Monthly Notices of the ROYAL ASTRONOMICAL SOCIETY

Mon. Not. R. Astron. Soc. **401,** 2406–2418 (2010) doi:10.1111/j.1365-2966.2009.15820.x

A new approach to probing primordial non-Gaussianity

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Accepted 2009 October 2. Received 2009 October 2; in original form 2009 May 6

ABSTRACT

We address the dual challenge of estimating deviations from Gaussianity arising in models of the early Universe, whilst retaining information necessary to assess whether a detection of non-Gaussianity is primordial. We do this by constructing a new statistic, the bispectrum-related power spectrum, which is constructed from a map of the cosmic microwave background. The estimator is optimized for primordial non-Gaussianity detection, but can also be useful in distinguishing primordial non-Gaussianity from secondary non-Gaussianity, such as may arise from unsubtracted point sources, or residuals from component separation. Extending earlier studies we present unbiased non-Gaussianity estimators optimized for partial sky coverage and inhomogeneous noise associated with realistic scan strategies, but which retain the ability to assess foreground contamination.

Key words: methods: analytical – methods: numerical – methods: statistical – cosmic microwave background – cosmology: theory – large-scale structure of Universe.

1 INTRODUCTION

The statistical properties of fluctuations in the cosmic microwave background (CMB) radiation can be used to probe the very earliest stages of the Universe's history, and provide valuable information on the mechanisms which ultimately gave rise to the existence of structure within the Universe. This may include evidence for inflation, the process by which the rapid expansion of the Universe is thought to have arisen. In standard inflationary models, the fields in the early Universe should be very close to random Gaussian fields, so a detection of large non-Gaussianity would be highly significant, and may indicate a different history, such as warm inflation or multiple-field inflation, or a completely different mechanism such as those arising from topological defects. A difficulty for methods designed to detect non-Gaussianity in the CMB is that other processes can contribute, such as gravitational lensing, unsubtracted point sources and imperfect subtraction of galactic foreground emission (e.g. Goldberg & Spergel 1999; Cooray & Hu 2000; Verde & Spergel 2002; Castro 2004; Babich & Pierpaoli 2008). The challenge therefore is to provide evidence that any detection of non-Gaussianity is primordial in origin, and not a result of these other effects. The aim of this paper is to provide an optimized framework not just for detecting non-Gaussianity, but also for assessing the contributions from various sources.

Non-Gaussianity from simplest inflationary models based on a single slowly rolling scalar field is typically very small (Salopek & Bond 1990, 1991; Falk et al. 1993; Gangui et al. 1994; Acquaviva et al. 2003; Maldacena 2003; see Bartolo, Matarrese & Riotto 2006 and references there in for more details). Variants of simple inflationary models such as multiple scalar fields (Linde & Mukhanov 1997; Lyth, Ungarelli & Wands 2003), features in the inflationary potential, non-adiabatic fluctuations, non-standard kinetic terms, warm inflation (Gupta, Berera & Heavens 2002; Moss & Xiong 2007) or deviations from Bunch–Davies vacuum can all lead to much higher level of non-Gaussianity. Early observational work on the bispectrum from *COBE* (Komatsu et al. 2002) and *MAXIMA* (Santos et al. 2003) was followed by much more accurate analysis with *Wilkinson Microwave Anisotropy Probe* (*WMAP*; Komatsu et al. 2003; Creminelli et al. 2007a; Spergel et al. 2007). With the recent claim of a detection of non-Gaussianity (Yadav & Wandelt 2008) in the *WMAP* 5-year (*WMAP*5) sky maps, interest in non-Gaussianity has obtained a tremendous boost. Much of the interest in primordial non-Gaussianity has focused on a phenomenological '*local* $f_{\rm NL}$' parametrization in terms of the perturbative non-linear coupling in the primordial curvature perturbation (Verde et al. 2000):

$$\Phi(x) = \Phi_{\rm L}(x) + f_{\rm NL}\left[\Phi^2_{\rm L}(x) - \left\langle \Phi^2_{\rm L}(x)\right\rangle\right], \quad (1)$$

where $\Phi_{\rm L}(x)$ denotes the linear Gaussian part of the Bardeen curvature and $f_{\rm NL}$ is the non-linear coupling parameter. A number of models have non-Gaussianity which can be approximated by this form. The leading order non-Gaussianity therefore is at the level of the bispectrum, or in configuration space at the three-point level. Many studies involving primordial non-Gaussianity have used the bispectrum, motivated by

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the fact that it contains all the information about $f_{\rm NL}$ (Babich 2005). It has been extensively studied (Creminelli 2003; Komatsu, Spergel & Wandelt 2005; Cabella et al. 2006; Creminelli et al. 2006; Medeiros & Contaldo 2006; Liguori et al. 2007; Smith, Senatore & Zaldarriaga 2009), with most of these measurements providing convolved estimates of the bispectrum. Optimized three-point estimators were introduced by Heavens (1998), and have been successively developed (Smith, Zahn & Dore 2000; Komatsu et al. 2005; Creminelli et al. 2006; Smith & Zaldarriaga 2006; Creminelli, Senatore & Zaldarriaga 2007b) to the point where an estimator for $f_{\rm NL}$ which saturates the Cramer–Rao bound exists for partial sky coverage and inhomogeneous noise (Smith et al. 2009). Approximate forms also exist for *equilateral* non-Gaussianity, which may arise in models with non-minimal Lagrangian with higher derivative terms (Chen et al. 2007; Chen, Easther & Lim 2007). In these models, the largest signal comes from spherical harmonic modes with $\ell_1 \simeq \ell_2 \simeq \ell_3$, whereas for the local model, the signal is highest when one ℓ is much smaller than the other two – the so-called *squeezed* configuration.

Reducing the CMB data to a single loss less estimate for $f_{\rm NL}$ is extremely elegant, but it suffers from the disadvantage that a single number loses the ability to determine the extent to which the estimate has been contaminated by non-primordial signals. What we seek to perform a less aggressive data compression of the CMB data, not to a single number, but to a function, which has known expected form for primordial models, and for which the contributions from other signals can be estimated. The purpose of this is to be able to demonstrate that a non-Gaussian signal is indeed primordial, or alternatively accounted for by non-primordial signals. We do this in a way which is still optimal for local or equilateral $f_{\rm NL}$ models, although the formalism is general. The function we choose is the integrated cross-power spectrum of pair of maps constructed from the CMB data. Mathematically, it is closely related to previous estimators, but the interpretation of the output is different, and offers very significant advantages.

This paper is organized as follows. Section 2 provides a small review of available models of primordial non-Gaussianity. Section 3 relates the projected bispectrum to the corresponding primordial bispectrum. Section 4 presents the optimized bispectrum-related power spectrum estimator for the idealized case of an all-sky survey with homogeneous noise. This is not optimal for partial sky coverage or inhomogeneous noise, but is straightforward and shows the connection with the $f_{\rm NL}$ estimator of Komatsu et al. (2005). Here we present the theoretical expectation for the local $f_{\rm NL}$ model, and show the link between the $f_{\rm NL}$ estimator based on one-point statistics and its two-point counterpart. Section 5 provides a method which can handle partial sky coverage and non-uniform noise in an approximate way using Monte Carlo simulations. In Section 6 we provide the full optimal weights for bispectrum analysis in the presence of sky cuts and the inhomogeneous noise. The full inverse covariance of the data is introduced which makes the estimator an optimal one. The final section is devoted to discussion and future plans for numerical implementation.

2 MODELS OF PRIMORDIAL NON-GAUSSIANITY

Deviations from pure Gaussian statistics can provide direct clues regarding inflationary dynamics. The single-field slow-roll model of inflation provides a very small level of departure from Gaussianity, far below present experimental detection limits (Acquaviva et al. 2003; Maldacena 2003). Many other variants however will produce a much higher level of non-Gaussianity which will be within the reach of all-sky experiments such as Planck.

In general models can be distinguished by the way they predict coupling between different Newtonian potential modes:

$$\langle \phi(\boldsymbol{k}_1)\phi(\boldsymbol{k}_2)\phi(\boldsymbol{k}_3)\rangle = (2\pi)^3\delta^{3\rm D}(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)F(k_1,k_2,k_3). \quad (2)$$

The function F encodes the information about mode–mode coupling and $\delta^{3\rm D}(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)$ ensures triangular equality in k-space. Different models of inflation are typically divided into two different groups. The first class of model is known as the 'local model', where the contribution from $F(k_1, k_2, k_3)$ is largest when the wavevectors are in the so-called 'squeezed' configuration, where e.g. $k_1 \ll k_2, k_3$. The local form of non-Gaussianity is predominant in models where there is non-linear coupling between the field driving inflation (the inflaton) and the curvature perturbations (Salopek & Bond 1990, 1991; Gangui et al. 1994), such as the curvaton model (Lyth et al. 2003) and the ekpyrotic model (Koyama et al. 2007; Buchbinder, Khoury & Ovrut 2008). The primordial bispectrum in the local model can be written as

$$F^{\rm loc}(k_1,k_2,k_3) = f_{\rm NL}^{\rm loc}\left[\frac{\Delta_\phi^2}{k_1^3k_2^3} + {\rm cyc.perm.}\right], \quad (3)$$

where the power spectrum of inflationary curvature perturbations is given by $P_\Phi = \Delta_\Phi/k^3$, in general for deviation from Harrison–Zeldovich power spectra one has $P_\Phi = \Delta_\Phi/k^{4-n_s}$.

The other main class consists of models where the contribution from $F(k_1, k_2, k_3)$ is maximum for configurations where $k_1 \sim k_2 \sim k_3$. The equilateral form appears from non-canonical kinetic terms such as Dirac–Born–Infield (DBI) action (Alishahiha, Silverstein & Tong 2004), the ghost condensation (Arkani-Hamed et al. 2004) or various single-field models where the scalar field acquires a low sound speed (Chen et al. 2007; Cheung et al. 2008). The equilateral model is not a separable function of the k_i, which complicates the analysis considerably, but it was shown by Creminelli et al. (2006) that the following approximate form can model the equilateral case very accurately:

$$F^{\rm eq}(k_1,k_2,k_3) = f_{\rm NL}^{\rm eq}\left[-3\frac{\Delta_\phi^2}{k_1^3k_2^3} - 2\frac{\Delta_\phi^2}{k_1^2k_2^2k_3^2} + 6\frac{\Delta_\phi^2}{k_1k_2^2k_3^3} + {\rm cyc.perm.}\right]. \quad (4)$$

Secondary non-Gaussianity resulting from various sources e.g. coupling of lensing and the Sunyaev–Zel'dovich effect, or lensing and the integrated Sachs–Wolfe effect can also contribute to the observed bispectrum (Spergel & Goldberg 1999a,b). We will present a general analysis of secondary bispectra as well as the one induced by quasi-linear evolution of gravitational perturbations (Munshi, Souradeep & Starobinsky 1995).

3 ANGULAR CMB BISPECTRUM WITH PRIMORDIAL NON-GAUSSIANITY

The angular bispectrum can be defined as the three-point correlation in the harmonic domain. With temperature fluctuations as a function of solid angle $\hat{\Omega}$, $\Delta T(\hat{\Omega})$,

$$a_{lm} \equiv \int \mathrm{d}\hat{\Omega} \frac{\Delta T(\hat{\Omega})}{T} Y^*_{lm}(\hat{\Omega}) \tag{5}$$

and the three-point function may be written

$$\langle a_{lm} a_{l'm'} a_{l''m''} \rangle \equiv B_{ll'l''} \begin{pmatrix} l & l' & l'' \\ m & m' & m'' \end{pmatrix}. \tag{6}$$

This form preserves the rotational invariance of the three-point correlation function in the harmonic domain. The quantity in parentheses is the Wigner 3j-symbol, which is non-zero only for triplets (l_1, l_2, l_3) which satisfy the triangle rule, including that the sum $l_1 + l_2 + l_3$ is even, ensuring the parity invariance of the bispectrum. The reduced bispectrum $b_{ll'l''}$ was introduced by Komatsu & Spergel (2001) which will be helpful (see Babich, Creminelli & Zaldarriaga 2004 for elaborate discussion):

$$B_{ll'l''} \equiv \sqrt{\frac{(2l+1)(2l'+1)(2l'+1)}{4\pi}} \begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix} b_{ll'l''} \equiv I_{ll'l''} b_{ll'l''}. \tag{7}$$

The reduced bispectrum $b_{ll'l''}$ can be expressed in terms of the kernel $F(k_1, k_2, k_3)$ for various models that we will be considering:

$$b_{l_1l_2l_3} = \left(\frac{2}{\pi}\right)^3 \int \mathrm{d}r r^2 \int k_1^2 \mathrm{d}k_1 j_{l_1}(k_1 r) \Delta^T_{l_1}(k_1 r) \int k_2^2 \mathrm{d}k_2 j_{l_2}(k_2 r) \Delta^T_{l_2}(k_2 r) \int k_3^2 \mathrm{d}k_3 j_{l_3}(k_3 r) \Delta^T_{l_3}(k_3 r) F(k_1, k_2, k_3), \tag{8}$$

where $\Delta^T_l(k)$ denotes the transfer function which relates the inflationary potential Φ to the spherical harmonics a_{lm} of the temperature perturbation in the sky (e.g. Wang & Kamionkowski 2000):

$$a_{lm} = 4\pi(-i)^l \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \Phi(\boldsymbol{k}) \Delta^T_l(k) Y^*_{lm}(\hat{\boldsymbol{k}}). \tag{9}$$

Using these definitions one can express the reduced bispectra for the local and equilateral case as follows:

$$b_{l_1l_2l_3} = 2 f^{\mathrm{loc}}_{\mathrm{NL}} \int r^2 \mathrm{d}r \left[\alpha_{l_1}(r)\beta_{l_2}(r)\beta_{l_3}(r) + \text{cyc.perm.}\right], \tag{10}$$

$$b_{l_1l_2l_3} = 6 f^{\mathrm{eq}}_{\mathrm{NL}} \int r^2 \mathrm{d}r \left[-\alpha_{l_1}(r)\beta_{l_2}(r)\beta_{l_3}(r) - 2\delta_{l_1}(r)\delta_{l_2}(r)\delta_{l_3}(r) + \beta_{l_1}(r)\gamma_{l_2}(r)\delta_{l_3}(r) + \text{cyc.perm.}\right]. \tag{11}$$

We will use these forms to construct associated fields A, B etc. from temperature fields with appropriate weighting to optimize our estimator. We list the explicit expressions for the functions $\alpha_l(r)$, $\beta_l(r)$ etc. for completeness (Creminelli et al. 2006):

$$\alpha_l(r) \equiv \frac{2}{\pi} \int_0^\infty k^2 \mathrm{d}k \, \Delta_l(k) j_l(kr),$$
$$\beta_l(r) \equiv \frac{2}{\pi} \int_0^\infty k^2 \mathrm{d}k \, P_\Phi(k) \Delta_l(k) j_l(kr),$$
$$\gamma_l(r) \equiv \frac{2}{\pi} \int_0^\infty k^2 \mathrm{d}k \, P_\Phi^{1/3}(k) \Delta_l(k) j_l(kr),$$
$$\delta_l(r) \equiv \frac{2}{\pi} \int_0^\infty k^2 \mathrm{d}k \, P_\Phi^{2/3}(k) \Delta_l(k) j_l(kr). \tag{12}$$

Numerical evaluations of these functions can be performed by using the publicly available software such as CAMB or CMBFAST.

4 ALL SKY ANALYSIS WITH HOMOGENEOUS NOISE

In this section, we compute the main statistic which will be used to estimate primordial non-Gaussianity, and which can also be used to assess whether a non-Gaussian signal is indeed primordial. We call the statistic the *bispectrum-related power spectrum*, as it derives from the cross-power spectrum of certain maps constructed from the CMB map data, and which, as we will see, is related to the primordial non-Gaussianity.

The analysis in this section is optimal for detecting primordial non-Gaussianity in the case of all-sky coverage and homogeneous noise. These assumptions will not hold in practice, but we present this simpler case for clarity first, and to show the connection with previous work. We relax the assumptions later, and give optimized estimators for realistic cases in later sections.

4.1 Local model

Following Komatsu et al. (2005), we first construct the 3D fields $A(r, \hat{\Omega})$ and $B(r, \hat{\Omega})$ from the expansion coefficients of the observed CMB map, a_{lm}. The harmonics here $A_{lm}(r)$ and $B_{lm}(r)$ are simply weighted spherical harmonics of the temperature field a_{lm} with weights constructed from the CMB power spectrum C_l and the functions $\alpha_l(r)$ and $\beta_l(r)$, respectively:

$$A(r, \hat{\Omega}) \equiv \sum_{lm} Y_{lm}(\hat{\Omega}) A_{lm}(r),\ A_{lm}(r) \equiv \frac{\alpha_l(r)}{C_l} b_l a_{lm}; \tag{13}$$

$$B(r, \hat{\Omega}) \equiv \sum_{lm} Y_{lm}(\hat{\Omega}) B_{lm}(r),\ B_{lm}(r) \equiv \frac{\beta_l(r)}{C_l} b_l a_{lm}. \tag{14}$$

The function b_l represents beam smoothing, and from here onward we will absorb it into the harmonic transforms. Using these definitions Komatsu et al. (2005) define the one-point mixed-skewness for the fields $A(r, \hat{\Omega})$ and $B(r, \hat{\Omega})$:

$$S_3^{\rm loc} = S_3^{AB^2} \equiv \int r^2 {\rm d}r \int {\rm d}\hat{\Omega} A(r, \hat{\Omega}) B^2(r, \hat{\Omega}). \tag{15}$$

S_3 can be used to estimate $f_{\rm NL}^{\rm loc}$, but such radical data compression to a single number loses the ability to estimate contamination of the estimator by other sources of non-Gaussianity. As a consequence, we construct a less radical compression, to a function of l which can be used to estimate $f_{\rm NL}^{\rm loc}$, but which can also be analysed for contamination by, for example, foregrounds. We do this by constructing the integrated cross-power spectrum of the maps $A(r, \hat{\Omega})$ and $B^2(r, \hat{\Omega})$. Expanding B^2 in spherical harmonics gives

$$\begin{aligned} B_{lm}^{(2)}(r) &\equiv \int {\rm d}\hat{\Omega} B^2(r, \hat{\Omega}) Y_{lm}(\hat{\Omega}) \\ &= \sum_{l'm'} \sum_{l''m''} \frac{\beta_{l'}(r)}{C_{l'}} \frac{\beta_{l''}(r)}{C_{l''}} \sqrt{\frac{(2l+1)(2l'+1)(2l''+1)}{4\pi}} \begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l' & l'' \\ m & m' & m'' \end{pmatrix} a_{l'm'} a_{l''m''}, \end{aligned} \tag{16}$$

and we define the cross-power spectrum $S_l^{A,B^2}(r)$ at a radial distance r as

$$S_l^{A,B^2}(r) = \frac{1}{2l+1} \sum_m {\rm Real} \left\{ A_{lm}(r) B_{lm}^{(2)}(r) \right\}. \tag{17}$$

Integrating this over r gives

$$S_l^{A,B^2} \equiv \int r^2 {\rm d}r\ S_l^{A,B^2}(r). \tag{18}$$

This integrated cross-power spectrum of $B^2(r, \hat{\Omega})$ and $A(r, \hat{\Omega})$ carries information about the underlying bispectrum $B_{ll'l''}$ as follows:

$$\begin{aligned} S_l^{A,B^2} = \frac{1}{2l+1} \sum_m \sum_{l'm'} \sum_{l''m''} & \int r^2 {\rm d}r \left\{ \frac{\alpha_l(r)}{C_l} \frac{\beta_{l'}(r)}{C_{l'}} \frac{\beta_{l''}(r)}{C_{l''}} \right\} a_{lm} a_{l'm'} a_{l''m''} \\ & \times \sqrt{\frac{(2l+1)(2l'+1)(2l''+1)}{4\pi}} \begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l' & l'' \\ m & m' & m'' \end{pmatrix}. \end{aligned} \tag{19}$$

Similarly we can construct the cross-power spectrum of the product map $AB(r, \hat{\Omega})$ and $B(r, \hat{\Omega})$, which we denote as $C_l^{B,AB}$:

$$\begin{aligned} S_l^{AB,B} = \frac{1}{2l+1} \sum_m \sum_{l'm'} \sum_{l''m''} & \int r^2 {\rm d}r \left\{ \frac{\beta_l(r)}{C_l} \frac{\alpha_{l'}(r)}{C_{l'}} \frac{\beta_{l''}(r)}{C_{l''}} \right\} a_{lm} a_{l'm'} a_{l''m''} \\ & \times \sqrt{\frac{(2l+1)(2l'+1)(2l''+1)}{4\pi}} \begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l' & l'' \\ m & m' & m'' \end{pmatrix}. \end{aligned} \tag{20}$$

Using these expressions, and the following relation, we can write this more compactly in terms of the estimated CMB bispectrum:

$$\hat{B}_{ll'l''} = \sum_{mm'm''} \begin{pmatrix} l & l' & l'' \\ m & m' & m'' \end{pmatrix} a_{lm} a_{l'm'} a_{l''m''} \tag{21}$$

from which we compute our new statistic, the *bispectrum-related power spectrum*, $C_l^{\rm loc}$, as

$$S_l^{\rm loc} \equiv \left(S_l^{A,B^2} + 2 S_l^{AB,B} \right) = \frac{2 \hat{f}_{\rm NL}^{\rm loc}}{(2l+1)} \sum_{l'} \sum_{l''} \left\{ \frac{B_{ll'l''}^{\rm loc} \hat{B}_{ll'l''}}{C_l C_{l'} C_{l''}} \right\}, \tag{22}$$

where $B_{ll'l''}^{\rm loc}$ is the bispectrum for the local $f_{\rm NL}$ model, normalized to $f_{\rm NL}^{\rm loc} = 1$. We can now use standard statistical techniques to estimate $f_{\rm NL}^{\rm loc}$. Note that if we sum over all l values then we recover the estimator $S_{\rm prim}$ of Komatsu et al. (2005), which is the cross-skewness of ABB:

$$S_3^{\rm loc} \equiv S_3^{AB^2} = \sum_l (2l+1) \left(S_l^{A,B^2} + 2 S_l^{AB,B} \right) = 2 \hat{f}_{\rm NL}^{\rm loc} \sum_l \sum_{l'} \sum_{l''} \left\{ \frac{B_{ll'l''}^{\rm loc} \hat{B}_{ll'l''}}{C_l C_{l'} C_{l''}} \right\}. \tag{23}$$

We show in Fig. 1 the form of the bispectrum-related power spectrum for the local model. If the signal observed is inconsistent with this form, it would indicate a departure from this form of primordial non-Gaussianity, and/or a significant contamination by foreground sources.

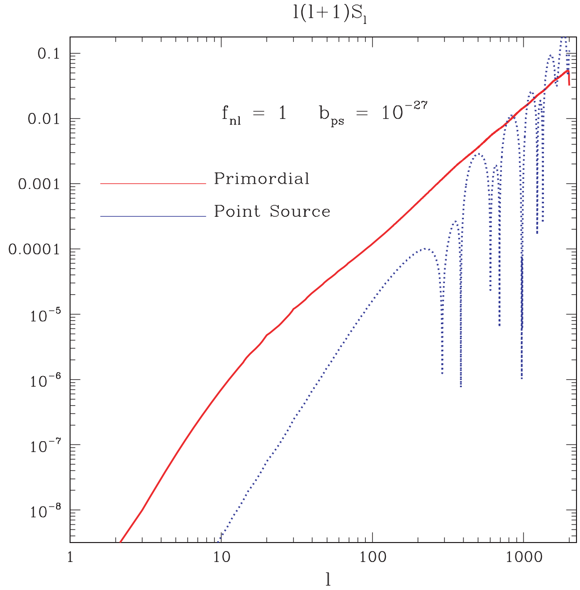

Figure 1. The bispectrum-related power spectrum is plotted as function of angular scale l. The dotted curve corresponds to the point source cross-contamination $b_{\rm ps} = 10^{-27}$ μK, and the solid curve corresponds to local model with $f_{\rm NL} = 1$. A *WMAP*5 background cosmology was assumed. See text for details.

In the right-hand panel we show the expected form of contamination of the $C_l^{\rm loc}$ statistic by a simple foreground source: unsubtracted point sources, randomly distributed. The contamination scales with the number density. This contamination is expected to be very low (Komatsu & Spergel 2001), but we show it as an illustration of how foreground effects may be detected. Since its structure is very different from the local primordial signal, it should be relatively easy to decouple. Note that the point source contamination here is the contribution to the local model bispectrum-related power spectrum – i.e. it is equation (22) with one local B and one point source B (constant $b_{ll'l''}$), not two point source B terms.

4.2 Equilateral model

The form for the reduced bispectrum for the equilateral model as mentioned above is an approximation to the real bispectrum generated in theories with higher order derivatives in the Lagrangian. In addition to the quantities A and B we have constructed analogous fields C and D with corresponding weights $\gamma_l(r)$ and $\delta_l(r)$:

$$C(r,\hat\Omega) \equiv \sum_{lm} Y_{lm}(\hat\Omega) C_{lm}(r),\; C_{lm}(r) \equiv \frac{\gamma_l(r)}{C_l} b_l a_{lm}; \tag{24}$$

$$D(r,\hat\Omega) \equiv \sum_{lm} Y_{lm}(\hat\Omega) D_{lm}(r),\; D_{lm}(r) \equiv \frac{\delta_l(r)}{C_l} b_l a_{lm}. \tag{25}$$

From these fields and using A and B previously defined a one-point statistic can be constructed (Creminelli et al. 2006):

$$S_3^{\rm eq} = -18\left\{ S_3^{AB^2} - \frac{2}{3} S_3^{D^3} - 2S_3^{BCD} \right\} = -18 \int r^2 {\rm d}r \int {\rm d}\hat\Omega \left[A(r,\hat\Omega)B(r,\hat\Omega)^2 + \frac{2}{3} D(r,\hat\Omega)^3 - 2B(r,\hat\Omega)C(r,\hat\Omega)D(r,\hat\Omega)\right]. \tag{26}$$

The associated power spectrum will have a composite structure with many contributing terms:

$$S_l^{\rm eq} = -18\left\{ \left(S_l^{A,B^2} + 2S_l^{AB,B}\right) - 2S_l^{D,D^2} - 2\left(S_l^{B,CD} + S_l^{C,BD} + S_l^{D,BC}\right)\right\}. \tag{27}$$

Following the same procedure outlined above one can now write

$$S_l^{\rm eq} = \frac{\hat f_{\rm NL}^{\rm eq}}{(2l+1)} \sum_{l'} \sum_{l''} \left\{ \frac{B_{ll'l''}^{\rm eq} \hat B_{ll'l''}}{C_l C_{l'} C_{l''}} \right\}, \tag{28}$$

and finally we can recover the one-point statistic or the cross-skewness of Komatsu et al. (2005):

$$S_3^{\rm eq} = \sum_l (2l+1) S_l^{\rm eq} = \hat f_{\rm NL}^{\rm eq} \sum_l \sum_{l'} \sum_{l''} \left\{ \frac{B_{ll'l''}^{\rm eq} \hat B_{ll'l''}}{C_l C_{l'} C_{l''}} \right\}. \tag{29}$$

Individual contributions to the final skewness following relations, which can be useful diagnostics for numerical checks:

$$\sum_l (2l+1) S_l^{A,B^2} = \sum_l (2l+1) S_l^{AB,B}; \qquad \sum_l (2l+1) S_l^{B,CD} = \sum_l (2l+1) S_l^{C,BD} = \sum_l (2l+1) S_l^{D,BC}. \tag{30}$$

Given the signal-to-noise ratio of current estimates of S_3 or equivalently $f_{\rm NL}$ from *WMAP* surveys, it may not be possible to use many narrow bins to evaluate the S_ls associated with the primordial bispectra. However, with the increase in experimental sensitivity of future CMB experiments, it will be possible to divide the l range in narrower bins. The important point here is that the data must show consistency with the theoretical $S_l^{\mathrm{loc/eq}}$ to make a convincing case that the non-Gaussianity is primordial. However, these expressions are only optimal for all-sky coverage and homogeneous noise; we relax these assumptions in the next sections.

5 PARTIAL SKY COVERAGE AND NON-UNIFORM NOISE: AN APPROXIMATE TREATMENT

It was pointed out in Babich (2005), Creminelli et al. (2006) and Yadav et al. (2008) that in the presence of partial sky coverage, e.g. due to the presence of a mask or because of galactic foregrounds and bright point sources, as well as, in the case of non-uniform noise, spherical symmetry is destroyed. The estimator introduced above will then have to be modified by adding terms linear in the observed map. The linear terms for the local model can be written in the following form:

$$\hat{S}^{\mathrm{linear}}_{\mathrm{loc}} = -\frac{1}{f_{\mathrm{sky}}}\int r^2 \mathrm{d}r \int \mathrm{d}\hat{\Omega}\left\{2B(r,\hat{\Omega})\langle A(r,\hat{\Omega})B(r,\hat{\Omega})\rangle_{\mathrm{sim}} + A(r,\hat{\Omega})\langle B^2(r,\hat{\Omega})\rangle_{\mathrm{sim}}\right\}. \tag{31}$$

The linear terms therefore are constructed from correlating the Monte Carlo (MC) averaged $\langle A(n,r)B(n,r)\rangle_{\mathrm{sim}}$ product maps with the input B map. The mask and the noise that are used in constructing the MC averaged product map are exactly same as the observed maps and the ones derived from them such as A or B:

$$\begin{aligned}\hat{S}^{\mathrm{linear}}_{\mathrm{eq}} = & -\frac{18}{f_{\mathrm{sky}}}\int r^2 \mathrm{d}r \int \mathrm{d}\hat{\Omega}\Big\{2B(r,\hat{\Omega})\langle A(r,\hat{\Omega})B(\hat{\Omega},r)\rangle_{\mathrm{sim}} + A(r,\hat{\Omega})\langle B^2(r,\hat{\Omega})\rangle_{\mathrm{sim}} + 2D(r,\hat{\Omega})\langle D(r,\hat{\Omega})^2\rangle \\ & -2B(r,\hat{\Omega})\langle C(r,\hat{\Omega})D(\hat{\Omega},r)\rangle_{\mathrm{sim}} - 2C(r,\hat{\Omega})\langle B(r,\hat{\Omega})D(r,\hat{\Omega})\rangle_{\mathrm{sim}} - 2D(r,\hat{\Omega})\langle B(r,\hat{\Omega})C(r,\hat{\Omega})\rangle\Big\}.\end{aligned} \tag{32}$$

Mode–mode coupling is important at low angular modes, and we consider the full case later, but for higher frequency modes, we can approximate the linear correction to the local shape:

$$S_l^{\mathrm{loc}} = \frac{1}{f_{\mathrm{sky}}}\left\{S_l^{A,B^2} - 2S_l^{\langle A,B\rangle B} - S_l^{A,\langle B^2\rangle}\right\} + \frac{2}{f_{\mathrm{sky}}}\left\{S_l^{AB,B} - S_l^{\langle AB\rangle,B} - S_l^{B\langle A,B\rangle} - S_l^{A\langle B,B\rangle}\right\}, \tag{33}$$

where f_{sky} is the sky fraction observed.

The S_ls such as $S_l^{\langle AB\rangle,B}$ describe the cross-power spectra associated with MC averaged product maps $\langle A(n,r)B(n,r)\rangle$ constructed with the same mask and the noise model as the observed map B. Likewise, the term $S_l^{A\langle B,B\rangle}$ denotes the average cross-correlation computed from MC averaging, of the product map constructed from the observed map $A(\Omega,r)$ multiplied with a MC realization of map $B(\Omega,r)$ against the same MC realization $B(\Omega,r)$:

$$\begin{aligned}S_l^{\mathrm{eq}} = & -18\Bigg[\frac{1}{f_{\mathrm{sky}}}\left\{S_l^{A,B^2} - 2S_l^{\langle A,B\rangle B} - S_l^{A,\langle B^2\rangle}\right\} + \frac{2}{f_{\mathrm{sky}}}\left\{S_l^{AB,B} - S_l^{\langle AB\rangle,B} - S_l^{B\langle A,B\rangle} - S_l^{A\langle B,B\rangle}\right\} \\ & + \frac{2}{f_{\mathrm{sky}}}\left\{S_l^{D,D^2} - 2S_l^{\langle D,D\rangle D} - S_l^{D,\langle D^2\rangle}\right\} - \frac{2}{f_{\mathrm{sky}}}\left\{S_l^{B,CD} - 2S_l^{\langle B,C\rangle D} - S_l^{B,\langle CD\rangle}\right\} \\ & - \frac{2}{f_{\mathrm{sky}}}\left\{S_l^{D,CB} - 2S_l^{\langle D,C\rangle B} - S_l^{D,\langle CB\rangle}\right\} - \frac{2}{f_{\mathrm{sky}}}\left\{S_l^{C,BD} - 2S_l^{\langle C,B\rangle D} - S_l^{C,\langle BD\rangle}\right\}\Bigg].\end{aligned} \tag{34}$$

Creminelli et al. (2006) showed via numerical analysis that the linear terms are less important in the equilateral case than in the local model. The use of such MC maps to model the effect of mask and noise greatly improves the speed compared to full bispectrum analysis.

The use of linear terms was found to greatly reduce the scatter of the estimator, thereby improving its optimality. The estimator was used in Yadav & Wandelt (2008) also to compute the f_{NL} from combined T and E maps. The analysis presented above for both the one-point statistics and the power-spectral analysis is approximate, because it uses a crude f_{sky} approximation to deconvolve the estimated power spectrum to compare with analytical prediction. A more accurate analysis should take into account the mode–mode coupling which can dominate at low l. The general expression which includes the mode–mode coupling will be presented in the next section. However, it was found out by Yadav & Wandelt (2008) that removing low ls from the analysis can be efficient way to bypass the mode–mode coupling. A complete numerical treatment for the case of two-point statistics such as S_l^{A,B^2} will be presented elsewhere.

6 GENERALIZATION OF OPTIMAL ESTIMATORS FOR REALISTIC SURVEY STRATEGY: EXACT ANALYSIS

The general expression for the bispectrum estimator was developed by Babich (2005) for arbitrary sky coverage and inhomogeneous noise. The estimator includes a cubic term, which by matched-filtering maximizes the response for a specific type of input map bispectrum. The linear terms vanish in the absence of anisotropy but should be included for realistic noise to reduce the scatter in the estimates (see Babich 2005 for details). We define the optimal estimator as

$$\begin{aligned}\hat{E}_L[a] = & \sum_{L'}[N^{-1}]_{LL'}\Bigg[\frac{1}{6}\sum_{MM'}\sum_{ll'l_imm'm_i} B_{L'll'}\begin{pmatrix} L' & l & l' \\ M' & m & m' \end{pmatrix} \\ & \times\Big\{\left(C^{-1}_{L'M',l_1m_1}a_{l_1m_1}\right)\left(C^{-1}_{lm,l_2m_2}a_{l_2m_2}\right)\left(C^{-1}_{l'm',l_3m_3}a_{l_3m_3}\right) \\ & - C^{-1}_{lm,l'm'}\left(C^{-1}_{L'M',l_2m_2}a_{l_2m_2}\right) - 2C^{-1}_{LM,lm}\left(C^{-1}_{l'm',l_2m_2}a_{l_2m_2}\right)\Big\}\Bigg],\end{aligned} \tag{35}$$

where $N_{LL'}$ is a normalization to be discussed later. A factor of $1/(2l+1)$ can be introduced with the sum $\sum_M$, if we choose not to introduce the $N_{LL'}$ normalization constant. This will make the estimator equivalent to the one introduced in the previous section. Clearly as the data is weighted by $C^{-1} = (S+N)^{-1}$, or the inverse covariance matrix, addition of higher modes will reduce the variance of the estimator. In contrast, the performance of suboptimal estimators can degrade with resolution, due to the presence of inhomogeneous noise or a galactic mask. However, a wrong noise covariance matrix cannot only make the estimator suboptimal but also it will make the estimator biased too. The noise model will depend on the specific survey scan strategy. Numerical implementation of such inverse-variance weighting or multiplication of a map by C^{-1} can be carried out by conjugate gradient inversion techniques. Taking clues from Smith & Zaldarriaga (2006), we extend their estimators for the case of the bispectrum-related power spectrum. We will be closely following their notation whenever possible. First we define $Q_L[a]$ and its derivative $\partial_{lm} Q_L[a]$. The required input harmonics a_{lm} are denoted as a:

$$\hat{Q}_L[a] \equiv \sum_M a_{LM} \sum_{l'm',l''m''} B_{Ll'l''} \begin{pmatrix} L & l' & l'' \\ M & m' & m'' \end{pmatrix} a_{l'm'} a_{l''m''}, \tag{36}$$

$$\partial_{lm} \hat{Q}_L[a] \equiv \delta_{Ll} \sum_{l'm',l''m''} B_{Ll'l''} \begin{pmatrix} L & l' & l'' \\ m & m' & m'' \end{pmatrix} a_{l'm'} a_{l''m''} + 2 \sum_M a_{LM} \sum_{l'm'} B_{Lll'} \begin{pmatrix} L & l & l' \\ M & m & m' \end{pmatrix} a_{l'm'}. \tag{37}$$

These expressions differ from that of one-point estimators by the absence of an extra summation index. $Q_L[a]$ therefore represents a map as well as $\partial_{lm} Q_L[a]$, however $Q_L[a]$ is cubic in input maps a_{lm} where as $\partial_{lm} Q_L[a]$ is quadratic in input.

The bispectrum-related power spectrum can then be written as (summation convention for the next two equations)

$$\hat{E}_L[a] = [N^{-1}]_{LL'} \left\{ Q_{L'}[C^{-1}a] - [C^{-1}a]_{lm} \langle \partial_{lm} Q_{L'}[C^{-1}a'] \rangle_{\rm MC} \right\}. \tag{38}$$

Here $\langle\rangle_{\rm MC}$ denotes the Monte Carlo averages. The inverse covariance matrix in harmonic domain $C^{-1}_{l_1m_1,l_2m_2} = \langle a_{l_1m_1} a_{l_2m_2} \rangle^{-1}$ encodes the effects of noise and the mask. For all sky and signal-only limit, it reduces to the usual $C^{-1}_{l_1m_1,l_2m_2} = (1/C_l)\delta_{ll'}\delta_{mm'}$. The normalization of the estimator which ensures unit response can be written as

$$N_{LL'} = \frac{1}{3} \left\langle \left\{ \partial_{l_1m_1} Q_L[C^{-1}a] \right\} C^{-1}_{l_1m_1,l_2m_2} \left\{ \partial_{l_2m_2} Q_{L'}[C^{-1}a] \right\} \right\rangle - \frac{1}{3} \left\{ \left\langle \partial_{l_1m_1} Q_L[C^{-1}a] \right\rangle \right\} C^{-1}_{l_1m_1,l_2m_2} \left\{ \langle \partial_{l_2m_2} Q_{L'}[C^{-1}a] \rangle \right\}. \tag{39}$$

We will be using the following identity in our derivation:

$$\left\langle [C^{-1}a]_{l_1m_1} [C^{-1}a]_{l_2m_2} \right\rangle = C^{-1}_{l_1m_1,l_2m_2}. \tag{40}$$

The Fisher matrix, encapsulating the errors and covariances on the E_L, for a general survey associated with a specific form of bispectrum can finally be written as

$$F_{LL'} = \sum_{MM'} \sum_{l_i l'_i m_i m'_i} B_{Ll_1l'_1} B_{L'l_2l'_2} \begin{pmatrix} L & l_1 & l'_1 \\ M & m_1 & m'_1 \end{pmatrix} \begin{pmatrix} L' & l_2 & l'_2 \\ M' & m_2 & m'_2 \end{pmatrix}$$
$$\times \frac{1}{6} \left\{ 2C^{-1}_{LM,L'M'} C^{-1}_{l_1m_1,l'_1m'_1} C^{-1}_{l_2m_2,l'_2m'_2} + 4C^{-1}_{LM,l_2m_2} C^{-1}_{l_1m_1,L'M'} C^{-1}_{l'_1m'_1,l'_2m'_2} \right\} = \frac{1}{36} \left\{ 2\alpha^{PP}_{LL'} + 4\alpha^{QQ}_{LL'} \right\}. \tag{41}$$

Using the following expressions which are extension of Smith & Zaldarriaga (2006), we find that the Fisher matrix can be written as sum of two α terms α^{PP} and α^{QQ}. The alpha terms correspond to coupling only of modes that appear in different $3j$ symbols. Self-couplings are represented by the beta terms. The subscripts describes the coupling of various l and L indices. The superscript PP correspond to coupling of free indices, i.e. one free index L_1 with another free index L_2 and similar coupling for indices that are summed over such as l_1, l_2 etc. Similarly for superscript QQ the free indices are coupled with summed indices. Couplings are represented by the inverse covariance matrices in harmonic domain e.g. $C^{-1}_{lm,LM}$ denotes coupling of mode LM with lm:

$$\alpha^{PP}_{L_1L_2} = \sum_{M_1,M_2} \sum_{l_i l'_i m_i m'_i} B_{L_1l_1l'_1} B_{L_2l_2l'_2} \begin{pmatrix} L_1 & l_1 & l'_1 \\ M_1 & m_1 & m'_1 \end{pmatrix} \begin{pmatrix} L_2 & l_2 & l'_2 \\ M_2 & m_2 & m'_2 \end{pmatrix} C^{-1}_{L_1M_1,L_2M_2} C^{-1}_{l_1m_1,l_2m_2} C^{-1}_{l'_1m'_1,l'_2m'_2}, \tag{42}$$

$$\alpha^{QQ}_{L_1L_2} = \sum_{M_1,M_2} \sum_{l_i l'_i m_i m'_i} B_{L_1l_1l'_1} B_{L_2l_2l'_2} \begin{pmatrix} L_1 & l_1 & l'_1 \\ M_1 & m_1 & m'_1 \end{pmatrix} \begin{pmatrix} L_2 & l_2 & l'_2 \\ M_2 & m_2 & m'_2 \end{pmatrix} C^{-1}_{L_1M_1,l_2m_2} C^{-1}_{l_1m_1,L_2M_2} C^{-1}_{l'_1m'_1,l'_2m'_2}, \tag{43}$$

$$\alpha^{PP}_{L_1L_2} = (L_1 l_1 l'_1)(L_2 l_2 l'_2); \quad \alpha^{QQ}_{L_1L_2} = (L_1 l_1 l'_1)(L_2 l_2 l'_2). \tag{44}$$

These results will reduce to those of Smith & Zaldarriaga (2006) when further summations over L_1 and L_2 are introduced to collapse the two-point object to the corresponding one-point quantity. The beta terms that denote cross-coupling can be written as

$$\beta^{PP}_{L_1L_2} = \sum_{M_1,M_2} \sum_{l_i l'_i m_i m'_i} B_{L_1l_1l'_1} B_{L_2l_2l'_2} \begin{pmatrix} L_1 & l_1 & l'_1 \\ M_1 & m_1 & m'_1 \end{pmatrix} \begin{pmatrix} L_2 & l_2 & l'_2 \\ M_2 & m_2 & m'_2 \end{pmatrix} C^{-1}_{L_1M_1,L_2M_2} C^{-1}_{l_1m_1,l'_1m'_1} C^{-1}_{l_2m_2,l'_2m'_2}, \tag{45}$$

$$\beta^{PQ}_{L_1L_2} = \sum_{M_1,M_2} \sum_{l_i l'_i m_i m'_i} B_{L_1l_1l'_1} B_{L_2l_2l'_2} \begin{pmatrix} L_1 & l_1 & l'_1 \\ M_1 & m_1 & m'_1 \end{pmatrix} \begin{pmatrix} L_2 & l_2 & l'_2 \\ M_2 & m_2 & m'_2 \end{pmatrix} C^{-1}_{L_1M_1,l_2m_2} C^{-1}_{l_1m_1,l'_1m'_1} C^{-1}_{L_2M_2,l'_2m'_2}, \tag{46}$$

$$\beta^{QQ}_{L_1L_2} = \sum_{M_1,M_2}\sum_{(l_i l'_i m_i m'_i} B_{L_1 l_1 l'_1} B_{L_2 l_2 l'_2} \begin{pmatrix} L_1 & l_1 & l'_1 \\ M_1 & m_1 & m'_1 \end{pmatrix} \begin{pmatrix} L_2 & l_2 & l'_2 \\ M_2 & m_2 & m'_2 \end{pmatrix} C^{-1}_{L_1M_1,l_1m_1} C^{-1}_{L_2M_2,l_2m_2} C^{-1}_{l'_1m'_1,l'_2m'_2}, \tag{47}$$

$$\beta^{PP}_{L_1L_2} = (L_1 l_1 l'_1)(L_2 l_2 l'_2), \quad \beta^{PQ}_{L_1L_2} = (L_1 l_1 l'_1)(L_2 l_2 l'_2), \quad \beta^{QQ}_{L_1L_2} = (L_1 l_1 l'_1)(L_2 l_2 l'_2). \tag{48}$$

No summation over repeated indices is assumed. Using these expressions one can finally show that

$$[F^{-1}]_{LL'} = \langle \hat{E}_L \hat{E}_{L'} \rangle = \langle AA + BB + CC + 2AB + 2BC + 2AC \rangle_{LL'}, \tag{49}$$

where

$$AA_{LL'} = \left\{ \frac{2}{36}\alpha^{PP}_{LL} + \frac{4}{36}\alpha^{PP}_{LL'} + \frac{2}{36}\beta^{PP}_{LL} + \frac{4}{36}\beta^{PP}_{LL'} + \frac{4}{36}\beta^{PP}_{LL'} \right\}, \quad BB_{LL'} = \beta^{PP}_{LL'}, \quad CC_{LL'} = 4\beta^{QQ}_{LL'}, \tag{50}$$

$$2AB_{LL'} = -2\left(\beta^{PP}_{LL} + 2\beta^{QP}_{LL}\right), \quad 2AC_{LL'} = -4\left(2\beta^{QQ}_{LL'} + \beta^{PQ}_{LL'}\right), \quad 2BC_{LL'} = 4\beta^{PQ}_{LL'}. \tag{51}$$

The final expression can be written in terms of only α terms as the β terms cancel out:

$$F_{LL'} = \left\{ \frac{2}{36}\alpha^{PP}_{LL'} + \frac{4}{36}\alpha^{QQ}_{LL'} \right\}. \tag{52}$$

If we sum over LL' the Fisher matrix reduces to a scalar $F = \sum_{LL'} F_{LL'}$ with $\alpha^{PP}_{LL'} = \alpha^{QQ}_{LL'} = \alpha$ and $\beta^{PP}_{LL'} = \beta^{PQ}_{LL'} = \beta^{QQ}_{LL'} = \beta$, where α, β and F are exactly the same as introduced in Smith & Zaldarriaga (2006).

6.1 Joint estimation of multiple bispectrum-related power spectra

The estimation technique described above can be generalized to take cover the bispectrum-related power spectrum associated with different set of bispectra (X, Y):

$$\hat{E}_L[a] = [F^{-1}]^{XY}_{LL'}\{Q^Y_{L'}[C^{-1}a] - [C^{-1}a]_{lm}\langle \partial_{lm} Q^Y_{L'}[C^{-1}a]\rangle_{\rm MC})\}. \tag{53}$$

The associated Fisher matrix now will consist of sectors $F^{XX}_{LL'}$, $F^{YY}_{LL'}$ and $F^{XY}_{LL'}$. The sector XX and YY will in general will be related to errors associated with estimation of bispectra of X and Y types, whereas the sector XY will correspond to their cross-correlation:

$$F^{XY}_{LL'} = \left\{ \frac{2}{36}\left[\alpha^{PP}_{LL'}\right]^{XY} + \frac{4}{36}\left[\alpha^{QQ}_{LL'}\right]^{XY} \right\}, \tag{54}$$

where we have

$$\left[\alpha^{PP}_{LL'}\right]^{XY} = \sum_{MM'}\sum_{l_i l'_i m_i m'_i} B^X_{Ll_1l'_1} B^Y_{L'l_2l'_2} \begin{pmatrix} L & l_1 & l'_1 \\ M & m_1 & m'_1 \end{pmatrix} \begin{pmatrix} L' & l_2 & l'_2 \\ M' & m_2 & m'_2 \end{pmatrix} C^{-1}_{LM,L'M'} C^{-1}_{l_1m_1,l_2m_2} C^{-1}_{l'_1m'_1,l'_2m'_2} \tag{55}$$

and a similar expression holds for $[\alpha^{QQ}_{LL'}]^{XY}$.

6.2 Generalization to non-optimal weights

Although the estimator as constructed is fully optimal – its true usefulness is determined by the affordability of the construction of the C^{-1} matrix as well as the availability of a fast method to multiply it with CMB maps in a MC chain. A more general class of estimator which is suboptimal can be constructed by replacing the inverse covariance weighting of the data $[C^{-1}a]$ by $[Ra]$, where $[R]$ is an arbitrary filter function. In this case the estimator with unit response can be written as

$$\hat{E}^R_L[a] = \sum_{L'}[F^{-1}]_{LL'}\{Q_{L'}[Ra] - [Ra]_{lm}\langle \partial_{lm} Q_{L'}[Ra]\rangle_{\rm MC}\}, \tag{56}$$

where the normalization and the variance of the estimator can be constructed in a similar manner:

$$F_{LL'} = \langle (\hat{E}_L)(\hat{E}_{L'})\rangle - \langle \hat{E}_L\rangle\langle \hat{E}_{L'}\rangle^{-1} = \frac{1}{3}\langle\{\partial_{lm} Q_L[Rs]\}\, R\,\{\partial_{lm} Q_{L'}[Rs]\}\rangle_{\rm MC}$$
$$= \frac{1}{3}\langle\{\partial_{l_1m_1} Q_l[Ra]\}[FCF]_{l_1m_1,l_2m_2}\{\partial_{l_1m_1} Q_{l'}[Ra]\}\rangle - \frac{1}{3}\{\langle\partial_{l_1m_1} Q_l[Ra]\rangle\}C^{-1}_{l_1m_1,l_2m_2}\{\langle\partial_{l_2m_2} Q_{l'}[Ra]\rangle\}. \tag{57}$$

The optimal weighting can be replaced by an arbitrary weight or no weighting at all ($R = I$). However, in this case the estimator though unbiased clearly becomes a suboptimal one.

6.3 Recovery of all-sky homogeneous noise model

In the all-sky limit we recover the usual expression

$$F_{LL'} = \frac{1}{36}\left\{ 2\sum_{ll'} \frac{B^2_{Lll'}}{C_l C_L C_{L'}}\delta_{LL'} + 4\sum_l \frac{B^2_{LL'l}}{C_l C_L C_{L'}} \right\}. \tag{58}$$

In the case of joint analysis as before we can write down the off-diagonal blocks of the Fisher matrix as

$$F^{XY}_{LL'} = \frac{1}{36}\left\{ 2\sum_{ll'} \frac{B^X_{Lll'}B^Y_{Lll'}}{C_L C_l C_{l'}}\delta_{LL'} + 4\sum_{l} \frac{B^X_{LL'l}B^Y_{LL'l}}{C_L C_{L'} C_l} \right\}. \tag{59}$$

For $X = Y$ we recover the diagonal blocks of the Fisher matrix for the independent estimations derived before. The errors for independent estimates are given by $\sqrt{(F^{XX}_{LL})^{-1}}$, where as for the joint estimation the errors are $\sqrt{(F^{YY}_{LL})^{-1}}$.

6.4 More general bispectra

A more general bispectrum can be written as a sum of individual product terms:

$$b_{l_1l_2l_3} = \frac{1}{6}\sum_{i}^{N_{\rm fact}} A^i_{l_1}B^i_{l_2}C^i_{l_3} + \text{symm.perm.} \tag{60}$$

This can be seen as a generalization of the type of bispectra introduced for the equilateral case. It may also possible to approximate the bispectrum $b_{l_1l_2l_3}$ to a smaller number of optimum factorizable terms $N_{\rm opt}$ with suitable weight factors w_i as given in Smith & Zaldarriaga (2006). In this case the bispectrum is expressed as

$$b_{l_1l_2l_3} = \frac{1}{6}\sum_{i}^{N_{\rm opt}} w_i A^i_{l_1}B^i_{l_2}C^i_{l_3} + \text{symm.perm.} \tag{61}$$

Clearly significant computational gain can only be achieved if $N_{\rm opt} \ll N_{\rm fact}$. In the following discussion we generalize the description in previous sections to such composite bispectra. Following the same analytical reasoning we can show that the Fisher matrix elements for such a composite bispectrum can be written as

$$\left[F^{XY}_{LL'}\right]_{ij} = \frac{1}{36}\Bigg\{2\delta_{LL'}\sum_{ll'}\frac{(2L+1)(2l+1)(2l'+1)}{144\pi}\begin{pmatrix} L & l & l' \\ 0 & 0 & 0 \end{pmatrix}^2 \frac{1}{C_L C_l C_{l'}}\left[A^i_L B^i_l C^i_{l'} + \cdots\right]^X \left[[A^j_L B^j_l C^j_{l'} + \cdots\right]^Y$$
$$+ 4\sum_{l}\frac{(2L+1)(2L'+1)(2l+1)}{144\pi}\begin{pmatrix} L & L' & l \\ 0 & 0 & 0 \end{pmatrix}^2 \frac{1}{C_L C_{L'} C_l}\left[A^i_L B^i_{L'} C^i_l + \cdots\right]^X \left[[A^j_L B^j_{L'} C^j_l + \cdots\right]^Y\Bigg\}. \tag{62}$$

The symbols A^i, B^i etc denotes the ith term in the factorized representation of a specific type of the bispectrum of type X or Y. The total contribution of all terms will constitute the final Fisher matrix:

$$F^{XY}_{LL'} = \sum_{ij}\left[F^{XY}_{LL'}\right]_{ij}; \quad F^{XY} = \sum_{LL'} F^{XY}_{LL'}. \tag{63}$$

Using the following identity we can project this expression on to real space:

$$\int_{-1}^{1} {\rm d}z\, P_{l_1}(z)P_{l_2}(z)P_{l_3}(z) = 2\begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}^2, \tag{64}$$

$$[F_{LL'}]_{ij} = \int_{-1}^{1} {\rm d}z \left[\left(\xi^{A^iA^j}_L(z)\xi^{B^iB^j}(z)\xi^{C^iC^j}(z) + \cdots\right)\delta_{LL'} + \left(\xi^{A^iA^j}_L(z)\xi^{B^iB^j}_{L'}(z)\xi^{C^iC^j}(z) + \cdots\right)\right]. \tag{65}$$

The first term describes the diagonal entries of the Fisher matrix and the second term relates to the off-diagonal terms:

$$\xi^{A^iA^j}(z) = \sum_l \xi^{A^iA^j}_l(z), \quad \text{where} \quad \xi^{A^iA^j}_l(z) \equiv \frac{2l+1}{4\pi}\frac{A^i_l A^j_l}{C_l}P_l(z). \tag{66}$$

In case of cross-correlational studies the A^i and A^j will come from two different factorization of distinct bispectra denoted before by X and Y. The case of weighted sum can also be derived in an exactly similar manner.

7 MODELS FOR NON-GAUSSIANITY

We will use two commonly used specific models for the primordial non-Gaussianity as well as one foreground source of contamination, i.e. extragalactic point sources to demonstrate the power of our statistics in this section.

7.1 Local or squeezed model

Using the specific form for $b^{\rm loc}_{l_1l_2l_3}$ the Fisher matrix elements for the local model can be expressed in terms of the functions α and β and the power spectrum C_l:

$$\alpha^{PP}_{L_1L_2} = \frac{f^2_{\rm NL}}{4\pi}(2L_1+1)(2L_2+1)\sum_l \begin{pmatrix} L_1 & L_2 & l \\ 0 & 0 & 0 \end{pmatrix}^2 \frac{1}{C_{L_1}C_{L_2}C_l}$$
$$\times\left\{\int r^2{\rm d}r\left[2\alpha_{L_1}(r)\beta_{L_2}(r)\beta_l(r) + \alpha_l(r)\beta_{L_1}(r)\beta_{L_2}(r)\right]\right\}^2, \tag{67}$$

$$\alpha^{QQ}_{L_1L_2} = \delta_{L_1L_2}\frac{f_{\rm NL}^2}{4\pi}(2L_1+1)\sum_{l_1l_2}(2l_1+1)(2l_2+1)\begin{pmatrix} L_1 & l_1 & l_2 \\ 0 & 0 & 0 \end{pmatrix}^2 \frac{1}{C_{L_1}C_{l_1}C_{l_2}} \times \left\{\int r^2{\rm d}r\left[\alpha_{L_1}(r)\beta_{l_1}(r)\beta_{l_2}(r)+\alpha_{l_1}(r)\beta_{l_2}(r)\beta_{L_1}(r)+\alpha_{l_2}(r)\beta_{l_1}(r)\beta_{L_1}(r)\right]\right\}^2. \quad (68)$$

7.2 Equilateral model

Using the equilateral model the contribution to the Fisher matrix can be expressed as

$$\alpha^{PP}_{L_1L_2} = \frac{f_{\rm NL}^2}{4\pi}(2L_1+1)(2L_2+1)\sum_{l}(2l+1)\begin{pmatrix} L_1 & l_1 & l_2 \\ 0 & 0 & 0 \end{pmatrix}^2 \frac{1}{C_{L_1}C_{L_2}C_l} \times \left\{\int r^2{\rm d}r\left[-\alpha_{L_1}(r)\beta_{L_2}(r)\beta_l(r)+\delta_{L_1}(r)\delta_{L_2}(r)\delta_l(r)+\beta_{L_1}(r)\gamma_{L_2}(r)\delta_l(r)+\text{cyc.perm.}\right]\right\}^2, \quad (69)$$

$$\alpha^{QQ}_{L_1L_2} = \delta_{L_1L_2}(2L_1+1)\frac{f_{\rm NL}^2}{4\pi}\sum_{l_1l_2}(2l_1+1)(2l_2+1)\begin{pmatrix} L_1 & l_1 & l_2 \\ 0 & 0 & 0 \end{pmatrix}^2 \frac{1}{C_{L_1}C_{l_1}C_{l_2}} \times \left\{\int r^2{\rm d}r\left[-\alpha_{L_1}(r)\beta_{l_1}(r)\beta_{l_2}(r)+\delta_{L_1}(r)\delta_{l_2}(r)\delta_{l_3}(r)+\beta_{L_1}(r)\gamma_{l_1}(r)\gamma_{l_2}(r)+\text{cyc.perm.}\right]\right\}^2. \quad (70)$$

The power spectra C_l appearing in the denominator take contributions both from the pure signal or CMB and the detector noise. It is possible to bin the estimates in large enough bins to report uncorrelated estimates which may be possible for an experiment such as Planck with very high sky coverage. A detailed analysis of the singularity structure of the error-covariance matrix will be presented elsewhere.

7.3 Point sources

The bispectrum from residual point sources which are assumed random can be modelled as $b_{l_1l_2l_3} = b_{\rm ps}$. The exact value depends on the flux limit and the mask used in the survey. The accuracy of such an approximation can indeed be extended by adding contributions from correlation terms:

$$\alpha^{PP}_{L_1L_2} = \frac{b_{\rm ps}^2}{4\pi}(2L_1+1)(2L_2+1)\sum_{l}\begin{pmatrix} L_1 & L_2 & l \\ 0 & 0 & 0 \end{pmatrix}^2 \frac{1}{C_{L_1}C_{L_2}C_l}, \quad (71)$$

$$\alpha^{QQ}_{L_1L_2} = \delta_{L_1L_2}\frac{b_{\rm ps}^2}{4\pi}(2L_1+1)\sum_{l_1,l_2}\begin{pmatrix} L_1 & l_1 & l_2 \\ 0 & 0 & 0 \end{pmatrix}^2 \frac{1}{C_{L_1}C_{l_1}C_{l_2}}. \quad (72)$$

Similar computations can be done for cross-correlation among various contributions e.g. contamination due to point sources or estimation of a specific type of non-Gaussianity. A joint estimation is useful for finding out also the level of cross-contamination from one theoretical model while another is being estimated.

8 CONCLUSIONS

We have addressed the problem of finding an estimator of primordial non-Gaussianity from microwave background data. The new feature of this analysis is that the technique presented here allows one to make an assessment of whether any non-Gaussian signal is primordial or not. The issue here is that if one finds an estimate of a level of primordial non-Gaussianity which is inconsistent with zero, then it is very difficult at present to make a convincing case that it is indeed primordial and not simply contamination by any number of other effects which might lead to a non-Gaussian CMB map. The method does this by performing a less aggressive data compression than previous analyses. Rather than compressing the data to a single number (typically an estimate of $f_{\rm NL}$), it reduces the data to a function, the *bispectrum-related power spectrum*. This is an average cross-power spectrum of certain maps constructed from the CMB data and has some features in common with the near optimal T^2–T correlation power spectra as shown by Cooray (2001) and Chen & Szapudi (2007). By doing this construction, one retains the ability to assess the contributions from different sources, such as residual point sources, incomplete foreground subtraction and so on (see e.g. Serra & Cooray 2008). As an example, we have computed the expected bispectrum-related power spectrum optimized for local non-Gaussianity (we also consider the equilateral type), and calculated the contribution expected from an unclustered population of point sources. Indeed, one can use standard statistical methods to estimate the amplitude of components of non-Gaussianity, and since these contributors have quite different harmonic dependences, the estimators will be largely decoupled. The power of the technique will depend on the level of the primordial signal, but if it is at the level claimed by Yadav & Wandelt (2008), then it will be possible with Planck data to construct a large number of band-power estimates of the bispectrum-related power spectrum to see if it is primordial. We also include polarization in the analysis (see Appendix A). The work draws extensively on previous studies, in particular generalizing the Komatsu et al.

(2005) analysis for the all-sky, homogeneous noise case. For the more realistic case of partial sky coverage and inhomogeneous noise, we present optimized estimators of the bispectrum-related power spectrum, including linear terms and extending the work of Babich (2005), Smith & Zaldarriaga (2006) and Smith et al. (2009). For studies such as this, it is normal to assume the background cosmology is known from the power spectrum, but uncertainties will propagate into the $f_{\rm NL}$ estimates (Ligouri & Riotto 2008) and will be investigated elsewhere. Note that the techniques here could be generalized to higher order statistics such as the trispectrum, should the bispectrum vanish for some symmetry reason.

ACKNOWLEDGMENTS

DM acknowledges financial support from an STFC rolling grant at the University of Edinburgh. DM acknowledges useful discussions with Patrick Valageas and Anthony Lewis at various stages of the work. We particularly grateful to Michele Liguori for his help with the plot. Use of the CMBFAST package for calculating the transfer function is acknowledged.

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APPENDIX A: JOINT ANALYSIS OF TEMPERATURE AND POLARIZATION

Most current constraints on non-Gaussianity still come from temperature maps, but with *WMAP* and Planck, the situation is changing. Similar calculations can be performed for joint temperature and E-type polarization analysis, and the estimators discussed above can be generalized to include E-type polarization to tighten the constraints. The functions α_l and β_l that we discussed in the main text needs to be generalized for both temperature, T, and E-type polarization. We follow the discussion in Yadav, Komatsu & Wandelt (2007), see also Babich & Zaldarriaga (2004) and Liguori et al. (2007) for related discussions:

$$\alpha_l^X(r) \equiv \frac{2}{\pi}\int_0^\infty k^2 \mathrm{d}k\, P_\Phi(k)\Delta_l^X(k) j_l(kr),\; \beta_l^X(r) \equiv \frac{2}{\pi}\int_0^\infty k^2 \mathrm{d}k\, \Delta_l^X(k) j_l(kr). \quad \text{(A1)}$$

The index X can be T or E. Similar calculations can in principle be performed for the equilateral case. We will focus however here only on the local or squeezed model. The power spectra now can be a temperature (TT)-only power spectrum or an electric–electric (EE) polarization spectra which we define below, with specific choice of Δ_l:

$$C_l^{XY}(r) \equiv \frac{2}{\pi}\int_0^\infty k^2 \mathrm{d}k\, P_\Phi(k)\Delta_l^X(k)\Delta_l^Y(k). \quad \text{(A2)}$$

We arrange the all-sky covariance matrix (in the harmonic domain) in a matrix, which takes the following form in terms of the C_l defined above:

$$[C]_l = \begin{bmatrix} C_l^{TT} & C_l^{TE} \\ C_l^{TE} & C_l^{EE} \end{bmatrix}. \quad \text{(A3)}$$

We can now construct the $A(r, \hat{\Omega})$ and $B(r, \hat{\Omega})$ fields now can be constructed using the full covariance matrix instead of only temperature data:

$$A(r, \hat{\Omega}) = \sum_{lm}\sum_{ip} \left[C^{-1}\right]_l^{ip} a_{lm}^i \alpha_l^p(r) Y_{lm}(\hat{\Omega}), \quad \text{(A4)}$$

$$B(r, \hat{\Omega}) = \sum_{lm}\sum_{ip} \left[C^{-1}\right]_l^{ip} a_{lm}^i \beta_l^p(r) Y_{lm}(\hat{\Omega}). \quad \text{(A5)}$$

The construction now follows exactly the same steps as depicted for the temperature case. We compute the cross-correlation of $B^2(r, \hat{\Omega})$ with the $A(r, \hat{\Omega})$. The $B^2(r, \hat{\Omega})$ can be decomposed in terms of the T and E harmonics both a_{lm}^j (j here takes values T or E):

$$B_{lm}^{(2)}(r) = \int \mathrm{d}\Omega B^2(r, \Omega) Y_{lm}(\Omega)$$
$$= \sum_{l'm'}\sum_{l''m''} \beta_{l'}(r)[C^{-1}]_{l'}^{jq}\beta_{l''}(r)[C^{-1}]_{l''}^{kr}\sqrt{\frac{(2l+1)(2l'+1)(2l''+1)}{4\pi}}\begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} l & l' & l'' \\ m & m' & m'' \end{pmatrix} a_{l'm'}^j a_{l''m''}^k. \quad \text{(A6)}$$

The cross-correlation of the associated A and B field will contain information both from temperature and polarization maps:

$$S_l^{A,B^2} = \int r^2 \mathrm{d}r\, S_l^{A,B^2}(r) = \int r^2 \mathrm{d}r\, B_{lm}^2(r) A_{lm}(r) = \frac{1}{2l+1}\sum_m\sum_{l'm'}\sum_{l''m''}\int r^2 \mathrm{d}r\, \alpha_l(r)[C^{-1}]_l^{ip}\beta_{l'}(r)[C^{-1}]_{l'}^{jq}\beta_{l''}(r)[C^{-1}]_{l''}^{kr}$$
$$\times\sqrt{\frac{(2l+1)(2l'+1)(2l''+1)}{4\pi}}\begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} l & l' & l'' \\ m & m' & m'' \end{pmatrix} a_{lm}^i a_{l'm'}^j a_{l''m''}^k, \quad \text{(A7)}$$

$$B_{ll'l''}^{ijk} = \sum_{mm'm''}\begin{pmatrix} l & l' & l'' \\ m & m' & m'' \end{pmatrix} a_{lm}^i a_{l'm'}^j a_{l''m''}^k. \quad \text{(A8)}$$

The mixed bispectrum, $B_{ll'l''}^{pqr}$, contains information about three-point correlation in harmonic space with the possibility that the harmonics a_{lm}^i can either be of T or temperature type or E or electric-polarization type. The theoretical model for such bispectrum depends on functions $\alpha_l^p(r)$ and $\beta^q(r)$ where again depending on the superscript the functions can be of temperature or the electric type:

$$B_{ll'l''}^{pqr} = \sqrt{\frac{(2l+1)(2l'+1)(2l''+1)}{4\pi}}\begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix}\int r^2 \mathrm{d}r \left\{\beta_l^p(r)\beta_{l'}^q(r)\alpha_{l''}^r(r) + \beta_l^p(r)\alpha_{l'}^q(r)\beta_{l''}^r(r) + \alpha_l^p(r)\beta_{l'}^q(r)\beta_{l''}^r(r)\right\}. \quad \text{(A9)}$$

Finally in an analogous way we can express the power spectrum related to the mixed bispectrum as follows:

$$(2l+1)\left(S_l^{A,B^2} + 2S_l^{AB,B}\right) = \hat{f}^{\mathrm{NL}}\sum_{l'}\sum_{l''}\left\{B_{ll'l''}^{ijk}[C^{-1}]_l^{ip}[C^{-1}]_{l'}^{jq}[C^{-1}]_{l''}^{kr}B_{ll'l''}^{pqr}\right\}. \quad \text{(A10)}$$

We have assumed summation of repeated indices which denote the polarization types, i.e. i, j, k and p, q, r in the above expression. The one-point mixed skewness which is related to the above power spectrum is analogously written as follows:

$$S_{TE}^{\rm prim} = \sum_l (2l+1)\left(S_l^{A,B^2} + 2S_l^{AB,B}\right) = \hat{f}^{\rm NL} \sum_l \sum_{l'} \sum_{l''} \left\{ B_{ll'l''}^{ijk} [C^{-1}]_l^{ip} [C^{-1}]_{l'}^{jq} [C^{-1}]_{l''}^{kr} B_{ll'l''}^{pqr} \right\}. \quad \text{(A11)}$$

This generalizes the temperature-only power spectrum estimator introduced in the text of this paper and can be extended to take into account other models, sky coverage and secondary anisotropies in an analogous manner.

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