots, L$. Conversely, if (m^*, V^*) is a solution to (15), (16), then clearly, by defining k^* co–ordinate wise as $k_\ell^* = \sum_{\ell'=1}^L A_{\ell'} p\left(\ell, \ell'\right) \int_0^{\bar{z}} z m_{\ell'}(z)\, dz$, $k^* \in [0, \zeta]^L$ is a fixed point of Φ. Furthermore, by uniqueness of (15), (16), $\left(m^{k^*}, V^{k^*}\right) = (m^*, V^*)$. So, existence and uniqueness of solutions to the innovation MFG (14) is equivalent to existence and uniqueness of fixed points of Φ. Hence, there exists a solution to the innovation MFG. Furthermore, this solution is unique, provided $A_\ell P_\ell < \frac{1}{\underline{C}}$ for every $\ell = 1, \ldots, L$. ■

Remark 8 *In practical terms we can guarantee the condition $A_\ell P_\ell < \frac{1}{\underline{C}}$ holds for every $\ell = 1, \ldots, L$ provided L is large enough. This is because $\sum_{\ell=1}^{L} A_\ell = 1$. So, for fixed $\|PA\|_1$, when L is sufficiently large we can take A_ℓ to be sufficiently small so that $A_\ell P_\ell < \frac{1}{\underline{C}}$*

4 Numerical simulations

4.1 Consumers

In the previous analysis, we assumed that consumers play a passive role in the model. In particular, the constant B has been fixed. However, in doing so we have not modelled the active nature of consumers in determining the price index R, to include this when implementing our numerical methods we return to (6b). Then, as the number of firms in each sector goes to infinity,

$$B = (1-\alpha)\left[1 + \left(\frac{\alpha}{\psi}\right)^{\frac{\alpha}{1-\alpha}} \left(\sum_{\ell=1}^{L} A_\ell \int_0^{\bar{z}} z^{\frac{\alpha}{1-\alpha}} m_\ell(z)\ dz\right)\right]^{-1} \left(\frac{\alpha}{\psi}\right)^{\frac{\alpha}{1-\alpha}} Y.$$

Note that this now needs to be solved as a fixed point, as m_ℓ itself depends on B.

4.2 Simulations

We computed simulations with synthetic data, using the numerical method outlined in Appendix A. From an economics perspective it is important to understand how the model affects the sector–level productivity. The purpose of the simulations is to provide initial insights into the role of the modelling parameters and of the network configuration.

4.2.1 Parameter effects

The MFG depends on the parameters σ, w, α, γ, ρ, Y, and ψ. Recall that $\sigma > 0$ is the strength of noise in an individual's dynamics, $w > 0$ is the wage paid to employees, $\alpha \in (0,1)$ is a parameter in the consumer optimisation problem which ensures convexity, and $\gamma \in (0,1)$ is the returns to labour i.e. the inefficiency in converting one unit of labour to one unit of knowledge, it also ensures convexity of the firm–level optimisation problem. Furthermore, Y denotes the total value of the economy, and ψ is the marginal cost of production for firms.

In order to separate the parameter effects from any effects caused by the sector network, we ran simulations with just a single sector. We fixed $\bar{z} = 2$, $A = A_1 = 1$ and $P = 0.1$, where $\bar{z}$ is the maximum productivity level, A_1 is the proportion of firms in sector 1 and P is the strength of connection from sector 1 to itself. For baseline values, we took $\sigma = 1$,

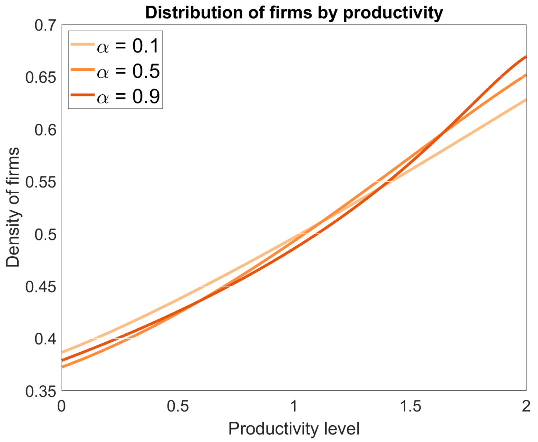

Figure 1: Distribution of firms by productivity level for varying α

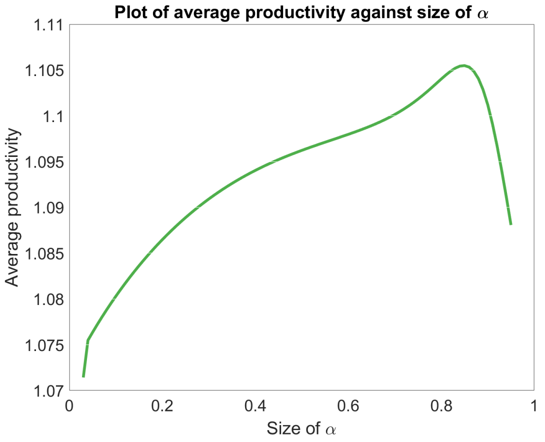

Figure 2: Average productivity vs α

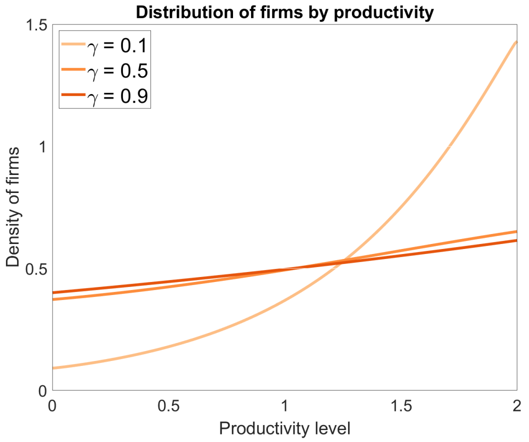

Figure 3: Distribution of firms by productivity level for varying γ

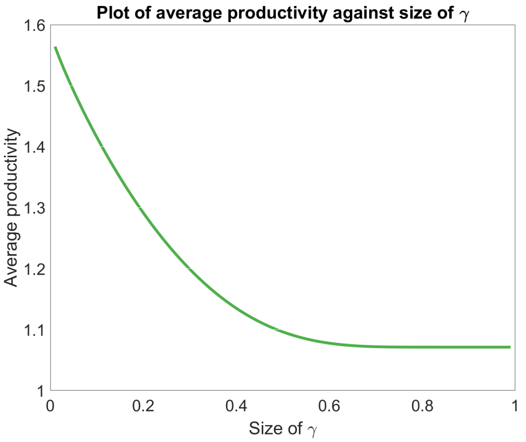

Figure 4: Average productivity vs γ

$w = 1$, $\rho = 1$, $\gamma = 0.5$, $\alpha = 0.5$, $Y = 1$, and $\psi = 1$. For each simulation, we varied one parameter while keeping all others at the baseline level. Figures 1 and 2 show that the relationship between α and the distribution of firms is a complex one. There is some $\alpha^* \in (0, 1)$ where the average productivity reaches a maximum, while on $(0, \alpha^*]$ average productivity is monotonically increasing, and on $[\alpha^*, 1)$ average productivity is monotonically decreasing. Note that, for fixed productivity level and firm distribution, a firm's revenue is $\pi_\ell = BZ_\ell^{\frac{\alpha}{1-\alpha}}$. The parameter B consists of a term that increases with respect to α multiplied by terms that decrease with respect to α. This results in a competing effect between α and a firm's revenue, which in turn affects a firm's return on investment, and therefore its level of investment in labour. Since labour investment has an increasing effect

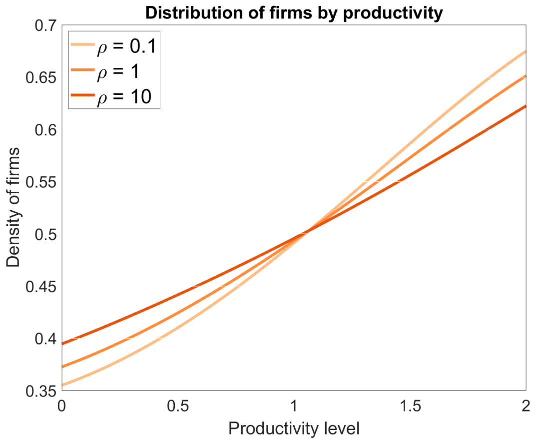

Figure 5: Distribution of firms by productivity level for varying ρ

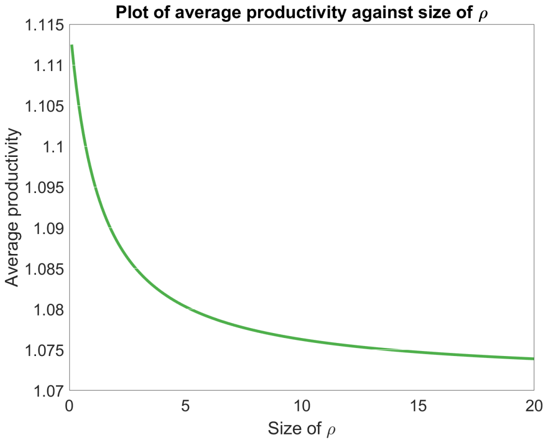

Figure 6: Average productivity vs ρ

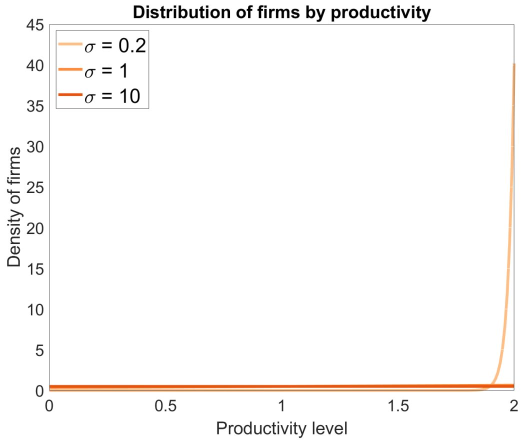

Figure 7: Distribution of firms by productivity level for varying σ

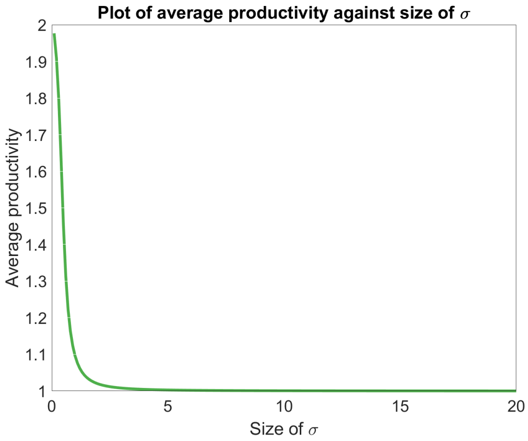

Figure 8: Average productivity vs σ

on average productivity, the competing terms in the revenue equation directly correspond to the behaviour exhibited in figure 2.

Figures 3 and 4 show the effect of γ on the sector–level productivity. Figure 4 shows that as γ increases, the average productivity decreases. Since γ relates to the inefficiency of converting one unit of labour to one unit of productive work, it seems counter–intuitive at first that average productivity would be a decreasing function of γ. Recall that the optimal level of employment is given by $h^* = \left(\frac{\gamma}{w}\max(0, V')\right)^{\frac{1}{1-\gamma}}$, which increases productivity at a rate $(h^*)^\gamma$. Then, h^* is increasing with respect to γ for fixed V' if and only if $V' \geq \frac{w}{\gamma}e^{\frac{\gamma-1}{\gamma}}$ and $(h^*)^\gamma$ is increasing if and only if $V' \geq \frac{w}{\gamma}e^{\gamma-1}$. Hence, the effect of γ on the average productivity depends on V' and how it changes with respect to γ.

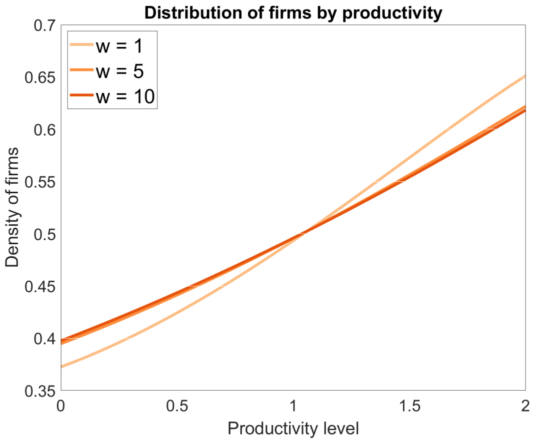

Figure 9: Distribution of firms by productivity level for varying w

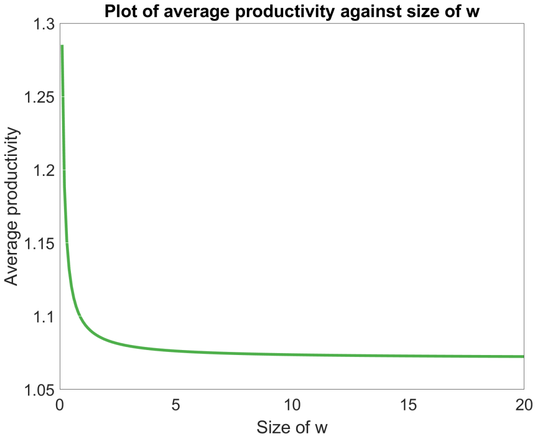

Figure 10: Average productivity vs w

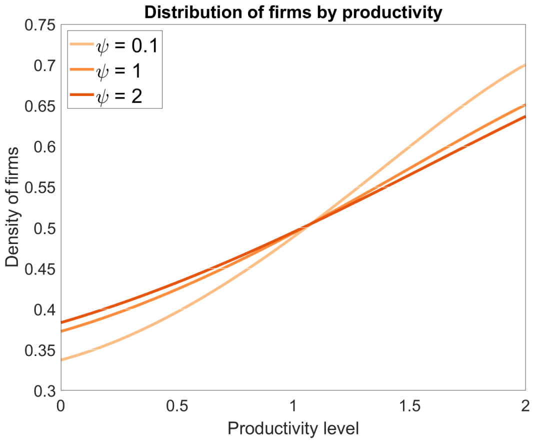

Figure 11: Distribution of firms by productivity level for varying ψ

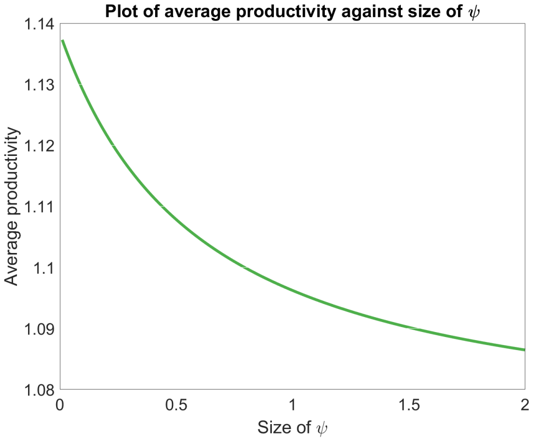

Figure 12: Average productivity vs ψ

The effects of ρ and σ on the average productivity, shown in Figures 5, 6 and 7, 8 respectively, show the same trend: average productivity decreases as each parameter increases. The size of ρ is the extent to which a firm discounts future profits. As ρ increases, firms care less about the future state of the system and so they are less willing to invest in labour; it is an investment whose effect is only on the future value of productivity. This results in reduced average productivity in the long run, which can be seen in Figure 6. As σ increases, the randomness in productivity evolution of each firm increases. So, the impact of labour on productivity decreases with increasing σ, and this is reflected in Figure 8.

Figures 9 and 10 shows that average productivity also decreases with increasing wage, w. The wage rate increases the cost of labour. So, we can directly see that as the wage

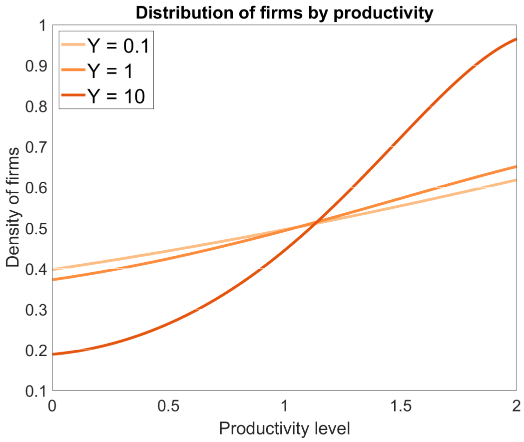

Figure 13: Distribution of firms by productivity level for varying Y

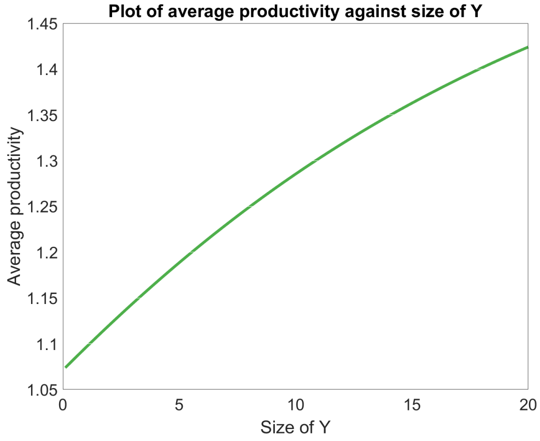

Figure 14: Average productivity vs Y

increases, the optimal level of employment, and hence the average productivity, decreases.

Finally, figures 13, 14 and 11, 12 show the effect of Y and ψ on productivity, respectively. In the case of ψ the effect of increasing ψ is decreasing productivity. This can be understood by noting that higher Ψ means greater unit costs of production, which in turn results in lower profits and less to invest in R&D. The resulting effect is a reduction in a company's knowledge, and hence productivity too. In contrast, increasing Y increases the productivity of a company. This relationship can be understood from the fact that increasing Y increases the budget available to the representative household. With more money available, more will be spent, increasing a firm's total profits and allowing more to be reinvested in productivity–increasing research.

4.2.2 Spillover size effects

The sector–level network, encoded by the vertex weights A_ℓ for sector ℓ, and the edge weights $p(\ell, \ell')$ for a transfer of knowledge from sector ℓ' to sector ℓ, is called the spillover network as it describes how knowledge and productivity spills over from one sector to another. A path in the spillover network is called a spillover path, or just spillover if there is no ambiguity. A path of length 1 from sector ℓ' to sector ℓ is called a direct spillover, a path of length 2 or greater from sector ℓ' to sector ℓ is called an indirect spillover, and in both cases sector ℓ is called the receiving sector and sector ℓ' is the originating sector.

In most economic literature focussing on measuring the impact of the spillover effect, only direct spillovers have been modelled and we are aware of no models that pay attention to the effect indirect spillovers have on economic productivity. In this subsection, we begin investigating how the productivity of a sector is affected by the structure of the spillover network, and in particular the effect of indirect spillovers on productivity. To undertake this investigation, we conducted three types of simulations. The first simulations were to model the six networks in Figures 15–20, to provide initial insight into how indirect spillover paths affect the distribution of firms. In the second simulations, we randomly generated spillover networks in models with three sectors and used the collected data to hypothesise a relationship between the average productivity of a sector and the size of spillovers (direct and indirect) it received. In the final simulations, we tested our hypothesis on more randomly generated spillover networks, this time for models with 10 sectors, which more closely resembles the number of sectors in the real economy. We showed that the hypothesis developed accurately describes the relationship between the spillover network and the average productivity of firms, moreover there was a 16% reduction in root mean squared error when direct and indirect spillovers were taken into account, compared with when only direct spillovers were considered. Therefore, our conclusion from this preliminary investigation is that indirect spillovers have a significant effect on economic productivity in our model and they should not be ignored.

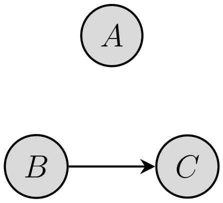

Figure 15: Network 1

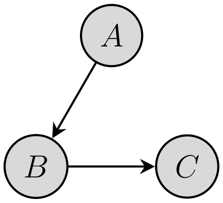

Figure 16: Network 2

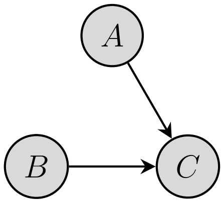

Figure 17: Network 3

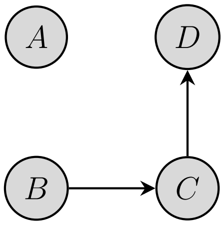

Figure 18: Network 4

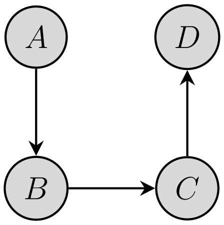

Figure 19: Network 5

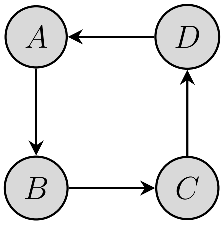

Figure 20: Network 6

The networks in Figure 15–17 provide insight into how indirect spillovers affect the distribution of firms, in comparison to direct spillovers. In network 1, sector C has one direct spillover, in network 2 it has one direct spillover and one indirect spillover of length 2, and in network 3 it has two direct spillovers. So, the difference in productivity in sector C between network 2 and network 1 will show the effect of an indirect spillover compared with having no spillover, and the difference between networks 3 and 2 will show the effect of an indirect spillover compared with a direct spillover. The differences in density of sector C are plotted in Figure 21. From the plots, it can be seen that the density of firms is larger at high productivity levels in network 2 compared with network 1 and the density is lower at low productivity levels. This means that the indirect spillover from sector A to sector C has a positive effect on sector C, skewing the distribution towards higher productivity levels. The same behaviour can be seen when we compare sector C in network 3 to network 2, however the effect is an order of magnitude larger. Therefore, although an indirect spillover path has some positive effect compared with no path at all, the effect is less strong than a direct spillover path.

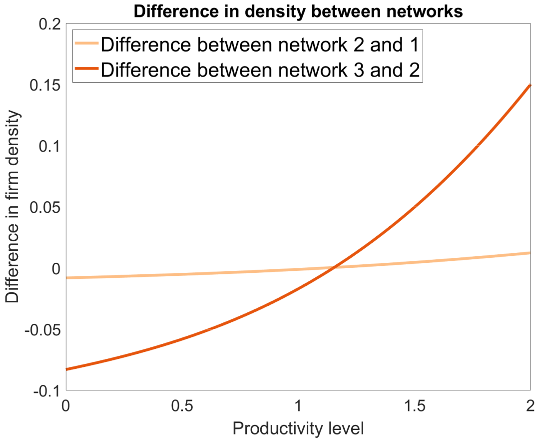

Figure 21: Difference of distributions of firms by productivity of sector C between networks 2 and 1, and networks 3 and 2

In Figure 22, sector D of networks four to six were modelled. For sector D: in network 4 there is one indirect spillover with path length 2; network 5 has one indirect spillover with path length 2 and one with path length 3; finally network 6 has an infinite number of indirect spillovers, one for every path length. We have plotted the difference in density of sector D between network 5 and network 4 and between networks 6 and 5. The difference between network 5 and network 4 shows the effect of an indirect spillover of length 3, while the difference between network 6 and network 5 shows the effect of indirect spillovers of all lengths greater than 3. For the difference between network 5 and network 4, the same qualitative result as the difference between network 2 and network 1, in Figure 21, is observed. This suggests that having spillover paths of greater length do have positive impacts on productivity, but with reduced impact for increased path lengths. Interestingly, sector D in network 6 is less productive than sector D in network 5. Further investigation showed that if B is fixed, rather than the solution of a fixed point problem, then the effect that more paths result in greater productivity returns, see figure 23. The reason for this is not immediately obvious and warrants further study. Since the observed change is very small, it can't be ruled out that this result is an artefact from simplifications in the model.

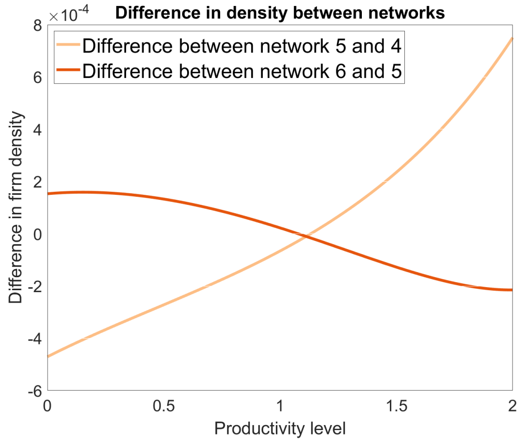

Figure 22: Difference of distributions of firms by productivity of sector D between networks 5 and 4, and networks 6 and 5

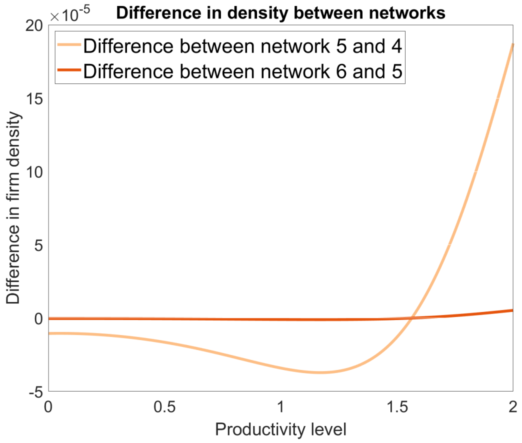

Figure 23: Difference of distributions of firms by productivity of sector D between networks 5 and 4, and networks 6 and 5, with fixed $B = 1$

In the second set of simulations, we took a closer look at how the spillover network structure affects the average productivity within each sector. Recall that if, given a network, we know the value of the fixed point, k^*, of the function Φ defined in Definition 4. Then the average productivity in sector ℓ is $\int_0^{\bar{z}} z m_\ell(z)\ dz = \int_0^{\bar{z}} z m^{k_\ell^*}(z)\ dz$. So, to understand the relationship between average productivity and the network, we first need to understand the relationship between $\int_0^{\bar{z}} z m^{k_\ell}(z)\ dz$ and k_ℓ, for any $k_\ell \geq 0$. Then, we also need to understand the relationship between k_ℓ^* and the $L \times L$ matrix S with entries defined by $S_{\ell,\ell'} = A_{\ell'} p(\ell, \ell')$, because

$$k^* = S \left(\int_0^{\bar{z}} z m^{k_\ell^*}(z)\ dz \right)_{\ell=1}^{L},$$

by the fact that k^* is a fixed-point of the map Φ. In Figure 24, we have plotted $\int_0^{\bar{z}} z m^k(z)\ dz$ as a function of $k \in [0, \infty)$, where m^k is the solution to the auxiliary Fokker–Planck equation (16), which depends on the nonnegative real parameter k. The relationship appears to approximately follow

$$\int_0^{\bar{z}} z m^k(z)\ dz = \bar{z} - \frac{b_0}{k^{b_1} + b_2}, \tag{28}$$

for some $b_0, b_1, b_2 > 0$, as can be seen by the second line in Figure 24. To understand the relationship between the fixed point of Φ and the matrix S, we considered networks of three vertices, with $A_\ell = 1/3$ for all ℓ. We created a random network between the vertices by choosing a connection probability p, and making a directed edge between vertices with probability p. We then weighted each directed edge with a random weight, chosen from a uniform distribution on $\left[0, \frac{1}{2}\right]$. We repeated this 100 times for each connection probability, and recorded both the size of direct spillovers to each sector and the value of the fixed point of Φ. Figures 25 and 26 show a scatter plot of k^*_ℓ, the ℓ^{th} co–ordinate of the fixed point of Φ, against the sum of direct spillover strengths $\sum_{\ell'=1}^{L} S_{\ell,\ell'}$. In the simulations with a high connection probability, Figure 25, there is a strong linear relationship between k^*_ℓ and $\sum_{\ell'=1}^{L} S_{\ell,\ell'}$. However, with low connection probabilities, Figure 26, the simulations tend to follow one of two weaker linear relationships with the row sum.

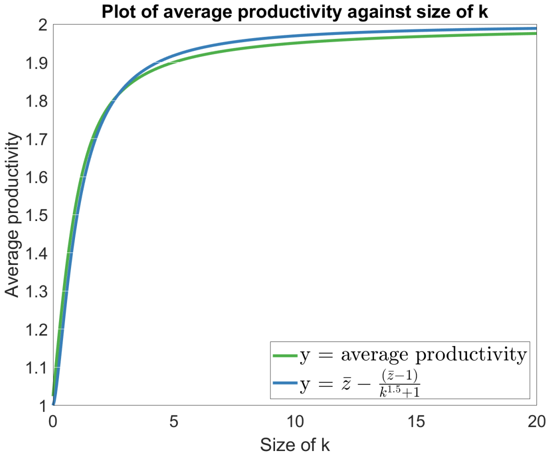

Figure 24: Average productivity $\int_0^{\bar{z}} z m^k(z)\ dz$ as a function of k (where m^k is the solution to the auxiliary Fokker–Planck equation (16)) and plot of $y = \bar{z} - \frac{(\bar{z}-1.4)}{k^2+1}$ for comparison

To understand the relationships further, we can look at the equation that $k^*_\ell \in [0, \infty)$ implicitly satisfies: $k^*_\ell = \sum_{\ell'=1}^{L} S_{\ell,\ell'} \int_0^{\bar{z}} z m^{k^*_{\ell'}}\ dz$, where m^{k_ℓ} is defined by (18). So, if sector ℓ receives no spillovers then $k^*_\ell = 0$. If it has only direct spillovers, then it is only connected to sectors with no spillovers. So, by defining $f(k) = \int_0^{\bar{z}} z m^k\ dz$, we get

$$k^*_\ell = f(0) \sum_{\ell'=1}^{L} S_{\ell,\ell'}\,. \tag{29}$$

We can see this linear relationship between k^*_ℓ and $\sum_{\ell'=1}^{L} S_{\ell,\ell'}$ in Figures 27 and 28, where we have taken the simulated points in Figure 26, and split the data into those points which have only direct spillovers and those that have indirect spillovers as well. In Figure 27, where sectors with only direct spillover paths are considered, the linear relationship described

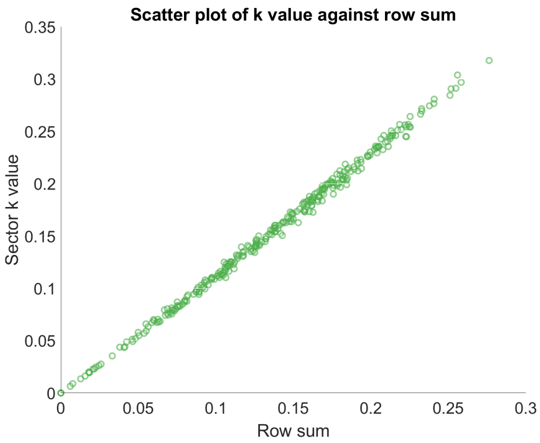

Figure 25: Scatter plot of k^*_ℓ, the ℓ^{th} co–ordinate of the fixed point of Φ, against the sum of direct spillover strengths $\sum_{\ell'=1}^{L} S_{\ell,\ell'}$ for 100 randomly generated three-node networks. Case where the probability of a directed edge between two nodes is equal to 0.8.

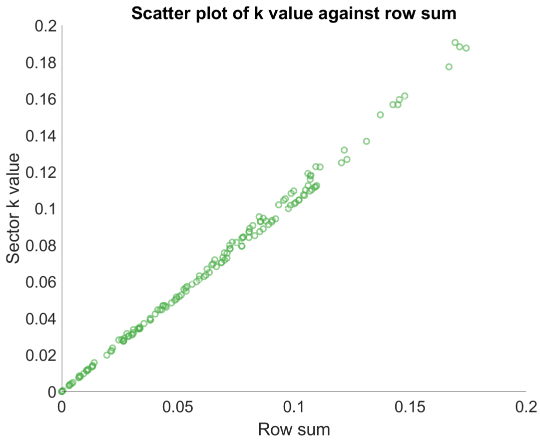

Figure 26: Same as Fig. 25 but in the case where the probability of a directed edge between two nodes is equal to 0.2

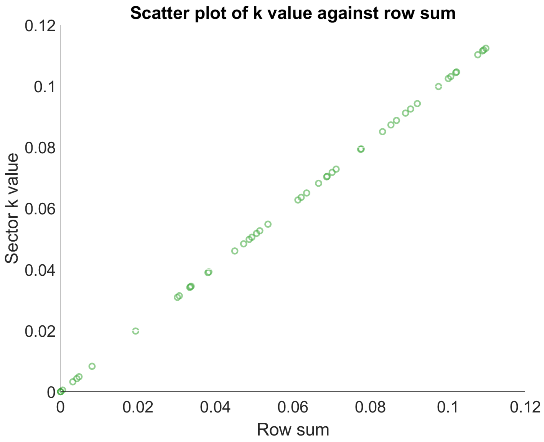

Figure 27: Same as Fig. 26 but restricted to sectors ℓ that have only direct spillovers

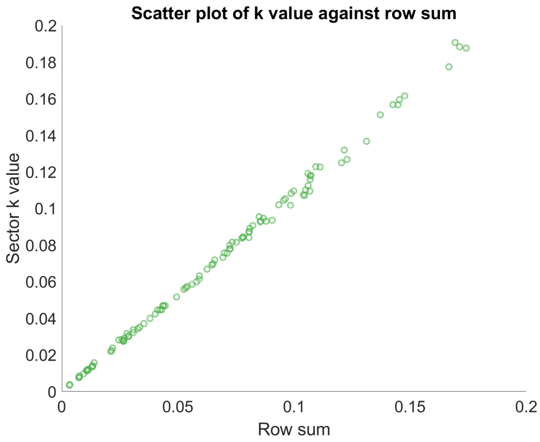

Figure 28: Same as Fig. 26 but restricted to sectors ℓ that have at least one indirect spillover

by (29) can be clearly seen.

To understand how the value of k^*_ℓ depends on the matrix S in the case of indirect spillovers, we can return to the definition of the spillover size and $f(k)$. If we assume that

Table 1: Table of regression results related to linear regression (32)

Variable	Coefficient estimate	Standard error	t stat	p value
f_0	1.58	1.16×10^{-3}	1360	0
f_1	0.648	4.46×10^{-4}	1450	0
b_0	1.13	2.23×10^{-3}	509	0
b_1	1.08	1.13×10^{-3}	954	0
b_2	1.16	2.25×10^{-3}	518	0

f is approximately linear for sectors with indirect spillovers, i.e. $f(k) = f_0 + f_1 k$, then

$$k^*_\ell = \left(S\left(f_0 \mathbf{1} + f_1 k^*\right)\right)_\ell \ , \tag{30}$$

where $\mathbf{1}$ is the vector of length L with ones in every entry. Using the identity $\left(I + f_1 S\right)^{-1} = \sum_{n=0}^{\infty} f_1^n S^n$, we can rearrange (30)

$$k^*_\ell = f_0 \sum_{n=0}^{\infty} f_1^n \left(S^{n+1}\mathbf{1}\right)_\ell \ , \tag{31}$$

which gives a way to estimate the value k^*_ℓ directly from the initial data. Therefore, combining estimates (28) and (31), we can estimate the value of average productivity from the

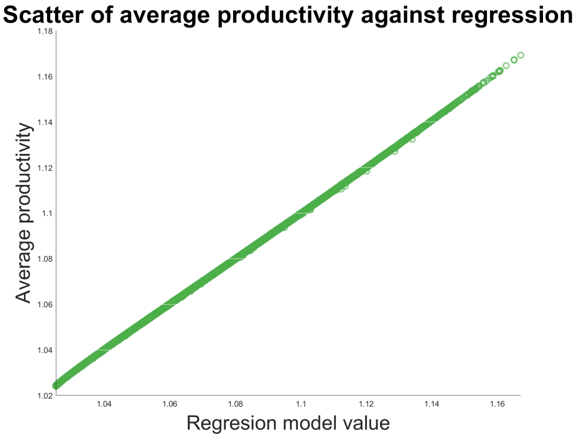

Figure 29: Scatter plot of average productivity (left-hand side of (32)) against right hand side of (32) with optimal values for f_i, b_i, for 1000 randomly-generated 10-nodes networks

Figure 30: Same as Figure 29, but against right hand side of (33) with optimal values for $\bar{f}_0, \bar{b}_i$

matrix S by

$$\int_0^{\bar{z}} z m_\ell(z)\ dz = \bar{z} - \frac{b_0}{\left(f_0 \sum_{\ell'=1}^{L} \sum_{n=0}^{\infty} f_1^n \left(S^{n+1}\right)_{\ell,\ell'}\right)^{b_1} + b_2}\,. \tag{32}$$

The relationship suggests that the average productivity depends on S^n for every n i.e. on indirect spillovers of every path length. Moreover, if f_1 is small enough, the effect of a spillover path is decreasing by an order of magnitude for every increase in path length, which agrees with our initial simulations of networks 1–6.

In order to verify the hypothesis, in the final simulations we ran a regression to estimate the parameters f_0, f_1, b_0, b_1, b_2 and provide evidence that approximation (32) is accurate. We performed 1000 simulations on networks of ten vertices, with connection probability chosen randomly and uniformly distributed in $[0,1]$, with connection strength chosen randomly and uniformly in $[0,3]$, and with sector sizes A_ℓ also randomly chosen. We ran a nonlinear regression, of the form (31), on sectors with indirect spillovers, to obtain optimal values of f_0 and f_1. Then, using the optimal values of f_0 and f_1 we ran a second nonlinear regression, of the form (32), to find the optimal values of b_0, b_1 and b_2. Table 1 gives estimates for the parameters f_i and b_i. We found that average productivity does behave approximately according to (32), with table 1 suggesting a statistically significant result. Visually, this can be seen in Figure 30, where we plotted (32) using the optimal values of f_i and b_i. We also computed estimates for the model

$$\int_0^{\bar{z}} z m_\ell(z)\ dz = \bar{z} - \frac{\bar{b}_0}{\left(\bar{f}_0 \sum_{\ell'=1}^{L} S_{\ell,\ell'}\right)^{\bar{b}_1} + \bar{b}_2}\,, \tag{33}$$

which assumes average productivity depends on direct spillovers only, and plotted the result in Figure 30. Comparing plots 29 and 30 shows that the model (32), which includes the effects of indirect spillovers, provides a more accurate estimate for average productivity than model (33), which only accounts for the effect of direct spillovers. This is reconfirmed by the 16% reduction in root mean squared error when indirect spillover paths are included in the model. Therefore, indirect spillover paths can not be ignored as a factor determining a sector's productivity.

5 Conclusion and future research

We have developed an MFG model of firm–level innovation from a microscopic formulation. The model can be calibrated to fit economic data of spillovers, so its economic validity can be verified. We have been able to prove existence of solutions and, under a smallness assumption on the data, uniqueness. We have investigated numerically how the modelling parameters and the spillover network affects the sector–level productivity, through the development of a simple algorithm that takes advantage of the structure of the proof of existence and uniqueness.

In future work, we hope to compare the MFG model with the socially optimal behaviour, as described by the mean field optimal control problem. We will also use patent–level data to calibrate and test the two models for their accuracy. We hope the comparison between the social optimum and the competitive equilibrium will suggest a method for implementing socially optimal subsidy policies for R&D.

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Appendix:

A Numerical Methods

The numerical method we designed to solve (14) is informed by the structure of the proof of existence and uniqueness. The method of proof relies on the contraction mapping theorem to find a fixed point of the map Φ, defined in Definition 4. We are also required to solve a fixed point problem to find the value of the parameter B. In light of this, our numerical method proceeds as follows, after choosing an initial guess $k^0 \in [0,\infty)^L$, $B^0 \in [0,\infty)$ and tolerances δ_1, δ_2.

1. Given $k^i \in [0,\infty)^L$ and $B^i \in [0,\infty)$, solve (15b), (15c) using the following method, based on a Newton–Raphson method in a Banach space.
   (a) Define $F(v) = -\frac{\sigma^2}{2}v'' + \rho v - kv' - (1-\gamma)\left(\frac{\gamma}{w}\right)^{\frac{\gamma}{1-\gamma}} (v')^{\frac{1}{1-\gamma}} - Bz^{\frac{\alpha}{1-\alpha}}$. We want to find zeros of $F(v)$.
   (b) We define $dF(v)(u) = -\frac{\sigma^2}{2}u'' + \rho u - ku' - \left(\frac{\gamma}{w}v'\right)^{\frac{\gamma}{1-\gamma}} u'$, which is the Fréchet derivative of F.
   (c) Denote by $V_0^{k^i_\ell, B^i}$ the initial guess for the ℓth component of the solution to (15b), (15c) with $k = k^i_\ell$ and $B = B^i$.
   (d) Given $V_n^{k^i_\ell, B^i}$, we compute the next iteration, $V_{n+1}^{k^i_\ell, B^i}$, using a Newton–Raphson method: $V_{n+1}^{k^i_\ell, B^i} = V_n^{k^i_\ell, B^i} - dF\left(V_n^{k^i_\ell, B^i}\right)^{-1}\left(F\left(V_n^{k^i_\ell, B^i}\right)\right)$.
   (e) Continue iteratively until $\left\|F\left(V_n^{k^i_\ell, B^i}\right)\right\|_1 \leq \delta_1$ and define $V^{i,\ell} = V_n^{k^i_\ell, B^i}$
2. Given V^i, compute the solution to (16b) using (18) and denote it by m^i
3. Define $k^{i+1} = \Phi(k^i)$
4. Define
$$B^{i+1} = Y(1-\alpha)\left(\frac{\alpha}{\psi}\right)^{\frac{\alpha}{1-\alpha}} \left[1 + \left(\frac{\alpha}{\psi}\right)^{\frac{\alpha}{1-\alpha}} \left(\sum_{\ell=1}^{L} A_\ell \int_0^{\bar{z}} z^{\frac{\alpha}{1-\alpha}} m^i_\ell(z)\, dz\right)\right]^{-1}$$
5. If $\|k^{i+1} - k^i\|_1 + |B^{i+1} - B^i| \leq \delta_2$ then stop the iteration process and define the MFG solution $(m, V) = (m^i, V^i)$. Otherwise return to Step 1.