Change of variable formulas for non-anticipative functionals on path space

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Abstract

We derive a functional change of variable formula for *non-anticipative* functionals defined on the space of $\mathbb{R}^d$-valued right continuous paths with left limits. The functional is only required to possess certain directional derivatives, which may be computed pathwise. Our results lead to functional extensions of the Ito formula for a large class of stochastic processes, including semimartingales and Dirichlet processes. In particular, we show the stability of the class of semimartingales under certain functional transformations.

Keywords: change of variable formula, functional derivative, functional calculus, stochastic integral, stochastic calculus, quadratic variation, Ito formula, Dirichlet process, semimartingale, Wiener space, Föllmer integral, Ito integral, cadlag functions.

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In his seminal paper *Calcul d'Ito sans probabilités* [7], Hans Föllmer proposed a non-probabilistic version of the Ito formula [9]: Föllmer showed that if a real-valued cadlag (right continuous with left limits) function x has finite quadratic variation along a sequence $\pi_n = (t_k^n)_{k=0..n}$ of subdivisions of $[0,T]$ with step size decreasing to zero, in the sense that the sequence of discrete measures

$$\sum_{k=0}^{n-1} \|x(t_{k+1}^n) - x(t_k^n)\|^2 \delta_{t_k^n}$$

converges vaguely to a Radon measure with Lebesgue decomposition $\xi + \sum_{t\in[0,T]} |\Delta x(t)|^2 \delta_t$ then for $f \in C^1(\mathbb{R})$ one can define the pathwise integral

$$\int_0^T f(x(t)) d^\pi x = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(x(t_i^n)).(x(t_{i+1}^n) - x(t_i^n)) \tag{1}$$

as a limit of Riemann sums along the subdivision $\pi = (\pi_n)_{n\geq 1}$. In particular if $X = (X_t)_{t\in[0,T]}$ is a semimartingale [5, 12, 13], which is the classical setting for stochastic calculus, the paths of X have finite quadratic variation along such subsequences: when applied to the paths of X, Föllmer's integral (1) then coincides, with probability one, with the Ito stochastic integral $\int_0^T f(X)dX$ with respect to the semimartingale X. This construction may in fact be carried out for a more general class of processes, including the class of Dirichlet processes [4, 7, 8, 11].

Of course, the Ito stochastic integral with respect to a semimartingale X may be defined for a much larger class of integrands: in particular, for a caglad process Y defined as a *non-anticipative functional* $Y(t) = F_t(X(u), 0 \leq u \leq t)$ of X, the stochastic integral $\int_0^T Y dX$ may be defined as a limit of non-anticipative Riemann sums [13].

Using a notion of directional derivative for functionals proposed by Dupire [6], we extend Föllmer's pathwise change of variable formula to non-anticipative functionals on the space $D([0,T],\mathbb{R}^d)$ of cadlag paths (Theorem 3). The requirement on the functionals is to possess certain directional derivatives which may be computed pathwise. Our construction allows to define a pathwise integral $\int F_t(x)dx$, defined as a limit of Riemann sums, for a class of functionals F of a cadlag path x with finite quadratic variation. Our results lead to functional extensions of the Ito formula for semimartingales (Section 6) and Dirichlet processes (Section 5). In particular, we show the stability of the the class of semimartingales under functional transformations verifying a regularity condition. These results yield a non-probabilistic proof for functional Ito formulas obtained in [2, 3, 6] using probabilistic methods and extend them to the case of discontinuous semimartingales.

Notation

For a path $x \in D([0,T],\mathbb{R}^d)$, denote by $x(t)$ the value of x at t and by $x_t = (x(u), 0 \leq u \leq t)$ the restriction of x to $[0,t]$. Thus $x_t \in D([0,t],\mathbb{R}^d)$. For a stochastic process X we shall similarly denote $X(t)$ its value at t and $X_t = (X(u), 0 \leq u \leq t)$ its path on $[0,t]$.

1 Non-anticipative functionals on spaces of paths

Let $T > 0$, and $U \subset \mathbb{R}^d$ be an open subset of $\mathbb{R}^d$ and $S \subset \mathbb{R}^m$ be a Borel subset of $\mathbb{R}^m$. We call "U-valued cadlag function" a right-continuous function $f : [0,T] \mapsto U$ with left limits such that for each $t \in]0,T]$, $f(t-) \in U$. Denote by $\mathcal{U}_t = D([0,t],U)$ (resp. $\mathcal{S}_t = D([0,t],S)$ the space of U-valued cadlag functions (resp. S), and $C_0([0,t],U)$ the set of continuous functions with values in U.

When dealing with functionals of a path $x(t)$ indexed by time, an important class is formed by those which are *non-anticipative*, in the sense that they only depend on the past values of x. A family $Y : [0,T] \times \mathcal{U}_T \mapsto \mathbb{R}$ of functionals is said to be *non-anticipative* if, for all $(t,x) \in [0,T] \times \mathcal{U}_T$, $Y(t,x) = Y(t,x_t)$ where $x_t = x_{|[0,t]}$ denotes the restriction of the path x to $[0,t]$. A non-anticipative functional may thus be represented as $Y(t,x) = F_t(x_t)$ where $(F_t)_{t\in[0,T]}$ is a family of maps $F_t : \mathcal{U}_t \mapsto \mathbb{R}$. This motivates the following definition:

Definition 1 (Non-anticipative functionals on path space)**.** A non-anticipative functional on $\mathcal{U}_T$ is a family $F = (F_t)_{t\in[0,T]}$ of maps

$$F_t : \mathcal{U}_t \to \mathbb{R}$$

Y is said to be *predictable*[1] if, for all $(t,x) \in [0,T] \times \mathcal{U}_T$, $Y(t,x) = Y(t,x_{t-})$ where x_{t-} denotes the function defined on $[0,t]$ by

$$x_{t-}(u) = x(u) \quad u \in [0,t[\qquad x_{t-}(t) = x(t-)$$

Typical examples of predictable functionals are integral functionals, e.g.

$$Y(t,x) = \int_0^t G_s(x_s)ds$$

where G is a non-anticipative, locally integrable, functional.

If Y is predictable then Y is non-anticipative, but predictability is a stronger property. Note that x_{t-} is cadlag and should *not* be confused with the caglad path $u \mapsto x(u-)$.

We consider throughout this paper non-anticipative functionals

$$F = (F_t)_{t\in[0,T]} \qquad F_t : \mathcal{U}_t \times \mathcal{S}_t \to \mathbb{R}$$

where F has a predictable dependence with respect to the second argument:

$$\forall t \le T, \quad \forall (x,v) \in \mathcal{U}_t \times \mathcal{S}_t, \quad F_t(x_t,v_t) = F_t(x_t,v_{t-}) \tag{2}$$

F can be viewed as a functional on the vector bundle $\Upsilon = \bigcup_{t\in[0,T]} \mathcal{U}_t \times \mathcal{S}_t$. We will also consider non-anticipative functionals $F = (F_t)_{t\in[0,T[}$ indexed by $[0,T[$.

1.1 Horizontal and vertical perturbation of a path

Consider a path $x \in D([0,T]),U)$ and denote by $x_t \in \mathcal{U}_t$ its restriction to $[0,t]$ for $t < T$. For $h \ge 0$, the *horizontal* extension $x_{t,h} \in D([0,t+h],\mathbb{R}^d)$ of x_t to $[0,t+h]$ is defined as

$$x_{t,h}(u) = x(u) \qquad u \in [0,t] \; ; \qquad x_{t,h}(u) = x(t) \qquad u \in]t,t+h] \tag{3}$$

For $h \in \mathbb{R}^d$ small enough, we define the *vertical* perturbation x_t^h of x_t as the cadlag path obtained by shifting the endpoint by h:

$$x_t^h(u) = x_t(u) \qquad u \in [0,t[\quad x_t^h(t) = x(t) + h \tag{4}$$

[1] This notion coincides with the usual definition of predictable process when the path space $\mathcal{U}_T$ is endowed with the filtration of the canonical process, see Dellacherie & Meyer [5, Vol. I].

or in other words $x_t^h(u) = x_t(u) + h1_{t=u}$. By convention, $x_{t,h}^u = (x_t^u)_{t,h}$, ie the vertical perturbation precedes the horizontal extension.

We now define a distance between two paths, not necessarily defined on the same time interval. For $T \geq t' = t+h \geq t \geq 0$, $(x,v) \in \mathcal{U}_t \times S_t^+$ and $(x',v') \in D([0,t+h],\mathbb{R}^d) \times \mathcal{S}_{t+h}$ define

$$d_\infty(\ (x,v),(x',v')\ ) = \sup_{u\in[0,t+h]} |x_{t,h}(u) - x'(u)| + \sup_{u\in[0,t+h]} |v_{t,h}(u) - v'(u)| +\ h \tag{5}$$

If the paths $(x,v),(x',v')$ are defined on the same time interval, then $d_\infty((x,v),(x',v'))$ is simply the distance in supremum norm.

1.2 Classes of non-anticipative functionals

Using the distance d_∞ defined above, we now introduce various notions of continuity for non-anticipative functionals.

Definition 2 (Continuity at fixed times)**.** A non-anticipative functional $F = (F_t)_{t\in[0,T]}$ is said to be continuous at fixed times if and only if for any $t \leq T$, $F_t : \mathcal{U}_t \times \mathcal{S}_t \mapsto \mathbb{R}$ is continuous for the supremum norm.

Definition 3 (Left-continuous functionals)**.** Define $\mathbb{F}_l^\infty$ as the set of functionals $F = (F_t, t \in [0,T])$ which verify:

$$\begin{aligned} &\forall t \in [0,T], \quad \forall \epsilon > 0, \forall (x,v) \in \mathcal{U}_t \times \mathcal{S}_t, \quad \exists \eta > 0, \forall h \in [0,t], \\ &\forall (x',v') \in \mathcal{U}_{t-h} \times \mathcal{S}_{t-h}, \qquad d_\infty((x,v),(x',v')) < \eta \Rightarrow |F_t(x,v) - F_{t-h}(x',v')| < \epsilon \end{aligned} \tag{6}$$

Definition 4 (Right-continuous functionals)**.** Define $\mathbb{F}_r^\infty$ as the set of functionals $F = (F_t, t \in [0,T[)$ which verify

$$\begin{aligned} &\forall t \in [0,T], \quad \forall \epsilon > 0, \forall (x,v) \in \mathcal{U}_t \times \mathcal{S}_t, \quad \exists \eta > 0, \forall h \in [0,T-t], \\ &\forall (x',v') \in \mathcal{U}_{t+h} \times \mathcal{S}_{t+h}, \qquad d_\infty((x,v),(x',v')) < \eta \Rightarrow |F_t(x,v) - F_{t+h}(x',v')| < \epsilon \end{aligned} \tag{7}$$

We denote $\mathbb{F}^\infty = \mathbb{F}_r^\infty \cap \mathbb{F}_r^\infty$ the set of continuous non-anticipative functionals.

We call a functional "boundedness preserving" if it is bounded on each bounded set of paths:

Definition 5 (Boundedness-preserving functionals)**.** Define $\mathbb{B}$ as the set of non-anticipative functionals F such that for every compact subset K of U, every $R > 0$, there exists a constant $C_{K,R}$ such that:

$$\forall t \leq T, \forall (x,v) \in D([0,t],K) \times \mathcal{S}_t, \sup_{s\in[0,t]} |v(s)| < R \Rightarrow |F_t(x,v)| < C_{K,R} \tag{8}$$

In particular if $F \in \mathbb{B}$, it is "locally" bounded in the neighborhood of any given path i.e.

$$\forall (x,v) \in \mathcal{U}_T \times \mathcal{S}_T, \quad \exists C > 0, \eta > 0, \qquad \forall t \in [0,T], \quad \forall (x',v') \in \mathcal{U}_t \times \mathcal{S}_t,$$
$$d_\infty((x_t,v_t),(x',v')) < \eta \Rightarrow \forall t \in [0,T], |F_t(x',v')| \leq C \tag{9}$$

The following result describes the behavior of paths generated by the functionals in the above classes:

Proposition 1 (Pathwise regularity)**.**

1. *If $F \in \mathbb{F}_l^\infty$ then for any $(x,v) \in \mathcal{U}_T \times \mathcal{S}_T$, the path $t \mapsto F_t(x_{t-}, v_{t-})$ is left-continuous.*
2. *If $F \in \mathbb{F}_r^\infty$ then for any $(x,v) \in \mathcal{U}_T \times \mathcal{S}_T$, the path $t \mapsto F_t(x_t, v_t)$ is right-continuous.*
3. *If $F \in \mathbb{F}^\infty$ then for any $(x,v) \in \mathcal{U}_T \times \mathcal{S}_T$, the path $t \mapsto F_t(x_t, v_t)$ is cadlag and continuous at all points where x and v are continuous.*
4. *If $F \in \mathbb{F}^\infty$ further verifies (2) then for any $(x,v) \in \mathcal{U}_T \times \mathcal{S}_T$, the path $t \mapsto F_t(x_t, v_t)$ is cadlag and continuous at all points where x is continuous.*
5. *If $F \in \mathbb{B}$, then for any $(x,v) \in \mathcal{U}_T \times \mathcal{S}_T$, the path $t \mapsto F_t(x_t, v_t)$ is bounded.*

Proof. 1. Let $F \in \mathbb{F}_l^\infty$ and $t \in [0,T)$. For $h > 0$ sufficiently small,

$$d_\infty((x_{t-h}, v_{t-h}), (x_{t-}, v_{t-})) = \sup_{u \in (t-h,t)} |x(u) - x(t-)| + \sup_{u \in (t-h,t)} |v(u) - v(t-)| + h \tag{10}$$

Since x and v are cadlag, this quantity converges to 0 as $h \to 0+$, so

$$F_{t-h}(x_{t-h}, v_{t-h}) - F_t(x_{t-}, v_{t-}) \overset{h \to 0^+}{\to} 0$$

so $t \mapsto F_t(x_{t-}, v_{t-})$ is left-continuous.

2. Let $F \in \mathbb{F}_r^\infty$ and $t \in [0,T)$. For $h > 0$ sufficiently small,

$$d_\infty((x_{t+h}, v_{t+h}), (x_t, v_t)) = \sup_{u \in [t,t+h)} |x(u) - x(t)| + \sup_{u \in [t,t+h)} |v(u) - v(t)| + h \tag{11}$$

Since x and v are cadlag, this quantity converges to 0 as $h \to 0+$, so

$$F_{t+h}(x_{t+h}, v_{t+h}) - F_t(x_t, v_t) \overset{h \to 0^+}{\to} 0$$

so $t \mapsto F_t(x_t, v_t)$ is right-continuous.

3. Assume now that F is in $\mathbb{F}^\infty$ and let $t \in]0,T]$. Denote $(\Delta x(t), \Delta v(t))$ the jump of (x,v) at time t. Then

$$d_\infty((x_{t-h}, v_{t-h}), x_t^{-\Delta x(t)}, v_t^{-\Delta v(t)})) = \sup_{u \in [t-h,t)} |x(u) - x(t)| + \sup_{u \in [t-h,t)} |v(u) - v(t)| + h$$

and this quantity goes to 0 because x and v have left limits. Hence the path has left limit $F_t(x_t^{-\Delta x(t)}, v_t^{-\Delta v(t)})$ at t. A similar reasoning proves that it has right-limit $F_t(x_t, v_t)$.

4. If $F \in \mathbb{F}^\infty$ verifies (2), for $t \in]0,T]$ the path $t \mapsto F_t(x_t, v_t)$ has left-limit $F_t(x_t^{-\Delta x(t)}, v_t^{-\Delta v(t)})$ at t, but (2) implied that this left-limit equals $F_t(x_t^{-\Delta x(t)}, v_t)$.

□

1.3 Measurability properties

Consider, on the path space $\mathcal{U}_T \times \mathcal{S}_T$, the filtration $(\mathcal{F}_t)$ generated by the canonical process

$$\begin{aligned} (X,V): \mathcal{U}_T \times \mathcal{S}_T \times [0,T] &\mapsto U \times S \\ (x,v),t &\to (X,V)((x,v),t) = (x(t),v(t)) \end{aligned} \tag{12}$$

$\mathcal{F}_t$ is the smallest sigma-algebra on $\mathcal{U}_T \times \mathcal{S}_T$ such that all coordinate maps $(X(.,s),V(.,s)), s \in [0,t]$ are $\mathcal{F}_t$-measurable.

The following result, proved in Appendix B, clarifies the measurability properties of processes defined by functionals in $\mathbb{F}_l^\infty, \mathbb{F}_r^\infty$:

Theorem 2. *If F is continuous at fixed time, then the process Y defined by $Y((x,v),t) = F_t(x_t,v_t)$ is $\mathcal{F}_t$-adapted. If $F \in \mathbb{F}_l^\infty$ or $F \in \mathbb{F}_r^\infty$, then:*

1. *the process Y defined by $Y((x,v),t) = F_t(x_t,v_t)$ is optional.*
2. *the process Z defined by $Z((x,v),t) = F_t(x_{t-},v_{t-})$ is predictable.*

2 Pathwise derivatives of non-anticipative functionals

2.1 Horizontal derivative

We now define a pathwise derivative for a non-anticipative functional $F = (F_t)_{t\in[0,T]}$, which may be seen as a "Lagrangian" derivative along the path x.

Definition 6 (Horizontal derivative)**.** The *horizontal derivative* at $(x,v) \in \mathcal{U}_t \times \mathcal{S}_t$ of non-anticipative functional $F = (F_t)_{t\in[0,T[}$ is defined as

$$\mathcal{D}_t F(x,v) = \lim_{h\to 0^+} \frac{F_{t+h}(x_{t,h},v_{t,h}) - F_t(x,v)}{h} \tag{13}$$

if the corresponding limit exists. If (13) is defined for all $(x,v) \in \Upsilon$ the map

$$\begin{aligned} \mathcal{D}_t F: \mathcal{U}_t \times \mathcal{S}_t &\mapsto \mathbb{R}^d \\ (x,v) &\to \mathcal{D}_t F(x,v) \end{aligned} \tag{14}$$

defines a non-anticipative functional $\mathcal{D}F = (\mathcal{D}_t F)_{t\in[0,T[}$, the *horizontal derivative* of F.

We will occasionally use the following "local Lipschitz property" that is weaker than horizontal differentiability:

Definition 7. A non-anticipative functional F is said to have the horizontal local Lipschitz property if and only if:

$$\begin{aligned} &\forall (x,v) \in \mathcal{U}_T \times \mathcal{S}_T, \exists C > 0, \eta > 0, \forall t_1 < t_2 \le T, \forall (x',v') \in \mathcal{U}_{t_1} \times \mathcal{S}_{t_1}, \\ &d_\infty((x_{t_1},v_{t_1}),(x',v')) < \eta \Rightarrow |F_{t_2}(x'_{t_1,t_2-t_1},v'_{t_1,t_2-t_1}) - F_{t_1}((x'_{t_1},v'_{t_1}))| < C(t_2-t_1) \end{aligned} \tag{15}$$

2.2 Vertical derivative

Dupire [6] introduced a pathwise spatial derivative for non-anticipative functionals, which we now introduce. Denote $(e_i, i = 1..d)$ the canonical basis in $\mathbb{R}^d$.

Definition 8. A non-anticipative functional $F = (F_t)_{t\in[0,T]}$ is said to be *vertically differentiable* at $(x, v) \in D([0, t]), \mathbb{R}^d) \times D([0, t], S_d^+)$ if

$$\begin{array}{rcl} \mathbb{R}^d & \mapsto & \mathbb{R} \\ e & \to & F_t(x_t^e, v_t) \end{array}$$

is differentiable at 0. Its gradient at 0

$$\nabla_x F_t\ (x, v) = (\partial_i F_t(x, v),\ i = 1..d) \qquad \text{where} \quad \partial_i F_t(x, v) = \lim_{h\to 0} \frac{F_t(x_t^{he_i}, v) - F_t(x, v)}{h} \tag{16}$$

is called the *vertical derivative* of F_t at (x, v). If (16) is defined for all $(x, v) \in \Upsilon$, the *vertical derivative*

$$\begin{array}{rcl} \nabla_x F : \mathcal{U}_t \times \mathcal{S}_t & \mapsto & \mathbb{R}^d \\ (x, v) & \to & \nabla_x F_t(x, v) \end{array} \tag{17}$$

define a non-anticipative functional $\nabla_x F = (\nabla_x F_t)_{t\in[0,T]}$ with values in $\mathbb{R}^d$.

Remark 1. If a vertically differentiable functional verifies (2), its vertical derivative also verifies (2).

Remark 2. $\partial_i F_t(x, v)$ is simply the directional derivative of F_t in direction $(1_{\{t\}} e_i, 0)$. Note that this involves examining cadlag perturbations of the path x, even if x is continuous.

Remark 3. If $F_t(x, v) = f(t, x(t))$ with $f \in C^{1,1}([0, T[\times\mathbb{R}^d)$ then we retrieve the usual partial derivatives:

$$\mathcal{D}_t F(x, v) = \partial_t f(t, x(t)) \qquad \nabla_x F_t(x_t, v_t) = \nabla_x f(t, x(t)).$$

Remark 4. Note that the assumption (2) that F is predictable with respect to the second variables entails that for any $t \in [0, T]$, $F_t(x_t, v_t^e) = F_t(x_t, v_t)$ so an analogous notion of derivative with respect to v would be identically zero under assumption (2).

If F admits a horizontal (resp. vertical) derivative $\mathcal{D}F$ (resp. $\nabla_x F$) we may iterate the operations described above and define higher order horizontal and vertical derivatives.

Definition 9. Define $\mathbb{C}^{j,k}$ as the set of functionals F which are

- continuous at fixed times,
- admit j horizontal derivatives and k vertical derivatives at all $(x, v) \in \mathcal{U}_t \times \mathcal{S}_t$, $t \in [0, T[$
- $\mathcal{D}^m F, m \leq j, \nabla_x^n F, n \leq k$ are continuous at fixed times.

3 Change of variable formula for functionals of a continuous path

We now state our first main result, a functional change of variable formula which extends the Itô formula without probability due to Föllmer [7] to functionals. We denote here S_d^+ the set of positive symmetric $d \times d$ matrices.

Definition 10. Let $\Pi_n = (t_0^n, \ldots, t_{k(n)}^n)$, where $0 = t_0^n \leq t_1^n \leq \ldots \leq t_{k(n)}^n = T$, be a sequence of subdivisions of $[0, T]$ with step decreasing to 0 as $n \to \infty$. $f \in C_0([0, T], \mathbb{R})$ is said to have finite quadratic variation along (π_n) if the sequence of discrete measures:

$$\xi^n = \sum_{i=0}^{k(n)-1} (f(t_{i+1}^n) - f(t_i^n))^2 \delta_{t_i^n} \tag{18}$$

where δ_t is the Dirac measure at t, converge vaguely to a Radon measure ξ on $[0, T]$ whose atomic part is null. The increasing function $[f]$ defined by

$$[f](t) = \xi([0, t])$$

is then called the quadratic variation of f along the sequence (π_n).
$x \in C_0([0, T], U)$ is said to have finite quadratic variation along the sequence (π_n) if the functions $x_i, 1 \leq i \leq d$ and $x_i + x_j, 1 \leq i < j \leq d$ do. The quadratic variation of x along (π_n) is the S_d^+-valued function x defined by:

$$[x]_{ii} = [x_i], [x]_{ij} = \frac{1}{2}([x_i + x_j] - [x_i] - [x_j]), i \neq j \tag{19}$$

Theorem 3 (Change of variable formula for functionals of continuous paths)**.** *Let* $(x, v) \in C_0([0, T], U) \times \mathcal{S}_T$ *such that* x *has finite quadratic variation along* (π_n) *and verifies* $\sup_{t \in [0,T] - \pi_n} |v(t) - v(t-)| \to 0$. *Denote:*

$$x^n(t) = \sum_{i=0}^{k(n)-1} x(t_{i+1}) 1_{[t_i, t_{i+1}[}(t) + + x(T) 1_{\{T\}}(t)$$

$$v^n(t) = \sum_{i=0}^{k(n)-1} v(t_i) 1_{[t_i, t_{i+1}[}(t) + v(T) 1_{\{T\}}(t), \qquad h_i^n = t_{i+1}^n - t_i^n \tag{20}$$

Then for any non-anticipative functional $F \in \mathbb{C}^{1,2}$ *satisfying the following assumptions:*

1. $F, \nabla_x F, \nabla_x^2 F \in \mathbb{F}_l^\infty$
2. $\nabla_x^2 F, \mathcal{D}F$ *satisfy the local boundedness property* (9)

the following limit

$$\lim_{n \to \infty} \sum_{i=0}^{k(n)-1} \nabla_x F_{t_i^n}(x_{t_i^n -}^n, v_{t_i^n -}^n)(x(t_{i+1}^n) - x(t_i^n)) \tag{21}$$

exists. Denoting this limit by $\int_0^T \nabla_x F(x_u, v_u) d^\pi x$ *we have*

$$F_T(x_T, v_T) - F_0(x_0, v_0) = \int_0^T \mathcal{D}_t F_t(x_u, v_u) du + \int_0^T \frac{1}{2} \mathrm{tr}\left({}^t\nabla_x^2 F_t(x_u, v_u) d[x](u)\right) + \int_0^T \nabla_x F(x_u, v_u) d^\pi x \quad (22)$$

Remark 5 (Föllmer integral). The limit (21), which we call the *Föllmer integral*, was defined in [7] for integrands of the form $f(X(t))$ where $f \in C^1(\mathbb{R}^d)$. It depends a priori on the sequence π of subdivisions, hence the notation $\int_0^T \nabla_x F(x_u, v_u) d^\pi x$. We will see in Section 6 that when x is the sample path of a *semimartingale*, the limit is in fact almost-surely independent of the choice of π.

Remark 6. The regularity conditions on F are given independently of (x, v) and of the sequence of subdivisions (π_n).

Proof. Denote $\delta x_i^n = x(t_{i+1}^n) - x(t_i^n)$. Since x is continuous hence uniformly continuous on $[0, T]$, and using Lemma 8 for v, the quantity

$$\eta_n = \sup\{|v(u) - v(t_i^n)| + |x(u) - x(t_i^n)| + |t_{i+1}^n - t_i^n|, 0 \le i \le k(n) - 1, u \in [t_i^n, t_{i+1}^n)\} \quad (23)$$

converges to 0 as $n \to \infty$. Since $\nabla_x^2 F, \mathcal{D}F$ satisfy the local boundedness property (9), for n sufficiently large there exists $C > 0$ such that

$$\forall t < T, \forall (x', v') \in \mathcal{U}_t \times \mathcal{S}_t, \qquad d_\infty((x_t, v_t), (x', v')) < \eta_n \Rightarrow |\mathcal{D}_t F_t(x', v')| \le C, |\nabla_x^2 F_t(x', v')| \le C$$

Denoting $K = \overline{\{x(u), s \le u \le t\}}$ which is a compact subset of U, and $U^c = \mathbb{R} - U$ its complement, one can also assume n sufficiently large so that $d(K, U^c) > \eta_n$.

For $i \le k(n) - 1$, consider the decomposition:

$$\begin{aligned} F_{t_{i+1}^n}(x_{t_{i+1}^n-}^n, v_{t_{i+1}^n-}^n) - F_{t_i^n}(x_{t_i^n-}^n, v_{t_i^n-}^n) &= F_{t_{i+1}^n}(x_{t_{i+1}^n-}^n, v_{t_i^n, h_i^n}^n) - F_{t_i^n}(x_{t_i^n}^n, v_{t_i^n}^n) \\ &+ F_{t_i^n}(x_{t_i^n}^n, v_{t_i^n-}^n) - F_{t_i^n}(x_{t_i^n-}^n, v_{t_i^n-}^n) \end{aligned} \quad (24)$$

where we have used property (2) to have $F_{t_i^n}(x_{t_i^n}^n, v_{t_i^n}^n) = F_{t_i^n}(x_{t_i^n}^n, v_{t_i^n-}^n)$. The first term can be written $\psi(h_i^n) - \psi(0)$ where:

$$\psi(u) = F_{t_i^n+u}(x_{t_i^n, u}^n, v_{t_i^n, u}^n) \quad (25)$$

Since $F \in \mathbb{C}^{1,2}([0, T])$, ψ is right-differentiable, and moreover by lemma 4, ψ is left-continuous, so:

$$F_{t_{i+1}^n}(x_{t_i^n, h_i^n}^n, v_{t_i^n, h_i^n}^n) - F_{t_i^n}(x_{t_i^n}^n, v_{t_i^n}^n) = \int_0^{t_{i+1}^n - t_i^n} \mathcal{D}_{t_i^n+u} F(x_{t_i^n, u}^n, v_{t_i^n, u}^n) du \quad (26)$$

The second term can be written $\phi(\delta x_i^n) - \phi(0)$, where:

$$\phi(u) = F_{t_i^n}(x_{t_i^n-}^{n,u}, v_{t_i^n-}^n) \quad (27)$$

Since $F \in \mathbb{C}^{1,2}([0, T])$, ϕ is well-defined and C^2 on the convex set $B(x(t_i^n), \eta_n) \subset U$, with:

$$\begin{aligned} \phi'(u) &= \nabla_x F_{t_i^n}(x_{t_i^n-}^{n,u}, v_{t_i^n-}^n) \\ \phi''(u) &= \nabla_x^2 F_{t_i^n}(x_{t_i^n-}^{n,u}, v_{t_i^n-}^n) \end{aligned} \quad (28)$$

So a second order Taylor expansion of ϕ at $u=0$ yields:

$$
\begin{aligned}
F_{t_i^n}(x^n_{t_i^n}, v^n_{t_i^n-}) - F_{t_i^n}(x^n_{t_i^n-}, v^n_{t_i^n-}) &= \nabla_x F_{t_i^n}(x^n_{t_i^n-}, v^n_{t_i^n-})\delta x_i^n \\
&+\frac{1}{2}\mathrm{tr}\left(\nabla_x^2 F_{t_i^n}(x^n_{t_i^n-}, v^n_{t_i^n-}) \quad {}^t\delta x_i^n \delta x_i^n\right) + r_i^n
\end{aligned} \tag{29}
$$

where r_i^n is bounded by

$$
K|\delta x_i^n|^2 \sup_{x\in B(x(t_i^n),\eta_n)} |\nabla_x^2 F_{t_i^n}(x^{n,x-x(t_i^n)}_{t_i^n-}, v^n_{t_i^n-}) - \nabla_x^2 F_{t_i^n}(x^n_{t_i^n-}, v^n_{t_i^n-})| \tag{30}
$$

Denote $i^n(t)$ the index such that $t\in[t^n_{i^n(t)}, t^n_{i^n(t)+1})$. We now sum all the terms above from $i=0$ to $k(n)-1$:.

- The left-hand side of (24) yields $F_T(x^n_{T-}, v^n_{T-}) - F_0(x_0, v_0)$, which converges to $F_T(x_{T-}, v_{T-}) - F_0(x_0, v_0)$ by left-continuity of F, and this quantity equals $F_T(x_T, v_T) - F_0(x_0, v_0)$ since x is continuous and F is predictable in the second variable.
- The first line in the right-hand side can be written:

$$
\int_0^T \mathcal{D}_u F(x^n_{t^n_{i^n(u)}, u-t^n_{i^n(u)}}, v^n_{t^n_{i^n(u)}, u-t^n_{i^n(u)}})du \tag{31}
$$

  where the integrand converges to $\mathcal{D}_u F(x_u, v_{u-})$ and is bounded by C. Hence the dominated convergence theorem applies and (31) converges to:

$$
\int_0^T \mathcal{D}_u F(x_u, v_{u-})du = \int_0^T \mathcal{D}_u F(x_u, v_u) \tag{32}
$$

  since $v_u = v_{u-}$, du-almost everywhere.
- The second line can be written:

$$
\sum_{i=0}^{k(n)-1} \nabla_x F_{t_i^n}(x^n_{t_i^n-}, v^n_{t_i^n-})(x_{t^n_{i+1}} - x_{t_i^n}) + \sum_{i=0}^{k(n)-1} \frac{1}{2} tr[\nabla_x^2 F_{t_i^n}(x^n_{t_i^n-}, v^n_{t_i^n-})]^t \delta x_i^n \delta x_i^n] + \sum_{i=0}^{k(n)-1} r_i^n \tag{33}
$$

  $[\nabla_x^2 F_{t_i^n}(x^n_{t_i^n-}, v^n_{t_i^n-})]1_{t\in]t_i^n, t^n_{i+1}]}$ is bounded by C, and converges to $\nabla_x^2 F_t(x_t, v_{t-})$ by left-continuity of $\nabla_x^2 F$, and the paths of both are left-continuous by lemma 4. Since x and the subdivision (π_n) are as in definition 10, lemma 12 in appendix C applies and gives as limit:

$$
\int_0^T \frac{1}{2}\mathrm{tr}[{}^t\nabla_x^2 F_t(x_u, v_{u-})]d[x](u)] = \int_0^T \frac{1}{2}\mathrm{tr}[{}^t\nabla_x^2 F_t(x_u, v_u)]d[x](u)] \tag{34}
$$

  since $\nabla_x^2 F$ is predictable in the second variable i.e. verifies (2). Using the same lemma, since $|r_i^n|$ is bounded by $\epsilon_i^n |\delta x_i^n|^2$ where ϵ_i^n converges to 0 and is bounded by $2C$, $\sum_{i=i^n(s)+1}^{i^n(t)-1} r_i^n$ converges to 0.

Since all other terms converge, the limit:

$$
\lim_n \sum_{i=0}^{k(n)-1} \nabla_x F_{t_i^n}(x^n_{t_i^n-}, v^n_{t_i^n-})(x(t^n_{i+1}) - x(t_i^n)) \tag{35}
$$

exists, and the result is established.

□

4 Change of variable formula for functionals of a cadlag path

We will now extend the previous result to functionals of cadlag paths. The following definition is a taken from Föllmer [7]:

Definition 11. Let $\pi_n = (t_0^n, \ldots, t_{k(n)}^n)$, where $0 = t_0^n \leq t_1^n \leq \ldots \leq t_{k(n)}^n = T$ be a sequence of subdivisions of $[0,T]$ with step decreasing to 0 as $n \to \infty$. $f \in D([0,T], \mathbb{R})$ is said to have finite quadratic variation along (π_n) if the sequence of discrete measures:

$$\xi^n = \sum_{i=0}^{k(n)-1} (f(t_{i+1}^n) - f(t_i^n))^2 \delta_{t_i^n} \tag{36}$$

where δ_t is the Dirac measure at t, converge vaguely to a Radon measure ξ on $[0,T]$ such that

$$[f](t) = \xi([0,t]) = [f]^c(t) + \sum_{0<s\leq t} (\Delta f(s))^2 \tag{37}$$

where $[f]^c$ is the continuous part of $[f]$. $[f]$ is called quadratic variation of f along the sequence (π_n). $x \in \mathcal{U}_T$ is said to have finite quadratic variation along the sequence (π_n) if the functions $x_i, 1 \leq i \leq d$ and $x_i + x_j, 1 \leq i < j \leq d$ do. The quadratic variation of x along (π_n) is the S_d^+-valued function x defined by:

$$[x]_{ii} = [x_i], \qquad [x]_{ij} = \frac{1}{2}([x_i + x_j] - [x_i] - [x_j]), \quad i \neq j \tag{38}$$

Theorem 4 (Change of variable formula for functionals of discontinuous paths)**.** *Let $(x, v) \in \mathcal{U}_T \times \mathcal{S}_T$ where x has finite quadratic variation along (π_n) and*

$$\sup_{t \in [0,T] - \pi_n} |x(t) - x(t-)| + |v(t) - v(t-)| \to 0 \tag{39}$$

Denote

$$x^n(t) = \sum_{i=0}^{k(n)-1} x(t_{i+1}-) 1_{[t_i, t_{i+1})}(t) + x(T) 1_{\{T\}}(t)$$

$$v^n(t) = \sum_{i=0}^{k(n)-1} v(t_i) 1_{[t_i, t_{i+1})}(t) + v(T) 1_{\{T\}}(t), \qquad h_i^n = t_{i+1}^n - t_i^n \tag{40}$$

Then for any non-anticipative functional $F \in \mathbb{C}^{1,2}$ satisfying the following assumptions:

1. *F is predictable in the second variable in the sense of* (2)
2. *$\nabla_x^2 F$ and $\mathcal{D}F$ have the local boundedness property* (9)
3. *$F, \nabla_x F, \nabla_x^2 F \in \mathbb{F}_l^\infty$*
4. *$\nabla_x F$ has the horizontal local Lipschitz property* (15) *(Definition 7)*

the following limit exists

$$\int_{]0,T]} \nabla_x F_t(x_{t-}, v_{t-}) d^\pi x := \lim_{n\to\infty} \sum_{i=0}^{k(n)-1} \nabla_x F_{t_i^n}(x^{n,\Delta x(t_i^n)}_{t_i^n -}, v^n_{t_i^n -})(x(t^n_{i+1}) - x(t^n_i)) \tag{41}$$

and

$$F_T(x_T, v_T) - F_0(x_0, v_0) = \int_{]0,T]} \mathcal{D}_t F_t(x_{u-}, v_{u-}) du + \int_{]0,T]} \frac{1}{2} \mathrm{tr}\left({}^t\nabla_x^2 F_t(x_{u-}, v_{u-}) d[x]^c(u)\right)$$
$$+ \int_{]0,T]} \nabla_x F_t(x_{t-}, v_{t-}) d^\pi x + \sum_{u\in]0,T]} [F_u(x_u, v_u) - F_u(x_{u-}, v_{u-}) - \nabla_x F_u(x_{u-}, v_{u-}).\Delta x(u)] \tag{42}$$

Remark 7. Condition (39) simply means that the subdivision asymptotically contains all discontinuity points of (x, v). Since a cadlag function has at most a countable set of discontinuities, this can always be achieved by adding e.g. the discontinuity points $\{t \in [0,T], \max(|\Delta x(t)|, |\Delta v(t)|) \geq 1/n\}$ to π_n.

Proof. Denote $\delta x_i^n = x(t^n_{i+1}) - x(t^n_i)$. Denote also

$$\eta_n = \sup\{|v(u) - v(t_i^n)| + |x(u) - x(t_i^n)| + |t^n_{i+1} - t^n_i|, 0 \leq i \leq k(n) - 1, u \in [t^n_i, t^n_{i+1})\} \tag{43}$$

and note that this quantity converges to 0 as $n \to \infty$, thanks to lemma 8. We assume n sufficiently large so that for some $C > 0$ such that, for any $t < T$, for any $(x', v') \in \mathcal{U}_t \times \mathcal{S}_t$, $d_\infty((x_t, v_t), (x', v')) < \eta_n \Rightarrow |\mathcal{D}_t F_t(x', v')| \leq C, |\nabla_x^2 F_t(x', v')| \leq C$, using the local boundedness property satisfied by these derivatives.

For $\epsilon > 0$, we separate the jump times of x in two sets: a finite set $C_1(\epsilon)$ and a set $C_2(\epsilon)$ such that $\sum_{s\in C_2(\epsilon)} |\Delta x_s|^2 < \epsilon^2$. We also separate the indices $0 \leq i \leq k(n) - 1$ in two sets: a set $I_1^n(\epsilon)$ such that $(t_i, t_{i+1}]$ contains at least a time in $C_1(\epsilon)$, and its complementary $I_2^n(\epsilon)$. Denoting $K = \overline{\{x(u), s \leq u \leq t\}}$ which is a compact subset of U, and $U^c = \mathbb{R} - U$, one may choose ϵ sufficiently small and n sufficiently large so that $d(K, U^c) > \epsilon + \eta_n$.
Denote $i^n(t)$ the index such that $t \in [t^n_i, t^n_{i+1})$. Property (39) implies that for n sufficiently $C_1(\epsilon) \subset \{t^n_{i+1}, i = 1..k(n)\}$ so

$$\sum_{0\leq i\leq k(n)-1, i\in I_1^n(\epsilon)} F_{t^n_{i+1}}(x^{n,\Delta x(t^n_{i+1})}_{t^n_{i+1}-}, v^n_{t^n_{i+1}-}) - F_{t^n_i}(x^{n,\Delta x(t^i_n)}_{t^n_i -}, v^n_{t^n_i -}) \to \sum_{u\in]0,T]\cup C_1(\epsilon)} F_u(x_u, v_u) - F_u(x_{u-}, v_{u-}) \tag{44}$$

as $n \to \infty$, by left-continuity of F.
Let us now consider, for $i \in I_2^n(\epsilon), i \leq k(n) - 1$, the decomposition:

$$\begin{aligned} F_{t^n_{i+1}}(x^{n,\Delta x(t^n_{i+1})}_{t^n_{i+1}-}, v^n_{t^n_{i+1}-}) - F_{t^n_i}(x^{n,\Delta x(t^n_i)}_{t^n_i -}, v^n_{t^n_i -}) &= F_{t^n_{i+1}}(x^{n,\Delta x(t^n_{i+1})}_{t^n_{i+1}-}, v^n_{t^n_{i+1}-}) - F_{t^n_{i+1}}(x^n_{t^n_{i+1}-}, v^n_{t^n_{i+1}-}) \\ &+ F_{t^n_{i+1}}(x^n_{t^n_{i+1}-}, v^n_{t^n_i, h^n_i}) - F_{t^n_i}(x^n_{t^n_i}, v^n_{t^n_i}) \\ &+ F_{t^n_i}(x^n_{t^n_i}, v^n_{t^n_i -}) - F_{t^n_i}(x^{n,\Delta x(t^n_i)}_{t^n_i -}, v^n_{t^n_i -}) \end{aligned} \tag{45}$$

where we have used the property (2) to obtain $F_{t^n_i}(x^n_{t^n_i}, v^n_{t^n_i}) = F_{t^n_i}(x^n_{t^n_i}, v^n_{t^n_i -})$. The second line in the right-hand side can be written $\psi(h^n_i) - \psi(0)$ where:

$$\psi(u) = F_{t^n_i + u}(x^n_{t^n_i, u}, v^n_{t^n_i, u}) \tag{46}$$

Since $F \in \mathbb{C}^{1,2}([0,T])$, ψ is right-differentiable, and moreover by lemma 4, ψ is continuous, so:

$$F_{t_{i+1}^n}(x^n_{t_i^n,h_i^n}, v^n_{t_i^n,h_i^n}) - F_{t_i^n}(x^n_{t_i^n}, v^n_{t_i^n}) = \int_0^{t_{i+1}^n - t_i^n} \mathcal{D}_{t_i^n+u} F(x^n_{t_i^n,u}, v^n_{t_i^n,u}) du \tag{47}$$

The third line can be written $\phi(x(t_{i+1}^n -) - x(t_i^n)) - \phi(0)$, where:

$$\phi(u) = F_{t_i^n}(x_{t_i^n-}^{n,\Delta x(t_i^n)+u}, v^n_{t_i^n-}) \tag{48}$$

Since $F \in \mathbb{C}^{1,2}([0,T])$, ϕ is well-defined and C^2 on the convex set $B(x(t_i^n), \eta_n + \epsilon) \subset U$, with:

$$\phi'(u) = \nabla_x F_{t_i^n}(x_{t_i^n-}^{n,\Delta x(t_i^n)+u}, v^n_{t_i^n-}) \phi''(u) = \nabla_x^2 F_{t_i^n}(x_{t_i^n-}^{n,\Delta x(t_i^n)+u}, v^n_{t_i^n-}) \tag{49}$$

So a second order Taylor expansion of ϕ at $u = 0$ yields:

$$\begin{aligned} F_{t_i^n}(x^n_{t_i^n}, v^n_{t_i^n-}) - F_{t_i^n}(x_{t_i^n-}^{n,\Delta x(t_i^n)}, v^n_{t_i^n-}) &= \nabla_x F_{t_i^n}(x_{t_i^n-}^{n,\Delta x(t_i^n)}, v^n_{t_i^n-})(x(t_{i+1}^n-) - x(t_i^n)) \\ &+ \frac{1}{2} tr[\nabla_x^2 F_{t_i^n}(x_{t_i^n-}^{n,\Delta x(t_i^n)}, v^n_{t_i^n-})]^t (x(t_{i+1}^n-) - x(t_i^n))(x(t_{i+1}^n-) - x(t_i^n))] + r_i^n \end{aligned} \tag{50}$$

where $r_{i,1}^n$ is bounded by

$$K|(x(t_{i+1}^n-) - x(t_i^n))|^2 \sup_{x \in B(x(t_i^n), \eta_n+\epsilon)} |\nabla_x^2 F_{t_i^n}(x_{t_i^n-}^{n,x-x(t_i^n)}, v^n_{t_i^n-}) - \nabla_x^2 F_{t_i^n}(x^n_{t_i^n-}, v^n_{t_i^n-})| \tag{51}$$

Similarly, the first line can be written $\phi(x(t_{i+1}^n) - x(t_i^n)) - \phi(x(t_{i+1}^n-) - x(t_i^n))$ where $\phi(u) = F_{t_{i+1}^n}(x_{t_i^n-,h_i^n}^{n,\Delta x(t_i^n)+u}, v^n_{t_{i-}^n})$. So, a second order Taylor expansion of ϕ at $u = 0$ yields:

$$\begin{aligned} F_{t_{i+1}^n}(x_{t_{i+1}^n-}^{n,\Delta x(t_{i+1}^n)}, v^n_{t_{i+1}^n-}) - F_{t_{i+1}^n}(x^n_{t_{i+1}^n-}, v^n_{t_{i+1}^n-}) &= \nabla_x F_{t_{i+1}^n}(x_{t_i^n-,h_i^n}^{n,\Delta x(t_i^n)}, v^n_{t_i^n,h_i^n}) \Delta x(t_{i+1}^n) \\ &+ \frac{1}{2} tr[\nabla_x^2 F_{t_{i+1}^n}(x_{t_i^n-,h_i^n}^{n,\Delta x(t_i^n)}, v^n_{t_i^n,h_i^n})]^t \Delta x(t_{i+1}^n) \Delta x(t_{i+1}^n) + r_{i,2}^n \end{aligned} \tag{52}$$

where $r_{i,2}^n$ is bounded by

$$K|\Delta x(t_{i+1}^n)|^2 \sup_{x \in B(x(t_i^n), \eta_n+\epsilon)} |\nabla_x^2 F_{t_i^n}(x_{t_i^n-}^{n,x-x(t_i^n-)}, v^n_{t_i^n-}) - \nabla_x^2 F_{t_i^n}(x_{t_i^n-}^{n,\Delta x(t_i^n)}, v^n_{t_i^n-})| \tag{53}$$

Using the horizontal local Lipschitz property (15) for $\nabla_x F$, for n sufficiently large:

$$|\nabla_x F_{t_{i+1}^n}(x_{t_i^n-,h_i^n}^{n,\Delta x(t_i^n)}, v^n_{t_i^n,h_i^n}) - \nabla_x F_{t_i^n}(x_{t_i^n-}^{n,\Delta x(t_i^n)}, v^n_{t_i^n-})| < C(t_{i+1}^n - t_i^n) \tag{54}$$

On other hand, since $\nabla_x^2 F$ is bounded by C on all paths considered:

$$\begin{aligned} | \quad & \mathrm{tr}\left(\nabla_x^2 F_{t_i^n}(x_{t_i^n-}^{n,\Delta x(t_i^n)}, v^n_{t_i^n-})^t (x(t_{i+1}^n-) - x(t_i^n))(x(t_{i+1}^n-) - x(t_i^n))\right) \\ &+ \mathrm{tr}\left(\nabla_x^2 F_{t_{i+1}^n}(x_{t_i^n-,h_i^n}^{n,\Delta x(t_i^n)}, v^n_{t_i^n,h_i^n})]^t \Delta x(t_{i+1}^n) \Delta x(t_{i+1}^n)\right) \\ &- \mathrm{tr}\left(\nabla_x^2 F_{t_i^n}(x_{t_i^n-}^{n,\Delta x(t_i^n)}, v^n_{t_i^n-})]^t \delta x_i^n \delta x_i^n\right) | < 2C|\Delta x(t_{i+1}^n)|^2 \end{aligned} \tag{55}$$

Hence, we have shown that:

$$F_{t_{i+1}^n}(x^{n,\Delta x(t_{i+1}^n)}_{t_{i+1}^n-}, v^n_{t_{i+1}^n-}) - F_{t_{i+1}^n}(x^n_{t_{i+1}^n-}, v^n_{t_{i+1}^n-}) + F_{t_i^n}(x^n_{t_i^n}, v^n_{t_i^n-}) - F_{t_i^n}(x^{n,\Delta x(t_i^n)}_{t_i^n-}, v^n_{t_i^n-}) =$$
$$\nabla_x F_{t_i^n}(x^{n,\Delta x(t_i^n)}_{t_i^n-}, v^n_{t_i^n-})\delta x_i^n + \frac{1}{2} tr[\nabla_x^2 F_{t_i^n}(x^{n,\Delta x(t_i^n)}_{t_i^n-}, v^n_{t_i^n-})]^t \delta x_i^n \delta x_i^n] + r_i^n + q_i^n$$

where r_i^n is bounded by:

$$4K|\delta x_i^n|^2 \sup_{x\in B(x(t_i^n),\eta_n+\epsilon)} |\nabla_x^2 F_{t_i^n}(x^{n,x-x(t_i^n-)}_{t_i^n-}, v^n_{t_i^n-}) - \nabla_x^2 F_{t_i^n}(x^{n,\Delta x(t_i^n)}_{t_i^n-}, v^n_{t_i^n-})| \tag{56}$$

and q_i^n is bounded by:

$$C'(h_i^n|\Delta x(t_i^n)| + |\Delta x(t_i^n)|^2) \tag{57}$$

Denote $i^n(t)$ the index such that $t \in [t^n_{i^n(t)}, t^n_{i^n(t)+1}[$. Summing all the terms above for $i \in C_2(\epsilon) \cap \{0, 1, ..k(n) - 1\}$:

- The left-hand side of (45) yields

$$F_T(x_T^n, v_T^n) - F_0(x_0, v_0) - \sum_{0\le i\le k(n)-1, i\in I_1^n(\epsilon)} F_{t_{i+1}^n}(x^n_{t_{i+1}^n}, v^n_{t_{i+1}^n}) - F_{t_i^n}(x^n_{t_i^n}, v^n_{t_i^n}) \tag{58}$$

  which converges to

$$F_T(x_T, v_T) - F_0(x_0, v_0) - \sum_{u\in]0,T]\cup C_1(\epsilon)} F_u(x_u, v_u) - F_u(x_{u-}, v_{u-}) \tag{59}$$

- The sum of the first and third lines of (45) the right-hand side can be written:

$$\sum_{0\le i\le k(n)-1, i\in I_2^n(\epsilon)} \nabla_x F_{t_i^n}(x^{n,\Delta x(t_i^n)}_{t_i^n-}, v^n_{t_i^n-})\delta x_i^n$$
$$+ \sum_{0\le i\le k(n)-1, i\in I_2^n(\epsilon)} \frac{1}{2}\mathrm{tr}\left(\nabla_x^2 F_{t_i^n}(x^{n,\Delta x(t_i^n)}_{t_i^n-}, v^n_{t_i^n-})^t \delta x_i^n \delta x_i^n\right)$$
$$+ \sum_{0\le i\le k(n)-1, i\in I_2^n(\epsilon)} r_i^n + q_i^n \tag{60}$$

  Consider the measures $\mu_{ij}^n = \xi_{ij}^n - \sum_{0<s\le T, s\in C_2(\epsilon)} (\Delta f_{ij}(s))^2 \delta_s$, where $f_{ii} = x_i, 1 \le j \le d$ and $f_{ij} = x_i + x_j, 1 \le i < j \le d$ and ξ_{ij}^n is defined in Definition 11. The second line of (60) can be decomposed as:

$$A_n + \frac{1}{2} \sum_{0<u\le T, u\in C_2(\epsilon)} \mathrm{tr}\left(\nabla_x^2 F_{t_i^n}(x^{n,\Delta x(t_i^n)}_{t_i^n-}, v^n_{t_i^n-})^t \Delta x(u)\Delta x(u)\right) \tag{61}$$

  where

$$A_n = \mathrm{tr} \int_{]0,T]} \mu^n(dt) \sum_{0\le i\le k(n)-1, i\in I_2^n(\epsilon)} \nabla_x^2 F_{t^n_{i^n(t)}}(x^{n,\Delta x(t^n_{i^n(t)})}_{t^n_{i^n(t)}-}, v^n_{t^n_{i^n(t)}-}) \quad 1_{t\in(t_i^n, t_{i+1}^n]}$$

where μ^n denotes the matrix-valued measure with components μ^n_{ij} defined above. μ^n_{ij} converges vaguely to the atomless measure $[f_{ij}]^c$. Since $\sum_{0\le i\le k(n)-1, i\in I_2^n(\epsilon)} \nabla_x^2 F_{t^n_{i^n(t)}}(x^{n,\Delta x(t^n_{i^n(t)})}_{t^n_{i^n(t)}-}, v^n_{t^n_{i^n(t)}-})\, 1_{t\in(t^n_i,t^n_{i+1}]}$ is bounded by C and converges to $\nabla_x^2 F_t(x_{t-}, v_{t-})1_{t\notin C_1(\epsilon)}$ by left-continuity of $\nabla_x^2 F$, applying Lemma 12 to A_n and yields that A_n converges to:

$$\int_{]0,T]} \frac{1}{2}\mathrm{tr}\left({}^t\nabla_x^2 F_t(x_{u-}, v_{u-})d[x]^c(u)\right) \tag{62}$$

The second term in (61) has the lim sup of its absolute value bounded by $C\epsilon^2$. Using the same argument, since $|r^n_i|$ is bounded by $s^n_i|\delta x^n_i|^2$ for some s^n_i which converges to 0 and is bounded by some constant, $\sum_{i=0}^{k(n)-1}|r^n_i|$ has its lim sup bounded by $2C\epsilon^2$; similarly, the lim sup of $\sum_{i=0}^{k(n)-1}|q^n_i|$ is bounded by $C'(T\epsilon+\epsilon^2)$.
The term in the first line of (60) can be written:

$$\begin{aligned} &\sum_{i=0}^{k(n)-1} \nabla_x F_{t^n_i}(x^{n,\Delta x(t^n_i)}_{t^n_i-}, v^n_{t^n_i-})(x(t^n_{i+1})-x(t^n_i)) \\ -&\sum_{0\le i\le k(n)-1, i\in I_1^n(\epsilon)} \nabla_x F_{t^n_i}(x^{n,\Delta x(t^n_i)}_{t^n_i-}, v^n_{t^n_i-})(x_{t^n_{i+1}} - x_{t^n_i}) \end{aligned} \tag{63}$$

where the second term converges to $\sum_{0<u\le T, u\in C_1(\epsilon)} \nabla_x F_u(x_{u-}, v_{u-})\Delta x(u)$.

- The second line of (45):

$$\int_0^T \mathcal{D}_t F_u(x^n_{t^n_{i^n(u)}, u-t^n_{i^n(u)}}, v^n_{t^n_{i^n(u)}, u-t^n_{i^n(u)}}) 1_{i^n(u)\in I_2^n(\epsilon)} du \tag{64}$$

where the integrand converges to $\mathcal{D}_t F_u(x_{u-}, v_{u-})1_{u\notin C_1(\epsilon)}$ and is bounded by C, hence by dominated convergence this term converges to:

$$\int_0^T \mathcal{D}_t F_t(x_{u-}, v_{u-})du \tag{65}$$

Summing up, we have established that the difference between the lim sup and the lim inf of:

$$\sum_{i=0}^{k(n)-1} \nabla_x F_{t^n_i}(x^{n,\Delta x(t^n_i)}_{t^n_i-}, v^n_{t^n_i-})(x(t^n_{i+1})-x(t^n_i))1_{s<t^n_i\le t} \tag{66}$$

is bounded by $C''(\epsilon^2+T\epsilon)$. Since this is true for any ϵ, this term has a limit.
Let us now write the equality we obtained for a fixed ϵ:

$$\begin{aligned} F_T(x_T, v_{tT}) - F_0(x_0, v_0) &= \int_{]0,T]} \mathcal{D}_t F_t(x_{u-}, v_{u-})du + \int_{]0,T]} \frac{1}{2}\mathrm{tr}[{}^t\nabla_x^2 F_t(x_{u-}, v_{u-})d[x]^c(u)] \\ &+ \lim_n \sum_{i=0}^{k(n)-1} \nabla_x F_{t^n_i}(x^{n,\Delta x(t^n_i)}_{t^n_i-}, v^n_{t^n_i-})(x(t^n_{i+1})-x(t^n_i)) \\ &+ \sum_{u\in]0,T]\cup C_1(\epsilon)} [F_u(x_u, v_u) - F_u(x_{u-}, v_{u-}) - \nabla_x F_u(x_{u-}, v_{u-})\Delta x(u)] + \alpha(\epsilon) \end{aligned}$$

where $\alpha(\epsilon) \leq C''(\epsilon^2 + T\epsilon)$. The only point left to show is that:

$$\sum_{u \in]0,T] \cup C_1(\epsilon)} [F_u(x_u, v_u) - F_u(x_{u-}, v_{u-}) - \nabla_x F_u(x_{u-}, v_{u-})\Delta x(u)] \tag{67}$$

converges to:

$$\sum_{u \in]0,T]} [F_u(x_u, v_u) - F_u(x_{u-}, v_{u-}) - \nabla_x F_u(x_{u-}, v_{u-})\Delta x(u)] \tag{68}$$

which is to say that the sum above is absolutely convergent.
We can first choose $d(K, U^c) > \eta > 0$ such that:

$$\forall u \in [0,T], \quad \forall (x', v') \in \mathcal{U}_u \times \mathcal{S}_u, d_\infty((x_t, v_t), (x', v')) \leq \eta \Rightarrow |\nabla_x^2 F_u(x(u), v(u))| < C \tag{69}$$

The jumps of x of magnitude greater than η are in finite number. Then, if u is a jump time of x of magnitude less than η, then $x(u-) + h\Delta x(u) \in U$ for $h \in [0,1]$, so that we can write:

$$F_u(x_u, v_u) - F_u(x_{u-}, v_{u-}) - \nabla_x F_u(x_{u-}, v_{u-})\Delta x(u) =$$
$$\int_0^1 (1-v)[{}^t\nabla_x^2 F_u(x_{u-}^{h\Delta x(u)}, v_{u-})^t \Delta x(u) \Delta x(u)] \ \leq \frac{1}{2} C |\Delta x(u)|^2$$

Hence, the theorem is established.

□

Remark 8. If the vertical derivatives are right-continuous instead of left-continuous, and $\nabla_x F$ not necessarily locally Lipschitz in time, define:

$$x^n(t) = \sum_{i=0}^{k(n)-1} x(t_i) 1_{[t_i, t_{i+1})}(t) + x(T) 1_{\{T\}}(t)$$
$$v^n(t) = \sum_{i=0}^{k(n)-1} v(t_i) 1_{[t_i, t_{i+1})}(t) + v(T) 1_{\{T\}}(t) \qquad h_i^n = t_{i+1}^n - t_i^n \tag{70}$$

Following the same argument than in the proof with the decomposition:

$$\begin{aligned} F_{t_{i+1}^n}(x_{t_{i+1}^n}^n, v_{t_{i+1}^n}) - F_{t_i^n}(x_{t_i^n}^n, v_{t_i^n}) &= F_{t_{i+1}^n}(x_{t_{i+1}^n}^n, v_{t_{i+1}^n}) - F_{t_{i+1}^n}(x_{t_{i+1}^n}^n, v_{t_i^n, h_i^n}) \\ &+ F_{t_{i+1}^n}(x_{t_{i+1}^n}^n, v_{t_i^n, h_i^n}) - F_{t_{i+1}^n}(x_{t_i^n, h_i^n}^n, v_{t_i^n, h_i^n}) \\ &+ F_{t_{i+1}^n}(x_{t_i^n, h_i^n}^n, v_{t_i^n, h_i^n}) - F_{t_i^n}(x_{t_i^n}^n, v_{t_i^n}) \end{aligned} \tag{71}$$

leads to the formula with the following Riemann sum:

$$\lim_n \sum_{i=0}^{k(n)-1} \nabla_x F_{t_{i+1}^n}(x_{t_i^n, h^i}^n, v_{t_i^n, h_i^n}^n)(x(t_{i+1}^n) - x(t_i^n)) \tag{72}$$

5 Functionals of Dirichlet processes

A Dirichlet process [8, 4], or finite energy process, on a filtered probability space $(\Omega, \mathcal{B}, (\mathcal{B}_t), \mathbb{P})$ is an adapted cadlag process that can be represented as the sum of a semimartingale and an adapted continuous process with zero quadratic variation along dyadic subdivisions.

For continuous Dirichlet processes, a pathwise Itô calculus was introduced by H. Föllmer in [7, 8, 11]. Coquet, Mémin and Slominski [4] extended these results to discontinuous Dirichlet processes [14]. Using Theorem 4 we can extend these results to functionals of Dirichlet processes; this yields in particular a pathwise construction of stochastic integrals for functionals of a Dirichlet process.

Let $Y(t) = X(t) + B(t)$ be a U-valued Dirichlet process defined as the sum of a semimartingale X on some filtered probability space $(\Omega, \mathcal{B}, \mathcal{B}_t, \mathbb{P})$ and B an adapted continuous process B with zero quadratic variation along the dyadic subdivision. We denote by $[X]$ the quadratic variation process associated to X, $[X]^c$ the continuous part of $[X]$, and $\mu(dt\ dz)$ the integer-valued random measure describing the jumps of X (see [10] for definitions).

Let A be an adapted process with S-valued cadlag paths. Note that A need not be a semimartingale.

We call $\Pi_n = \{0 = t_0^n < t_1^n < \ldots < t_{k(n)}^n = T\}$ a *random subdivision* if the t_i^n are stopping times with respect to $(\mathcal{B}_t)_{t\in[0,T]}$.

Proposition 5 (Change of variable formula for Dirichlet processes)**.** *Let $\Pi_n = \{0 = t_0^n < t_1^n < \ldots < t_{k(n)}^n = T\}$ be any sequence of random subdivisions of $[0,T]$ such that*
(i) X has finite quadratic variation along Π_n and B has zero quadratic variation along Π_n almost-surely,

$$(ii) \qquad \sup_{t\in[0,T]-\Pi_n} |Y(t) - Y(t-)| + |A(t) - A(t-)| \stackrel{n\to\infty}{\to} 0 \quad \mathbb{P} - a.s.$$

Then there exists $\Omega_1 \subset \Omega$ with $\mathbb{P}(\Omega_1) = 1$ such that for any non-anticipative functional $F \in \mathbb{C}^{1,2}$ satisfying

1. *F is predictable in the second variable in the sense of* (2)
2. *$\nabla_x^2 F$ and $\mathcal{D}F$ satisfy the local boundedness property* (9)
3. *$F, \nabla_x F, \nabla_x^2 F \in \mathbb{F}_l^\infty$*
4. *$\nabla_x F$ has the horizontal local Lipschitz property* (15)*,*

the following equality holds on Ω_1 for all $t \le T$:

$$\begin{aligned} F_t(Y_t, A_t) - F_0(Y_0, A_0) = \int_{]0,t]} \mathcal{D}_u F(Y_{u-}, A_{u-})du + \int_{]0,t]} \frac{1}{2}\mathrm{tr}[{}^t\nabla_x^2 F_u(Y_{u-}, A_{u-})d[X]^c(u)] \\ + \int_{]0,t]}\int_{R^d} [F_u(Y_{u-}^z, A_u-) - F_u(Y_{u-}, A_{u-}) - z\nabla_x F_u(Y_{u-}, A_{u-})]\mu(du, dz) \\ + \int_{]0,t]} \nabla_x F_u(Y_{u-}, A_{u-}).dY(u) \quad a.s. \end{aligned} \tag{73}$$

where the last term is the Föllmer integral (41) *along the subdivision Π_n, defined for $\omega \in \Omega_1$ by:*

$$\int_{]0,t]} \nabla_x F_u(Y_{u-}, A_{u-}).dY(u) := \lim_n \sum_{i=0}^{k(n)-1} \nabla_x F_{t_i^n}(Y_{t_i^n-}^{n,\Delta Y(t_i^n)}, A_{t_i^n-}^n)(Y(t_{i+1}^n) - Y(t_i^n))1_{]0,t]} \tag{74}$$

where (Y^n, A^n) *are the piecewise constant approximations along* Π_n*, defined as in* (40).

Moreover, the Föllmer integral with respect to any other random subdivision verifying (i)–(ii), is almost-surely equal to (74).

Remark 9. Note that the convergence of (74) holds over a set Ω_1 which may be chosen independently of the choice of $F \in \mathbb{C}^{1,2}$.

Proof. Let (Π_n) be a sequence of random subdivisions verifying (i)–(ii). Then there exists a set Ω_1 with $\mathbb{P}(\Omega_1) = 1$ such that for $\omega \in \Omega_1$ (X, A) is a cadlag function and (i)-(ii) hold pathwise. Applying Theorem 4 to $(Y(., \omega), A(., \omega))$ along the subdivision $\Pi_n(\omega)$ shows that (73) holds on Ω_1.

To show independence of the limit in (74) from the chosen subdivision, we note that if Π_n^2 another sequence of random subdivisions satisfies (i)–(ii), there exists $\Omega_2 \subset \Omega$ with $\mathbb{P}(\Omega_2) = 1$ such that one can apply Theorem 4 pathwise for $\omega \in \Omega_2$. So we have

$$\int_{]0,t]} \nabla_x F_u(Y_{u-}, A_{u-}).d^{\Pi^2} Y(u) = \int_{]0,t]} \nabla_x F_u(Y_{u-}, A_{u-}).d^{\Pi} Y(u)$$

on $\Omega_1 \cap \Omega_2$. Since $\mathbb{P}(\Omega_1 \cap \Omega_2) = 1$ we obtain the result. □

6 Functionals of semimartingales

Proposition 5 holds of course when X is a semimartingale. We will now show that in this case, under an additional assumption, the pathwise integral coincides almost-surely with the stochastic integral $\int Y dX$.

6.1 Cadlag semimartingales

Let X be a cadlag semimartingale and A an adapted cadlag process on $(\Omega, \mathcal{B}, \mathcal{B}_t, \mathbb{P})$. We use the notations $[X]$, $[X]^c$, $\mu(dt\ dz)$ defined in Section 5.

Theorem 4 yields an Itô formula for functionals of X: under the additional assumption $\nabla_x F \in \mathbb{B}$, the pathwise Föllmer integral coincides with the stochastic integral.

Proposition 6 (Functional Itô formula for a semimartingale)**.** *Let* $F \in \mathbb{C}^{1,2}$ *be a non-anticipative functional satisfying*

1. *F is predictable in the second variable, i.e. verifies* (2)*,*
2. $\nabla_x F,\ \nabla_x^2 F,\ \mathcal{D}F \in \mathbb{B}$,
3. $F, \nabla_x F, \nabla_x^2 F \in \mathbb{F}_l^\infty$,
4. *$\nabla_x F$ has the horizontal local Lipschitz property 15.*

Then:

$$F_t(X_t, A_t) - F_0(X_0, A_0) =$$
$$\int_{]0,t]} \mathcal{D}_u F(X_{u-}, A_{u-})du + \int_{]0,t]} \frac{1}{2}\mathrm{tr}[{}^t\nabla_x^2 F_u(X_{u-}, A_{u-})d[X]^c(u)] + \int_{]0,t]} \nabla_x F_u(X_{u-}, A_{u-}).dX(u)$$
$$+ \int_{]0,t]}\int_{R^d} [F_u(X_{u-}^z, A_u-) - F_u(X_{u-}, A_{u-}) - z.\nabla_x F_u(X_{u-}, A_{u-})]\mu(du, dz), \mathbb{P}\text{-}a.s. \tag{75}$$

where the stochastic integral is the Itô integral with respect to a semimartingale.
In particular, $Y(t) = F_t(X_t, A_t)$ is a semimartingale.

Remark 10. These results yield a non-probabilistic proof for functional Ito formulas obtained for continuous semimartingales [2, 3, 6] using probabilistic methods.

Proof. Assume first that the process X does not exit a compact set $K \subset U$, and that A is bounded by some constant $R > 0$. We define the following sequence of stopping times:

$$\tau_0^n = 0$$
$$\tau_k^n = \inf\{u > \tau_{k-1}^n | 2^n u \in \mathbb{N} \text{ or } \quad |A(u) - A(u-)| \vee |X(u) - X(u-)| > \frac{1}{n}\} \wedge T \tag{76}$$

Then the coordinate processes X_i and their sums $X_i + X_j$ satisfy the property:

$$\sum_{\tau_i < s} (Z(\tau_i) - Z(\tau_{i-1}))^2 \underset{n\to\infty}{\overset{\mathbb{P}}{\to}} [Z](s) \tag{77}$$

in probability. There exists a subsequence of subdivisions such that the convergence happens almost surely for all s rational, and hence it happens almost surely for all s because both sides of (77) are right-continuous. Let Ω_1 be the set on which this convergence happens, and on which the paths of X and A are U-valued cadlag functions. For $\omega \in \Omega_1$, Theorem 4 applies and yields

$$F_t(X_t, A_t) - F_0(X_0, A_0) = \int_{]0,t]} \mathcal{D}_u F(X_{u-}, A_{u-})du + \int_{]0,t]} \frac{1}{2}\mathrm{tr}[{}^t\nabla_x^2 F_u(X_{u-}, A_{u-})d[X]^c(u)] \tag{78}$$
$$+ \int_{]0,t]}\int_{R^d} [F_u(X_{u-}^z, A_u-) - F_u(X_{u-}, A_{u-}) - z.\nabla_x F_u(X_{u-}, A_{u-})]\mu(du, dz)$$
$$+ \lim_{n\to\infty} \sum_{i=0}^{k(n)-1} \nabla_x F_{\tau_i^n}(X_{\tau_i^n -}^{n,\Delta X(\tau_i^n)}, A_{\tau_i^n -}^n)(X(\tau_{i+1}^n) - X(\tau_i^n))$$

It remains to show that the last term, which may also be written as

$$\lim_{n\to\infty} \int_{]0,t]} \sum_{i=0}^{k(n)-1} 1_{]\tau_i^n, \tau_{i+1}^n]}(t) \quad \nabla_x F_{\tau_i^n}(X_{\tau_i^n -}^{n,\Delta X(\tau_i^n)}, A_{\tau_i^n -}^n).dX(t) \tag{79}$$

coincides with the (Ito) stochastic integral of $\nabla_x F(X_{u-}, A_{u-})$ with respect to the semimartingale X.

First, we note that since X, A are bounded and $\nabla_x F \in \mathbb{B}$, $\nabla_x F(X_{u-}, A_{u-})$ is a bounded predictable process (by Theorem 2) hence its stochastic integral $\int_0^{\cdot} \nabla_x F(X_{u-}, A_{u-}).dX(u)$ is well-defined. Since the integrand in (79) converges almost surely to $\nabla_x F_t(X_{t-}, A_{t-})$, and is bounded independently of n by a deterministic constant C, the dominated convergence theorem for stochastic integrals [13, Ch.IV Theorem32] ensures that (79) converges in probability to $\int_{]0,t]} \nabla_x F_u(X_{u-}, A_{u-}).dX(u)$. Since it converges almost-surely by proposition 5, by almost-sure uniqueness of the limit in probability, the limit has to be $\int_{]0,t]} \nabla_x F_u(X_{u-}, A_{u-}).dX(u)$.

Now we consider the general case where X and A may be unbounded. Let $U^c = \mathbb{R}^d - U$ and denote $\tau_n = \inf\{s < t | d(X(s), U^c) \le \frac{1}{n} \text{ or } |X(s)| \ge n \text{ or } |A(s)| \ge n\} \wedge t$, which are stopping times. Applying the previous result to the stopped processes $(X^{\tau_n -}, A^{\tau_n -}) = (X(t \wedge \tau_n -), A(t \wedge \tau_n -))$ leads to:

$$\begin{aligned} F_t(X_t^{\tau_n -}, A_t^{\tau_n -}) &= \int_{]0,\tau^n)} [\mathcal{D}_u F(X_u, A_u) du + \frac{1}{2} \mathrm{tr}[{}^t \nabla_x^2 F_u(X_u, A_u) d[X]^c(u)] \\ &+ \int_{]0,\tau_n)} \nabla_x F_u(X_u, A_u).dX(u) \\ &+ \int_{]0,\tau_n)} \int_{R^d} [F_u(X_{u-}^x, A_u -) - F_u(X_{u-}, A_{u-}) - z.\nabla_x F_u(X_{u-}, A_{u-})] \mu(du\ dz) \\ &+ \int_{(\tau^n, t)} \mathcal{D}_u F(X_u^{\tau_n}, A_u^{\tau_n}) du \end{aligned} \tag{80}$$

Since almost surely $t \wedge \tau_n = t$ for n sufficiently large, taking the limit $n \to \infty$ yields:

$$\begin{aligned} F_t(X_{t-}, A_{t-}) &= \int_{]0,t)} [\mathcal{D}_u F(X_u, A_u) du + \frac{1}{2} \mathrm{tr} \left({}^t \nabla_x^2 F_u(X_u, A_u) d[X]^c(u) \right) \\ &+ \int_{]0,t)} \nabla_x F_u(X_u, A_u).dX(u) \\ &+ \int_{]0,t)} \int_{R^d} [F_u(X_{u-}^x, A_u -) - F_u(X_{u-}, A_{u-}) - z.\nabla_x F_u(X_{u-}, A_{u-})] \mu(du\ dz) \end{aligned} \tag{81}$$

Adding the jump $F_t(X_t, A_t) - F_t(X_{t-}, A_{t-})$ to both the left-hand side and the third line of the right-hand side, and adding $\nabla_x F_t(X_{t-}, A_{t-}) \Delta X(t)$ to the second line and subtracting it from the third, leads to the desired result. □

Example 1 (Doléans exponential). Let X be a scalar cadlag semimartingale, such that the continuous part of its quadratic variation can be represented as:

$$[X]^c(t) = \int_0^t A(s) ds \tag{82}$$

for some cadlag adapted process A. Consider the non-anticipative functional:

$$F_t(x_t, v_t) = e^{x(t) - \frac{1}{2} \int_0^t v(s) ds} \prod_{s \le t} (1 + \Delta x(s)) e^{-\Delta x(s)} \tag{83}$$

Then $F \in \mathbb{C}^{1,\infty}$ with:

$$\mathcal{D}_t F(x_t, v_t) = -\frac{1}{2} v(t) F_t(x_t, v_t) \tag{84}$$

and

$$\nabla_x^k F_t(x_t, v_t) = F_t(x_t, v_t), k \geq 1 \tag{85}$$

and satisfies the assumptions of Proposition 6. The process

$$Y(t) = F_t(X_t, A_t) = e^{X(t) - \frac{1}{2}[X]^c(t)} \prod_{s \leq t} (1 + \Delta X(s)) e^{-\Delta X(s)} \tag{86}$$

is the Doléans exponential of the semimartingale X and Proposition 6 yields the well-known relation

$$Y(t) = \int_0^t Y(s-) dX(s).$$

6.2 Continuous semimartingales

In the case of a continuous semimartingale X and a continuous adapted process A, an Itô formula may also be obtained for functionals whose vertical derivative isright-continuous rather than left-continuous.

Proposition 7 (Functional Itô formula for a continuous semimartingale)**.** *Let X be a continuous semimartingale with quadratic variation process $[X]$, and A a continuous adapted process, on some filtered probability space $(\Omega, \mathcal{B}, \mathcal{B}_t, \mathbb{P})$. Then for any non-anticipative functional $F \in \mathbb{C}^{1,2}$ satisfying*

1. *F is predictable in the second variable, i.e. verifies* (2)*,*
2. *$\nabla_x F,\ \nabla_x^2 F,\ \mathcal{D} F \in \mathbb{B}$,*
3. *$F, \nabla_x F, \nabla_x^2 F \in \mathbb{F}_l^\infty$,*
4. *$F \in \mathbb{F}_l^\infty$*
5. *$\nabla_x F, \nabla_x^2 F \in \mathbb{F}_r^\infty$*

we have

$$F_t(X_t, A_t) - F_0(X_0, A_0) = \int_0^t \mathcal{D}_u F(X_u, A_u) du$$
$$+ \int_0^t \frac{1}{2} \mathrm{tr}[{}^t\nabla_x^2 F_u(X_u, A_u) d[X](u)] + \int_0^t \nabla_x F_u(X_u, A_u).dX(u), \qquad \mathbb{P}\text{-}a.s.$$

where last term is the Itô stochastic integral with respect to the X.

Proof. Assume first that X does not exit a compact set $K \subset U$ and that A is bounded by some constant $R > 0$. Let $0 = t_0^n \leq t_1^n \ldots \leq t_{k(n)}^n = t$ be a deterministic subdivision of $[0, t]$. Define the approximates (X^n, A^n) of (X, A) as in remark 8, and notice that, with the same notations:

$$\sum_{i=0}^{k(n)-1} \nabla_x F_{t_{i+1}^n}(X^n_{t_i^n, h_i^n}, A^n_{t_i^n, h_i^n})(X(t_{i+1}^n) - X(t_i^n)) = \int_{]0,t]} \nabla_x F_{t_{i+1}^n}(X^n_{t_i^n, h_i^n}, A^n_{t_i^n, h_i^n}) 1_{]t_i^n, t_{i+1}^n]}(t) dX(t)$$

which is a well-defined stochastic integral since the integrand is predictable (left-continuous and adapted by theorem 2), since the times t_i^n are *deterministic*; this would not be the case if we had to include jumps of X and/or A in the subdivision as in the case of the proof of proposition 6. By right-continuity of $\nabla_x F$, the integrand converges to $\nabla_x F_t(X_t, A_t)$. It is moreover bounded independently of n and ω since $\nabla_x F$ is assumed to be boundedness-preserving. The dominated convergence theorem for the stochastic integrals [13, Ch.IV Theorem32] ensures that it converges in probability to $\int_{]0,t]} \nabla_x F_u(X_{u-}, A_{u-}).dX(u)$. Using remark 8 concludes the proof.
Consider now the general case. Let K_n be an increasing sequence of compact sets with $\bigcup_{n \geq 0} K_n = U$ and denote

$$\tau_n = \inf\{s < t | X_s \notin K^n \text{ or } |A_s| > n\} \wedge t$$

which are optional times. Applying the previous result to the stopped process $(X_{t\wedge\tau_n}, A_{t\wedge\tau_n})$ leads to:

$$F_t(X_{t\wedge\tau_n}, A_{t\wedge\tau_n}) - F_0(X_0, A_0) = \int_0^{t\wedge\tau_n} \mathcal{D}_u F_u(X_u, A_u) du + \frac{1}{2} \int_0^{t\wedge\tau_n} \operatorname{tr}\left({}^t\nabla_x^2 F_u(X_u, A_u) d[X](u)\right)$$
$$+ \int_0^{t\wedge\tau_n} \nabla_x F_u(X_u, A_u).dX + \int_{t\wedge\tau^n}^{t} \mathcal{D}_u F(X_{u\wedge\tau_n}, A_{u\wedge\tau_n}) du \quad (87)$$

The terms in the first line converges almost surely to the integral up to time t since $t \wedge \tau_n = t$ almost surely for n sufficiently large. For the same reason the last term converges almost surely to 0. ☐

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A Some results on cadlag functions

For a cadlag function $f : [0,T] \mapsto \mathbb{R}^d$ we shall denote $\Delta f(t) = f(t) - f(t-)$ its discontinuity at t.

Lemma 8. *For any cadlag function* $f : [0,T] \mapsto \mathbb{R}^d$

$$\nabla\epsilon > 0, \quad \exists\eta > 0, \quad |x-y| \leq \eta \Rightarrow |f(x) - f(y)| \leq \epsilon + \sup_{t\in[x,y]} \{|\Delta f(t)|\} \tag{88}$$

Proof. Assume the conclusion does not hold. Then there exists a sequence $(x_n, y_n)_{n\geq 1}$ such that $x_n \leq y_n$, $y_n - x_n \to 0$ but $|f(x_n) - f(y_n)| > \epsilon + \sup_{t\in[x_n,y_n]}\{|\Delta f(t)|\}$. We can extract a convergent subsequence $(x_{\psi(n)})$ such that $x_{\psi(n)} \to x$. Noting that either an infinity of terms of the sequence are less than x or an infinity are more than x, we can extract *monotone* subsequences $(u_n, v_n)_{n\geq 1}$ of (x_n, y_n) which converge to x. If $(u_n), (v_n)$ both converge to x from above or from below, $|f(u_n) - f(v_n)| \to 0$ which yields a contradiction. If one converges from above and the other from below, $\sup_{t\in[u_n,v_n]}\{|\Delta f(t)|\} > |\Delta f(x)|$ but $|f(u_n) - f(v_n)| \to |\Delta f(x)|$, which results in a contradiction as well. Therefore (88) must hold. $\square$

The following lemma is a consequence of lemma 8:

Lemma 9 (Uniform approximation of cadlag functions by step functions)**.**
Let h be a cadlag function on $[0,T]$. If $(t^n_k)_{n\geq 0,k=0..n}$ is a sequence of subdivisions $0 = t^n_0 < t_1 < ... < t^n_{k_n} = t$ of $[0,T]$ such that:

$$\sup_{0\leq i\leq k-1} |t^n_{i+1} - t^n_i| \to_{n\to\infty} 0 \qquad \sup_{u\in[0,T]\setminus\{t^n_0,\ldots,t^n_{k_n}\}} |\Delta f(u)| \to_{n\to\infty} 0$$

then

$$\sup_{u\in[0,T]}|h(u)-\sum_{i=0}^{k_n-1}h(t_i)1_{[t_i^n,t_{i+1}^n)}(u)+h(t_{k_n}^n)1_{\{t_{k_n}^n\}}(u)|\to_{n\to\infty}0 \tag{89}$$

B Proof of theorem 2

Lemma 10. *Consider the canonical space $\mathcal{U}_T$ endowed with the natural filtration of the canonical process $X(x,t)=x(t)$. Let $\alpha\in\mathbb{R}$ and σ be an optional time. Then the following functional:*

$$\tau(x)=\inf\{t>\sigma,\quad |x(t)-x(t-)|>\alpha\} \tag{90}$$

is a stopping time.

Proof. We can write that:

$$\{\tau(x)\le t\}=\bigcup_{q\in\mathbb{Q}\bigcap[0,t)}(\{\sigma\le t-q\}\bigcap\{\sup_{t\in(t-q,t]}|x(u)-x(u-)|>\alpha\} \tag{91}$$

and

$$\{\sup_{u\in(t-q,t]}|x(u)-x(u-)|>\alpha\}=\bigcup_{n_0>1}\bigcap_{n>n_0}\{\sup_{1\le i\le 2^n}|x(t-q\frac{i-1}{2^n})-x(t-q\frac{i}{2^n})|>\alpha\} \tag{92}$$

thanks to the lemma 8 in Appendix A. □

We can now prove Theorem 2 using lemma 8 from Appendix A.
Proof of Theorem 2: Let's first prove point 1.; by lemma 4 it implies point 2. for right-continuous functionals and point 3. for left-continuous functionals. Introduce the following random subdivision of $[0,t]$:

$$\tau_0^N(x,v)=0$$
$$\tau_k^N(x,v)=\inf\{t>\tau_{k-1}^N(x,v)|2^Nt\in\mathbb{N}\text{ or }\quad |v(t)-v(t-)|\vee|x(t)-x(t-)|>\frac{1}{N}\}\wedge t \tag{93}$$

From lemma 10, those functionals are stopping times for the natural filtration of the canonical process. We define the stepwise approximations of x_t and v_t along the subdivision of index N:

$$x^N(s)=\sum_{k=0}^{\infty}x_{\tau_k^N(x,v)}1_{[\tau_k^N(x,v),\tau_{k+1}^N(x,v)[}(s)+x(t)1_{\{t\}}(s)$$
$$v^N(s)=\sum_{k=0}^{\infty}v_{\tau_k^N(x,v)}1_{[\tau_k^N(x,v),\tau_{k+1}^N(x,v)[}(t)+v(t)1_{\{t\}}(s) \tag{94}$$

as well as their truncations of rank K:

$${}_Kx^N(s)=\sum_{k=0}^{K}x_{\tau_k^N}1_{[\tau_k^N,\tau_{k+1}^N[}(s)$$
$${}_Kv^N(t)=\sum_{k=0}^{K}v_{\tau_k^N}1_{[\tau_k^N,\tau_{k+1}^N[}(t) \tag{95}$$

First notice that:

$$F_t(x_t^N, v_t^N) = \lim_{K\to\infty} F_t({}_K x_t^N, {}_K v_t^N) \tag{96}$$

because $({}_K x_t^N, {}_K v_t^N)$ coincides with (x_t^N, v_t^N) for K sufficiently large. The truncations

$$F_t^n({}_K x_t^N, {}_K v_t^N)$$

are $\mathcal{F}_t$-measurable as they are continuous functionals of the measurable functions:

$$\{(x(\tau_k^N(x,v)), v(\tau_k^N(x,v))), k \le K\}$$

so their limit $F_t(x_t^N, v_t^N)$ is also $\mathcal{F}_t$-measurable. Thanks to lemma 9, x_t^N and v_t^N converge uniformly to x_t and v_t, hence $F_t(x_t^N, v_t^N)$ converges to $F_t(x_t, v_t)$ since F is continuous at fixed times.

Now to show optionality of $Y(t)$ for a left-continuous functional, we will exhibit it as limit of right-continuous adapted processes. For $t \in [0,T]$, define $i^n(t)$ to be the integer such that $t \in [\frac{iT}{n}, \frac{(i+1)T}{n})$. Define the process: $Y^n((x,v),t) = F_{\frac{i^n(t)T}{n}}(x_{\frac{(i^n(t))T}{n}}, v_{\frac{(i^n(t))T}{n}})$, which is piecewise-constant and has right-continuous trajectories, and is also adapted by the first part of the theorem. Now, by d_∞ left-continuity of F, $Y^n(t) \to Y(t)$, which proves that Y is optional.
We similarly prove predictability of $Z(t)$ for a right-continuous functional. We will exhibit it as a limit of left-continuous adapted processes. For $t \in [0,T]$, define $i^n(t)$ to be the integer such that $t \in (\frac{iT}{n}, \frac{(i+1)T}{n}]$. Define the process: $Z^n((x,v),t) = F_{\frac{(i^n(t)+1)T}{n}}(x_{t-, \frac{(i^n(t)+1)T}{n}-t}, v_{t-, \frac{(i^n(t)+1)T}{n}-t})$, which has left-continuous trajectories since as $s \to t-$, $t-s$ sufficiently small, $i^n(s) = i^n(t)$ and $(x_{s-, \frac{(i^n(s)+1)T}{n}-s}, v_{s-, \frac{(i^n(s)+1)T}{n}-s})$ converges to $(x_{t-, \frac{(i^n(t)+1)T}{n}-t}, v_{t-, \frac{(i^n(t)+1)T}{n}-t})$ for d_∞. Moreover, $Z^n(t)$ is $\mathcal{F}_t$-measurable by the first part of the theorem, hence $Z^n(t)$ is predictable. Since $F \in \mathbb{F}_r^\infty$, $Z^n(t) \to Z(t)$, which proves that Y is predictable.

C Measure-theoretic lemmas used in the proof of theorem 3 and 4

Lemma 11. *Let f be a bounded left-continuous function defined on $[0,T]$, and let $\mu(n)$ be a sequence of Radon measures on $[0,T]$ such that μ_n converges vaguely to a Radon measure μ with no atoms. Then for all $0 \le s < t \le T$, with $\mathcal{I}$ being $[s,t]$, $(s,t]$,$[s,t)$ or (s,t):*

$$\lim_n \int_{\mathcal{I}} f(u) d\mu_n(u) = \int_{\mathcal{I}} f(u) d\mu(u) \tag{97}$$

Proof. Let M be an upper bound for $|f|$, $F_n(t) = \mu_n([0,t])$ and $F(t) = \mu([0,t])$ the cumulative distribution functions associated to μ_n and μ. For $\epsilon > 0$ and $u \in (s,t]$, define:

$$\eta(u) = \inf\{h > 0 || f(u-h) - f(u)| \ge \epsilon\} \wedge u \tag{98}$$

and we have $\eta(u) > 0$ by right-continuity of f. Define similarly $\theta(u)$:

$$\theta(u) = \inf\{h > 0 || f(u-h) - f(u)| \ge \frac{\epsilon}{2}\} \wedge u \tag{99}$$

By uniform continuity of F on $[0,T]$ there also exists $\zeta(u)$ such that $\forall v \in [T-\zeta(u),T], F(v+\zeta(u))-F(v) < \epsilon\eta(u)$. Take a finite covering

$$[s,t] \subset \bigcup_{i=0}^{N} (u_i - \theta(u_i), u_i + \zeta(u_i)) \tag{100}$$

where the u_i are in $[s,t]$, and in increasing order, and we can choose that $u_0 = s$ and $u_N = t$. Define the decreasing sequence v_j as follow: $v_0 = t$, and when v_j has been constructed, choose the minimum index $i(j)$ such that $v_j \in (u_{i(j)}, u_{i(j)+1}]$, then either $u_{i(j)} \le v_j - \eta(v_j)$ and in this case $v_{j+1} = u_{i(j)}$, else $u_{i(j)} > v_j - \eta(v_j)$, and in this case $v_{j+1} = \max(v_j - \eta(v_j), s)$. Stop the procedure when you reach s, and denote M the maximum index of the v_j. Define the following piecewise constant approximation of f on $[s,t]$:

$$g(u) = \sum_{j=0}^{M-1} f(v_j) 1_{(v_{j+1},v_j]}(u) \tag{101}$$

Denote J_1 the set of indices j where v_{j+1} has been constructed as in the first case, and J_2 its complementary. If $j \in J_1$, $|f(u)-g(u)| < \epsilon$ on $[v_j - \eta(v_j), v_j]$, and $v_j - \eta(u_{i(j)})) - v_{j+1} < \zeta(u_{i(j)+1}) = \zeta(v_{j+1})$, because of the remark that $v_j - \eta_{v_j} < u_{i(j)} - \theta(u_{i(j)})$. Hence:

$$\int_{(v_j,v_{j+1}]} |f(u) - g(u)| d\mu(u) \le \epsilon[F(v_{j+1}) - F(v_j)] + 2M\epsilon\eta(v_{j+1}) \tag{102}$$

If $j \in J_2$, $|f(u) - g(u)| < \epsilon$ on $[v_{j+1}, v_j]$. So that summing up all terms we have the following inequality:

$$\int_{[s,t]} |f(u) - g(u)| d\mu(u) \le \epsilon \left(F(t) - F(s) + 2M(t-s)\right) \tag{103}$$

because of the fact that: $\eta(v_j) \le v_j - v_{j+1}$ for $j < M$. The same argument applied to μ_n yields:

$$\begin{aligned} \int_{[s,t]} |f(u) - g(u)| d\mu_n(u) &\le \epsilon[F_n(t) - F_n(s-)] \\ &+2M \sum_{j=0}^{M-1} F_n(v_{j+1}) - F_n(v_{j+1} - \zeta(v_{j+1})) \end{aligned} \tag{104}$$

so that the lim sup satisfies (103) since $F_n(u)$ converges to $F(u)$ for every u.

On other hand, it is immediately observed that

$$\lim_n \int_{\mathcal{I}} g(u) d\mu_n(u) = \int_{\mathcal{I}} g(u) d\mu(u) \tag{105}$$

since $F_n(u)$ and $F_n(u-)$ both converge to $F(u)$ since μ has no atoms (g is a linear combination of indicators of intervals). So the lemma is established.

□

Lemma 12. *Let $(f_n)_{n\geq 1}, f$ be left-continuous functions defined on $[0,T]$, satisfying:*

$$\forall t \in [0,T], \lim_n f_n(t) = f(t) \qquad \forall t \in [0,T], f_n(t) \leq K \tag{106}$$

Let also μ_n be a sequence of Radon measures on $[0,T]$ such that $\mu(n)$ converges vaguely to a Radon measure μ with no atoms. Then for all $0 \leq s < t \leq T$, with $\mathcal{I}$ being $[s,t]$, $(s,t]$,$[s,t)$ or (s,t):

$$\int_{\mathcal{I}} f_n(u) d\mu_n(u) \to_{n\to\infty} \int_s^t f(u) d\mu(u) \tag{107}$$

Proof. Let $\epsilon > 0$ and let n_0 such that $\mu(\{\sup_{m\geq n_0} |f_m - f| > \epsilon\}) < \epsilon$. The set $\{\sup_{m\geq n_0} |f_m - f| > \epsilon\}$ is a countable union of disjoint intervals since the functionals are left-continuous, hence it is a continuity set of μ since μ has no atoms; hence, since μ_n converges vaguely to μ [1]:

$$\lim_n \mu_n(\{\sup_{m\geq n_0} |f_m - f| > \epsilon\}) = \mu(\{\sup_{m\geq n_0} |f_m - f| > \epsilon\}) < \epsilon \tag{108}$$

since μ_n converges vaguely to μ which has no atoms.

So we have, for $n \geq n_0$:

$$\int_{\mathcal{I}} |f_n(u) - f(u)| d\mu_n(u) \leq 2K\mu_n \left(\{\sup_{n\geq n_0} |f_n - f| > \epsilon\}\right) + \epsilon\mu_n(\mathcal{I}) \tag{109}$$

Hence the lim sup of this quantity is less or equal to:

$$2K\mu(\{\sup_{m\geq n_0} |f_m - f| > \epsilon\} + \epsilon\mu(\mathcal{I}) \leq (2K + \mu(\mathcal{I}))\epsilon \tag{110}$$

On other hand:

$$\lim_n \int_{\mathcal{I}} f(u) d\mu_n(u) = \int_{\mathcal{I}} f(u) d\mu(u) \tag{111}$$

by application of lemma 11.

□