Solving Constrained Mean-Field Linear Quadratic and Other Stochastic Optimal Control Problems

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Abstract

We study a few constrained Stochastic Optimal Control Problems. First, we look at problems with terminal constraints. For various convex problems with constrained control, such as Linear Quadratic Mean-Field problem or Non-Markovian problem with stochastic coefficients, we draw equivalence relationship between the Fritz John condition and Karush–Kuhn–Tucker (KKT) conditions. Then we construct an unconstrained problem with the Lagrange Multiplier derived from Fritz John condition. Finally, we show the equivalence between the optimality of the unconstrained problem and its original problem. Furthermore, we look at the Duality of Linear Quadratic Mean-Field control problems and find an equivalence relationship between the primal and dual problems in the absence of control constraints. Lastly we compare the Riccati solutions to the Linear Quadratic Mean-Field control problem and the empirical solutions to the Mean-Field Forward Backward Stochastic Differential Equations (FBSDEs) using Deep Learning to verify our results.

Keywords: Duality, Lagrange Multiplier, Fritz John Condition, KKT conditions, Linear Quadratic Mean-Field Control, Deep Learning, Mean-Field Forward Backward Equations

To my family.

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Notation

Notation

- $\mathbb{R}^d$: the d-dimensional Eucledean space
- $\mathbb{R}^{m\times n}$: the set of all $m \times n$ real matrices
- $\mathbb{S}^n$: the set of all $n \times n$ symmetric real matrices
- $\mathbb{S}^n_+$: the set of all $n \times n$ symmetric positive definite real matrices
- $\bar{\mathbb{S}}^n_+$: the set of all $n \times n$ symmetric positive semi-definite real matrices
- I_n: the $n \times n$ identity matrix which is simply denoted as I if no confusion arises
- $M^\intercal$: the transpose of matrix M
- $M^\dagger$: the Moore-Penrose pseudoinverse of matrix M
- $\mathrm{tr}(M)$: the trace of matrix M
- $\langle \cdot, \cdot \rangle$: the inner product in a Hilbert Space, particularly for matrices in $\mathbb{R}^{m\times n}$, $\langle M, N\rangle = \mathrm{tr}(M^\intercal N)$
- $|M|$: the Forbenius norm of matrix M which is $\sqrt{\langle M, M\rangle}$
- $(\Omega, \mathcal{F}, \mathbb{P})$: A complete probability space
- $\{W_t, t \in [0, T]\}$: a $\mathbb{R}^d$-valued standard Brownian Motion on $(\Omega, \mathcal{F}, \mathbb{P})$

- $\{\mathcal{F}_t : 0 \le t \le T\}$: the natural filtration generated from $\{W_t, t \in [0,T]\}$

- $\mathbb{F}$: the usual augmentation of $\{\mathcal{F}_t : 0 \le t \le T\}$

Let $\mathbb{H}$ be a Euclidean Space

- $L^p_{\mathcal{F}_t}(0,T;\mathbb{H})$: the set of all $\mathcal{F}_t$-measurable $\mathbb{H}$-valued random variables ξ such that $\mathbb{E}[|\xi|^2] < \infty$

- $L^p(t,T;\mathbb{H})$: the set of all $\mathbb{H}$-valued functions that are p-th power Lebesgue integrable on $[t,T]$, especially, $L^\infty(t,T;\mathbb{H})$ denote the set of $\mathbb{H}$-valued Lebesgue measurable functions that are essentially bounded

- $L^p_{\mathbb{F}}(t,T;\mathbb{H})$: the set of all $\mathbb{H}$-valued $\mathbb{F}$-progressively measurable processes $\phi : [t,T]\times\Omega \to \mathbb{H}$ such that $\mathbb{E}[\int_0^T |\phi(s)|^p ds] < \infty$

- $\bar{L}^p_{\mathbb{F}}(t,T;\mathbb{H})$: the set of all $\mathbb{H}$-valued $\mathbb{F}$-progressively measurable processes $\phi : [t,T]\times\Omega \to \mathbb{H}$ such that $\mathbb{E}[\sup_{s\in[t,T]} |\phi(s)|^p] < \infty$

- $\bar{L}^p_{\mathbb{F}}(C(t,T;\mathbb{H}))$: the set of all $\mathbb{H}$-valued $\mathbb{F}$-adapted continuous processes $\phi : [t,T]\times\Omega \to \mathbb{H}$ such that $\mathbb{E}[\sup_{s\in[t,T]} |\phi(s)|^p] < \infty$

For any random variable X or stochastic process $\{Y(t)\}_{t\in[0,T]}$:

- $\bar{X}$: $\mathbb{E}[X]$

- $\tilde{X}$: $X - \mathbb{E}[X]$

- $\bar{Y}(\cdot)$: $\mathbb{E}[Y(\cdot)]$

- $\tilde{Y}(\cdot)$: $Y(\cdot) - \mathbb{E}[Y(\cdot)]$

1

Introduction

1.1 Overview

Due to its purpose of obtaining optimal values, stochastic optimal control has naturally become a commonly studied topic, especially in mathematical finance. There is extensive research tackling systems involving stochastic differential equations (SDEs) which can be converted to equally solving Forward and Backward SDEs (FBSDEs). In the late 1970s, Merton [47, 48] introduced the idea of stochastic control to solve a problem in a Markovian setting. With the findings of Black and Scholes [10], the concept was reinforced and paved the way for more complicated problems. Most of the research that deals with optimal control problems can be divided into 2 methods: the Bellman dynamic programming approach (Hamilton-Jacobi-Bellman or HJB) in [6] and the Pontryagin (stochastic) maximum principle (SMP) approach in [11].

HJB provides a necessary and sufficient condition for optimality of the problem with respect to its loss function in the form of a Partial Differential Equation (PDE). When the problem is linear quadratic, by assuming the value function to be quadratic, the HJB equation can be simplified into a Riccati equation. On the other hand, SMP tries to maximise the Hamiltonian to establish a necessary condition for optimality. While HJB may seem to be a more powerful tool, SMP is equally important in the Stochastic Programming analysis.The role of SMP becomes more prominent when people are no longer satisfied with Markovian settings and start investigating problems outside of Markovian settings such as when the coefficients are stochastic like those in [9, 53, 20]. Similarly, in the case of linear quadratic problem, by using a decoupling method, the problem can also be simplified to solve the Riccati equations. Refer to [39] and references therein for more details. On the other hand, HJB becomes less studied, as it requires generalising PDEs into stochastic PDEs, which are much more difficult to solve.

But, of course, mathematicians never stop at the simplest problems. Inspired by real-life scenarios such as risk minimisation in finance and cost reduction in transportation, people are also interested in problems with extra constraints, such as constrained controls and terminal constraints. For example, Pham [51] discusses a problem involving CRRA uitility model with convex cone-constrained control. When the coefficients are stochastic, [29] gives a general solution represented by Extended Stochastic Riccati Equations to problems with controls in a closed cone. There are even more studies on constraints on State process, either at terminal or across the whole time period. Altman [4] listed many publications and examples of constraint Markovian dynamic programming problems. Risk constraints through probability expectations, variance, or value constraints are very common in financial sectors, such as in [52, 19, 26, 43].

In another field of mathematics, game theories grow extensively in the late 90s and early 20s. When studying high-dimensional games, many consider stochastic differential games and N Nash points like in [12, 46, 31, 41]. By letting N tend to infinity, the concept of Mean-Field games emerges together with the limit, McKean–Vlasov equation. Inspired by the concept, Mean-Field Backward Stochastic Differential Equations (MFBSDEs) were studied in [14] as the limit of a system containing N interacting agents and further developed in [15]. Based on the new concept, people start to investigate Mean-Field Forward Backward Stochastic Differential Equations (MFFBSDEs) and optimal control problems involving them such as in [42, 22, 17].

If, in addition, the problem is convex, then duality used in convex optimaization is still relevant to the analysis. Duality in SDEs was first proposed by Bismut in [8]. Then the idea is further developed in [20, 21, 40]. As mentioned in [55], from a heuristic point of view, the dual process is a Lagrangian multiplier which aims to optimise the Lagrangian. Although not always easier to solve the original problem, the dual problem can sometimes provide an explicit solution while its primal counterpart cannot, opening up more opportunities to tackle the problems.

However, if the problem is more complicated, more often than not, an analytical solution may not exist. Due to rapid development in recent years, Deep Learning techniques can be used to find empirical solutions. The method can be applied to various problems from SDEs to PDEs and related BSDEs and from fully coupled FBSDEs to MFFBSDEs as exemplified in [56, 23, 33, 18].

1.2 Outline of the thesis

Chapter 2 studies the Stochastic Optimal Control problems with terminal constraints under different settings. The first problem involves stochastic coefficients and a constrained control in a convex cone. The second problem is a MFSDE with, again, a constrained control in a convex cone. The main approach is to draw an equivalent relationship between Fritz John condition (FJ) and Karush–Kuhn–Tucker conditions (KKT). Then use FJ to show the existence of the Lagrange multiplier so that the problem with constrained terminal state can be converted to an unconstrained problem. Lastly, show that the optimality to the unconstrained problem with KKT conditions is equivalent to the optimality to the original problem.

Chapter 3 studies a type of Mean-Field Stochastic Optimal Control problem with initial value of the state process involved in the cost function. It lists the assumptions required for the existence and uniqueness to the optimal control, as well as the adjoint process. Then it investigates the optimality condition using the SMP method.

Chapter 4 studies the duality of a linear quadratic mean-field stochastic optimal control problem. Then investigates the equivalent relationship between the primal and dual problem when the control is set in the whole space.

Chapter 5 tries to verify the results of previous chapters by comparing the results of analytical solutions and empirical values of the deep learning method.

Finally, in conclusion, the limitations and future work are discussed.

2

Lagrange Multiplier

2.1 Introduction

Constrained Stochastic Optimal Control has always been a popular area of study. Different methods are used to tackle different constraints. Such as duality [34], HJB [16, 13], and most naturally Lagrange multiplier. Flam [27] gave a proof of necessary conditions of KKT in terms of subdifferential for discrete Markovian problem followed by a correction in [28]. Since then, more people have used the Lagrange multiplier method to convert a constrained problem into an unconstrained one and apply the established method to solve problems, such as those in [38, 7, 36]. While many of the results can be applied to more general problems, which are non-convex and non-Frechet differentiable, they often complicate the problem. Moreover, due to generality, the local extrema obtained using KKT are often not global extrema.

As such, inspired by the Fritz John condition mentioned in [25], which does not contain any requirement on the differentiability of the problem, we generalised the results there from discrete problems into continuous time problems and draw relationships between the FJ condition and KKT conditions, the later of which can be shown to be necessary and sufficient to the optimality of original problem under nice enough conditions.

In this section, we studied two problem. The first one is inspired by problem in [44], which is linear quadratic, containing stochastic coefficients and convex cone constrained control. When the terminal constraint also follows a similar quadratic form, the unconstrained problem can be solved using the same method proposed in the paper and hence give an optimal solution to the original constrained problem. The second problem is inspired by [59], which is mean-field.

2.2 Lagrange Multiplier for Stochastic Optimal Control Problem with additional terminal inequality constraints

Let $T > 0$ be a fixed terminal time, $\{W_t, t \in [0, T]\}$ a $\mathbb{R}^N$-valued standard Brownian Motion with entries $W_m, m = 1, 2...N$, on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Lemma 2.2.1 *$L^2_{\mathbb{F}}(0, T; \mathbb{R}^N)$ forms a Hilbert space with inner product $\langle x, y\rangle$ defined as*

$$\langle x, y\rangle = \mathbb{E}[\int_0^T x^\intercal(t) y(t) dt].$$

Lemma 2.2.2 *Every bounded sequence in Hilbert space has a weakly convergent subsequence.*

Proof. For example refer to [35]. $\square$

Theorem 2.2.3 (Banach Sak's Thoerem) *In a uniformly convex Banach space, any weakly converging sequence has a subsequence whose Cesaros-mean converges strongly, where the Cesaros-mean denotes the arithmetic mean.*

Proof. For example refer to [37]. □

Assume that $r, Q \in L^1_{\mathbb{F}}(0,T;\mathbb{R})$, $\theta, S \in L^1_{\mathbb{F}}([0,T],\mathbb{R}^N)$, $\sigma, R \in L^1_{\mathbb{F}}(0,T;\mathbb{R}^{N\times N})$ are uniformly bounded. For all $(z,\omega,t) \in \mathbb{R}^N \times \Omega \times [0,T]$, there exists a constant $k > 0$ such that

$$z^\intercal \sigma(\omega,t)\sigma^\intercal(\omega,t) z \geq k|z|^2,$$

R symmetric and the matrix $\begin{pmatrix} Q(t) & S^\intercal(t) \\ S(t) & R(t) \end{pmatrix}$ is positive definite for almost all $(\omega,t) \in \Omega \times [0,T]$.

Lemma 2.2.4 (Schur's Complement)

$$\begin{aligned} &\left(\begin{smallmatrix} Q(t) & S^\intercal(t) \\ S(t) & R(t) \end{smallmatrix}\right) \succ 0 \\ \Longleftrightarrow\ & Q(t) \succ 0, R(t) - S(t)Q^{-1}(t)S(t)^\intercal \succ 0 \\ \Longleftrightarrow\ & Q(t) - S(t)^\intercal R(t)^{-1} S(t) \succ 0, R \succ 0. \end{aligned}$$

Proof. For example, refer to Section 1.4 Theorem 1.12 from [60]. □

Lemma 2.2.5 (Fatou's Lemma) *Given a measure space $(\Omega, \mathcal{F}, \mu)$ and a set $X \in \mathcal{F}$, let $\{f_n\}$ be a sequence of $(\mathcal{F}, \mathcal{B})$-measurable non-negative functions $f_n : X \to [0, +\infty]$. Define the function $f : X \to [0, +\infty]$ by setting $f(x) = \liminf_{n\to\infty} f_n(x)$, for every $x \in X$. Then f is $(\mathcal{F}, \mathcal{B})$-measureable, and also $\int_X f d\mu \leq \liminf_{n\to\infty} \int_X f_n d\mu$, hwere the integrals may be infinite.*

Proof. This is a direct application of Monotonic Convergence Theorem. Refer to Theorem 2 from section 5.2 in [57]. □

Theorem 2.2.6 (Jensen's Inequality) *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, X an integrable real-valued random variable and f a convex function. Then:*

$$f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)].$$

Proof. Refer to [50] for a general proof on an infinite-dimensional space. $\square$

Let a, c, m, n, k be bounded variables valued in $\mathbb{R}$ with a, m being positive. Under the quadratic and convex setting, for $x \in \mathbb{R}, \pi \in \mathbb{R}^N$, let

$$\begin{cases} f(\omega, t, x, \pi) & = \frac{1}{2}(Q(t)x^2 + 2\pi^\intercal S(t)x + \pi^\intercal R(t)\pi) \\ g(\omega, x) & = \frac{1}{2}(ax^2 + 2cx) \\ h(\omega, x) & = \frac{1}{2}(mx^2 + 2nx + k) \end{cases}$$

Then for the state process $X^\pi \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^N)$ and control $\pi \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^N)$, the optimization problem is as the following:

$$\underset{\pi \in \mathcal{A}}{\textbf{minimize}} \quad J(\pi) = \mathbb{E}\left[\int_0^T f(t, X^\pi(t), \pi(t))dt + g(X^\pi(T))\right] \tag{2.1}$$

(P1)

$$\textbf{subject to} \quad H(\pi) = \mathbb{E}\left[h(X^\pi(T))\right] \leq 0 \tag{2.2}$$

where

$$dX^\pi(t) = (r(t)X^\pi(t) + \pi^\intercal(t)\sigma(t)\theta(t))dt + \pi^\intercal(t)\sigma(t)dW_t;\ X^\pi(0) = x_0 \tag{2.3}$$

and

$$\mathcal{A} := \{\pi \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^N) : \pi(t) \in K \text{ for all } t \in [0, T] \text{ a.e.}\},$$

with $K \subseteq \mathbb{R}^N$ a closed convex set containing 0.

Define

$$\mathcal{B} := \{\pi \in \mathcal{A} : H(\pi) \leq 0\}.$$

We call π admissible if it is in $\mathcal{B}$ and the pair (π, X^{π}) is called admissible if X^{π} is a strong solution to (2.3). Let $V(x_0)$ denote the value function of (P1), i.e. $V(x_0) = \inf_{\pi \in \mathcal{B}} J(\pi)$.

Lemma 2.2.7 *$\mathcal{B}$ is convex.*

Proof. $\mathcal{B} = \mathcal{A} \cap \{\pi : H(\pi) \leq 0\}$ and if it can be shown that $\mathcal{B}$ is a union of 2 convex sets, then it is also convex. It is straightforward to see that $\mathcal{A}$ is convex. To see that $\{\pi : H(\pi) \leq 0\}$ is convex, first observe (2.3) is linear about π and X. Therefore $X^{\tilde{\pi}} = X^{\mu\pi_1 + (1-\mu)\pi_2} = X^{\mu\pi_1} + X^{(1-\mu)\pi_2} = \mu X^{\pi_1} + (1-\mu)X^{\pi_2}$ for $\mu \in [0,1]$ and (π_i, X^{π_i}) admissible. Given h convex, we can conclude that

$$H(\tilde{\pi}) = \mathbb{E}\left[h(X^{\tilde{\pi}}(T))\right] \leq \mathbb{E}\left[\mu h(X^{\pi_1}(T)) + (1-\mu)h(X^{\pi_2}(T))\right] \leq 0.$$

Hence $\tilde{\pi} \in \{\pi : H(\pi) \leq 0\}$ and so $\mathcal{B}$ is convex. $\square$

If the solution to the non-constrained problem (2.1) already satisfies the constraint (2.2), then we are done here. Thus it is advisable to solve the problem without additional constraint first.

For a more general solution, let's introduce a Langrange Multiplier $\lambda \geq 0$ to reduce constrained problem (P1) to an unconstrained case:

$$\text{for each } \lambda \ \underset{\pi \in \mathcal{A}}{\textbf{minimize}} \quad J(\pi, \lambda) = J(\pi) + \lambda H(\pi), \tag{2.4}$$

(P2)

$$\text{then find } \lambda \text{ such that} \quad \begin{cases} H(\pi) \leq 0, \\ \lambda H(\pi) = 0, \\ \lambda \geq 0. \end{cases} \tag{2.5}$$

While we can also draw equivalent relationship between Karush–Kuhn–Tucker conditions and Fritz John condition similar to the deterministic optimal control, in stochastic case, we shall use an alternative version of Fritz John (FJ) condition mentioned in [25].

Define

$$C := \{(r, s) \in \mathbb{R} \times \mathbb{R} : J(\pi) \leq V(x_0) + r, H(\pi) \leq s, \text{ for some } \pi \in \mathcal{A}\}.$$

Then the following assumption ensures that we will arrive at FJ condition:

(FJ1) The convex hull of C, $\mathbf{Conv}C$ has non-empty interior and $(0, 0)$ lies on the boundary of $\mathbf{Conv}C$.

Lemma 2.2.8 *C is convex.*

Proof. Suppose $(r_1, s_1), (r_2, s_2) \in C$, so there exists $\pi_1, \pi_2 \in \mathcal{A}$ s.t.

$$J(\pi_i) \leq V(x_0) + r_i, H(\pi_i) \leq s_i, \text{ for } i \in \{1, 2\}.$$

Then for any real number $\mu \in [0, 1]$, $\tilde{\pi} = \mu\pi_1 + (1 - \mu)\pi_2$, $\tilde{r} = \mu r_1 + (1 - \mu)r_2$ and $\tilde{s} = \mu s_1 + (1 - \mu)s_2$,

$$J(\tilde{\pi}) \leq \mu J(\pi_1)+(1-\mu)J(\pi_2) \leq \mu V(x_0)+\mu r_1+(1-\mu)V(x_0)+(1-\mu)r_2 \leq V(x_0)+\tilde{r},$$

$$H(\tilde{\pi}) \leq \mu H(\pi_1) + (1 - \mu)H(\pi_2) \leq \mu s_1 + (1 - \mu)s_2 \leq \tilde{s}.$$

Hence $\tilde{\pi} \in C$ and C is convex. □

Also, as long as $\left|V(x_0)\right| < \infty$, C is non-empty so it only suffices to assume the origin is on the boundary. As such assumption (FJ1) is equivalent to the following condition under the current problem setting:

(FJ2) $\left|V(x_0)\right| < \infty$ and origin is on the boundary of C.

Theorem 2.2.9 (Fritz John Condition) *With (FJ2) satisfied, one can find a non-zero pair* $(r^*, s^*) \in \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0}$ *such that*

$$r^* V(x_0) = \inf_{\pi \in \mathcal{A}} \{r^* J(\pi) + s^* H(\pi)\}. \tag{2.6}$$

Proof. From (FJ2), the origin is on the boundary of C, as such, we can find a supporting hyperplane to the convex hull through the origin. Hence there exists a non-zero pair $(r^*, s^*) \in \mathbb{R} \times \mathbb{R}$ such that $r^* r + s^* s \geq 0$, for any (r, s) in C. If $r^* < 0$, for any pair (r, s) we can pick arbitrarily large $\tilde{r}$ such that $(\tilde{r}, s)$ is still in C. Then there exits $\tilde{r}$ such that $\tilde{r}^* r + s^* s < 0$ and contradicts the result. Hence $r^* \geq 0$. With a similar argument, $s^* \geq 0$. Furthermore, by definition of C, for any $\pi \in \mathcal{A}$, $(J(\pi) - V(x_0), H(\pi)) \in C$ and

$$r^*(J(\pi) - V(x_0)) + s^* H(\pi) \geq 0.$$

Hence

$$r^* V(x_0) \leq \inf_{\pi \in \mathcal{A}} \{r^* J(\pi) + s^* H(\pi)\}.$$

The reverse is trivially true since $s^* \geq 0$ and $H(\pi) \leq 0$ for any $\pi \in \mathcal{B}$. Then,

$$r^* V(x_0) = \inf_{\pi \in \mathcal{B}} r^* J(\pi) \geq \inf_{\pi \in \mathcal{B}} \{r^* J(\pi) + s^* H(\pi)\} \geq \inf_{\pi \in \mathcal{A}} \{r^* J(\pi) + s^* H(\pi)\}.$$

Hence,

$$r^* V(x_0) = \inf_{\pi \in \mathcal{A}} \{r^* J(\pi) + s^* H(\pi)\}.$$

□

To ensure the existence of the Lagrange multiplier, another assumption has to be made:

(SF1) There exists $\pi \in \mathcal{A}$ s.t. $H(\pi) < 0$.

Theorem 2.2.10 *If both (FJ2) and (SF1) are satisfied, then we can always find a Lagrange multiplier λ such that*

$$V(x_0) = \inf_{\pi \in \mathcal{A}} \{J(\pi) + \lambda H(\pi)\}.$$

Proof. Suppose for a contradiction for all pair $(r^*, s^*) \in C$, $r^* = 0$, then by non-zeroness, $s^* > 0$, together with (SF1)

$$0 = s^* \inf_{\pi \in \mathcal{A}} H(\pi) \leq s^* \inf_{\pi \in \mathcal{B}} H(\pi) < 0,$$

which leads to a contradiction. Hence we can find such $r^* > 0$. Divide r^* on both sides of (2.6) to obtain

$$V(x_0) = \inf_{\pi \in \mathcal{A}} \{J(\pi) + \frac{s^*}{r^*} H(\pi)\}.$$

We can conclude $\lambda = \frac{s^*}{r^*}$ as the non-negative Lagrange multiplier. $\square$

Next, we want to show the Lagrange multiplier indeed satisfies (2.5).

Theorem 2.2.11 *Under (FJ2) and (SF1), we can find λ such that*

$$V(x_0) = \inf_{\pi \in \mathcal{A}} \{J(\pi) + \lambda H(\pi)\}$$

and (2.5) *is true.*

Proof. From Theorem 2.2.10, we can find a non-negative Lagrange multiplier λ s.t.

$$V(x_0) = \inf_{\pi \in \mathcal{A}} \{J(\pi) + \lambda H(\pi)\} \leq \inf_{\pi \in \mathcal{B}} \{J(\pi) + \lambda H(\pi)\}.$$

As $\lambda \geq 0$, for any $\pi \in \mathcal{B}$, $J(\pi) + \lambda H(\pi) \leq J(\pi)$, so

$$V(x_0) \leq \inf_{\pi \in \mathcal{B}} \{J(\pi) + \lambda H(\pi)\} \leq \inf_{\pi \in \mathcal{B}} J(\pi) = V(x_0).$$

Hence

$$\inf_{\pi \in \mathcal{B}} \{J(\pi) + \lambda H(\pi)\} = \inf_{\pi \in \mathcal{B}} J(\pi) = V(x_0). \tag{2.7}$$

Then, by the definition of infimum, we can find a sequence $\{\pi_n\}$ s.t. $\pi_n \in \mathcal{A}, H(\pi_n) \leq 0$ and

$$V(x_0) \leq J(\pi_n) \leq V(x_0) + \frac{1}{n}.$$

By completing the square in the running cost

$$f(t, X^{\pi_n}, \pi_n) = \frac{1}{2} Q \left(X_n^{\pi} + Q^{-1} S^{\intercal} \pi_n \right)^2 + \frac{1}{2} Q \pi_n^{\intercal} (R - S Q^{-1} S^{\intercal}) \pi_n.$$

From Lemma 2.2.4, $Q > 0$ and $R - SQ^{-1}S^{\intercal} \succ 0$. By the Min-Max Theorem, $\pi_n^{\intercal}(t)(R - SQ^{-1}S^{\intercal})\pi_n(t) \geq \lambda_{\min}\left|\pi_n(t)\right|^2$. Hence, $f(t, X^{\pi_n}, \pi_n) \geq \lambda_{\min}\left|\pi_n(t)\right|^2$.
The terminal cost is quadratic with quadratic coefficient $(a + \lambda m) > 0$. Therefore, it is bounded from below by some number κ. Then $J(\pi_n) \geq \mathbb{E}[\int_0^T \lambda_{\min}\left|\pi_n(t)\right|^2 dt] + \kappa$. As $J(\pi_n) \leq V(x_0) + \frac{1}{n}$ and $\left|V(x_0)\right| < \infty$, $\{\pi_n\} \subseteq \bar{L}^2_{\mathbb{F}}(0, T; \mathbb{R}^N)$, a bounded sequence in the Hilbert space $L^2_{\mathbb{F}}(0, T; \mathbb{R}^N)$. By Lemma 2.2.2, there exists a weakly convergent subsequence $\{\pi_i, i \in I_1\}$. The by Thoerem 2.2.3, we can again find a subsequence $\{\pi_i, i \in I_2 \subseteq I_1\}$ such that, after reordering the subsequence, the Cesaro-mean converges strongly to $\tilde{\pi}$, i.e.

$$\lim_{i \to \infty} \| \frac{1}{i} \sum_{j=1}^{i} \pi_{n_j} - \tilde{\pi} \| \to 0.$$

As $L^2_{\mathbb{F}}(0, T; \mathbb{R}^N)$ is a Hilbert space, the closed subset of Hilbert space is complete, hence the limit $\tilde{\pi} \in \mathcal{B}$.
Additionally note $J(\pi)$ is convex and bounded from below, then by Fatou's Lemma 2.2.5 and Jensen's inequality 2.2.6:

$$
\begin{aligned}
V(x_0) &\le J(\tilde{\pi}) \\
&= J(\lim_{i\to\infty} \frac{1}{i}\sum_{j=1}^{i} \pi_{n_j}) \\
&\le \lim_{i\to\infty} J(\frac{1}{i}\sum_{j=1}^{i} \pi_{n_j}) \\
&\le \lim_{i\to\infty} \frac{1}{i}\sum_{j=1}^{i} J(\pi_{n_j}) \\
&= V(x_0) + \lim_{i\to\infty} \frac{1}{i}\sum_{j=1}^{i} \frac{1}{n_j} \\
&\le V(x_0) + \lim_{i\to\infty} \frac{1}{i}\sum_{j=1}^{i} \frac{1}{j} \\
&= V(x_0).
\end{aligned}
$$

Substitute $\tilde{\pi}$ to (2.7) we have

$$
J(\tilde{\pi}) = \inf_{\pi\in\mathcal{B}} \{J(\pi) + \lambda H(\pi)\} \le J(\tilde{\pi}) + \lambda H(\tilde{\pi}) \le J(\tilde{\pi})
$$

Therefore, $\lambda H(\tilde{\pi}) = 0$. $\square$

On the other hand, for sufficiency,

Theorem 2.2.12 *Suppose that there exists a pair* (π^*, λ^*) *that solves (P2). Then the value function to* (2.4)*,* $J(\pi^*, \lambda^*)$ *equals the value function of (P1). Furthermore, the corresponding* (X^{π^*}, π^*) *is optimal to the original constrained problem (P1), that is*

$$
J(\pi^*, \lambda^*) = J(\pi^*) = V(x_0).
$$

Proof. As (π^*, λ^*) is optimal to the unconstrained problem (P2), for any $\pi \in \mathcal{B}$, we

have

$$J(\pi, \lambda^*) \geq J(\pi^*, \lambda^*).$$

From (2.4) and (2.5),

$$J(\pi) + \lambda^* H(\pi) = J(\pi, \lambda^*) \geq J(\pi^*, \lambda^*) = J(\pi^*) + \lambda^* H(\pi^*) = J(\pi^*).$$

Therefore, for any $\pi \in \mathcal{B}$,

$$J(\pi) = J(\pi^*) - \lambda^* H(\pi) \geq J(\pi^*),$$

that is

$$J(\pi^*) \leq \inf_{\pi \in \mathcal{B}} J(\pi) = V(x_0) \leq J(\pi^*).$$

Hence we can show that

$$J(\pi^*, \lambda^*) = J(\pi^*) = V(x_0).$$

□

Corollary 2.2.13 *Under the same setting as in Theorem 2.2.12 above, we can find a non-zero pair* $(r^*, s^*) \in \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0}$ *satisfy* (2.6).

Proof. The proof is straight forward from results of Theorem 2.2.12.

$$V(x_0) = J(\pi^*, \lambda^*) = \inf_{\pi \in \mathcal{A}} \{J(\pi) + \lambda^* H(\pi)\}.$$

Time both sides by some positive number to arrive at (2.6). □

Therefore, we have shown that under (FJ2) and (SF1), the FJ condition is equivalent to the KKT conditions. Furthermore, Theorem 2.2.11 shows that with an optimal solution to the constrained problem (P1), there exists an optimal pair of control and Lagrange multiplier to solve the unconstrained problem (P2) while Theorem 2.2.12

shows that the KKT conditions are sufficient for the optimal solution to the non-constrained problem (P2) to match the optimal solution to the constrained problem (P1). As such, we have built the necessary and sufficient conditions between the optimality of constrained and unconstrained problems. Lastly, it is natural to ask what would happen when (FJ2) and (SF1) are not satisfied.

Lemma 2.2.14 *If* $(0, 0) \in C$*, then it will be on the boundary.*

Proof. Suppose for a contradiction, then the origin will be in the interior of C which means we can find $\varepsilon > 0$ such that the ball $\mathbb{B}((0, 0), \varepsilon) \subseteq C$. Hence we can pick $(\mu, 0) \in C$ where $-\varepsilon < \mu < 0$ which by definition of C says there exists a $\pi \in \mathcal{A}$ such that $J(\pi) \leq V(x_0) + \mu < V(x_0)$ contradicting the fact $V(x_0)$ is the infimum. $\square$

Therefore, if $(FJ2)$ is not satisfied, either $\left|V(x_0)\right| = \infty$, which is a trivial case, or $(0, 0) \notin C$.

Lemma 2.2.15 *If* C *is not empty, then* $(0, 0) \in C$*.*

Proof. Suppose for a contradiction, if $(0, 0) \notin C$, then there exists $\varepsilon > 0$ such that $\mathbb{B}((0, 0), \varepsilon) \subseteq C^{\complement}$, the complement of C. So for all $\pi \in \mathcal{A}$ and $(r, s) \in \mathbb{B}((0, 0), \varepsilon)$, $J(\pi) > V(x_0) + r$ or $H(\pi) > s$. Pick $r, s > 0$, then for all $\pi \in \mathcal{B}$, $J(\pi) > V(x_0) + r$ which means $V(x_0) = \inf_{\pi \in \mathcal{B}} J(\pi) \geq V(x_0) + r > V(x_0)$, leading to a contradiction. $\square$

Lemma 2.2.16 *If (SF1) is not satisfied, then the constrained problem is equivalent to* (2.1) *subject to* $H(\pi) = 0$*.*

Proof. If (SF1) is not satisfied, then for any $\pi \in \mathcal{B}$, $H(\pi) \geq 0$. Combining with constraint (2.2), the new constraint would be $H(\pi) = 0$. $\square$

Remark *Although Theorem 2.2.12 says that the optimal control for the unconstrained problem matches that of the constrained problem, we still need a way to find the optimal control and optimal value. The steps are to first fix a Lagrange* *multiplier λ, then solve* (2.4)*, for example, using the established method in [44]. Define*

$$\tilde{H}(\pi) = \mathbb{E}[\frac{1}{2}mX^2 + nX]$$

as an alternative version of (2.3) *without the constant term. Then, for any fixed $\lambda \geq 0$, $J(\pi)+\lambda\tilde{H}(\pi)$ can be treated as the cost function in [44] and solved accordingly. Subsequently, pick λ^* that satisfies* (2.5) *which definitely exists by Theorem 2.2.11.*

2.3 Lagrange Multiplier for Mean-Field Stochastic Optimal Control Problem with additional terminal inequality constraints

In the section we change the setting to a mean-field problem. Most of the proofs will be similar to 2.2, so they will be omitted. However, because most of the studies on mean-field stochastic optimal control, especially linear-quadratic, center around deterministic coefficients, we will use new settings similar to those in [59]:

(H1) $A, \bar{A} \in L^1(0,T;\mathbb{R}^{n\times n})$, $B, \bar{B} \in L^2(0,T;\mathbb{R}^{n\times m})$, $C, \bar{C} \in L^2(0,T;\mathbb{R}^{n\times n})$ and $D, \bar{D} \in L^\infty(0,T;\mathbb{R}^{n\times m})$.

(H2) $Q, \bar{Q} \in L^1(0,T;\mathbb{S}^n)$,$S, \bar{S} \in L^2(0,T;\mathbb{R}^{m\times n})$, $R, \bar{R} \in L^\infty(0,T;\mathbb{S}^m)$, $G_T, \bar{G}_T \in \mathbb{S}^n$ and $g_T, \bar{g}_T \in \mathbb{R}^n$.

Let $u \in \mathcal{U}[t,T] := L^2_{\mathbb{F}}(t,T;\mathbb{R}^m)$ and the initial pair (t,ξ) from

$$\mathcal{D} := \{(t,\xi) : t \in [0,T], \xi \in L^2_{\mathcal{F}_t}(\mathbb{R}^n)\}.$$

Define the state process as

$$
\begin{cases}
dX(s) & = \big(A(s)X(s) + \bar{A}(s)\bar{X}(s) + B(s)u(s) + \bar{B}(s)\bar{u}(s)\big)\, ds \\
& \qquad + \big(C(s)X(s) + \bar{C}(s)\bar{X}(s) + D(s)u(s) + \bar{D}(s)\bar{u}(s)\big)\, dW(s) \\
X(t) & = \xi,
\end{cases}
$$

and the objective cost function being

$$
\begin{aligned}
& J(t, \xi; u) \\
= & \mathbb{E}[\int_t^T f(s, X(s), u(s), \bar{X}(s), \bar{u}(s))ds + g(X(T), X\bar{(T)})] \\
= & \mathbb{E}\big[\langle G_T X(T), X(T)\rangle + 2\langle g_T, X(T)\rangle + \langle \bar{G}_T \bar{X}(T), \bar{X}(T)\rangle + 2\langle \bar{g}_T, \bar{X}(T)\rangle \\
& + \int_t^T \langle \begin{pmatrix} Q(s) & S^{\intercal}(s) \\ S(s) & R(s) \end{pmatrix} \begin{pmatrix} X(s) \\ u(s) \end{pmatrix}, \begin{pmatrix} X(s) \\ u(s) \end{pmatrix} \rangle + \langle \begin{pmatrix} \bar{Q}(s) & \bar{S}^{\intercal}(s) \\ \bar{S}(s) & \bar{R}(s) \end{pmatrix} \begin{pmatrix} \bar{X}(s) \\ \bar{u}(s) \end{pmatrix}, \begin{pmatrix} \bar{X}(s) \\ \bar{u}(s) \end{pmatrix} \rangle ds]
\end{aligned}
$$

(H3) There exists a constant $\delta > 0$ such that

$$
J(t, 0; u) \geq \delta \mathbb{E}[\int_t^T \big|u(s)\big|^2 ds].
$$

As (H1), (H2) and (H3) are the same as (A1), (A2) and (A4) from section 3 in [59],(2.8) admits a unique $X^u \in L^2_{\mathbb{F}}(C(t, T; \mathbb{R}^n))$ and the problem

$$
\underset{u \in \mathcal{U}[t,T]}{\textbf{minimize}} \quad J(t, \xi; u) \tag{2.8}
$$

has a unique solution $u^* \in \mathcal{U}[t, T]$ for each pair $(t, \xi) \in \mathcal{D}$.

Additionally, it is also pointed out that (H3) is equivalent to $J(t, \xi; u)$ being uniformly (strongly) convex.

Extending from Yong and Sun's work, suppose there is an additional constraint at the terminal time,

$$H(u) = \mathbb{E}\left[\langle H_T X(T), X(T)\rangle + 2\langle h_T, X(T)\rangle + \langle \bar{H}_T \bar{X}(T), \bar{X}(T)\rangle + 2\langle \bar{h}_T, \bar{X}(T)\rangle + c\right] \leq 0,$$

where H_T and $\bar{H}_T \in \mathbb{S}^n_+$ while h_T, $\bar{h}_T$ and c are constant. Then the constrained problem will be

$$\underset{u\in\mathcal{U}[t,T]}{\textbf{minimize}} \quad J(t,\xi;u)$$

(P3)

$$\textbf{subject to} \quad H(u) \leq 0$$

Let $\tilde{\mathcal{U}}[t,T] = \{u \in \mathcal{U}[t,T] : H(u) \leq 0\}$ and $V(t,\xi)$ denotes the value function so

$$V(t,\xi) := \inf_{u\in\tilde{\mathcal{U}}[t,T]} J(t,\xi;u).$$

Similarly to Section 2.2, introduce the Lagrange multiplier and construct an unconstrained problem. Define $J(t,\xi;u,\lambda) = J(t,\xi;u) + \lambda H(u)$ then for each λ, the unconstrained problem becomes

$$\underset{u\in\mathcal{U}[t,T]}{\textbf{minimize}} \quad J(t,\xi;u,\lambda) \tag{2.9}$$

Let its value function be $V(t,\xi;\lambda)$, so

$$V(t,\xi;\lambda) := \inf_{u\in\mathcal{U}[t,T]} J(t,\xi;u,\lambda).$$

The KKT condition is finding λ such that

$$\begin{cases} H(u) \leq 0, \\ \lambda H(u) = 0, \\ \lambda \geq 0. \end{cases} \tag{2.10}$$

Define

$$C := \{(r,s) \in \mathbb{R}^2 : J(t,\xi;u) \leq V(t,\xi) + r, H(u) \leq s, \text{ for some } u \in \mathcal{U}[t,T]\},$$

and the assumption about C will be still be (FJ1).

Lemma 2.3.1 *C is convex.*

Proof. Since J is convex and the SDE (2.8) is linear, the proof is the same as in 2.2.8. ☐

Assuming $\left|V(t,\xi)\right| < \infty$, we can again simplify (FJ1) to

(FJ3) $\left|V(t,\xi)\right| < \infty$ and the origin is on the boundary of C.

Theorem 2.3.2 *With (FJ3) satisfied, one can find a non-zero pair $(r^*, s^*) \in \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0}$ such that*

$$r^* V(t,\xi) = \inf_{u \in \mathcal{U}[t,T]} \{r^* J(t,\xi;u) + s^* H(u)\}.$$

Proof. The proof is the same as Theorem 2.2.9. ☐

The strict feasibility says

(SF2) There exists $u \in \mathcal{U}$ s.t. $H(u) < 0$.

Theorem 2.3.3 *If both (FJ3) and (SF2) are satisfied, then we can always find a Lagrange multiplier λ such that*

$$V(t,\xi) = \inf_{u \in \mathcal{U}[t,T]} \{J(t,\xi;u) + \lambda H(u)\}.$$

Proof. Proof same as Theorem 2.2.10. ☐

Theorem 2.3.4 *Under (FJ3) and (SF2), we can find λ such that*

$$V(t,\xi) = \inf_{u \in \mathcal{U}[t,T]} \{J(t,\xi;u) + \lambda H(u)\}$$

and (2.10) *is true.*

Proof. According to (3.3.3) from [59], for each pair $(t, \xi) \in \mathcal{D}$, $J(t, \xi; u)$ can be expressed as quadratic equation on u and strictly convex from $(H3)$. Hence it is bounded from below by $\int_t^T \alpha \big|u(s)\big| ds + \kappa$ for some $\alpha > 0$ and $\kappa \in \mathbb{R}$. Also $\mathcal{U}[t, T]$ is a Hilbert space so the proof will be the same as that in Theorem 2.2.11. □

Theorem 2.3.5 *For any pair* $(t, \xi) \in \mathcal{D}$*, suppose that there exists a pair* (u^*, λ^*) *that solves* (2.9) *and satisfies* (2.10)*. Then the value function* $J(t, \xi; u^*, \lambda^*)$ *equals the value function of (P3). Furthermore, the corresponding* (X^{u^*}, u^*) *is optimal to the original constrained problem (P3), that is*

$$V(t, \xi; \lambda^*) = J(t, \xi; u^*, \lambda^*) = J(t, \xi; u^*) = V(t, \xi).$$

Proof. The proof is the same as Theorem 2.2.12. □

Remark *For each* λ*, we can treat the unconstrained problem* (2.9) *as an example in [59] to solve. Then find* λ^* *that satisfies* (2.10)*.*

2.4 Conclusion

Notice that most of the proofs only require finiteness of the value function and convexity of the problem. As such it is possible to apply the results to a more general settings, for example, the cost functions do not have to be limited to quadratic. Any convex function bounded below can be used instead.

Lemma 2.2.16 points out what would happen if the Strict Feasiblity condition is violated. The additional constraint $H(\pi)$ would only be zero and the problem becomes finding an optimal solution with the equality constraint, given that the additional constraint is always non-negative. Although the problem becomes limited, it would still be interesting to see what the solution would look like.

Lastly, the problem proposed in [59] does not have any constraint on the control. Therefore, a new method would be needed to address the problem. The following chapters show that using a duality approach, the constrained primal problem can be converted to an equivalent dual problem which could be easier to solve than its primal counterpart.

3

MFSDE

3.1 Introduction

Similar to usual stochastic optimal control problem, we are interested in whether there exists an easy way to characterise the optimality conditions of a Mean-Field Stochstic Optimal Control problem. With the work in [14], MFFBSDEs can be rigorously derived. As such we can properly discuss the existence of adjoint processes which can help to formulate Hamiltonian for the optimal control problem and recover the relevant Maxmum Principle conditions for optimality. Many articles like [5] and [42] study this problem, but the setting usually involves no term directly depending on the distribution of the control. Furthermore, since the dual problem in [44] contains the initial value of the state process in the cost function, it is natural to believe that the dual problem to Mean-Field primal problem would have a similar cost function. As such, it is an incentive to study the optimal condition when the cost function contains initial values. Although [2] has a similar setting, in its proof of sufficiency, the concavity condition missed out the requirement that h must be increasing. Furthermore, their problem is one-dimensional, whereas our problem has to be multidimensional with control constraints.

We have divided the proof into two parts. The first part shows the existence and uniqueness of optimal control as well as adjoint processes, while the second part shows the necessary and sufficient conditions for the optimality for the constrained problem.

3.2 Adjoint processes to Mean-Field Stochastic Control

Let $|\cdot|$ denote the Euclidean norm (a.k.a Forbenius norm) for vectors and matrices. So for any matrix (or vector) $A := (a_{ij})_{1\leq i\leq n_1, 1\leq j\leq n_2} \in \mathbb{R}^{n_1\times n_2}$, it has a norm

$$|A| := (\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} a_{ij}^2)^{\frac{1}{2}}.$$

Suppose $\chi \in \mathbb{R}^{k_0}$, $\mathbf{u} = (u_1, u_2, ..., u_n)^\intercal \in K$ where $K \subseteq \mathbb{R}^{\sum_0^n k_i}$ is a closed convex set, $u_i \in \mathbb{R}^{k_i}$ and $k_i \in \mathbb{N}^+$ for $i \in \{1, 2, ..., n\}$, $\mathbf{m} = (m_0, m_1, ..., m_n)^\intercal$ where $m_i \in \mathbb{R}^{k_i}$ for $i \in \{0, 1, ..., n\}$ and functions

$$b = b(t) = b(t, x, \mathbf{u}, \mathbf{m}) : [0, T] \times \mathbb{R}^{k_0} \times \mathbb{R}^{\sum_1^n k_i} \times \mathbb{R}^{\sum_0^n k_i} \mapsto \mathbb{R}^{k_0},$$

and

$$\sigma = \sigma(t) = \sigma(t, x, \mathbf{u}, \mathbf{m}) : [0, T] \times \mathbb{R}^{k_0} \times \mathbb{R}^{\sum_1^n k_i} \times \mathbb{R}^{\sum_0^n k_i} \mapsto \mathbb{R}^{k_0\times d}$$

,

satisfying the following assumptions,

(A1) Both b and σ are continuous and differentiable with respect to x and m_0 with finite derivatives.

(A2) For any fixed $\mathbf{u}$ and fixed $\{m_i\}_{1\leq i\leq r}$, functions b and σ are Lipschitz continuous, i.e. there exists a constant $C \geq 0$ such that

$$\big|b(t,x',\mathbf{u},\mathbf{m}') - b(t,x,\mathbf{u},\mathbf{m})\big| + \big|\sigma(t,x',\mathbf{u},\mathbf{m}') - \sigma(t,x,\mathbf{u},\mathbf{m})\big| \leq C(\big|x'-x\big| + \big|m_0' - m_0\big|)$$

for almost all $t \in [0,T]$ and $x, x', \mathbf{m} = (m_0, m_1, ..., m_n)^\intercal, \mathbf{m}' = (m_0', m_1, ..., m_n)^\intercal$.

(A3) For any fixed $\mathbf{u}$ and fixed $\{m_i\}_{1\leq i\leq r}$, functions b and σ are such that

$$\|b(t,0,\mathbf{u},\mathbf{m}_0)\|_{L_2}^2 + \|\sigma(t,0,\mathbf{u},\mathbf{m}_0)\|_{L_2}^2 = \mathbb{E}[\int_0^T \big|b(t,0,\mathbf{u},\mathbf{m})\big|^2 + \big|\sigma(t,0,\mathbf{u},\mathbf{m})\big|^2 dt] < \infty,$$

where $\mathbf{m}_0 = (m_0, 0, ..., 0)^\intercal \in \mathbb{R}^{\sum_0^n k_i}$.

Now suppose $\mathbf{u} \in L^2_{\mathbb{F}}(0,T;K)$, then the following SDE admits a unique solution $X = X^{\mathbf{u},\chi} \in L^2_{\mathbb{F}}(C([0,T],\mathbb{R}^{k_0}))$,

$$\begin{cases} dX(t) &= b(t,X(t),\mathbf{u}(t),\mathbf{M}(t))dt + \sigma(t,X(t),\mathbf{u}(t),\mathbf{M}(t))dW(t) \\ X(0) &= \chi, \end{cases}$$

where $\mathbf{M} = \mathbf{M}(t) = \mathbf{M}^{\mathbf{u},\chi}(t) = (M_0(t), M_1(t), ..., M_n(t))^\intercal$ is defined as

$$M_i(t) := \mathbb{E}[u_i(t)] \text{ for } 1 \leq i \leq n$$

while

$$M_0(t) := \mathbb{E}[X^{\mathbf{u},\chi}(t)].$$

Furthermore, we call such $\mathbf{u}$ an admissible control and let $\mathcal{U} \subseteq L^2_{\mathbb{F}}(0,T;K)$ denote the set of all admissible controls. The optimal control problem is to minimise the objective function:

$$J(\mathbf{u},\chi) = \mathbb{E}\left[\int_0^T f(t, X(t), \mathbf{u}(t), \mathbf{M}(t))dt + g(X(T), \mathbb{E}[X(T)], \chi)\right], \tag{3.1}$$

where

$$f = f(t) = f(t, x, \mathbf{u}, \mathbf{m}) : [0,T] \times \mathbb{R}^{k_0} \times \mathbb{R}^{\sum_1^n k_i} \times \mathbb{R}^{\sum_0^n k_i} \mapsto \mathbb{R},$$

and

$$g = g(x, m_0, \chi) : \mathbb{R}^{k_0} \times \mathbb{R}^{k_0} \times \mathbb{R}^{k_0} \mapsto \mathbb{R}.$$

From now on, $\sum_i$ and $\sum_j$ will be used as a shortened form for $\sum_{i=1}^n$ and $\sum_{j=0}^n$ if there is no confusion. For the running cost and terminal cost we need the following assumption throughout this section:

(A4) f and g are continuous and differentiable with respect to x and m_0. f is Frechet-differentiable with respect to $\mathbf{u}$ and $\mathbf{m}$. g is continuous and differentiable with respect to χ. The derivatives are bounded by a linear growth and Lipschitz. For example,

$$|\partial_{m_0} f| \leq C(1 + |x| + |\mathbf{u}| + |\mathbf{m}|)$$

and

$$\left|\partial_{m_0} f(t, x, \mathbf{u}, \mathbf{m}) - \partial_{m_0} f(t, x', \mathbf{u}', \mathbf{m}')\right| \leq C(1 + \left|x - x'\right| + \left|\mathbf{u} - \mathbf{u}'\right| + \left|\mathbf{m} - \mathbf{m}'\right|)$$

for some constant C, where ∂ denotes the partial derivative. For example, if $x = (x_1, x_2, ...x_{k_0})^\intercal$, $b = (b_1, b_2, ...b_{k_0})^\intercal$, $y = (y_1, y_2, ...y_{k_0})^\intercal$, $z = (z_{ij})_{1\leq i\leq k_0, 1\leq j\leq d}$ and $\sigma = (\sigma_{i,j})_{1\leq i\leq k_0, 1\leq j\leq d}$, then

$$\partial_x b := \begin{pmatrix} \partial_{x_1} b \\ ... \\ \partial_{x_{k_0}} b \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \partial_{x_1} b_1 \\ ... \\ \partial_{x_1} b_{k_0} \end{pmatrix} \\ ... \\ \begin{pmatrix} \partial_{x_{k_0}} b_1 \\ ... \\ \partial_{x_{k_0}} b_{k_0} \end{pmatrix} \end{pmatrix}, y^\intercal \partial_x b := \begin{pmatrix} y^\intercal \partial_{x_1} b \\ ... \\ y^\intercal \partial_{x_{k_0}} b \end{pmatrix}, \partial_x (y^\intercal) b := \begin{pmatrix} \partial_{x_1} (y^\intercal) b \\ ... \\ \partial_{x_{k_0}} (y^\intercal) b \end{pmatrix}$$

and

$$\partial_x \sigma := \begin{pmatrix} \partial_{x_1} \sigma \\ ... \\ \partial_{x_{k_0}} \sigma \end{pmatrix} = \begin{pmatrix} (\partial_{x_1} \sigma_{ij})_{1 \le i \le k_0, 1 \le j \le d} \\ ... \\ (\partial_{x_{k_0}} \sigma_{i,j})_{1 \le i \le k_0, 1 \le j \le d} \end{pmatrix}, \operatorname{tr}(z^\intercal \partial_x \sigma) := \begin{pmatrix} \operatorname{tr}(z^\intercal \partial_{x_1} \sigma) \\ ... \\ \operatorname{tr}\Big(z^\intercal \partial_{x_{k_0}} \sigma \Big) \end{pmatrix}$$

while $\partial_x b \cdot x := \sum_i \partial_{x_i} b x_i = \sum_i \begin{pmatrix} \partial_{x_i} b_1 x_i \\ ... \\ \partial_{x_i} b_{k_0} x_i \end{pmatrix} = (\partial_x b)^\intercal x.$

Define the Hamiltonian $\mathcal{H} : [0,T] \times \mathbb{R}^{k_0} \times \mathbb{R}^{\sum_1^n k_i} \times \mathbb{R}^{\sum_0^n k_i} \times \mathbb{R}^{k_0} \times \mathbb{R}^{k_0 \times d} \mapsto \mathbb{R}$ as

$$\mathcal{H} = \mathcal{H}(t) = \mathcal{H}(t, x, \mathbf{u}, \mathbf{m}, y, z) := f + y^\intercal b + \operatorname{tr}(z^\intercal \sigma).$$

For example, refer to [1], for

$$Y = Y(t) = Y(t, \omega) : [0,T] \times \Omega \mapsto \mathbb{R}^{k_0}$$

and

$$Z = Z(t) = Z(t, \omega) : [0,T] \times \Omega \mapsto \mathbb{R}^{k_0 \times d},$$

consider the adjoint BSDE,

$$\begin{cases} dY(t) &= -\left[\partial_x \mathcal{H}(t) + \mathbb{E}[\partial_{m_0} \mathcal{H}(t)]\right] dt + Z(t) dW(t) \\ &= -\Big[\partial_x f(t) + Y^\intercal(t)\partial_x b(t) + \operatorname{tr}\big(Z^\intercal(t)\partial_x \sigma(t)\big) + \mathbb{E}[\partial_{m_0} f(t)] + \mathbb{E}[Y^\intercal(t)\partial_{m_0} b(t)] \\ &\qquad + \mathbb{E}[\operatorname{tr}\big(Z^\intercal(t)\partial_{m_0}\sigma(t)\big)]\Big] dt + Z(t)dW(t) \\ Y(T) &= \partial_x g(X(T), \mathbb{E}[X(T)], \chi) + \mathbb{E}\left[\partial_{m_0} g(X(T), \mathbb{E}[X(T)], \chi)\right]. \end{cases} \tag{3.2}$$

Lemma 3.2.1 *Let* $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}}) = (\Omega \times \Omega, \mathcal{F} \otimes \mathcal{F}, \mathbb{P} \otimes \mathbb{P})$ *be the product of* $(\Omega, \mathcal{F}, \mathbb{P})$ *with itself. Let its filtration be* $\bar{\mathcal{F}}_t : \mathcal{F}_t \otimes \mathcal{F}_t$*, for* $0 \leq t \leq T$*. Any measurable random variable* ξ *originally defined on* Ω *can be extended to* $\bar{\Omega}$*:* $\xi'(\omega', \omega) := \xi(\omega')$*, where* $(\omega', \omega) \in \bar{\Omega}$*. Let* $\bar{L}^p(\mathbb{H})$ *denote the set of all pth integrable* $\mathbb{H}$*-valued random variables on probability space* $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$*. For any* $\theta \in \bar{L}^p(\mathbb{H})$*, the variable* $\theta(\cdot, \omega) : \Omega \mapsto \mathbb{H}$ *belongs to* $L^1(\mathbb{H})$*,* $\mathbb{P}(d\omega)$*-a.s. and we denote its expectation by*

$$\mathbb{E}'\left[\theta(\cdot, \omega)\right] = \int_\Omega \theta(\omega', \omega)\mathbb{P}(d\omega').$$

Note that $\mathbb{E}'[\theta] = \mathbb{E}'\left[\theta(\cdot, \omega)\right] \in L^1(\mathbb{H})$*, and*

$$\bar{\mathbb{E}}[\theta] = \int_{\bar{\Omega}} \theta d\bar{\mathbb{P}} = \int_\Omega \mathbb{E}'\left[\theta(\cdot, \omega)\right]\mathbb{P}(d\omega) = \mathbb{E}[\mathbb{E}'[\theta]].$$

Suppose the function

$$\phi = \phi(\omega', \omega, t, y', z', y, z) : \bar{\Omega} \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}$$

is $\bar{\mathcal{F}}_t$*-progressively measurable and there exists a constant* $C \geq 0$ *such that* $d\bar{\mathbb{P}}dt$*-a.s., for any* $y_1, y_2, y_1', y_2' \in \mathbb{R}, z_1, z_2, z_1', z_2' \in \mathbb{R}^d$*,*

$$\left|\phi(t, y_1', z_1', y_1, z_1) - \phi(t, y_2', z_2', y_2, z_2)\right| \leq C(\left|y_1' - y_2'\right| + \left|z_1' - z_2'\right| + |y_1 - y_2| + |z_1 - z_2|).$$

Also, $\phi(\cdot,0,0,0,0) \in L^2_{\mathbb{F}}(0,T;\mathbb{R})$. *Then for any square integrable random variable* ξ*, the MFBSDE*

$$Y(t) = \xi + \int_t^T \mathbb{E}'\left[\phi(s, Y'(s), Z'(s), Y(s), Z(s))\right] ds - \int_t^T Z(s)dW(s), 0 \le t \le T,$$

has a unique adapted solution $(Y,Z) \in L^2_{\mathbb{F}}(C([0,T],\mathbb{R}) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^d)$*, where*

$$\mathbb{E}'\left[\phi(s, Y'(s), Z'(s), Y_s, Z_s)\right](\omega) = \int_\Omega' \phi(s, Y(s,\omega'), Z(s,\omega'), Y(s,\omega), Z(s,\omega)) d\mathbb{P}(\omega').$$

Proof. Refer to Theorem 3.1 in [15]. □

Corollary 3.2.2 *With (A1) to (A4), the adjoint processes* $(Y,Z) \in L^2_{\mathbb{F}}(C([0,T],\mathbb{R}^{k_0}) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^{k_0\times d})$ *exist and are unique for any* $u \in \mathcal{U}$.

Proof. For any admissible control u and real value χ, with (A1) to (A3), there exists a unique state process $X \in L^2_{\mathbb{F}}(C([0,T],\mathbb{R}^{k_0}))$ that satisfies the forward SDE.

With appropriate substitutions, the results in [15] can be applied:

$$\begin{cases} \phi(\omega',\omega,t,Y',Z',Y,Z) := Y^\intercal \partial_x b(t, X(\omega,t), \mathbf{u}(\omega,t), \mathbf{M}(t)) + \operatorname{tr}\big(Z^\intercal \partial_x \sigma(t, X(\omega,t), \mathbf{u}(\omega,t), \mathbf{M}(t))\big) \\ \qquad + \partial_x f(t, X(\omega,t), \mathbf{u}(\omega,t), \mathbf{M}(t)) + \partial_{m_0} f(t, X(\omega',t), \mathbf{u}(\omega',t), \mathbf{M}(t)) \\ \qquad + Y'^\intercal \partial_{m_0} b(t, X(\omega',t), \mathbf{u}(\omega',t), \mathbf{M}(t)) + \operatorname{tr}\big((Z')^\intercal \partial_{m_0} \sigma(t, X(\omega',t), \mathbf{u}(\omega',t), \mathbf{M}(t))\big) \\ \xi := \partial_x g(X(T), \mathbb{E}[X(T)], \chi) + \mathbb{E}\left[\partial_{m_0} g(X(T), \mathbb{E}[X(T)], \chi)\right], \end{cases}$$

From (A1) and (A4), $\partial_x b, \partial_x \sigma$ and $\partial_{m_j} b, \partial_{m_j} \sigma$ are bounded. Therefore, ϕ is Lipschitz in y, y', z, z' Moreover, as $\partial_x f$ and $\partial_{m_j} f$ are bounded by linear growth of x, u and

m,

$$
\begin{aligned}
\|\phi(\cdot,\cdot,\cdot,0,0,0,0)\|_{L_2} :&= \sqrt{\int_{\bar{\Omega}}\int_0^T \left|\phi(\omega',\omega,t,0,0,0,0)\right|^2 dtd\bar{\mathbb{P}}} \\
&= \sqrt{\mathbb{E}\left[\int_0^T \left|\partial_x f + \mathbb{E}[\partial_{m_0} f]\right|^2 dt\right]} \\
&\leq \sqrt{(r+2)\mathbb{E}\left[\int_0^T |\partial_x f|^2 + |\partial_{m_0} f|^2 dt\right]} \\
&\leq \sqrt{C^2(r+2)^2\mathbb{E}\left[\int_0^T (1+\left|X(t)\right| + \left|\mathbf{u}(t)\right| + \left|\mathbf{M}(t)\right|)^2 dt\right]} \\
&\leq C'(1+\|X\|_{L2} + \|\mathbf{u}\|_{L2} + \|\mathbf{M}\|_{L2}),
\end{aligned}
$$

for some constant $C' = 2(r+2)C$ and fixed $\mathbf{u}, \chi$ and X, function $\phi(\cdot,\cdot,\cdot,0,0,0,0) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^{k_0})$. Hence conditions in Lemma 3.2.1 are satisfied and the result can be applied: There exists a unique pair $(Y,Z) \in L^2_{\mathbb{F}}(C([0,T],\mathbb{H}) \times L^2_{\mathbb{F}}(0,T;\mathbb{H})$ solving the backward equation.

$$
Y(t) = \xi + \int_t^T \mathbb{E}'\left[\phi(\cdot,\omega,s,Y'(s),Z'(s),Y(s),Z(s))\right] ds - \int_t^T Z(s)dW(s)
$$

which is equivalent to (3.2). □

3.3 Stochastic Maximum Conditions

To show the necessary condition for optimality, we need to use the Gateaux derivative and its product rule, as well as chain rule. Also note that differentiability implies Frechet which implies Gateaux diffenrentiabilty. Let $D_d F(u)$ denote the Gateaux derivative of F with respect to u if it exists, along the direction of d, that is,

$$
D_d F(u) := \lim_{r\to 0} \frac{F(u+rd) - F(u)}{r}.
$$

Theorem 3.3.1 *A necessary condition for* $\hat{\mathbf{u}}$ *to minimize* (3.1) *over* $u \in \mathcal{U}$ *is*

$$(\partial_{\mathbf{u}}\mathcal{H}|_{\mathbf{u}=\hat{\mathbf{u}}} + \mathbb{E}[\partial_{m_i}\mathcal{H}|_{\mathbf{u}=\hat{\mathbf{u}}}]) \cdot (\mathbf{u} - \hat{\mathbf{u}}) \geq 0 \tag{3.3}$$

for all $\mathbf{u} \in \mathcal{U}$ *almost surely, and*

$$Y(0) + \mathbb{E}\left[\partial_{\chi} g(X(T), \mathbb{E}[X(T)], \chi)\right] = 0. \tag{3.4}$$

In the case K *is the whole space,* (3.3) *can be simplified into*

$$\partial_{u_i}\mathcal{H}(t) + \mathbb{E}[\partial_{m_i}\mathcal{H}(t)] = 0 \tag{3.5}$$

almost surely for $t \in [0, T]$ *for all* i*,*

Proof. For any pair $(\mathbf{u}, \chi) \in \mathcal{U} \times \mathbb{R}^{k_0}$, fix χ and apply Gateaux derivative in direction of $\boldsymbol{\beta}$ to $\mathbf{u}$ both sides of the forward SDE. As the state process is in $L^2_{\mathbb{F}}(C([0,T], \mathbb{R}^{k_0})$, DCT applies. As such the Gateaux derivative can be taken inside the integral on both sides of the state process SDE as well as expectations. Write X in short of $X^{\mathbf{u},\chi}$ and M in short of $M^{\mathbf{u},\chi} = \mathbb{E}[X^{\mathbf{u},\chi}]$. Apply the Gateaux derivative to the forward SDE of X. Then, by linearity of the derivatives, continuity and chain rules,

$$\begin{aligned}
&d(D_{\boldsymbol{\beta}}X(t)) \\
&= D_{\boldsymbol{\beta}}b dt + D_{\boldsymbol{\beta}}\sigma dW(t) \\
&= (\partial_x b \cdot D_{\boldsymbol{\beta}}X + \sum_i \partial_{u_i} b \cdot \beta_i + \sum_j \partial_{m_j} b \cdot D_{\boldsymbol{\beta}}M_j) dt \\
&\quad + (\partial_x \sigma \cdot D_{\boldsymbol{\beta}}X + \sum_i \partial_{u_i}\sigma \cdot \beta_i + \sum_j \partial_{m_j}\sigma \cdot D_{\boldsymbol{\beta}}M_j) dW(t) \\
&= (\partial_x b \cdot D_{\boldsymbol{\beta}}X + \sum_i \partial_{u_i} b \cdot \beta_i + \partial_{m_0} b \cdot \mathbb{E}[D_{\boldsymbol{\beta}}X] + \sum_i \partial_{m_i} b \cdot \mathbb{E}[\beta_i]) dt \\
&\quad + (\partial_x \sigma \cdot D_{\boldsymbol{\beta}}X + \sum_i \partial_{u_i}\sigma \cdot \beta_i + \partial_{m_0}\sigma \cdot \mathbb{E}[D_{\boldsymbol{\beta}}X] + \sum_i \partial_{m_i}\sigma \cdot \mathbb{E}[\beta_i]) dW(t).
\end{aligned} \tag{3.6}$$

Next, take the Gateaux derivative of $J(\mathbf{u},\chi)$ along the direction of $\boldsymbol{\beta}$. From the assumptions, the integral will be bounded, hence we can take the limit inside the expectation and integral:

$$\begin{aligned}
D_{\boldsymbol{\beta}}J(\mathbf{u},\chi) &= \mathbb{E}\left[\int_0^T D_{\boldsymbol{\beta}}f dt + D_{\boldsymbol{\beta}}g(X(T),\mathbb{E}[X(T)],\chi)\right] \\
&= \mathbb{E}\Big[\int_0^T (\partial_x f \cdot D_{\boldsymbol{\beta}}X + \sum_i \partial_{u_i} f \cdot \beta_i + \sum_j [\partial_{m_j} f \cdot D_{\boldsymbol{\beta}}M_j])dt \\
&\quad + \partial_x g(X(T),\mathbb{E}[X(T)],\chi)\cdot D_{\boldsymbol{\beta}}X(T) + \partial_{m_0} g(X(T),\mathbb{E}[X(T)],\chi)\cdot \mathbb{E}[D_{\boldsymbol{\beta}}X(T)]\Big] \\
&= \mathbb{E}\Big[\int_0^T (\partial_x f \cdot D_{\boldsymbol{\beta}}X + \sum_i \partial_{u_i} f \cdot \beta_i + \partial_{m_0} f \cdot \mathbb{E}[D_{\boldsymbol{\beta}}X] + \sum_i \partial_{m_i} f \cdot \mathbb{E}[\beta_i])dt \\
&\quad + \partial_x g(X(T),\mathbb{E}[X(T)],\chi)\cdot D_{\boldsymbol{\beta}}X(T) + \mathbb{E}\left[\partial_{m_0} g(X(T),\mathbb{E}[X(T)],\chi)\right]\cdot D_{\boldsymbol{\beta}}X(T)\Big] \\
&= \mathbb{E}\Big[\int_0^T (\partial_x f \cdot D_{\boldsymbol{\beta}}X + \sum_i \partial_{u_i} f \cdot \beta_i + \partial_{m_0} f \cdot \mathbb{E}[D_{\boldsymbol{\beta}}X] + \sum_i \partial_{m_i} f \cdot \mathbb{E}[\beta_i])dt \\
&\quad + Y^\intercal(T)D_{\boldsymbol{\beta}}X(T)\Big]. \qquad (3.7)
\end{aligned}$$

The last line involves switching the two expectation on $\mathbb{E}\left[\partial_{m_0} g(X(T),\mathbb{E}[X(T)],\chi)\cdot \mathbb{E}[D_{\boldsymbol{\beta}}X(T)]\right]$. Apply Ito's Lemma to $Y^\intercal(t)D_{\boldsymbol{\beta}}X(t)$ and substitute with (3.2) and (3.6). Then we

have

$$
\begin{aligned}
&d(Y^\intercal(t) D_{\boldsymbol{\beta}} X(t)) \\
=\;& Y^\intercal(t) d(D_{\boldsymbol{\beta}} X(t)) + (dY(t))^\intercal D_{\boldsymbol{\beta}} X(t) + d\langle Y, D_{\boldsymbol{\beta}} X\rangle_t \\
=\;& \{Y^\intercal \left[\partial_x b \cdot D_{\boldsymbol{\beta}} X + \sum_i \partial_{u_i} b \cdot \beta_i + \sum_j [\partial_{m_j} b \cdot (D_{\boldsymbol{\beta}} M_j)]\right] \\
& - \left[\partial_x f + Y^\intercal \partial_x b + \mathrm{tr}(Z^\intercal \partial_x \sigma) + (\mathbb{E}[\partial_{m_0} f] + \mathbb{E}[Y^\intercal \partial_{m_0} b] + \mathbb{E}[\mathrm{tr}(Z^\intercal \partial_{m_0} \sigma)])\right]^\intercal D_{\boldsymbol{\beta}} X \\
& + \mathrm{tr}\left(Z^\intercal \left[\partial_x \sigma \cdot D_{\boldsymbol{\beta}} X + \sum_i \partial_{u_i} \sigma \cdot \beta_i + \sum_j [\partial_{m_j} \sigma \cdot (D_{\boldsymbol{\beta}} M_j)]\right]\right)\} dt + N_{\boldsymbol{\beta}} dW(t) \\
=\;& \{Y^\intercal \left[\partial_x b \cdot D_{\boldsymbol{\beta}} X + \sum_i \partial_{u_i} b \cdot \beta_i + \sum_j [\partial_{m_j} b \cdot (D_{\boldsymbol{\beta}} M_j)]\right] \\
& - \sum_{k=1}^{k_0} \left[\partial_{x_k} f + Y^\intercal \partial_{x_k} b + \mathrm{tr}\big(Z^\intercal \partial_{x_k} \sigma\big) + (\mathbb{E}[\partial_{m_{0k}} f] + \mathbb{E}[Y^\intercal \partial_{m_{0k}} b] + \mathbb{E}[\mathrm{tr}\big(Z^\intercal \partial_{m_{0k}} \sigma\big)])\right] D_{\boldsymbol{\beta}} X_k \\
& + \mathrm{tr}\left(Z^\intercal \left[\partial_x \sigma \cdot D_{\boldsymbol{\beta}} X + \sum_i \partial_{u_i} \sigma \cdot \beta_i + \sum_j [\partial_{m_j} \sigma \cdot (D_{\boldsymbol{\beta}} M_j)]\right]\right)\} dt + N_{\boldsymbol{\beta}} dW(t) \\
=\;& \{Y^\intercal \left[\partial_x b \cdot D_{\boldsymbol{\beta}} X + \sum_i \partial_{u_i} b \cdot \beta_i + \sum_j [\partial_{m_j} b \cdot (D_{\boldsymbol{\beta}} M_j)]\right] \\
& - \left[\partial_x f + (\mathbb{E}[\partial_{m_0} f] + \mathbb{E}[Y^\intercal \partial_{m_0} b] + \mathbb{E}[\mathrm{tr}(Z^\intercal \partial_{m_0} \sigma)])\right] \cdot D_{\boldsymbol{\beta}} X - Y^\intercal \partial_x b \cdot D_{\boldsymbol{\beta}} X - \mathrm{tr}\big(Z^\intercal \partial_x \sigma \cdot D_{\boldsymbol{\beta}} X\big) \\
& + \mathrm{tr}\left(Z^\intercal \left[\partial_x \sigma \cdot D_{\boldsymbol{\beta}} X + \sum_i \partial_{u_i} \sigma \cdot \beta_i + \sum_j [\partial_{m_j} \sigma \cdot (D_{\boldsymbol{\beta}} M_j)]\right]\right)\} dt + N_{\boldsymbol{\beta}} dW(t) \\
=\;& \{Y^\intercal \left[\sum_i \partial_{u_i} b \cdot \beta_i + \sum_j [\partial_{m_j} b \cdot (D_{\boldsymbol{\beta}} M_j)]\right] + \mathrm{tr}\left(Z^\intercal \left[\sum_i \partial_{u_i} \sigma \cdot \beta_i + \sum_j [\partial_{m_j} \sigma \cdot (D_{\boldsymbol{\beta}} M_j)]\right]\right) \\
& - \left[\partial_x f + (\mathbb{E}[\partial_{m_0} f] + \mathbb{E}[Y^\intercal \partial_{m_0} b] + \mathbb{E}[\mathrm{tr}(Z^\intercal \partial_{m_0} \sigma)])\right] \cdot D_{\boldsymbol{\beta}} X\} dt + N_{\boldsymbol{\beta}} dW(t)
\end{aligned}
\tag{3.8}
$$

For some process $N_{\boldsymbol{\beta}}$.

Remark

$$N_{\boldsymbol{\beta}} = Y^{\intercal}\left[\partial_x\sigma \cdot D_{\boldsymbol{\beta}}X + \sum_i \partial_{u_i}\sigma \cdot \beta_i + \sum_j [\partial_{m_j}\sigma \cdot (D_{\boldsymbol{\beta}}M_j)]\right] + (D_{\boldsymbol{\beta}}X)^{\intercal}Z$$

From assumption (A1) to (A3), $X \in L^2_{\mathbb{F}}(C([0,T],\mathbb{H}))$ and its adjoint process (Y,Z) are such that $\mathbb{E}\left[\sup_{s\in[0,T]}\left|Y(s)\right|^2\right] < \infty$ and $\mathbb{E}[\int_0^T \left|Z(s)\right|^2 ds] < \infty$.

So, by a similar argument in Section 5 of [44], using the BDG inequality, we can show that $\mathbb{E}\left[\sup_{s\in[0,T]}\left|N_{\boldsymbol{\beta}}(s)\right|^2\right] < \infty$. As such the volatility part is a martingale and equals to 0 when taking expectation.

Taking the expectation and by swapping the expectation ($\mathbb{E}\left[\mathbb{E}[X^{\intercal}]Y\right] = \mathbb{E}\left[X^{\intercal}\mathbb{E}[Y]\right]$), (3.8) can be simplified to

$$\begin{aligned}
&d\mathbb{E}[Y^{\intercal}(t)D_{\boldsymbol{\beta}}X(t)] \\
&= \mathbb{E}\Big[\{Y^{\intercal}(\sum_i \partial_{u_i}b \cdot \beta_i + \sum_j [\partial_{m_j}b \cdot D_{\boldsymbol{\beta}}M_j]) + \mathrm{tr}\left(Z^{\intercal}(\sum_i \partial_{u_i}\sigma \cdot \beta_i + \sum_j [\partial_{m_j}\sigma \cdot D_{\boldsymbol{\beta}}M_j])\right) \\
&\quad - (\partial_x f + \mathbb{E}[\partial_{m_0} f] + \mathbb{E}[Y^{\intercal}(t)\partial_{m_0}b(t)] + \mathbb{E}[\mathrm{tr}(Z^{\intercal}\partial_{m_0}\sigma)]) \cdot D_{\boldsymbol{\beta}}X\}dt\Big] \qquad (3.9) \\
&= \mathbb{E}\Big[\{Y^{\intercal}(\sum_i \partial_{u_i}b \cdot \beta_i + \partial_{m_0}b \cdot \mathbb{E}[D_{\boldsymbol{\beta}}X] + \sum_i \partial_{m_i}b \cdot \mathbb{E}[\beta_i]) \\
&\quad + \mathrm{tr}\left(Z^{\intercal}(\sum_i \partial_{u_i}\sigma \cdot \beta_i + \partial_{m_0}\sigma \cdot \mathbb{E}[D_{\boldsymbol{\beta}}X] + \sum_i \partial_{m_i}\sigma \cdot \mathbb{E}[\beta_i])\right) \\
&\quad - (\partial_x f + \mathbb{E}[\partial_{m_0} f]) \cdot D_{\boldsymbol{\beta}}X - (Y^{\intercal}(t)\partial_{m_0}b(t) + \mathrm{tr}(Z^{\intercal}\partial_{m_0}\sigma)) \cdot \mathbb{E}[D_{\boldsymbol{\beta}}X]\}dt\Big] \qquad (3.10) \\
&= \mathbb{E}\Big[\{Y^{\intercal}(\sum_i \partial_{u_i}b \cdot \beta_i + \sum_i \partial_{m_i}b \cdot \mathbb{E}[\beta_i]) + \mathrm{tr}\left(Z^{\intercal}(\sum_i \partial_{u_i}\sigma \cdot \beta_i + \sum_i \partial_{m_i}\sigma \cdot \mathbb{E}[\beta_i])\right) \\
&\quad - (\partial_x f + \mathbb{E}[\partial_{m_0} f]) \cdot D_{\boldsymbol{\beta}}X\}dt\Big]. \qquad (3.11)
\end{aligned}$$

Integrate the equation from 0 to T and note $D_{\boldsymbol{\beta}}X(0) = 0$:

$$
\begin{aligned}
&\mathbb{E}[Y^\intercal(T)D_{\boldsymbol{\beta}}X(T)] \\
&= \mathbb{E}\Big[\int_0^T \{Y^\intercal(\sum_i \partial_{u_i} b \cdot \beta_i + \sum_i \partial_{m_i} b \cdot \mathbb{E}[\beta_i]) + \operatorname{tr}\left(Z^\intercal(\sum_i \partial_{u_i}\sigma \cdot \beta_i + \sum_i \partial_{m_i}\sigma \cdot \mathbb{E}[\beta_i])\right) \\
&\quad - (\partial_x f + \mathbb{E}[\partial_{m_0} f]) \cdot D_{\boldsymbol{\beta}}X\}dt + Y^\intercal(0)D_{\boldsymbol{\beta}}X(0)\Big] \\
&= \mathbb{E}\Big[\int_0^T \{Y^\intercal(\sum_i \partial_{u_i} b \cdot \beta_i + \sum_j [\partial_{m_j} b \cdot \mathbb{E}[\beta_j]]) + \operatorname{tr}\left(Z^\intercal(\sum_i \partial_{u_i}\sigma \cdot \beta_i + \sum_j [\partial_{m_j}\sigma \cdot \mathbb{E}[\beta_j]])\right) \\
&\quad - (\partial_x f + \mathbb{E}[\partial_{m_0} f]) \cdot D_{\boldsymbol{\beta}}X\}dt\Big].
\end{aligned} \tag{3.12}
$$

Substituting (3.12) into (3.7), we have

$$
\begin{aligned}
&D_{\boldsymbol{\beta}}J(\mathbf{u},\chi) \\
&= \mathbb{E}\left[\int_0^T \{\partial_x f \cdot D_{\boldsymbol{\beta}}X + \sum_i \partial_{u_i} f \cdot \beta_i + \partial_{m_0} f \cdot \mathbb{E}[D_{\boldsymbol{\beta}}X] + \sum_i \partial_{m_i} f \cdot \mathbb{E}[\beta_i]\}dt\right] \\
&\quad + \mathbb{E}\Big[\int_0^T \{Y^\intercal(\sum_i \partial_{u_i} b \cdot \beta_i + \sum_i [\partial_{m_i} b \cdot \mathbb{E}[\beta_i]]) - (\partial_x f + \mathbb{E}[\partial_{m_0} f]) \cdot D_{\boldsymbol{\beta}}X \\
&\quad + \operatorname{tr}\left(Z^\intercal(\sum_i \partial_{u_i}\sigma \cdot \beta_i + \sum_i [\partial_{m_i}\sigma \cdot \mathbb{E}[\beta_i]])\right)\}dt\Big] \\
&= \mathbb{E}\left[\int_0^T \{\sum_i [(\partial_{u_i} f + Y^\intercal \partial_{u_i} b + \operatorname{tr}\big(Z^\intercal \partial_{u_i}\sigma\big)) \cdot \beta_i + (\partial_{m_i} f + Y^\intercal \partial_{m_i} b + \operatorname{tr}\big(Z^\intercal \partial_{m_i}\sigma\big)) \cdot \mathbb{E}[\beta_i]]\}dt\right] \\
&= \mathbb{E}\left[\int_0^T \{\sum_i [\partial_{u_i}\mathcal{H} \cdot \beta_i + \partial_{m_i}\mathcal{H} \cdot \mathbb{E}[\beta_i]]\}dt\right] \\
&= \mathbb{E}\left[\int_0^T \{\sum_i [(\partial_{u_i}\mathcal{H} + \mathbb{E}[\partial_{m_i}\mathcal{H}]) \cdot \beta_i]\}dt\right]
\end{aligned}
$$

where the last line involves switching two expectation signs.

If the pair $(\mathbf{u}, \chi)$ is optimal the Gateaux derivative should equal to 0 for all $\boldsymbol{\beta}$ such that $\mathbf{u} + \boldsymbol{\beta} \in L^2_{\mathbb{F}}(0, T; \mathring{K})$ where $\mathring{K}$ denotes the interior of K while greater or equal to 0 if $\mathbf{u}+\boldsymbol{\beta}$ is on the boundary. That is

$$\inf_{\boldsymbol{\beta}:\mathbf{u}+\boldsymbol{\beta}\in\mathcal{U}} \left[\partial_{u_i}\mathcal{H}+\mathbb{E}[\partial_{m_i}\mathcal{H}]\right]\cdot\beta_i \geq 0.$$

Since $\boldsymbol{\beta}$ is in the direction of $\mathbf{u}-\hat{\mathbf{u}}$, the above equation is equivalent to

$$(\partial_{\mathbf{u}}\mathcal{H}|_{\mathbf{u}=\hat{\mathbf{u}}}+\mathbb{E}[\partial_{m_i}\mathcal{H}|_{\mathbf{u}=\hat{\mathbf{u}}}])\cdot(\mathbf{u}-\hat{\mathbf{u}})\geq 0$$

for all $\mathbf{u}\in\mathcal{U}$ almost surely.

In the case K being the whole space, the Gateaux derivative should be 0 for all $\boldsymbol{\beta}$. Then we have

$$\partial_{u_i}\mathcal{H}+\mathbb{E}[\partial_{m_i}\mathcal{H}]=0,$$

for all i and almost all t.

Now, with a similar approach to that in (3.6), fix $\mathbf{u}$ and apply the Gateaux derivative in the direction of ζ to χ on both sides of the forward SDE.

$$\begin{aligned}
&d(D_\zeta X(t))\\
&=D_\zeta b dt+D_\zeta\sigma dW(t)\\
&=(\partial_x b\cdot D_\zeta X+\sum_j\partial_{m_j}b\cdot D_\zeta M_j)dt+(\partial_x\sigma\cdot D_\zeta X+\sum_j\partial_{m_j}\sigma\cdot D_\zeta M_j)dW(t)\\
&=(\partial_x b\cdot D_\zeta X+\partial_{m_0}b\cdot\mathbb{E}[D_\zeta X])dt+(\partial_x\sigma\cdot D_\zeta X+\partial_{m_0}\sigma\cdot\mathbb{E}[D_\zeta X])dW(t).
\end{aligned}$$

Next, take the Gateaux derivative of $J(\mathbf{u},\chi)$ along the direction of ζ. From the assumptions, the integral will be bounded, hence we can take the limit inside the expectation and integral:

$$
\begin{aligned}
& D_\zeta J(\mathbf{u},\chi) \\
&= \mathbb{E}\left[\int_0^T D_\zeta f dt + D_\zeta g(X(T),\mathbb{E}[X(T)],\chi)\right] \\
&= \mathbb{E}\Big[\int_0^T (\partial_x f \cdot D_\zeta X + \partial_{m_0} f \cdot \mathbb{E}[D_\zeta X])dt + \partial_x g(X(T),\mathbb{E}[X(T)],\chi)\cdot D_\zeta X(T) \\
&\quad + \partial_{m_0} g(X(T),\mathbb{E}[X(T)],\chi)\cdot \mathbb{E}[D_\zeta X(T)] + \partial_\chi g(X(T),\mathbb{E}[X(T)],\chi)\cdot\zeta\Big] \\
&= \mathbb{E}\Big[\int_0^T (\partial_x f \cdot D_\zeta X + \partial_{m_0} f \cdot \mathbb{E}[D_\zeta X])dt + \partial_\chi g(X(T),\mathbb{E}[X(T)],\chi)\cdot\zeta \\
&\quad + \partial_x g(X(T),\mathbb{E}[X(T)],\chi)\cdot D_\zeta X(T) + \mathbb{E}\left[\partial_{m_0} g(X(T),\mathbb{E}[X(T)],\chi)\right]\cdot D_\zeta X(T)\Big] \\
&= \mathbb{E}\left[\int_0^T (\partial_x f \cdot D_\zeta X + \partial_{m_0} f \cdot \mathbb{E}[D_\zeta X])dt + Y^\intercal(T) D_\zeta X(T) + \partial_\chi g(X(T),\mathbb{E}[X(T)],\chi)\cdot\zeta\right].
\end{aligned}
\tag{3.13}
$$

Applying Ito's lemma to $Y^\intercal(t)D_\zeta X(t)$, we have

$$
\begin{aligned}
& d(Y^\intercal(t)D_\zeta X(t)) \\
&= Y^\intercal(t)d(D_\zeta X(t)) + (D_\zeta X(t))^\intercal dY(t) + d\langle Y, D_\zeta X\rangle_t \\
&= \{Y^\intercal\left[\partial_x b\cdot D_\zeta X + \partial_{m_0} b\cdot\mathbb{E}[D_\zeta X]\right] \\
&\quad - \left[\partial_x f + Y^\intercal\partial_x b + \mathrm{tr}(Z^\intercal\partial_x\sigma) + \mathbb{E}[\partial_{m_0} f] + \mathbb{E}[Y^\intercal\partial_{m_0} b] + \mathbb{E}[\mathrm{tr}(Z^\intercal\partial_{m_0}\sigma)]\right]^\intercal D_\zeta X \\
&\quad + \mathrm{tr}\Big(Z^\intercal\left[\partial_x\sigma\cdot D_\zeta X + \partial_{m_0}\sigma\cdot\mathbb{E}[D_\zeta X]\right]\Big)\}dt + N_\zeta dW(t)
\end{aligned}
\tag{3.14}
$$

For some process N_ζ.

Taking expectation and swapping the expectations, (3.14) can be simplified to

$$
d\mathbb{E}[Y^\intercal(t)D_\zeta X(t)] = -\mathbb{E}\left[[(\partial_x f + \mathbb{E}[\partial_{m_0} f])\cdot D_\zeta X]dt\right]. \tag{3.15}
$$

Integrate (3.15) from 0 to T and substitute into (3.13) and note $D_\zeta X(0) = \zeta$, we

have

$$
\begin{aligned}
& D_\zeta J(\mathbf{u},\chi) \\
&= \mathbb{E}\left[\int_0^T (\partial_x f \cdot D_\zeta X + \partial_{m_0} f \cdot \mathbb{E}[D_\zeta X])dt + Y^\intercal(T) D_\zeta X(T) + \partial_\chi g(X(T), \mathbb{E}[X(T)], \chi)\cdot\zeta\right] \\
&= \mathbb{E}\Big[\int_0^T (\partial_x f \cdot D_\zeta X + \partial_{m_0} f \cdot \mathbb{E}[D_\zeta X])dt + Y^\intercal(0) D_\zeta X(0) \\
&\quad - \int_0^T (\partial_x f \cdot D_\zeta X + \partial_{m_0} f \cdot \mathbb{E}[D_\zeta X])dt + \partial_\chi g(X(T), \mathbb{E}[X(T)], \chi)\cdot\zeta\Big] \\
&= \mathbb{E}\left[Y^\intercal(0)\zeta + \partial_\chi g(X(T), \mathbb{E}[X(T)], \chi)\cdot\zeta\right] \\
&= \mathbb{E}\left[(Y(0) + \partial_\chi g(X(T), \mathbb{E}[X(T)], \chi))\cdot\zeta\right].
\end{aligned}
$$

By optimality, the equation equals 0 for all ζ, hence the condition is equivalent to

$$
Y(0) + \mathbb{E}\left[\partial_\chi g(X(T), \mathbb{E}[X(T)], \chi)\right] = 0
$$

□

(A5) Suppose function $\mathcal{H}$ and g are convex with respect to $x, \mathbf{u}, \mathbf{m}$ and there exist $\mathbf{u}^* \in \mathcal{U}$ and $\chi^* \in \mathbb{R}^{k_0}$ such that the corresponding state process X^* and adjoint processes (Y, Z) satisfy the following conditions:

$$
\mathcal{H}^* := \mathcal{H}(X^*, \mathbf{u}^*, \mathbf{M}^*, Y, Z) = \inf_{\mathbf{u}\in\mathcal{U}} \mathcal{H}(X^*, \mathbf{u}, (\mathbb{E}[X^*], \mathbb{E}[u_1], ..., \mathbb{E}[u_n])^\intercal, Y, Z) \tag{3.16}
$$

for almost all time t, $\mathbb{P}$-almost surely, and

$$
Y(0) + \mathbb{E}\left[\partial_\chi g^*\right] = 0,
$$

where $g^* := g(X^*(T), \mathbb{E}[X^*(T)], \chi^*)$

Theorem 3.3.2 *Suppose that the assumptions (A1) to (A5) are satisfied, then the pair $(\mathbf{u}^*, \chi^*)$ is optimal.*

In the case where the control set is the whole space, (3.16) *is equivalent to* (3.5)*,*

$$\partial_{u_i}\mathcal{H}(t) + \mathbb{E}[\partial_{m_i}\mathcal{H}(t)] = 0$$

for all i, for almost all t almost surely.

Proof. Denote $\Delta y^* := y - y^*$ for any variable or process.
First by convexity of g,

$$\begin{aligned}\Delta g^* = g - g^* &\geq \partial_x g^* \cdot \Delta X^*(T) + \partial_{m_0} g^* \cdot \Delta M_0^*(T) + \partial_\chi g^* \cdot \Delta\chi^* \\ &= \partial_x g^* \cdot \Delta X^*(T) + \partial_{m_0} g^* \cdot \mathbb{E}[\Delta X^*(T)] + \partial_\chi g^* \cdot \Delta\chi^*, \end{aligned} \tag{3.17}$$

similarly by convexity,

$$\begin{aligned}&\Delta\mathcal{H}^* \\ &= \mathcal{H} - \mathcal{H}^* \\ &\geq \partial_x \mathcal{H}^* \cdot \Delta X^* + \sum_i \partial_{u_i}\mathcal{H}^* \cdot \Delta u_i^* + \sum_j \partial_{m_j}\mathcal{H}^* \cdot \Delta M_j^* \\ &= \partial_x \mathcal{H}^* \cdot \Delta X^* + \sum_i \partial_{u_i}\mathcal{H}^* \cdot \Delta u_i^* + \partial_{m_0}\mathcal{H}^* \cdot \mathbb{E}[\Delta X^*] + \sum_i \partial_{m_i}\mathcal{H}^* \cdot \mathbb{E}[\Delta u_i^*] \text{ for all } t. \end{aligned} \tag{3.18}$$

Next, apply Ito's Lemma to $Y^\intercal \Delta X^*$:

$$\begin{aligned} d(Y^\intercal\Delta X^*) &= (dY)^\intercal \Delta X^* + Y^\intercal(d\Delta X^*) + d\langle Y, \Delta X^*\rangle \\ &= \left[-(\partial_x\mathcal{H}^*(t) + \mathbb{E}[\partial_{m_0}\mathcal{H}^*(t)])^\intercal \Delta X^* + Y^\intercal \Delta b^* + \mathrm{tr}(Z^\intercal \Delta\sigma^*)\right] dt + N_2 dW(t) \\ &= \left[-(\partial_x\mathcal{H}^*(t) + \mathbb{E}[\partial_{m_0}\mathcal{H}^*(t)]) \cdot \Delta X^* + Y^\intercal \Delta b^* + \mathrm{tr}(Z^\intercal \Delta\sigma^*)\right] dt + N_2 dW(t), \end{aligned}$$

for some square integrable process N_2. Integrate both sides from 0 to T and take expectation, again by Fubini-Tonelli's theorem the integration and expectation can be swapped and we could get:

$$
\begin{aligned}
&\mathbb{E}\left[\partial_x g(X^*(T), \mathbb{E}[X^*(T)], \chi^*) \cdot \Delta X^*(T) + \partial_{m_0} g(X^*(T), \mathbb{E}[X^*(T)], \chi \cdot \mathbb{E}\left[\Delta X^*(T)\right]\right] \\
&= \mathbb{E}\left[(\partial_x g(X^*(T), \mathbb{E}[X^*(T)], \chi^*) + \mathbb{E}\left[\partial_{m_0} g(X^*(T), \mathbb{E}[X^*(T)], \chi^*)\right]) \cdot \Delta X^*(T)\right] \\
&= \mathbb{E}\left[(\partial_x g(X^*(T), \mathbb{E}[X^*(T)], \chi^*) + \mathbb{E}\left[\partial_{m_0} g(X^*(T), \mathbb{E}[X^*(T)], \chi^*)\right])^\intercal \Delta X^*(T)\right] \\
&= \mathbb{E}[Y^\intercal(T)\Delta X^*(T)] \\
&= \mathbb{E}\left[Y^\intercal(0)\Delta X^*(0) + \int_0^T d(Y^\intercal(t)\Delta X^*(t))\right] \\
&= Y(0)\cdot\Delta\chi^* + \int_0^T (-\mathbb{E}[\partial_x \mathcal{H}^* \cdot \Delta X^*] - \mathbb{E}[\partial_{m_0}\mathcal{H}^*]\cdot\mathbb{E}[\Delta X^*] + \mathbb{E}[Y^\intercal \Delta b^* + \mathrm{tr}(Z^\intercal \Delta\sigma^*)])dt.
\end{aligned}
\tag{3.19}
$$

Also note by the definition of Hamiltonian,

$$
\Delta f^* = \Delta\mathcal{H}^* - Y^\intercal \Delta b^* - \mathrm{tr}(Z^\intercal \Delta\sigma^*). \tag{3.20}
$$

Substituting (3.17) (3.18) (3.19) and (3.20), we have

$$
\begin{aligned}
&\Delta J^*(\mathbf{u},\chi)\\
&= J(\mathbf{u},\chi) - J(\mathbf{u}^*,\chi^*)\\
&= \mathbb{E}\left[\int_0^T \Delta f^*(t)dt + \Delta g^*\right]\\
&= \mathbb{E}\left[\int_0^T [\Delta\mathcal{H}^*(t) - Y^\intercal(t)\Delta b^*(t) - \mathrm{tr}\big(Z^\intercal(t)\Delta\sigma^*(t)\big)]dt + \Delta g^*\right]\\
&\geq \mathbb{E}\Bigg[\int_0^T [\partial_x\mathcal{H}^*\cdot\Delta X^* + \sum_i \partial_{u_i}\mathcal{H}^*\cdot\Delta u_i^* + \partial_{m_0}\mathcal{H}^*\cdot\mathbb{E}[\Delta X^*] + \sum_i \partial_{m_i}\mathcal{H}^*\cdot\mathbb{E}[\Delta u_i^*]\\
&\qquad - Y^\intercal\Delta b^* - \mathrm{tr}(Z^\intercal\Delta\sigma^*)]dt + \partial_x g^*\cdot\Delta X^*(T) + \partial_{m_0}g^*\cdot\mathbb{E}[\Delta X^*(T)] + \partial_\chi g^*\cdot\Delta\chi^*\Bigg]\\
&= \mathbb{E}\Bigg[\int_0^T [\partial_x\mathcal{H}^*\cdot\Delta X^* + \sum_i \partial_{u_i}\mathcal{H}^*\cdot\Delta u_i^* + \partial_{m_0}\mathcal{H}^*\cdot\mathbb{E}[\Delta X^*] + \sum_i \partial_{m_i}\mathcal{H}^*\cdot\mathbb{E}[\Delta u_i^*]\\
&\qquad - Y^\intercal\Delta b^* - \mathrm{tr}(Z^\intercal\Delta\sigma^*)]dt\Bigg] + \int_0^T (-\mathbb{E}[\partial_x\mathcal{H}^*\cdot\Delta X^*] - \mathbb{E}[\partial_{m_0}\mathcal{H}^*]\cdot\mathbb{E}[\Delta X^*]\\
&\qquad + \mathbb{E}[Y^\intercal\Delta b^* + \mathrm{tr}(Z^\intercal\Delta\sigma^*)])dt + (Y(0) + \mathbb{E}[\partial_\chi g^*])\cdot\Delta\chi^*\\
&= \mathbb{E}\left[\int_0^T \sum_i(\partial_{u_i}\mathcal{H}^*\cdot\Delta u_i^* + \partial_{m_i}\mathcal{H}^*\cdot\mathbb{E}[\Delta u_i^*])dt\right].
\end{aligned}
\tag{3.21}
$$

Again since at $\mathbf{u}^*$, $\mathcal{H}(X^*,\mathbf{u},\mathbf{M}^*,Y,Z)$ reaches minimal, if it is in the interior then the equation in the last line is 0 and if it on the boundary the equation will be greater than 0. As such, $\Delta J^*(\mathbf{u},\chi)\geq 0$ for any admissible control $\mathbf{u}$. Therefore, $\mathbf{u}^*$ is the optimal solution to the generalised problem. The integral in the last line of

(3.21) can be simplified as

$$
\begin{aligned}
&\mathbb{E}\left[\sum_i (\partial_{u_i}\mathcal{H}^* \cdot \Delta u_i^* + \partial_{m_i}\mathcal{H}^* \cdot \mathbb{E}[\Delta u_i^*])\right] \\
&= \mathbb{E}\left[\sum_i (\partial_{u_i}\mathcal{H}^* \cdot \Delta u_i^* + \mathbb{E}[\partial_{m_i}\mathcal{H}^*] \cdot \Delta u_i^*)\right] \\
&= \mathbb{E}\left[\sum_i (\partial_{u_i}\mathcal{H}^* + \mathbb{E}[\partial_{m_j}\mathcal{H}^*]) \cdot \Delta u_i^*\right].
\end{aligned}
$$

In the case that the control space is the whole space, (3.16) is equivalent to (3.5). □

3.4 Conclusion

In this chapter, we have built the foundation for the Dual problem to Mean-Field Stochastic Optimal Control problem. One possible extension would be to look at a multidimensional Brownian motion. Then a summation $\sum_i \sigma_i dW_i$ will be required in the SDE. Another opportunity for extension would be to loosen the assumptions as it is shown that the drift and diffusion terms do not necessarily need to be differentiable, some Lipschitz conditions would be enough.

4

Duality

4.1 Introduction

Due to its simple structure, Linear Quadratic has always been a popular setting in a problem. Furthermore, as linear quadratic often guarantees smoothness as well as convexity, many results from optimal control can directly apply without need of imposing too many other conditions. [58],[24],[30] and many others built the foundation for solving a Mean-Field Linear Quadratic Stochastic Optimal problem. In addition, one particular property of expectations in linear quadratic problems is $\mathbb{E}\left[X\mathbb{E}[Y]\right] = \mathbb{E}\left[\mathbb{E}[X]\mathbb{E}[Y]\right] = \mathbb{E}[X]\mathbb{E}[Y]$. This property, together with convexity, opens the possibility of deriving the dual problem from the original problem. The setting of this chapter is mainly based on [59], in which value functions can be represented using solutions to the Riccati equations. Here we study the duality of the problem which could be helpful in solving the problem when it has additional constraints on controls. The results on optimality of primal and dual problems is a direct application of results in Chapter 3. Then, when there is no control constraint, we show that the dual problem can replicate the primal problem with given substitutions and vice versa, hence drawing an equivalent relationship between the primal and dual problems.

4.2 Dual Problem to a Linear Quadratic Mean-Filed Stochastic Control Problem

If the problem is linear quadratic and the control is in whole space, then for some control $\pi \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$ consider the state process following the SDE below on the interval $[0,T]$:

$$\underset{\pi\in\mathcal{U}}{\textbf{minimize}} \quad J_1(\pi) = \mathbb{E}\left[\int_0^T \tilde{f}_1(t, X_1(t), \bar{X}_1(t), \pi(t), \bar{\pi}(t))dt + \tilde{g}_1(X_1(T), \bar{X}_1(T))\right]$$

where

$$\begin{cases} dX_1(t) &= \big(A_1(t)X_1(t) + \bar{A}_1(t)\bar{X}_1(t) + B_1(t)\pi(t) + \bar{B}_1(t)\bar{\pi}(t)\big)\, ds \\ & \qquad + \big(C_1(t)X_1(t) + \bar{C}_1(t)\bar{X}_1(t) + D_1(t)\pi(t) + \bar{D}_1(t)\bar{\pi}(t)\big)\, dW(t) \\ X_1(0) &= x_1, \end{cases}$$

$$\tilde{f}_1(t,x,\bar{x},\pi,\bar{\pi}) = \frac{1}{2}\langle \begin{pmatrix} Q_1(t) & S_1^\intercal(t) \\ S_1(t) & R_1(t) \end{pmatrix} \begin{pmatrix} x \\ \pi \end{pmatrix}, \begin{pmatrix} x \\ \pi \end{pmatrix} \rangle + \frac{1}{2}\langle \begin{pmatrix} \bar{Q}_1(t) & \bar{S}_1^\intercal(t) \\ \bar{S}_1(t) & \bar{R}_1(t) \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{\pi} \end{pmatrix}, \begin{pmatrix} \bar{x} \\ \bar{\pi} \end{pmatrix} \rangle,$$

$$\tilde{g}_1(x,\bar{x}) = \frac{1}{2}\langle G_{T1}x, x\rangle + \langle g_{T1}, x\rangle + \frac{1}{2}\langle \bar{G}_{T1}\bar{x}, \bar{x}\rangle + \langle \bar{g}_{T1}, \bar{x}\rangle.$$

Define $\hat{\Lambda}_1 = \Lambda_1 + \bar{\Lambda}_1$ for $\Lambda \in \{A, B, C, D, Q, S, R, G_T, g_T\}$. Since for any $X, Y \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^m)$ we have

$$\mathbb{E}[\tilde{X}\tilde{Y}] = \mathbb{E}\left[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])\right] = \mathbb{E}\left[XY - \mathbb{E}[X]\mathbb{E}[Y]\right] = \mathbb{E}[XY - \bar{X}\bar{Y}],$$

the problem can be rewritten into equivalent form

$$\begin{cases} dX_1(t) &= \Big(A_1(t)\tilde{X}_1(t) + \hat{A}_1(t)\bar{X}_1(t) + B_1(t)\tilde{\pi}(t) + \hat{B}_1(t)\bar{\pi}(t)\Big)\,dt \\ &\qquad + \Big(C_1(t)\tilde{X}_1(t) + \hat{C}_1(t)\bar{X}_1(t) + D_1(t)\tilde{\pi}(t) + \hat{D}_1(t)\bar{\pi}(t)\Big)\,dW(t) \\ X_1(0) &= x_1. \end{cases}$$

Let $\mathcal{U}$ denotes the set of admissible controls. The primal problem becomes

$$\underset{\pi \in \mathcal{U}}{\textbf{minimize}} \quad J_1(\pi) = \mathbb{E}\left[\int_0^T f_1(t, \tilde{X}_1(t), \bar{X}_1(t), \tilde{\pi}(t), \bar{\pi}(t))dt + g_1(\tilde{X}_1(T), \bar{X}_1(T))\right] \tag{4.1}$$

where

$$f_1(t, \tilde{x}, \bar{x}, \tilde{\pi}, \bar{\pi}) = \frac{1}{2}\langle \begin{pmatrix} Q_1(t) & S_1^\intercal(t) \\ S_1(t) & R(t) \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{\pi} \end{pmatrix}, \begin{pmatrix} \tilde{x} \\ \tilde{\pi} \end{pmatrix} \rangle + \frac{1}{2}\langle \begin{pmatrix} \hat{Q}_1(t) & \hat{S}_1^\intercal(t) \\ \hat{S}_1(t) & \hat{R}_1(t) \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{\pi} \end{pmatrix}, \begin{pmatrix} \bar{x} \\ \bar{\pi} \end{pmatrix} \rangle, \tag{4.2}$$

$$g_1(\tilde{x}, \bar{x}) = \frac{1}{2}\langle G_{T1}\tilde{x}, \tilde{x}\rangle + \langle g_{T1}, \tilde{x}\rangle + \frac{1}{2}\langle \hat{G}_{T1}\bar{x}, \bar{x}\rangle + \langle \hat{g}_{T1}, \bar{x}\rangle. \tag{4.3}$$

Let $V_1(x_1) := \inf_{\pi \in \mathcal{U}} J_1(\pi)$ denotes the value function.

For any matrix $L \in \mathbb{R}^{m \times n}$, write $L^\dagger \in \mathbb{R}^{n \times m}$ as its Moore-Penrose inverse which equals its actual inverse when L is invertible. By definition, it satisfies the following conditions:

$$\begin{cases} LL^{\dagger}L = L \\ L^{\dagger}LL^{\dagger} = L^{\dagger} \\ (LL^{\dagger})^{\intercal} = LL^{\dagger} \\ (L^{\dagger}L)^{\intercal} = L^{\dagger}L. \end{cases}$$

Especially, if L is of full column rank, $L^{\dagger} = (L^{\intercal}L)^{-1}L^{\intercal}$.

Lemma 4.2.1 *Suppose L is an $m \times n$ matrix of coefficients, l an m-dimensional vector of constants and x an n-dimensional vector of unknowns.*
A solution to $Lx = l$ exists if and only if $LL^{\dagger}l = l$. Furthermore, if solution exists, then the complete set of solutions is given by

$$x = L^{\dagger}l + (I - L^{\dagger}L)w,$$

for an arbitrary n-dimensional vector and I an identity matrix. As such the solution x is unique if and only if $L^{\dagger}L = I$, that is L has full column rank.

Proof. Refer to Theorem 1 and 2 in [32]. □

Suppose the coefficients of the primal problem satisfy the following assumptions:

(A6) $A_1, \hat{A}_1 \in L^2(0,T;\mathbb{R}^{n\times n})$, $B_1, \hat{B}_1 \in L^1(0,T;\mathbb{R}^{n\times m})$, $C_1, \hat{C}_1 \in L^1(0,T;\mathbb{R}^{n\times n})$ and $D_1, \hat{D}_1 \in L^{\infty}(0,T;\mathbb{R}^{n\times m})$. Additionally, D_1 and $\hat{D}_1$ have full column rank.

(A7) $Q_1, \hat{Q}_1 \in L^{\infty}(0,T;\mathbb{S}^n_+)$,$S_1, \hat{S}_1 \in L^{\infty}(0,T;\mathbb{R}^{m\times n})$, $R_1, \hat{R}_1 \in L^{\infty}(0,T;\mathbb{S}^m_+)$, $G_{T1}, \hat{G}_{T1} \in \mathbb{S}^n_+$ and $g_{T1}, \hat{g}_{T1} \in \mathbb{R}^n$. $R_1 - S_1Q_1^{-1}S_1^{\intercal}$ and $\hat{R}_1 - \hat{S}_1\hat{Q}_1^{-1}\hat{S}_1^{\intercal}$ are positive definite.

For simplicity, define

$$O_1 := \begin{pmatrix} Q_1 & S_1^\intercal \\ S_1 & R_1 \end{pmatrix} \quad \bar{O}_1 := \begin{pmatrix} \bar{Q}_1 & \bar{S}_1^\intercal \\ \bar{S}_1 & \bar{R}_1 \end{pmatrix} \text{ and } \hat{O}_1 := O_1 + \bar{O}_1 = \begin{pmatrix} \hat{Q}_1 & \hat{S}_1^\intercal \\ \hat{S}_1 & \hat{R}_1 \end{pmatrix} \tag{4.4}$$

There exits $\delta > 0$ such that

$$\langle O_1(t) \begin{pmatrix} x(t) \\ \pi(t) \end{pmatrix}, \begin{pmatrix} x(t) \\ \pi(t) \end{pmatrix} \rangle \geq \delta \|\pi(t)\|_2^2$$

and

$$\langle \hat{O}_1(t) \begin{pmatrix} x(t) \\ \pi(t) \end{pmatrix}, \begin{pmatrix} x(t) \\ \pi(t) \end{pmatrix} \rangle \geq \delta \|\pi(t)\|_2^2$$

almost surely on $[0, T]$ for any $x \in L^2(0, T; \mathbb{R}^m)$ and $\pi \in L^2(0, T; \mathbb{R}^m)$.

Remark *From (A7),*

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \tilde{f}_1(t, X_1, \bar{X}_1, \pi, \bar{\pi}) dt\right] \\
&= \mathbb{E}\left[\int_0^T f_1(t, \tilde{X}_1, \bar{X}_1, \tilde{\pi}, \bar{\pi}) dt\right] \\
&= \frac{1}{2}\mathbb{E}\left[\int_0^T \langle O_1 \begin{pmatrix} \tilde{X}_1 \\ \tilde{\pi} \end{pmatrix}, \begin{pmatrix} \tilde{X}_1 \\ \tilde{\pi} \end{pmatrix} \rangle + \langle \hat{O}_1 \begin{pmatrix} \bar{X}_1 \\ \bar{\pi} \end{pmatrix}, \begin{pmatrix} \bar{X}_1 \\ \bar{\pi} \end{pmatrix} \rangle dt\right] \\
&\geq \frac{\delta}{2}\mathbb{E}[\int_0^T (\|\tilde{\pi}\|_2^2 + \|\bar{\pi}\|_2^2) dt] \\
&= \frac{\delta}{2}\mathbb{E}[\int_0^T \|\pi\|_2^2 dt].
\end{aligned}$$

As such, the assumption (A4) in [59] is satisfied. Since the coefficient conditions in this report is stricter than that in (A1) and (A2) in the book, results in Theorem 3.4.1 can be applied here.

Remark *Note that as g_{T1} is deterministic, $\mathbb{E}[\langle g_{T1}, \tilde{X}_1(T)\rangle] = 0$ and we can safely remove it from the terminal cost. From now on, the updated terminal function shall be*

$$g_1(\tilde{x}, \bar{x}) = \frac{1}{2}\langle G_{T1}\tilde{x}, \tilde{x}\rangle + \frac{1}{2}\langle \hat{G}_{T1}\bar{x}, \bar{x}\rangle + \langle \hat{g}_{T1}, \bar{x}\rangle.$$

Lemma 4.2.2 *For a invertible block matrix $P = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, we can express the inverse of P, P^{-1} in the following way:*
if D is invertible,

$$P^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)^{-1}BD^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & D^{-1} + D^{-1}C(A - BD^{-1}C)^{-1}BD^{-1} \end{pmatrix};$$

if A is invertible,

$$P^{-1} = \begin{pmatrix} A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix}$$

Note that if both A and D are invertible,

$$A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} = (A - BD^{-1}C)^{-1}, \tag{4.5}$$

$$D^{-1} + D^{-1}C(A - BD^{-1}C)^{-1}BD^{-1} = (D - CA^{-1}B)^{-1}.$$

and

$$(A - BD^{-1}C)^{-1}BD^{-1} = A^{-1}B(D - CA^{-1}B)^{-1}. \tag{4.6}$$

(4.6) *times D then apply* (4.5)*:*

$$(A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1})B = A^{-1}B(D - CA^{-1}B)^{-1}D. \tag{4.7}$$

Proof. For example, refer to [45]. $\square$

Theorem 4.2.3 *Suppose (A6) and (A7) are satisfied, then the dual problem would be*

$$\underset{\alpha,\beta,\chi}{\boldsymbol{minimize}} \quad J_2(\alpha,\beta,\chi) = \mathbb{E}\left[\int_0^T f_2(t,\tilde{\alpha},\tilde{\beta},\bar{\alpha},\bar{\beta})dt + g_2^*(\tilde{X}_2(T),\bar{X}_2(T),\chi)\right],$$

where

$$\begin{cases} dX_2(t) &= \left[A_2\tilde{X}_2 + \tilde{\alpha} + B_2\tilde{\beta} + \hat{A}_2\bar{X}_2 + \bar{\alpha} + \hat{B}_2\bar{\beta}\right]dt \\ &\quad + \left[C_2\tilde{X}_2 + D_2\tilde{\beta} + \hat{C}_2\bar{X}_2 + \hat{D}_2\bar{\beta}\right]dW(t), \\ X_2(0) &= \chi, \end{cases}$$

$$\begin{cases} f_2(t,\tilde{\alpha},\tilde{\beta},\bar{\alpha},\bar{\beta}) = \frac{1}{2}\langle R_2(t)\begin{pmatrix}\tilde{\alpha}\\ \tilde{\beta}\end{pmatrix}, \begin{pmatrix}\tilde{\alpha}\\ \tilde{\beta}\end{pmatrix}\rangle + \frac{1}{2}\langle \hat{R}_2(t)\begin{pmatrix}\bar{\alpha}\\ \bar{\beta}\end{pmatrix}, \begin{pmatrix}\bar{\alpha}\\ \bar{\beta}\end{pmatrix}\rangle \\ g_2^*(\tilde{y},\bar{y},\chi) = \frac{1}{2}\langle G_{T2}\tilde{y},\tilde{y}\rangle + \frac{1}{2}\langle \hat{G}_{T2}\bar{y},\bar{y}\rangle + \langle \hat{g}_{T2},\bar{y}\rangle + \frac{1}{2}\langle \hat{g}_{T2},\hat{G}_{T2}^{-1}\hat{g}_{T2}\rangle + x_1^\intercal\chi \end{cases}$$

and

$$\begin{cases} A_2 &= (-A_1 + B_1D_1^\dagger C_1)^\intercal \\ B_2 &= -C_1^\intercal(D_1^\dagger)^\intercal \\ C_2 &= -(D_1^\dagger)^\intercal B_1^\intercal \\ D_2 &= (D_1^\dagger)^\intercal \\ R_2 &= O_1^{-1} \\ G_{T2} &= G_{T1}^{-1} \end{cases} \qquad \begin{cases} \hat{A}_2 &= (-\hat{A}_1 + \hat{B}_1\hat{D}_1^\dagger\hat{C}_1)^\intercal \\ \hat{B}_2 &= -(\hat{D}_1^\dagger\hat{C}_1)^\intercal \\ \hat{C}_2 &= -(\hat{B}_1\hat{D}_1^\dagger)^\intercal \\ \hat{D}_2 &= (\hat{D}_1^\dagger)^\intercal \\ \hat{R}_2 &= \hat{O}_1^{-1} \\ \hat{G}_{T2} &= \hat{G}_{T1}^{-1} \\ \hat{g}_{T2} &= \hat{G}_{T1}^{-1}\hat{g}_{T1}. \end{cases} \tag{4.8}$$

Proof. Suppose

$$dX_2 = (\tilde{M} + \bar{M})dt + (\tilde{N} + \bar{N})dW(t),$$

where $\mathbb{E}[\tilde{M}] = \mathbb{E}[\tilde{N}] = 0$ and $\bar{M}, \bar{N}$ deterministic. Then we can rewrite the state processes as

$$\begin{cases} d\tilde{X}_1 & = (A_1\tilde{X}_1 + B_1\tilde{\pi})dt + (C_1\tilde{X}_1 + \hat{C}_1\bar{X}_1 + D_1\tilde{\pi} + \hat{D}_1\bar{\pi})dW(t) \\ d\bar{X}_1 & = (\hat{A}_1\bar{X}_1 + \hat{B}_1\bar{\pi})dt \\ d\tilde{X}_2 & = \tilde{M}dt + (\tilde{N} + \bar{N})dW(t) \\ d\bar{X}_2 & = \bar{M}dt. \end{cases}$$

Apply Ito's Lemma to $\tilde{X}_1^\intercal \tilde{X}_2 + \bar{X}_1^\intercal \bar{X}_2$:

$$\begin{aligned} & d(\tilde{X}_1^\intercal \tilde{X}_2 + \bar{X}_1^\intercal \bar{X}_2) \\ = & \big[\tilde{X}_1^\intercal \tilde{M} + (A_1\tilde{X}_1 + B_1\tilde{\pi})^\intercal \tilde{X}_2 + (C_1\tilde{X}_1 + \hat{C}_1\bar{X}_1 + D_1\tilde{\pi} + \hat{D}_1\bar{\pi})^\intercal(\tilde{N} + \bar{N}) + \bar{X}_1^\intercal \bar{M} \\ & + (\hat{A}_1\bar{X}_1 + \hat{B}_1\bar{\pi})^\intercal \bar{X}_2\big]dt + EdW(t) \\ = & \big[\tilde{X}_1^\intercal(\tilde{M} + A_1^\intercal \tilde{X}_2 + C_1^\intercal(\tilde{N} + \bar{N})) + \tilde{\pi}^\intercal(B_1^\intercal \tilde{X}_2 + D_1^\intercal(\tilde{N} + \bar{N})) \\ & + \bar{X}_1^\intercal(\hat{C}_1^\intercal(\tilde{N} + \bar{N}) + \bar{M} + \hat{A}_1^\intercal \bar{X}_2) + \bar{\pi}^\intercal(\hat{D}_1^\intercal(\tilde{N} + \bar{N}) + \hat{B}_1^\intercal \bar{X}_2)\big]dt + EdW(t), \end{aligned}$$

for some $E \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^n)$.

Taking expectation on both sides:

$$\begin{aligned} & d\mathbb{E}[X_1^\intercal X_2] \\ = & \, d\mathbb{E}[\tilde{X}_1^\intercal \tilde{X}_2 + \bar{X}_1^\intercal \bar{X}_2] \\ = & \, \mathbb{E}\Big[\tilde{X}_1^\intercal(\tilde{M} + A_1^\intercal \tilde{X}_2 + C_1^\intercal(\tilde{N} + \bar{N})) + \tilde{\pi}^\intercal(B_1^\intercal \tilde{X}_2 + D_1^\intercal(\tilde{N} + \bar{N})) \\ & + \bar{X}_1^\intercal(\hat{C}_1^\intercal(\tilde{N} + \bar{N}) + \bar{M} + \hat{A}_1^\intercal \bar{X}_2) + \bar{\pi}^\intercal(\hat{D}_1^\intercal(\tilde{N} + \bar{N}) + \hat{B}_1^\intercal \bar{X}_2)\Big]dt \\ = & \Big[\tilde{X}_1^\intercal(\tilde{M} + A_1^\intercal \tilde{X}_2 + C_1^\intercal \tilde{N}) + \tilde{\pi}^\intercal(B_1^\intercal \tilde{X}_2 + D_1^\intercal \tilde{N}) \\ & + \bar{X}_1^\intercal(\hat{C}_1^\intercal \bar{N} + \bar{M} + \hat{A}_1^\intercal \bar{X}_2) + \bar{\pi}^\intercal(\hat{D}_1^\intercal \bar{N} + \hat{B}_1^\intercal \bar{X}_2)\Big]dt. \end{aligned} \tag{4.9}$$

which means we can define:

$$\begin{cases} \tilde{\alpha} &= \tilde{M} + A_1^\intercal \tilde{X}_2 + C_1^\intercal \tilde{N}, \\ \tilde{\beta} &= B_1^\intercal \tilde{X}_2 + D_1^\intercal \tilde{N}, \\ \bar{\alpha} &= \hat{C}_1^\intercal \bar{N} + \bar{M} + \hat{A}_1^\intercal \bar{X}_2, \\ \bar{\beta} &= \hat{D}_1^\intercal \bar{N} + \hat{B}_1^\intercal \bar{X}_2. \end{cases} \tag{4.10}$$

Note $\mathbb{E}[\tilde{\alpha}] = \mathbb{E}[\tilde{\beta}] = 0$ which is desired.

By (A6), D_1 and $\hat{D}_1$ are of full column rank, so $D_1^\intercal (D_1^\intercal)^\dagger = \hat{D}_1^\intercal (\hat{D}_1^\intercal)^\dagger = I$ then by Lemma 4.2.1, and taking $w = 0$, we can rearrange (4.10) to obtain:

$$\begin{cases} \tilde{N} &= (D_1^\dagger)^\intercal (\tilde{\beta} - B_1^\intercal \tilde{X}_2) = -(B_1 D_1^\dagger)^\intercal \tilde{X}_2 + (D_1^\dagger)^\intercal \tilde{\beta} = C_2 \tilde{X}_2 + D_2 \tilde{\beta} \\ \tilde{M} &= \tilde{\alpha} - A_1^\intercal \tilde{X}_2 - C_1^\intercal \tilde{N} = \tilde{\alpha} - (D_1^\dagger C_1)^\intercal \tilde{\beta} + (-A_1 + B_1 D_1^\dagger C_1)^\intercal \tilde{X}_2 = A_2 \tilde{X}_2 + \tilde{\alpha} + B_2 \tilde{\beta} \\ \bar{N} &= (\hat{D}_1^\dagger)^\intercal (\bar{\beta} - \hat{B}_1^\intercal \bar{X}_2) = -(\hat{B}_1 \hat{D}_1^\dagger)^\intercal \bar{X}_2 + (\hat{D}_1^\dagger)^\intercal \bar{\beta} = \hat{C}_2 \bar{X}_2 + \hat{D}_2 \bar{\beta} \\ \bar{M} &= \bar{\alpha} - \hat{A}_1^\intercal \bar{X}_2 - \hat{C}_1^\intercal \bar{N} = (-\hat{A}_1 + \hat{B}_1 \hat{D}_1^\dagger \hat{C}_1)^\intercal \bar{X}_2 + \bar{\alpha} - (\hat{D}_1^\dagger \hat{C}_1)^\intercal \bar{\beta} = \hat{A}_2 \bar{X}_2 + \bar{\alpha} + \hat{B}_2 \bar{\beta}, \end{cases} \tag{4.11}$$

where

$$\begin{cases} A_2 &= (-A_1 + B_1 D_1^\dagger C_1)^\intercal \\ B_2 &= -(D_1^\dagger C_1)^\intercal \\ C_2 &= -(B_1 D_1^\dagger)^\intercal \\ D_2 &= (D_1^\dagger)^\intercal \end{cases} \qquad \begin{cases} \hat{A}_2 &= (-\hat{A}_1 + \hat{B}_1 \hat{D}_1^\dagger \hat{C}_1)^\intercal \\ \hat{B}_2 &= -(\hat{D}_1^\dagger \hat{C}_1)^\intercal \\ \hat{C}_2 &= -(\hat{B}_1 \hat{D}_1^\dagger)^\intercal \\ \hat{D}_2 &= (\hat{D}_1^\dagger)^\intercal. \end{cases}$$

Hence, the dual state process should follow

$$\begin{aligned} & dX_2 \\ &= (\tilde{M} + \bar{M})dt + (\tilde{N} + \bar{N})dW(t) \\ &= \left[A_2 \tilde{X}_2 + \tilde{\alpha} + B_2 \tilde{\beta} + \hat{A}_2 \bar{X}_2 + \bar{\alpha} + \hat{B}_2 \bar{\beta} \right] dt + \left[C_2 \tilde{X}_2 + D_2 \tilde{\beta} + \hat{C}_2 \bar{X}_2 + \hat{D}_2 \bar{\beta} \right] dW(t). \end{aligned}$$

Let $X_2(0) = \chi$ be another control.

Let the dual running cost be $f_2(t, \tilde{\alpha}, \tilde{\beta}, \bar{\alpha}, \bar{\beta})$ and the dual terminal cost be $g_2(\tilde{X}_2(T), \bar{X}_2(T))$ where

$$\begin{aligned}
g_2(\tilde{y}, \bar{y}) :&= \sup_{\tilde{x},\bar{x}}\{-\tilde{x}^\intercal \tilde{y} - \bar{x}^\intercal \bar{y} - g_1(\tilde{x}, \bar{x})\} \\
&= \frac{1}{2}\langle G_{T1}^{-1}\tilde{y}, \tilde{y}\rangle + \frac{1}{2}\langle \hat{G}_{T1}^{-1}\bar{y}, \bar{y}\rangle + \langle \hat{G}_{T1}^{-1}\hat{g}_{T1}, \bar{y}\rangle + \frac{1}{2}\langle \hat{G}_{T1}^{-1}\hat{g}_{T1}, \hat{g}_{T1}\rangle \\
&= \frac{1}{2}\langle G_{T2}\tilde{y}, \tilde{y}\rangle + \frac{1}{2}\langle \hat{g}_{T2}\bar{y}, \bar{y}\rangle + \langle \hat{g}_{T2}, \bar{y}\rangle + \frac{1}{2}\langle \hat{g}_{T2}, \hat{G}_{T2}^{-1}\hat{g}_{T2}\rangle,
\end{aligned}$$

$$\begin{aligned}
f_2(t, \tilde{\alpha}, \tilde{\beta}, \bar{\alpha}, \bar{\beta}) :&= \sup_{\tilde{x},\tilde{\pi},\bar{x},\bar{\pi}}\{\tilde{x}^\intercal\tilde{\alpha} + \tilde{\pi}^\intercal\tilde{\beta} + \bar{x}^\intercal\bar{\alpha} + \bar{\pi}^\intercal\bar{\beta} - f_1(t, \tilde{x}, \bar{x}, \tilde{\pi}, \bar{\pi})\} \\
&= \frac{1}{2}\langle R_2(t)\begin{pmatrix}\tilde{\alpha}\\ \tilde{\beta}\end{pmatrix}, \begin{pmatrix}\tilde{\alpha}\\ \tilde{\beta}\end{pmatrix}\rangle + \frac{1}{2}\langle \hat{R}_2(t)\begin{pmatrix}\bar{\alpha}\\ \bar{\beta}\end{pmatrix}, \begin{pmatrix}\bar{\alpha}\\ \bar{\beta}\end{pmatrix}\rangle,
\end{aligned}$$

and

$$\begin{cases} R_2 &= O_1^{-1} \\ G_{T2} &= G_{T1}^{-1} \end{cases} \qquad \begin{cases} \hat{R}_2 &= \hat{O}_1^{-1} \\ \hat{G}_{T2} &= \hat{G}_{T1}^{-1} \\ \hat{g}_{T2} &= \hat{G}_{T1}^{-1}\hat{g}_{T1}. \end{cases}$$

Integrate (4.9) from 0 to T and by definition of the dual cost functions:

$$\begin{aligned}
&\mathbb{E}[g_1(\tilde{X}_1(T), \bar{X}_1(T)] + \mathbb{E}[g_2(\tilde{X}_2(T), \bar{X}_2(T))] \\
&\geq -\mathbb{E}[\tilde{X}_1^\intercal(T)\tilde{X}_2(T) + \bar{X}_1^\intercal(T)\bar{X}_2(T)] \\
&= -x_1^\intercal\chi - \mathbb{E}[\int_0^T (\tilde{X}_1^\intercal\tilde{\alpha} + \tilde{\pi}^\intercal\tilde{\beta} + \bar{X}_1^\intercal\bar{\alpha} + \bar{\pi}^\intercal\bar{\beta})dt] \\
&\geq -x_1^\intercal\chi - \mathbb{E}[\int_0^T f_1(t, \tilde{X}_1(t), \bar{X}_1(t), \tilde{\pi}(t), \bar{\pi}(t))dt + \int_0^T f_2(t, \tilde{\alpha}(t), \tilde{\beta}(t), \bar{\alpha}(t), \bar{\beta}(t))dt]
\end{aligned} \tag{4.12}$$

Rearrange we would have for any π, α, β and χ

$$
\begin{aligned}
J_1(\pi) &= \mathbb{E}[\int_0^T f_1(t, \tilde{X}_1(t), \bar{X}_1(t), \tilde{\pi}(t), \bar{\pi}(t))dt + g_1(\tilde{X}_1(T), \bar{X}_1(T)] \\
&\geq -x_1^\intercal \chi - \mathbb{E}[\int_0^T f_2(t, \tilde{\alpha}(t), \tilde{\beta}(t), \bar{\alpha}(t), \bar{\beta}(t))dt + g_2(\tilde{X}_2(T), \bar{X}_2(T))]
\end{aligned}
$$

So the infimum of LHS is greater or equal to the supremum of RHS, which gives

$$
V_1(x_1) = \inf_\pi J_1(\pi) \geq - \inf_{\alpha,\beta,\chi} \mathbb{E}[\int_0^T f_2(t, \tilde{\alpha}(t), \tilde{\beta}(t), \bar{\alpha}(t), \bar{\beta}(t))dt + g_2^*(\tilde{X}_2(T), \bar{X}_2(T), \chi)] \tag{4.13}
$$

where

$$
g_2^*(\tilde{y}, \bar{y}, \chi) := g_2(\tilde{y}, \bar{y}) + x_1^\intercal \chi.
$$

Define

$$
J_2(\alpha, \beta, \chi) := \mathbb{E}[\int_0^T f_2(t, \tilde{\alpha}(t), \tilde{\beta}(t), \bar{\alpha}(t), \bar{\beta}(t))dt + g_2^*(\tilde{X}_2(T), \bar{X}_2(T), \chi)] \tag{4.14}
$$

and

$$
V_2(x_1) := \inf_{\alpha,\beta,\chi} J_2(\alpha, \beta, \chi), \tag{4.15}
$$

then we have the dual cost function and the value function. To obtain equality in the first inequality of (4.12), we would need to have

$$
(-\tilde{y}, -\bar{y}) = (\partial_{\tilde{x}} g_1, \partial_{\bar{x}} g_2) \tag{4.16}
$$

or

$$
(-\tilde{x}, -\bar{x}) = (\partial_{\tilde{y}} g_2, \partial_{\bar{y}} g_2).
$$

The equality in the second inequality of (4.12) can be achieved only if

$$
(\tilde{\alpha}, \tilde{\beta}, \bar{\alpha}, \bar{\beta}) = (\partial_{\tilde{x}} f_1, \partial_{\tilde{\pi}} f_1, \partial_{\bar{x}} f_1, \partial_{\bar{\pi}} f_1) \tag{4.17}
$$

or

$$(\tilde{x}, \tilde{\pi}, \bar{x}, \bar{\pi}) = (\partial_{\tilde{\alpha}} f_2, \partial_{\tilde{\beta}} f_2, \partial_{\bar{\alpha}} f_2, \partial_{\bar{\beta}} f_2).$$

In the case of (4.12), the conditions of (4.16) and (4.17) are as follows.

$$\begin{cases} -\tilde{X}_2(T) & = \partial_{\tilde{x}} g_1(\tilde{X}_1(T), \bar{X}_1(T)) = G_{T1}\tilde{X}_1(T), \\ -\bar{X}_2(T) & = \partial_{\bar{x}} g_1(\tilde{X}_1(T), \bar{X}_1(T)) = \hat{G}_{T1}\bar{X}_1(T) + \hat{g}_{T1}, \\ \begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{pmatrix} & = O_1 \begin{pmatrix} \tilde{x} \\ \tilde{\pi} \end{pmatrix}, \\ \begin{pmatrix} \bar{\alpha} \\ \bar{\beta} \end{pmatrix} & = \bar{O}_1 \begin{pmatrix} \bar{x} \\ \bar{\pi} \end{pmatrix}. \end{cases}$$

Note that as $\mathbb{E}[\tilde{\Lambda}] = 0$ for $\Lambda \in \{X_1, \pi, X_2, \alpha, \beta\}$, and as all the coefficients are deterministic, the equations above indeed work out.
Hence we can conclude that the dual problem is what is expected in Theorem 4.2.3.

□

Remark *Equivalently, the dual problem can be rewritten as*

$$\begin{cases} dX_2(t) & = \left[A_2X_2 + \alpha + B_2\beta + \bar{A}_2\bar{X}_2 + \bar{\alpha} + \bar{B}_2\bar{\beta}\right] dt \\ & \qquad + \left[C_2X_2 + D_2\beta + \bar{C}_2\bar{X}_2 + \bar{D}_2\bar{\beta}\right] dW(t) \\ X_2(0) & = \chi, \end{cases}$$

subject to

$$\underset{\alpha,\beta,\chi}{\boldsymbol{minimize}} \quad J_2(\alpha, \beta, \chi) = \mathbb{E}\left[\int_0^T \tilde{f}_2(t, \alpha, \beta, \bar{\alpha}, \bar{\beta})dt + \tilde{g}_2^*(X_2(T), \bar{X}_2(T), \chi)\right].$$

where $\bar{\Lambda}_2 := \hat{\Lambda}_2 - \Lambda_2$ *for* $\Lambda \in \{A, B, C, D, R, G_T, g_T\}$ *and*

$$\begin{cases} \tilde{f}_2(t,\alpha,\beta,\bar{\alpha},\bar{\beta}) = \frac{1}{2}\langle R_2(t)\begin{pmatrix}\alpha\\ \beta\end{pmatrix}, \begin{pmatrix}\alpha\\ \beta\end{pmatrix}\rangle + \frac{1}{2}\langle \bar{R}_2(t)\begin{pmatrix}\bar{\alpha}\\ \bar{\beta}\end{pmatrix}, \begin{pmatrix}\bar{\alpha}\\ \bar{\beta}\end{pmatrix}\rangle \\ \tilde{g}_2(y,\bar{y}) = \frac{1}{2}\langle G_{T2}y, y\rangle + \langle g_{T2}, \bar{y}\rangle + \frac{1}{2}\langle \bar{G}_{T2}\bar{y}, \bar{y}\rangle + \langle \bar{g}_{T2}, \bar{y}\rangle \\ \tilde{g}_2^*(y,\bar{y},\chi) = \tilde{g}_2(y,\bar{y}) + \frac{1}{2}\langle \hat{g}_{T2}, \hat{G}_{T2}^{-1}\hat{g}_{T2}\rangle + x_1^\intercal\chi. \end{cases}$$

4.3 Relationship Between Primal and Dual Problem

Theorem 4.3.1 *Suppose assumption (A6) and (A7) are true and the matrix* O_1 *from* (4.4) *is positive definite and* G_{T1} *is positive.*
The admissible control $\pi \in \mathcal{U}$ *is optimal if and only if the solution* (X_1, Y_1, Z_1) *to the primal FBSDEs*

$$\begin{cases} dX_1(t) &= (A_1\tilde{X}_1 + \hat{A}_1\bar{X}_1 + B_1\tilde{\pi} + \hat{B}_1\bar{\pi})dt + (C_1\tilde{X}_1 + \hat{C}_1\bar{X}_1 + D_1\tilde{\pi} + \hat{D}_1\bar{\pi})dW(t) \\ X_1(0) &= x_1 \\ dY_1(t) &= -(A_1^\intercal\tilde{Y}_1 + \hat{A}_1^\intercal\bar{Y}_1 + C_1^\intercal\tilde{Z}_1 + \hat{C}_1^\intercal\bar{Z}_1 + Q_1\tilde{X}_1 + S^\intercal\tilde{\pi} + \hat{Q}_1\bar{X}_1 + \hat{S}_1^\intercal\bar{\pi})dt + Z_1dW(t) \\ Y_1(T) &= G_{T1}\tilde{X}_1(T) + \hat{G}_{T1}\bar{X}_1(T) + \hat{g}_{T1} \end{cases} \tag{4.18}$$

satisfies the following condition:

$$B_1^\intercal\tilde{Y}_1 + \hat{B}_1^\intercal\bar{Y}_1 + D_1^\intercal\tilde{Z}_1 + \hat{D}_1^\intercal\bar{Z}_1 + S_1\tilde{X}_1 + \hat{S}_1\bar{X}_1 + R_1\tilde{\pi} + \hat{R}_1\bar{\pi} = 0. \tag{4.19}$$

Proof. Using the results of Theorem 3.3.1 and 3.3.2 Let

$$\begin{cases} u_1 := \pi \\ b := A_1x + \bar{A}_1m_0 + B_1u_1 + \bar{B}_1m_1 \\ \sigma := C_1x + \bar{C}_1m_0 + D_1u_1 + \bar{D}_1m_1 \\ f := \tilde{f}_1(t, x, m_0, u_1, m_1) \\ g := \tilde{g}_1(x, m_0) \end{cases}$$

The state process is linear, and the objective function is quadratic with positive-definite quadratic coefficients, hence Lipschitz, bounded by a linear growth, diffenrentiable, and convex. Then the Hamiltonian is also convex.

$$\begin{aligned} \mathcal{H} = &(A_1x + \bar{A}_1m_0 + B_1u_1 + \bar{B}_1m_1)^\intercal y + \operatorname{tr}\big((C_1x + \bar{C}_1m_0 + D_1u_1 + \bar{D}_1m_1)^\intercal z\big) \\ &+ \frac{1}{2}\langle \begin{pmatrix} Q_1 & S_1^\intercal \\ S_1 & R_1 \end{pmatrix} \begin{pmatrix} x \\ u_1 \end{pmatrix}, \begin{pmatrix} x \\ u_1 \end{pmatrix} \rangle + \frac{1}{2}\langle \begin{pmatrix} \bar{Q}_1 & \bar{S}_1^\intercal \\ \bar{S}_1 & \bar{R}_1 \end{pmatrix} \begin{pmatrix} m_0 \\ m_1 \end{pmatrix}, \begin{pmatrix} m_0 \\ m_1 \end{pmatrix} \rangle. \end{aligned}$$

So assumption (A1) to (A5) are satisfied. The adjoint process should be

$$\begin{cases} dY_1 & = -\left[\partial_x \mathcal{H} + \sum_j \mathbb{E}[\partial_{m_j}\mathcal{H}]\right] dt + Z_1 dW(t) \\ & = -(A_1^\intercal Y_1 + \bar{A}_1^\intercal \bar{Y}_1 + C_1^\intercal Z_1 + \bar{C}_1^\intercal \bar{Z}_1 + Q_1X_1 + \bar{Q}_1\bar{X}_1 + S_1^\intercal \pi + \bar{S}_1^\intercal \bar{\pi})dt + Z_1 dW(t) \\ & = -(A_1^\intercal \tilde{Y}_1 + \hat{A}_1^\intercal \bar{Y}_1 + C_1^\intercal \tilde{Z}_1 + \hat{C}_1^\intercal \bar{Z}_1 + Q_1\tilde{X}_1 + \hat{Q}_1\bar{X}_1 + S_1^\intercal \tilde{\pi} + \hat{S}_1^\intercal \bar{\pi})dt + Z_1 dW(t) \\ Y_1(T) & = \partial_x \tilde{g}_1(X_1(T), \bar{X}_1(T)) + \mathbb{E}\left[\partial_{m_0}\tilde{g}_1(X_1(T), \bar{X}_1(T))\right] \\ & = G_{T1}X_1(T) + \bar{G}_{T1}\tilde{X}_1(T) + \hat{g}_{T1} \\ & = G_{T1}\tilde{X}_1(T) + \hat{G}_{T1}\tilde{X}_1(T) + \hat{g}_{T1} \end{cases}$$

Since the terminal cost does not depend explicitly on x_1, the control is optimal if and only if (3.5) is true with $x = X_1, m_0 = \bar{X}_1, u_1 = \pi, m_1 = \bar{\pi}$:

$$
\begin{aligned}
0 &= \partial_{u_1}\mathcal{H} + \mathbb{E}[\partial_{m_1}\mathcal{H}] \\
&= \partial_{\pi}\mathcal{H} + \mathbb{E}[\partial_{m_1}\mathcal{H}] \\
&= B_1^{\intercal}Y_1 + \bar{B}_1^{\intercal}\bar{Y}_1 + D_1^{\intercal}Z_1 + \bar{D}_1^{\intercal}\bar{Z}_1 + S_1X_1 + \bar{S}_1\bar{X}_1 + R_1\pi + \bar{R}_1\bar{\pi} \\
&= B_1^{\intercal}\tilde{Y}_1 + \hat{B}_1^{\intercal}\bar{Y}_1 + D_1^{\intercal}\tilde{Z}_1 + \hat{D}_1^{\intercal}\bar{Z}_1 + S_1\tilde{X}_1 + \hat{S}_1\bar{X}_1 + R_1\tilde{\pi} + \hat{R}_1\bar{\pi}.
\end{aligned}
$$

□

Lemma 4.3.2 *Suppose assumption (A6) and (A7), then the coefficients in dual problem will once again satisfy the assumptions of (A6) and (A7).*

Proof. Suppose that assumption (A6) is satisfied, since $D_1 \in L^{\infty}(0, T; \mathbb{R}^{n\times m})$ and $D_1^{\dagger} = D_1^{\intercal}(D_1D_1^{\intercal})^{-1}$, $D_1^{\dagger} \in L^{\infty}(0, T; \mathbb{R}^{n\times m})$ as well. From (4.8), $A_2 = (-A_1 + B_1D_1^{\dagger}C_1)^{\intercal}$, since B and C are square-integrable, the product $B_1D_1^{\dagger}C_1$ is integrable like A_1. Therefore, A_2 is in $L^1(0, T; \mathbb{R}^{n\times n})$. The rest can be shown using a similar approach.

Suppose assumption (A7) is satisfied then $|O_1| \leq \Delta$ for some $\Delta \in \mathbb{R}$. R_2 is the inverse of matrix O_1 as defined in (4.4), since $\langle O_1(t)\begin{pmatrix} x(t) \\ \pi(t) \end{pmatrix}, \begin{pmatrix} x(t) \\ \pi(t) \end{pmatrix}\rangle \geq \delta\|\begin{pmatrix} x(t) \\ \pi(t) \end{pmatrix}\|_2^2$, $\delta \leq |O_1| \leq \Delta$, so its inverse is also essentially bounded and $\langle R_2\begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{pmatrix}, \begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{pmatrix}\rangle$ uniformly convex. Additionally, $G_{T2} = G_{T1}^{-1}$ is positive definite since G_{T1} is positive definite. A similar argument can be applied to $\hat{R}_2, \hat{G}_2$ and $\hat{g}_2$. As such, the dual problem satisfies assumptions (A6) and (A7). □

Theorem 4.3.3 *Suppose assumption (A6) and (A7) are true then the matrix R_2 from* (4.8) *and G_{T2} are positive definite.*

The admissible control pair and the initial value (χ, α, β) is optimal if and only if *the solution (X_2, Y_2, Z_2) to the dual FBSDEs:*

$$\begin{cases} dX_2(t) &= (A_2\tilde{X}_2 + \hat{A}_2\bar{X}_2 + \tilde{\alpha} + B_2\tilde{\beta} + \bar{\alpha} + \hat{B}_2\bar{\beta})dt \\ &\qquad + (C_2\tilde{X}_2 + \hat{C}_2\bar{X}_2 + D_2\tilde{\beta} + \hat{D}_2\bar{\beta})dW(t) \\ X_2(0) &= \chi \\ dY_2(t) &= -(A_2^\intercal\tilde{Y}_2 + \hat{A}_2^\intercal\bar{Y}_2 + C_2^\intercal\tilde{Z}_2 + \hat{C}_2^\intercal\bar{Z}_2)dt + Z_2dW(t) \\ Y_2(T) &= G_{T2}\tilde{X}_2(T) + \hat{G}_{T2}\bar{X}_2(T) + \hat{g}_{T2}. \end{cases} \tag{4.20}$$

satisfies the following conditions:

$$\begin{pmatrix} \tilde{Y}_2 \\ B_2^\intercal\tilde{Y}_2 \end{pmatrix} + \begin{pmatrix} \bar{Y}_2 \\ \hat{B}_2^\intercal\bar{Y}_2 \end{pmatrix} + \begin{pmatrix} 0 \\ D_2^\intercal\tilde{Z}_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \hat{D}_2^\intercal\bar{Z}_2 \end{pmatrix} + R_2\begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{pmatrix} + \hat{R}_2\begin{pmatrix} \bar{\alpha} \\ \bar{\beta} \end{pmatrix} = 0. \tag{4.21}$$

and

$$Y_2(0) = -x_1 \tag{4.22}$$

Proof. Consider the equivalent form of dual problem as in (4.20) and (4.21), apply the result in Theorems 3.3.1 and 3.3.2 by setting

$$\begin{cases} u_1 := \alpha \\ u_2 := \beta \\ b := A_2x + \bar{A}_2m_0 + u_1 + B_2u_2 + m_1 + \bar{B}_2m_2 \\ \sigma := C_2x + \bar{C}_2m_0 + D_2u_2 + \bar{D}_2m_2 \\ f := \tilde{f}_2(t, u_1, u_2, m_1, m_2) \\ g := \tilde{g}_2^*(x, m_0, \chi). \end{cases}$$

Then the Hamiltonian is

$$\mathcal{H} = (A_2x + \bar{A}_2m_0 + u_1 + B_2u_2 + m_1 + \bar{B}_2m_2)^\intercal y + \operatorname{tr}\Big((C_2x + \bar{C}_2m_0 + D_2u_2 + \bar{D}_2m_2)^\intercal z\Big)$$
$$+ \frac{1}{2}\langle R_2 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\rangle + \frac{1}{2}\langle \bar{R}_2 \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}\rangle.$$

From Lemma 4.3.2, the coefficients in the dual problem also satisfy Assumptions (A6) and (A7). Then by a similar argument as in Theorem 4.3.1, assumption (A1) to (A5) are satisfied. Hence the BSDE for adjoint processes (Y_2, Z_2) is

$$\begin{cases} dY_2 & = -\Big[\partial_x\mathcal{H} + \sum_j \mathbb{E}[\partial_{m_j}\mathcal{H}]\Big]\, dt + Z_2dW(t) \\ & = -(A_2^\intercal Y_2 + \bar{A}_2^\intercal \bar{Y}_2 + C_2^\intercal Z_2 + \bar{C}_2^\intercal \bar{Z}_2)dt + Z_2dW(t) \\ & = -(A_2^\intercal \tilde{Y}_2 + \hat{A}_2^\intercal \bar{Y}_2 + C_2^\intercal \tilde{Z}_2 + \hat{C}_2^\intercal \bar{Z}_2)dt + Z_2dW(t) \\ Y_2(T) & = \partial_x g_2^*(X_2(T), \bar{X}_2(T), \chi) + \mathbb{E}\left[\partial_{m_0} g_2^*(X_2(T), \bar{X}_2(T), \chi)\right] \\ & = G_{T2}X_2(T) + \bar{G}_{T2}\bar{X}_2(T) + \hat{g}_{T2} \\ & = G_{T2}\tilde{X}_2(T) + \hat{G}_{T2}\bar{X}_2(T) + \hat{g}_{T2} \end{cases}$$

and the control is optimal if and only if (3.5) is true with $x = X_2, m_0 = \bar{X}_2$, $m_1 = \bar{\alpha}$ and $m_2 = \bar{\beta}$:

$$\begin{aligned} 0 &= \partial_{u_i}\mathcal{H} + \mathbb{E}[\partial_{m_i}\mathcal{H}] \\ (u_1 = \alpha) &= \partial_\alpha\mathcal{H} + \mathbb{E}[\partial_{m_1}\mathcal{H}] \\ (u_2 = \beta) &= \partial_\beta\mathcal{H} + \mathbb{E}[\partial_{m_2}\mathcal{H}], \end{aligned}$$

combining the two equations,

$$
\begin{aligned}
0 &= \begin{pmatrix} \partial_\alpha \mathcal{H} \\ \partial_\beta \mathcal{H} \end{pmatrix} + \begin{pmatrix} \mathbb{E}[\partial_{m_1}\mathcal{H}] \\ \mathbb{E}[\partial_{m_2}\mathcal{H}] \end{pmatrix} \\
&= \begin{pmatrix} Y_2 \\ B_2^\intercal Y_2 \end{pmatrix} + \begin{pmatrix} \bar{Y}_2 \\ \bar{B}_2^\intercal \bar{Y}_2 \end{pmatrix} + \begin{pmatrix} 0 \\ D_2^\intercal Z_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \bar{D}_2^\intercal \bar{Z}_2 \end{pmatrix} + R_2 \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + \bar{R}_2 \begin{pmatrix} \bar{\alpha} \\ \bar{\beta} \end{pmatrix} \\
&= \begin{pmatrix} Y_2 \\ B_2^\intercal \tilde{Y}_2 \end{pmatrix} + \begin{pmatrix} \bar{Y}_2 \\ \hat{B}_2^\intercal \bar{Y}_2 \end{pmatrix} + \begin{pmatrix} 0 \\ D_2^\intercal \tilde{Z}_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \hat{D}_2^\intercal \bar{Z}_2 \end{pmatrix} + R_2 \begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{pmatrix} + \hat{R}_2 \begin{pmatrix} \bar{\alpha} \\ \bar{\beta} \end{pmatrix},
\end{aligned}
$$

and (3.4) gives:

$$0 = Y_2(0) + \partial_\chi \tilde{g}_2^* = Y_2(0) + x_1$$

□

Theorem 4.3.4 *Under Assumptions (A6) and (A7), suppose that π is optimal for the primal problem as in Theorem 4.3.1 with (X_1, Y_1, Z_1) as the solution to the corresponding FBSDEs. Suppose further*

$$(D_1^\intercal)^\dagger D_1^\intercal \tilde{Z}_1 = \tilde{Z}_1 \tag{4.23}$$

and

$$(\hat{D}_1^\intercal)^\dagger \hat{D}_1^\intercal \bar{Z}_1 = \bar{Z}_1. \tag{4.24}$$

Define

$$
\begin{cases}
\chi & := -Y_1(0) \\
\alpha & := Q_1 \tilde{X}_1 + S_1^\intercal \tilde{\pi} + \hat{Q}_1 \bar{X}_1 + \hat{S}_1^\intercal \bar{\pi} \\
\beta & := -D_1^\intercal \tilde{Z}_1 - B_1^\intercal \tilde{Y}_1 - \hat{D}_1^\intercal \bar{Z}_1 - \hat{B}_1^\intercal \bar{Y}_1.
\end{cases} \tag{4.25}
$$

Then (χ, α, β) is optimal for the dual problem as in Theorem 4.3.3, and the optimal *state and adjoint processes can be represented by*

$$\begin{cases} X_2 & := -Y_1 \\ Y_2 & := -X_1 \\ Z_2 & := -C_1\tilde{X}_1 - \hat{C}_1\bar{X}_1 - D_1\tilde{\pi} - \hat{D}_1\bar{\pi}. \end{cases} \tag{4.26}$$

Theorem 4.3.5 *Let assumption (A6) and (A7) be satisfied, suppose* (χ, α, β) *is optimal to the dual problem as in Theorem 4.3.3 with* (X_2, Y_2, Z_2) *as a solution to the corresponding FBSDEs. Suppose further*

$$D_1 D_1^{\dagger}(\tilde{Z}_2 - C_1\tilde{Y}_2) = \tilde{Z}_2 - C_1\tilde{Y}_2 \tag{4.27}$$

and

$$\hat{D}_1 \hat{D}_1^{\dagger}(\bar{Z}_2 - \hat{C}_1\bar{Y}_2) = \bar{Z}_2 - \hat{C}_1\bar{Y}_2 \tag{4.28}$$

Define

$$\pi := -B_2^{\intercal}\tilde{Y}_2 - D_2^{\intercal}\tilde{Z}_2 - \hat{B}_2^{\intercal}\bar{Y}_2 - \hat{D}_2^{\intercal}\bar{Z}_2, \tag{4.29}$$

then π *is optimal to the primal problem as in Theorem 4.3.1 and the state and adjoint processes can be represented by*

$$\begin{cases} X_1 & := -Y_2 \\ Y_1 & := -X_2 \\ Z_1 & := -C_2\tilde{X}_2 - \hat{C}_2\bar{X}_2 - D_2\tilde{\beta} - \hat{D}_2\bar{\beta}. \end{cases} \tag{4.30}$$

Remark *Assumptions* (4.23), (4.24) *and* (4.27), (4.28) *are necessary to build the equivalence relationship from the Primal problem to the Dual problem and vice versa. In addition,* (4.27), (4.28) *are necessary in deriving the expression of the Dual problem. One simple assumption for all these conditions to meet is to assume* D_1 *and* $\hat{D}_1$ *to be invertible.*

Corollary 4.3.6 *With the same setting as in Theorem 4.3.4, for the constructed dual problem,* (4.27), (4.28) *are satisfied.*

Proof. From (A.9) in Theorem 4.3.4,

$$\begin{cases} \tilde{Y}_2 = -\tilde{X}_1 \\ \bar{Y}_2 = -\bar{X}_1 \\ \tilde{Z}_2 = -C_1\tilde{X}_1 - D_1\tilde{\pi} \\ \bar{Z}_2 = -\hat{C}_1\bar{X}_1 - \hat{D}_1\bar{\pi}. \end{cases}$$

Then

$$D_1D_1^\dagger(\tilde{Z}_2 - C_1\tilde{Y}_2) = -D_1D_1^\dagger D_1\tilde{\pi} = -D_1\tilde{\pi} = \tilde{Z}_2 - C_1\tilde{Y}_2$$

and similarly for

$$\hat{D}_1\hat{D}_1^\dagger(\bar{Z}_2 - \hat{C}_1\bar{Y}_2) = \bar{Z}_2 - \hat{C}_1\bar{Y}_2.$$

□

Corollary 4.3.7 *With the same setting as in Theorem 4.3.5, for the constructed primal problem,* (4.23), (4.24) *are satisfied.*

Proof. From (A.10) in Theorem 4.3.5,

$$\begin{cases} \tilde{Z}_1 = -C_2\tilde{X}_2 - D_2\tilde{\beta} = (D_1^\intercal)^\dagger(B_1^\intercal\tilde{X}_2 - \tilde{\beta}) \\ \bar{Z}_1 = -\hat{C}_2\bar{X}_2 - \hat{D}_2\bar{\beta} = (\hat{D}_1^\intercal)^\dagger(\hat{B}_1^\intercal\bar{X}_2 - \bar{\beta}). \end{cases}$$

Then

$$(D_1^\intercal)^\dagger D_1^\intercal\tilde{Z}_1 = -(D_1^\intercal)^\dagger D_1^\intercal(D_1^\intercal)^\dagger(B_1^\intercal\tilde{X}_2 - \tilde{\beta}) = (D_1^\intercal)^\dagger(B_1^\intercal\tilde{X}_2 - \tilde{\beta}) = \tilde{Z}_1$$

and similarly for

$$(\hat{D}_1^\intercal)^\dagger\hat{D}_1^\intercal\bar{Z}_1 = \bar{Z}_1.$$

$\square$

Corollary 4.3.8 *Without conditions* (4.23) *and* (4.24), *if the dual problem is still defined as in Theorem 4.3.4, then* $\tilde{V}_1(x_1) \leq -\tilde{V}_2(x_1)$ *where the value functions are defined as in* (4.13) *and* (4.15).

Proof. As $D_1^\dagger D_1 D_1^\dagger = D_1^\dagger$ and $\hat{D}_1^\dagger \hat{D}_1 \hat{D}_1^\dagger = \hat{D}_1^\dagger$, if $\tilde{Z}_1 = (D_1^\intercal)^\dagger \tilde{v}$ and $\bar{Z}_1 = (\hat{D}_1^\intercal)^\dagger \bar{v}$ for some $v \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$ then

$$
\begin{aligned}
V_1(x_1) &= \inf_{\pi} J_1(\pi) \\
&\leq \inf\{J_1(\pi)|\pi : Z_1 = (D_1^\intercal)^\dagger \tilde{v}_1 + (\hat{D}_1^\intercal)^\dagger \bar{v}_1 \text{ for some } v_1\} \\
&= -\inf\{J_2(\alpha,\beta,\chi)|(\alpha,\beta) : \tilde{Z}_2 - C_1\tilde{Y}_2 = D_1\tilde{v}_2, \bar{Z}_2 - \hat{C}_1\bar{Y}_2 = \hat{D}_1\bar{v}_2 \text{ for some } v_2\} \\
&\leq -\inf_{\alpha,\beta,\chi} J_2(\alpha,\beta,\chi) \\
&= V_2(x_1) \qquad (4.31)
\end{aligned}
$$

$\square$

Example 4.3.9 *In the case where the running cost* $f_1 \equiv 0$ *the state process in the dual problem could be solved explicitly.*

Proof. For simplicity, everything will be assumed to be one-dimensional. Using a similar approach as in Theorem 4.2.3, let the dual running cost be

$$f_2(\tilde{\alpha},\tilde{\beta},\bar{\alpha},\bar{\beta}) = \sup_{\tilde{x},\tilde{\pi},\bar{x},\bar{\pi}} \{\tilde{x}^\intercal\tilde{\alpha} + \tilde{\pi}^\intercal\tilde{\beta} + \bar{x}^\intercal\bar{\alpha} + \bar{\pi}^\intercal\bar{\beta} - f_1(t,\tilde{x},\bar{x},\tilde{\pi},\bar{\pi})\}.$$

However, as $f_1 \equiv 0$, to make sure the dual running cost to be finite, we need to set dual controls α and β to be 0. As a result, so is f_2. Then the dual problem, with controls being 0 becomes

$$\begin{cases} dX_2(t) &= (A_2X_2 + \bar{A}_2\mathbb{E}[X_2])dt + (C_2X_2 + \bar{C}_2\mathbb{E}[X_2])dW(t), \\ X_2(0) &= \chi, \\ dY_2(t) &= -(A_2Y_2 + \bar{A}_2\mathbb{E}[Y_2] + C_2Z_2 + \bar{C}_2\mathbb{E}[Z_2])dt + Z_2dW(t), \\ Y_2(T) &= G_{T2}^{-1}X_2(T) + \bar{G}_{T2}\mathbb{E}[X_2(T)] + \hat{g}_{T2}. \end{cases}$$

So

$$d\mathbb{E}[X_2] = \hat{A}_2\mathbb{E}[X_2]ds,$$

and $\mathbb{E}[X_2] = \chi e^{\int_0^t \hat{A}_2 ds}$. With substitution into the Forward equation, the equation becomes a linear SDE that has an explicit solution and a general solution to X_2 would be

$$\begin{aligned} X_2(t) = \chi e^{\int_0^t (A_2 - \frac{C_2^2}{2})ds + \int_0^t C_2 dW(s)} \Big[\int_0^t [e^{-\int_0^s (A_2 - \frac{C_2^2}{2})dr - \int_0^s C_2 dW(r)} (\bar{A}_2 - C_2\bar{C}_2) e^{\int_0^s \hat{A}_2 dr}] ds \\ + \int_0^t [e^{-\int_0^s (A_2 - \frac{C_2^2}{2})dr - \int_0^s C_2 dW(r)} \bar{C}_2 e^{\int_0^s \hat{A}_2 dr}] dW(s) + 1 \Big]. \end{aligned} \tag{4.32}$$

Using the expression and considering $X_2(T)$ and $X_2(t)$, we have

$$\begin{aligned} & X_2(T)\chi^{-1} e^{-\int_0^T (A_2 - \frac{C_2^2}{2})ds - \int_0^T C_2 dW(s)} - X_2(t)\chi^{-1} e^{-\int_0^t (A_2 - \frac{C_2^2}{2})ds - \int_0^t C_2 dW(s)} \\ &= \int_t^T \Big[e^{-\int_0^s (A_2 - \frac{C_2^2}{2})dr - \int_0^s C_2 dW(r)} (\bar{A}_2 - C_2\bar{C}_2) e^{\int_0^s \hat{A}_2 dr}] ds \\ &\quad + \int_t^T [e^{-\int_0^s (A_2 - \frac{C_2^2}{2})dr - \int_0^s C_2 dW(r)} \bar{C}_2 e^{\int_0^s \hat{A}_2 dr} \Big] dW(s) \end{aligned} \tag{4.33}$$

By Ito's Lemma,

$$dX_2^2 = 2X_2(A_2X_2 + \bar{A}_2\mathbb{E}[X_2])dt + (C_2X_2 + \bar{C}_2\mathbb{E}[X_2])^2 dt + 2X_2(C_2X_2 + \bar{C}_2\mathbb{E}[X_2])dW(t),$$

since $X_2 \in L^2_{\mathbb{F}}(C([0,T],\mathbb{R}))$, taking the expectation on both sides to get

$$d\mathbb{E}[X_2^2] = (2A_2 + C_2^2)\mathbb{E}[X_2^2] + (2\bar{A}_2 + 2C_2\bar{C}_2 + \bar{C}_2^2)\mathbb{E}[X_2]^2 dt,$$

which means $\mathbb{E}[X_2^2(t)] = e^{\int_0^t 2A_2 + C_2^2 ds}\chi^2(1 + \int_0^t (2\bar{A}_2 + 2C_2\bar{C}_2 + \bar{C}_2^2)e^{\int_0^s 2\bar{A}_2 - C_2^2 dr} ds)$. The primal value function can then be derived from the following equation:

$$
\begin{aligned}
& V_1(x_1) && (4.34)\\
&= -V_2(x_1)\\
&= -\inf_{\chi}\{\mathbb{E}\left[\tilde{g}_2^*(X_2(T), \mathbb{E}[X_2(T)], \chi)\right]\}\\
&= -\inf_{\chi}\{G_{T2}\mathbb{E}[X_2^2(T)] + \bar{G}_{T2}\mathbb{E}[X_2(T)]^2 + \hat{g}_{T2}\mathbb{E}[X_2(T)] + x_1\chi\} - \frac{1}{2}\langle \hat{g}_{T2}, \hat{G}_{T2}^{-1}\hat{g}_{T2}\rangle\\
&= -\inf_{\chi}\Big\{ \left[G_{T2}e^{\int_0^t 2A_2 + C_2^2 ds}(1 + \int_0^t (2\bar{A}_2 + 2C_2\bar{C}_2 + \bar{C}_2^2)e^{\int_0^s 2\bar{A}_2 - C_2^2 dr} ds) + \bar{G}_{T2}e^{2\int_0^t \hat{A}_2 ds} \right] \chi^2\\
&\quad + \left[\hat{g}_{T2}e^{\int_0^t \hat{A}_2 ds} + x_1 \right]\chi \Big\} - \frac{1}{2}\langle \hat{g}_{T2}, \hat{G}_{T2}^{-1}\hat{g}_{T2}\rangle. && (4.35)
\end{aligned}
$$

□

4.4 Constrained Controls

The first part of the analysis in this section primarily comes from [54].

Definition 4.4.1 *A proper convex function is an extended real-valued convex function with non-empty domain that never takes $-\infty$ and is not identically equal to ∞.*

Definition 4.4.2 *A vector x^* is said to be a subgradient of a convex function f at a point x if*

$$f(z) \geq f(x) + \langle x^*, z - x\rangle, \forall z.$$

The set of all subgradients of f at x is called the subdifferential of f at x and is *denoted by ∂f.*

Theorem 4.4.3

$$\operatorname{dom}\partial f := \{x|\partial f(x) \neq \emptyset\} \subseteq \operatorname{dom} f := \{x|f(x) < \infty\}.$$

Proof. For example, refer to Theorem 23.4 in [54]. □

Definition 4.4.4 *A proper convex function f in $\mathbb{R}^n$ is said to be essentially strictly convex in C if f is strictly convex in all convex subsets of* $\operatorname{dom}\partial f$.

Definition 4.4.5 *An extended-real-valued proper convex function on $\mathbb{R}^n$ is said to be essentially smooth if it satisfied the following three conditions for $C := \operatorname{int}(\operatorname{dom} f)$:*

- *C is non-empty;*
- *f is differentiable throughout C;*
- *$\lim_{i\to\infty} \left|\nabla f(x_i)\right| = +\infty$ whenever $x_1, x_2, ...$ is a sequence in C converging to a boundary point x of C.*

Theorem 4.4.6 *A closed proper convex function is essentially strictly convex if and only if its conjugate is essentially smooth.*

Proof. Refer to Theorem 26.3 on page 253 of [54]. □

When the control is no longer a whole space in the primal problem, for example, if $\pi \in K$ for some closed convex space K, we can no longer treat $\tilde{\pi}$ and $\bar{\pi}$ independently when deriving the Legendre Transform of the running cost. That is, the dual running cost would become:

$$\begin{aligned} f_2(t, \tilde{\alpha}, \tilde{\beta}, \bar{\alpha}, \bar{\beta}) :&= \sup_{\tilde{x},\bar{x},\pi\in K} \{\tilde{x}^\intercal\tilde{\alpha} + \tilde{\pi}^\intercal\tilde{\beta} + \bar{x}^\intercal\bar{\alpha} + \bar{\pi}^\intercal\bar{\beta} - f_1(t, \tilde{x}, \bar{x}, \tilde{\pi}, \bar{\pi})\} \\ &= \sup_{\tilde{x},\bar{x},\pi\in K,\tilde{\pi}+\bar{\pi}\in K} \{\tilde{x}^\intercal\tilde{\alpha} + \tilde{\pi}^\intercal\tilde{\beta} + \bar{x}^\intercal\bar{\alpha} + \bar{\pi}^\intercal\bar{\beta} - f_1(t, \tilde{x}, \bar{x}, \tilde{\pi}, \bar{\pi})\}. \end{aligned}$$

which is complicated to represent. As such, an alternative way of representing the dual problem would be needed. We shall not use $\tilde{X}_1$ and $\bar{X}_1$ and so on in the equations, instead, X_1 and $\bar{X}_1$ and so on shall be used. Then the dual running cost would look like this:

$$\begin{aligned} f_2(t,\alpha,\beta,\bar{\alpha},\bar{\beta}) :&= \sup_{x,\bar{x},\pi\in K}\{x^\intercal\alpha+\nu\bar{x}^\intercal\bar{\alpha}+\pi^\intercal\beta+\rho\bar{\pi}^\intercal\bar{\beta}-f_1(x,\bar{x},\pi,\bar{\pi})\} \\ &= \sup_{x,\bar{x},\pi\in K,\bar{\pi}\in K}\{x^\intercal\alpha+\nu\bar{x}^\intercal\bar{\alpha}+\pi^\intercal\beta+\rho\bar{\pi}^\intercal\bar{\beta}-f_1(x,\bar{x},\pi,\bar{\pi})\} \\ &= \sup_{x,\bar{x},\pi,\bar{\pi}}\{x^\intercal\alpha+\nu\bar{x}^\intercal\bar{\alpha}+\pi^\intercal\beta+\rho\bar{\pi}^\intercal\bar{\beta}-\hat{f}_1(x,\bar{x},\pi,\bar{\pi})\}, \end{aligned}$$

for some ν and ρ to be determined and $\hat{f}_1 = f_1 + \Phi(\pi,\bar{\pi})$ with Φ the penalizing function for π and $\bar{\pi}$ on K. That is

$$\Phi(\pi,\bar{\pi}) = \begin{cases} 0 \text{ if } \pi,\bar{\pi}\in K \\ \infty \text{ otherwise }. \end{cases}$$

Since $\hat{f}_1$ is essentially strictly convex, its dual conjugate f_2 will be essentially smooth by Theorem 4.4.6. Hence the dual gap will be closed under a similar condition to (4.17):

$$(\alpha,\beta,\nu\bar{\alpha},\rho\bar{\beta}) = (\partial_x\hat{f}_1,\partial_\pi\hat{f}_1,\partial_{\bar{x}}\hat{f}_1,\partial_{\bar{\pi}}\hat{f}_1).$$

Furthermore, as $\bar{\alpha}=\mathbb{E}[\alpha]$ and $\bar{\beta}=\mathbb{E}[\beta]$, we need to pick ν and ρ so that

$$\partial_{\bar{x}}\hat{f}_1 = \nu\mathbb{E}[\partial_x\hat{f}_1]$$

and

$$\partial_{\bar{\pi}}\hat{f}_1 = \rho\mathbb{E}[\partial_\pi\hat{f}_1].$$

For the dual terminal cost, a similar condition to (4.16) would be

$$(-y,-\bar{y}) = (\partial_x g_1,\partial_{\bar{x}} g_1)$$

which means that

$$\mathbb{E}[\partial_x g_1] = \partial_{\tilde{x}} g_1.$$

An example of the primal problem that could close the duality gap when K is the whole space will be when $\bar{O}_1 = cO_1$, $\bar{G}_{T1} = \mu G_{T1}$ and $\bar{g}_{T1} = \mu g_{T1}$ for some non-negative c and μ.

4.5 Conclusion

In this chapter, we discuss the derivation of dual problem both in the case of unconstrained and constrained controls. Then we show that when there is no control constraints, the primal and dual problem are equivalent provided (4.23) and (4.24) or (4.27) and (4.28) are satisfied. When D_1 and $\hat{D}_1$ are non-degenerate square matrix, these conditions will always hold so there will be no duality gap. An interesting further study would be to discuss what would happen if, when expressing $\tilde{N}$ and $\bar{N}$ in (4.11), we keep w as any general vector. Then there will be many possible variations of dual problem. Although (4.12) shows that the derivation of the value function will be the same, it would still be nice to study the relationship between various dual problems. Another area for further study would be to derive equivalent relationships between the primal and dual problems when the control is constrained. It will involve Gateaux derivative of primal and dual running costs by considering the optimality condition for Legendre's transform. But because the running costs are inexplicit, a more analytical approach would be required.

5

Verification

5.1 Introduction

Ever since the introduction of Machine Learning, it has been trained to solve FBSDEs because there are often no explicit solutions to these problems.

- Starting from the decoupled problem, there are well-established methods for solving forward SDEs as well as mean-field ones by discretisation (Euler-Maruyama Method or Milstein Method). Sauer [56] has done a brief summary of the common methods used to solve SDEs. (Note that the mean-field terms are deterministic, as such will not interfere with higher order terms involving $W(t)$ in the expansion).
- However, it is rarer for methods to solve stand-alone BSDEs numerically. More commonly, the problem involves a decoupled forward SDE as the state process, and then the adjoint process would be of interest and depend on the

state process. Such an observation is mainly caused by the fact that BSDEs problems are usually derived from PDEs. [23] lists a forward discretisation method to solve BSDEs that approximates the function u by assuming that the adjoint processes are functions of the terminal values. Note that albeit the central problem in the paper is a BSDE, under Section 3.2, the general case applies to FBSDEs with decoupled state processes as well.

- On the other hand, solutions to FBSDEs have been extensively studied. There is a 4-step scheme method that converts the problem back into a PDE problem. In [33], three similar algorithms are discussed. The differences depend mainly on which variables are treated as controls.
- For coupled Mean-Field FBSDEs, the equations are usually derived from Mean-Field control problems such as the method proposed in [18].

In this chapter, we first apply the results of the Riccati equations from [59] to the primal and dual problems of Chapter 4 to verify our conclusions. Then, inspired by [3], we derive a representation of the solutions to MFBSDEs and show that some results in [3] could be mistaken. Lastly, with the help of deep learning, we find some empirical results for both the primal and dual FBSDEs problems and compare them with the analytical results derived from the Riccati solutions to check the performance.

5.2 Riccati Solution

Since both the primal and dual problems are in the Linear-Quadratic form as stated in [59], the method in their book can be adapted to verify the dual problem. However, note that the cost function used in it is twice that of the setting in Chapter 4. Setting $u_1 = \pi$, $u_2 = (\alpha, \beta)^\intercal$, $\mathcal{B}_1 = B_1$, $\mathcal{B}_2 = \begin{pmatrix} I & B_2 \end{pmatrix}$, $\bar{\mathcal{B}}_1 = \bar{B}_1$, $\bar{\mathcal{B}}_2 = \begin{pmatrix} \kappa I & \bar{B}_2 \end{pmatrix}$, $\mathcal{D}_1 = D_1$, $\mathcal{D}_2 = \begin{pmatrix} 0 & D_2 \end{pmatrix}$, $\bar{\mathcal{D}}_1 = \bar{D}_1$, $\bar{\mathcal{D}}_2 = \begin{pmatrix} 0 & \bar{D}_2 \end{pmatrix}$, we are able to simplify both the primal and dual problems. Suppose P_i and Π_i are the solutions to the following Ricatti equations:

$$\begin{cases} \dot{P}_i + P_i A_i + A_i^\intercal P_i + C_i^\intercal P_i C_i + Q_i \\ \qquad - (\mathcal{B}_i^\intercal P_i + \mathcal{D}_i^\intercal P_i C_i + S_i)^\intercal (R_i + \mathcal{D}_i^\intercal P_i \mathcal{D}_i)^{-1} (\mathcal{B}_i^\intercal P_i + \mathcal{D}_i^\intercal P_i C_i + S_i) = 0 \\ P_i(T) = G_{Ti} \end{cases}$$

$$\begin{cases} \dot{\Pi}_i + \Pi_i \hat{A}_i + \hat{A}_i^\intercal \Pi_i + \hat{C}_i^\intercal P_i \hat{C}_i + \hat{Q}_i \\ \qquad - (\hat{\mathcal{B}}_i^\intercal \Pi_i + \hat{\mathcal{D}}_i^\intercal P_i \hat{C}_i + \hat{S}_i)^\intercal (\hat{R}_i + \hat{\mathcal{D}}_i^\intercal P_i \hat{\mathcal{D}}_i)^{-1} (\hat{\mathcal{B}}_i^\intercal \Pi_i + \hat{\mathcal{D}}_i^\intercal P_i \hat{C}_i + \hat{S}_i) = 0 \\ \Pi_i(T) = \hat{G}_{Ti}. \end{cases}$$

Define

$$\mathcal{R}_i(P_i) = R_i + \mathcal{D}_i^\intercal P_i \mathcal{D}_i, \mathcal{S}_i(P_i) = \mathcal{B}_i^\intercal P_i + \mathcal{D}_i^\intercal P_i C_i + S_i$$

$$\hat{\mathcal{R}}_i(P_i) = \hat{R}_i + \hat{\mathcal{D}}_i{}^\intercal P_i \hat{\mathcal{D}}_i, \hat{\mathcal{S}}_i(P_i, \Pi_i) = \hat{\mathcal{B}}_i^\intercal \Pi_i + \hat{\mathcal{D}}_i^\intercal P_i \hat{C}_i + \hat{S}_i$$

and

$$\Theta_i = -\mathcal{R}_i^{-1}(P_i)\mathcal{S}_i(P_i), \hat{\Theta}_i = -\hat{\mathcal{R}}_i^{-1}(P_i)\hat{\mathcal{S}}_i(P_i, \Pi_i)$$

$$\varphi_i = -\mathcal{R}_i(P_i)^{-1}(\mathcal{B}_i^\intercal \eta_i + \mathcal{D}_i^\intercal \zeta_i), \bar{\varphi}_i = -\hat{\mathcal{R}}_i(P_i)^{-1}(\hat{\mathcal{B}}_i^\intercal \bar{\eta}_i + \hat{\mathcal{D}}_i^\intercal \mathbb{E}[\zeta_i]),$$

where (η, ζ) is the solution to

$$\begin{cases} d\eta_i(s) &= -\left[(A_i + \mathcal{B}_i \Theta_i)^\intercal \eta_i + (C_i + \mathcal{D}_i \Theta_i)^\intercal \zeta_i \right] ds + \zeta_i dW(s) \\ \eta_i(T) &= g_{Ti} \end{cases}$$

and $\bar{\eta}$ is the solution to

$$\begin{cases} \dot{\bar{\eta}}_i(s) + \left[(\hat{A}_i + \hat{\mathcal{B}}_i \hat{\Theta}_i)^\intercal \bar{\eta}_i + (\hat{C}_i + \hat{\mathcal{D}}_i \hat{\Theta}_i)^\intercal \mathbb{E}[\zeta_i] \right] = 0 \\ \bar{\eta}_i(T) = \mathbb{E}[g_{Ti}] + \bar{g}_{Ti}. \end{cases} \tag{5.1}$$

and the value function will be

$$
\begin{aligned}
2V_i(t,\xi_i) &= \mathbb{E}\langle P_i(t)(\xi_i - \mathbb{E}[\xi_i]) + 2\eta_i(t), \xi_i - \mathbb{E}[\xi_i]\rangle + \langle \Pi_i(t)\mathbb{E}[\xi_i] + 2\bar{\eta}_i(t), \mathbb{E}[\xi_i]\rangle \\
&\quad - \mathbb{E}\int_t^T \langle \mathcal{R}_i(P_i)(\varphi_i - \mathbb{E}[\varphi_i]), \varphi_i - \mathbb{E}[\varphi_i]\rangle + \langle \hat{\mathcal{R}}_i(P_i)\bar{\varphi}_i, \bar{\varphi}_i\rangle ds
\end{aligned}
$$

Note that $t = 0$ and since ξ_i(i.e. x_1 or χ) and g_{T1} are set to be deterministic, then $\zeta_i = 0$, φ_i is deterministic, $\bar{\eta}_i(T) = \hat{g}_{Ti}$ and the value function can be simplified to

$$
V_i(0,\xi_i) = \frac{1}{2}\langle \Pi_i(0)\xi_i + 2\bar{\eta}_i(0), \xi_i\rangle - \frac{1}{2}\int_0^T \langle\langle \hat{\mathcal{R}}_i(P_i)\bar{\varphi}_i, \bar{\varphi}_i\rangle ds,
$$

with $\bar{\varphi}_i = -\hat{\mathcal{R}}_i(P_i)^{-1}\hat{\mathcal{B}}_i^\intercal \bar{\eta}_i$.

Lemma 5.2.1 *Given two non-empty sets X, Y and $f : X \times Y \to \mathbb{R}$, we have*

$$
\inf_{x,y} f(x,y) = \inf_x(\inf_y f(x,y)).
$$

Proof.

$$
\forall (x,y) \in X \times Y : f(x,y) \geq \inf_y f(x,y) \geq \inf_x(\inf_y f(x,y)).
$$

Hence, $\inf_{x,y} f(x,y) \geq \inf_x(\inf_y f(x,y))$. On the other hand,

$$
\forall x \in X : \inf_y f(x,y) \geq \inf_{x,y} f(x,y).
$$

Therefore, $\inf_x(\inf_y f(x,y)) \geq \inf_{x,y} f(x,y)$. □

However as the terminal cost function in [59] does not explicitly depend on the initial value of the state process, (4.31) can be broken down into two parts by Lemma 5.2.1:

$$
\begin{aligned}
&\inf_{\pi} J_1(\pi) \\
&\leq V_2(x_1) \\
&= -\inf_{\chi,\alpha,\beta} \mathbb{E}\left[\int_0^T \tilde{f}_2(\alpha,\beta,\mathbb{E}[\alpha],\mathbb{E}[\beta])dt + \tilde{g}_2^*(X_2(T),\mathbb{E}[X_2(T)],\chi)\right] \\
&= -\inf_{\chi}\{\inf_{\alpha,\beta} \mathbb{E}\left[\int_0^T \tilde{f}_2(\alpha,\beta,\mathbb{E}[\alpha],\mathbb{E}[\beta])dt + \tilde{g}_2(X_2(T),\mathbb{E}[X_2(T)])\right] + x_1^\intercal \chi\} \\
&\quad - \frac{1}{2}\langle \hat{g}_{T2}, \hat{G}_{T2}^{-1}\hat{g}_{T2}\rangle \\
&= -\inf_{\chi}\{V_2(0,\chi) + x_1^\intercal \chi\} - \frac{1}{2}\langle \hat{g}_{T2}, \hat{G}_{T2}^{-1}\hat{g}_{T2}\rangle
\end{aligned}
$$

Hence, having no duality gap is equivalent to having

$$
\begin{aligned}
&- V_1(0,x_1) \\
&= -\frac{1}{2}\langle \Pi_1(0)x_1 + 2\bar{\eta}_1(0), x_1\rangle + \frac{1}{2}\int_0^T \langle\langle \hat{\mathcal{R}}_1(P_1)\bar{\varphi}_1, \bar{\varphi}_1\rangle ds \\
&= \inf_{\chi}\{x_1^\intercal \chi + V_2(0,\chi)\} + \frac{1}{2}\langle \hat{g}_{T2}, \hat{G}_{T2}^{-1}\hat{g}_{T2}\rangle \\
&= \inf_{\chi}\{x_1^\intercal \chi + \frac{1}{2}\langle \Pi_2(0)\chi + 2\bar{\eta}_2(0), \chi\rangle - \frac{1}{2}\int_0^T \langle\langle \hat{\mathcal{R}}_2(P_2)\bar{\varphi}_2, \bar{\varphi}_2\rangle ds\} + \frac{1}{2}\langle \hat{g}_{T2}, \hat{G}_{T2}^{-1}\hat{g}_{T2}\rangle \\
&= -\frac{1}{2}\langle \Pi_2^{-1}(0)(x_1 + 2\bar{\eta}_2(0)), x_1\rangle - \frac{1}{2}\int_0^T \langle \hat{\mathcal{R}}_2(P_2)\bar{\varphi}_2, \bar{\varphi}_2\rangle ds \\
&\quad - \frac{1}{2}\langle \Pi_2^{-1}(0)\bar{\eta}_2(0), \bar{\eta}_2(0)\rangle + \frac{1}{2}\langle \hat{g}_{T2}, \hat{G}_{T2}^{-1}\hat{g}_{T2}\rangle. \qquad (5.2)
\end{aligned}
$$

From Theorem 3.4.6 in [59], the optimality is obtained if and only if

$$
u_i = \Theta_i(X_i - \mathbb{E}[X_i]) + \hat{\Theta}_i\mathbb{E}[X_i] + \varphi_i - \mathbb{E}[\varphi]_i + \bar{\varphi}_i = \Theta_i(X_i - \mathbb{E}[X_i]) + \hat{\Theta}_i\mathbb{E}[X_i] + \bar{\varphi}_i.
$$

The equation above is true if and only if the following holds:

$$\begin{cases} \tilde{u}_i = \Theta_i \tilde{X}_i \\ \bar{u}_i = \hat{\Theta}_i \bar{X}_i + \bar{\varphi}_i. \end{cases} \tag{5.3}$$

Example 5.2.2 *Solve the problem as stated in Example 4.3.9 in the last section using the Riccati equations from primal problem then show that the value functions in the primal problem and the dual problem match. Furthermore, the optimality conditions in the primal and dual problems coincide.*

Proof. Without running cost, the primal Riccati equations become

$$\begin{cases} \quad \dot{P}_1 + 2A_1P_1 + C_1^2P_1 - (B_1 + C_1D_1)^2D_1^{-2}P_1 \\ = \dot{P}_1 - (2A_2 + C_2^2)P_1 = 0 \\ P_1(T) = G_{T1} \end{cases} \tag{5.4}$$

$$\begin{cases} \quad \dot{\Pi}_1 + 2\hat{A}_1\Pi_1 + \hat{C}_1^2P_1 - (\hat{B}_1\Pi_1 + \hat{C}_1\hat{D}_1P_1)^2\hat{D}_1^{-2}P_1^{-1} \\ = \dot{\Pi}_1 - 2\hat{A}_2\Pi - \hat{C}_2^2P_1^{-1}\Pi_1^2 = 0 \\ \Pi_1(T) = \hat{G}_{T1}. \end{cases} \tag{5.5}$$

$$\begin{cases} \dot{\bar{\eta}}_1(s) + (\hat{A}_1 + \hat{B}_1\hat{\Theta}_1)^{\intercal}\bar{\eta}_1 = 0 \\ \bar{\eta}_1(T) = \hat{g}_{T1}. \end{cases} \tag{5.6}$$

(5.4) gives

$$P_1(t) = G_{T1}e^{-\int_t^T 2A_2 + C_2^2 ds} \tag{5.7}$$

.

Let $\Gamma = \Pi_1^{-1}$, so $\dot{\Gamma} = -\Pi_1^{-2}\dot{\Pi}_1$ and (5.5) can be rewritten as

$$\begin{cases} \dot{\Gamma} + 2\hat{A}_2\Gamma + G_{T1}^{-1}\hat{C}_2^2 e^{\int_t^T 2A_2 + C_2^2 ds} = 0 \\ \Gamma(T) = \hat{G}_{T1}^{-1}, \end{cases}$$

which gives the solution $\Gamma(t) = e^{\int_t^T 2\hat{A}_2 ds}(\hat{G}_{T1}^{-1} + G_{T1}^{-1}\int_t^T \hat{C}_2^2 e^{\int_s^T -2\bar{A}_2 + C_2^2 dr} ds)$ and so

$$\Pi_1(t) = e^{-\int_t^T 2\hat{A}_2 ds}(\hat{G}_{T1}^{-1} + G_{T1}^{-1}\int_t^T \hat{C}_2^2 e^{\int_s^T -2\bar{A}_2 + C_2^2 dr} ds)^{-1}. \tag{5.8}$$

$\hat{\Theta}_1 = -\hat{\mathcal{R}}_1^{-1}(P_1)\hat{\mathcal{S}}_1(P_1, \Pi_1) = -\hat{D}_1^{-2}P_1^{-1}(\hat{B}_1\Pi_1 + \hat{C}_1\hat{D}_1P_1)$, so from (5.6),

$$\bar{\eta}_1 = \hat{g}_{T1}e^{\int_t^T \hat{A}_1 + \hat{B}_1\hat{\Theta}_1 ds} = \hat{g}_{T1}e^{\int_t^T \hat{A}_1 + \hat{B}_1(\hat{C}_2\Pi_1 P_1^{-1} + \hat{B}_2)ds} = \hat{g}_{T1}e^{-\int_t^T \hat{A}_2 + \hat{C}_2^2\Pi_1 P_1^{-1} ds} \tag{5.9}$$

Next, to solve the dual problem,

$$\begin{cases} \dot{P}_2 + 2A_2P_2 + C_2^2P_2 = 0 \\ P_2(T) = G_{T2} \\ \dot{\Pi}_2 + 2\hat{A}_2\Pi_2 + \hat{C}_2^2P_2 = 0 \\ \Pi_2(T) = \hat{G}_{T2} \\ \dot{\bar{\eta}}_2(s) + \hat{A}_2\bar{\eta}_2 = 0 \\ \bar{\eta}_2(T) = \hat{g}_{T2}. \end{cases} \tag{5.10}$$

Solving the system of linear ODEs in (5.10) gives

$$\begin{cases} P_2(t) & = G_{T2}e^{\int_t^T 2A_2+C_2^2dr} \\ & = G_{T1}^{-1}e^{\int_t^T 2A_2+C_2^2dr} \\ & = P_1^{-1}(t) \text{ by comparing with (5.7)} \\ \Pi_2(t) & = e^{\int_t^T 2\hat{A}_2dr}(\hat{G}_{T2} + G_{T2}\int_t^T \hat{C}_2^2 e^{\int_s^T -2\bar{A}_2+C_2^2dr}ds) \\ & = e^{\int_t^T 2\hat{A}_2dr}(\hat{G}_{T1}^{-1} + G_{T1}^{-1}\int_t^T \hat{C}_2^2 e^{\int_s^T -2\bar{A}_2+C_2^2dr}ds) \\ & = \Pi_1^{-1}(t) \text{ by comparing with (5.8)} \\ \bar{\eta}_2(t) & = \hat{g}_{T2}e^{\int_t^T \hat{A}_2ds}. \end{cases} \tag{5.11}$$

Therefore, the quadratic terms on x_1 in the primal and dual value functions (5.2) match. Let $F(t) := \int_t^T \hat{C}_2^2 e^{\int_s^T -2\bar{A}_2+C_2^2dr}ds$, so $F'(t) = -\hat{C}_2^2 e^{\int_t^T -2\bar{A}_2+C_2^2dr}$then

$$\begin{aligned} \hat{C}_2^2\Pi_1(t)P_1^{-1}(t) &= \hat{C}_2^2 P_2(t)\Pi_2^{-1}(t) \\ &= \hat{C}_2^2 G_{T1}^{-1} e^{\int_t^T 2A_2+C_2^2dr} e^{-\int_t^T 2\hat{A}_2dr}(\hat{G}_{T2} + G_{T2}\int_t^T \hat{C}_2^2 e^{\int_s^T -2\bar{A}_2+C_2^2dr}ds)^{-1} \\ &= -F'(t)(\hat{G}_{T2}G_{T2}^{-1} + F(t))^{-1} \end{aligned}$$

Together with (5.9),

$$\begin{aligned} &\frac{\Pi_2(t)\eta_1(t)}{\eta_2(t)} \\ &= e^{\int_t^T 2\hat{A}_2dr}(\hat{G}_{T2} + G_{T2}\int_t^T \hat{C}_2^2 e^{\int_s^T -2\bar{A}_2+C_2^2dr}ds)\hat{g}_{T1}e^{-\int_t^T \hat{A}_2+\hat{C}_2^2\Pi_1P_1^{-1}dr}\hat{g}_{T2}^{-1}e^{-\int_t^T \hat{A}_2ds} \\ &= e^{-\int_t^T \hat{C}_2^2\Pi_1P_1^{-1}ds}(1 + G_{T2}\hat{G}_{T2}^{-1}\int_t^T \hat{C}_2^2 e^{\int_s^T -2\bar{A}_2+C_2^2dr}ds) \\ &= e^{\int_t^T -F'(s)(\hat{G}_{T2}G_{T2}^{-1}+F(s))^{-1}ds}(1 + G_{T2}\hat{G}_{T2}^{-1}F(t)) \\ &= e^{\left[\log\left(\hat{G}_{T2}G_{T2}^{-1}+F(s)\right)\right]_t^T}(1 + G_{T2}\hat{G}_{T2}^{-1}F(t)) \\ &= 1. \end{aligned}$$

Hence the linear terms on x_1 in (5.2) match.

Lastly, consider

$$\begin{aligned}
\frac{d(\Pi_2^{-1}(t)\eta_2^2(t))}{dt} &= -\Pi_2^{-2}\dot{\Pi}_2\eta_2^2 + 2\Pi_2^{-1}\eta_2\dot{\eta}_2 \\
&= -\Pi_2^{-2}(-2\hat{A}_2\Pi_2 - \hat{C}_2^2 P_2)\eta_2^2 + 2\Pi_2^{-1}\eta_2(-2\hat{A}_2\eta_2) \\
&= \hat{C}_2^2 P_2(\Pi_2^{-1}\eta_2)^2 \\
&= \hat{C}_2^2 P_1^{-1}\eta_1^2 \\
&= \hat{\mathcal{R}}_1(P_1)\bar{\varphi}_1^2,
\end{aligned}$$

and note that $\Pi_2^{-1}(T)\eta_2^2(T) = G_{T2}^{-1}\hat{g}_{T2}^2$, we can conclude the constant terms in (5.2) match. Hence, we have shown that (5.2) holds.

Next to show optimality of the dual problem, note that as the dual problem is independent of controls, there is no partial derivative with respect to controls. So the SMP equation (4.21) becomes meaningless. We will need to prove the optimality of the primary problem by (5.3). Since $\Theta_1 = -\mathcal{R}_1^{-1}(P_1)\mathcal{S}_1(P_1) = -D_1^{-2}P_1^{-1}(B_1P_1 + C_1D_1P_1) = -(D_1^{-2}B_1 + D_1^{-1}C_1)$, from (4.29) the RHS of the first line of (5.3) is equivalent to

$$\begin{aligned}
-D_1^{-1}\tilde{Z}_2 + D_1^{-1}C_1\tilde{Y}_2 &= (D_1^{-2}B + D_1^{-1}C_1)\tilde{Y}_2 \\
\tilde{Z}_2 &= -D_1^{-1}B\tilde{Y}_2 = C_2\tilde{Y}_2.
\end{aligned} \tag{5.12}$$

$\hat{\Theta}_1 = -\hat{\mathcal{R}}_1^{-1}(P_1)\hat{\mathcal{S}}_1(P_1, \Pi_1) = -\hat{D}_1^{-2}P_1^{-1}(\hat{B}_1\Pi_1 + \hat{C}_1\hat{D}_1P_1)$. So the second line of (5.3) is equivalent to

$$\begin{aligned}
-\hat{D}_1^{-1}\bar{Z}_2 + \frac{1}{\mu+1}\hat{D}_1^{-1}\hat{C}\bar{Y}_2 &= \hat{D}_1^{-2}P_1^{-1}(\hat{B}_1\Pi_1 + \hat{C}_1\hat{D}_1P_1)\frac{1}{\mu+1}\bar{Y}_2 \\
\bar{Z}_2 &= -\frac{1}{\mu+1}\hat{D}_1^{-1}\hat{B}_1P_1^{-1}\Pi_1\bar{Y}_2.
\end{aligned} \tag{5.13}$$

From assumption of Riccati equation for dual problem, $\tilde{Y}_2 = P_2\tilde{X}_2$ and $\bar{Y}_2 = \Pi_2\bar{X}_2$, by Malliavin derivative we have

$$Z_2 = P_2(C_2X_2 + \bar{C}_2\bar{X}_2)$$

as such $\tilde{Z}_2 = P_2C_2\tilde{X}_2 = C_2\tilde{Y}_2$, which is indeed (5.12), and

$$\begin{aligned}
\bar{Z}_2 &= P_2\hat{C}_2\bar{X}_2 \\
&= \hat{C}_2P_2\Pi_2^{-1}\bar{Y}_2 \\
(5.10) &= \hat{C}_2\Pi_1P_1^{-1}\bar{Y}_2 \\
(4.8) &= -\hat{D}_1^{-1}\hat{B}_1P_1^{-1}\Pi_1\bar{Y}_2,
\end{aligned}$$

which is exactly (5.13). As such, the optimal solution of the dual problem is equivalent to the optimality of the primal problem. □

5.3 Solution to Linear Mean-Field BSDEs

In the first part of the section, steps in [3] to find solution to Linear Mean-Field BSDEs will be replicated and all the notations used are from the paper where $B(t)$ denotes the standard Brownian Motion. The second part is to dispute the proof with the examples used in the previous chapters and give a correct expression instead.

Theorem 5.3.1 (Page 6 Section 3.2 of [3]) *Consider a linear mean-field BSDE of the form*

$$\begin{cases} dY(t) &= -[\alpha_1(t)Y(t) + \beta_1(t)Z(t) + \alpha_2(t)\mathbb{E}[Y(t)] + \beta_2(t)\mathbb{E}[Z(t)]]dt + Z(t)dB(t), \\ Y(T) &= \xi, \end{cases}$$

$t \in [0,T]$, *where* $\alpha_1(t), \alpha_2(t), \beta_1(t).\beta_2(t)$ *are given deterministic functions,* $\xi \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R})$ *is a given* $\mathcal{F}_T$*-measurable random variable.*

It can be rewritten as

$$Y(t) = \mathbb{E}\left[\xi\Gamma(t,T) + \int_t^T \Gamma(t,s)\{\alpha_2(s)\mathbb{E}[Y(s)] + \beta_2(s)\mathbb{E}[Z(s)]\}ds|\mathcal{F}_t\right], t \in [0,T], \tag{5.14}$$

where $\Gamma(t,s)$ *is the solution of the following linear SDE*

$$\begin{cases} d\Gamma(t,s) & = \Gamma(t,s)\left[\alpha_1(t)dt + \beta_1 dB(t)\right], s \in [t,T], \\ \Gamma(t,t) & = 1. \end{cases} \tag{5.15}$$

In the case of one-dimensional, $\Gamma(t,s) = e^{\int_t^s \beta_1(r)dB(r) + \int_t^s (\alpha_1 - \frac{1}{2}(\beta_1(r))^2)dr}$ *and* $\mathbb{E}[\Gamma(t,s)] = e^{\int_t^s \alpha_1(r)dr}$.

Denoting $\bar{Y}(t) := \mathbb{E}[Y(t)]$ *and* $\bar{Z}(t) := \mathbb{E}[Z(t)]$, *we have*

$$\bar{Y}(t) = \mathbb{E}\left[\xi\Gamma(t,T) + \int_t^T \Gamma(t,s)\{\alpha_2(s)\bar{Y}(s) + \beta_2(s)\bar{Z}(s)\}ds\right], t \in [0,T]. \tag{5.16}$$

To find $\bar{Z}(t)$, *first note that* $Y(t)$ *can be expressed as a forward SDE:*

$$Y(t) = Y(0) + \int_0^t \left[\alpha_1(s)Y(s) + \alpha_2(s)\bar{Y}(s) + \beta_1(s)Z(s) + \beta_2(s)\bar{Z}(s)\right] ds + \int_0^t Z(s)dB(s),$$

$t \in [0,T]$, *for some deterministic initial value* $Y(0)$.

Remark *Next we will take Malliavin derivatives of the equation. The formal definition of the derivative require rigorous discussion starting from Wiener-Ito chaos expansion. For readers interested in the concept, we recommend reading the first 3 sections of Chapter 1 in [49]. However, as we are only using the derivative as a tool, we can heuristically treat it as the derivative with respect to the white noise* $dB(t)$.

We can compute the Malliavin derivative of $Y(t)$ *for all* $r < t$, *using the following properties (From Example 2.2 on [3]):*

Lemma 5.3.2 *Let* D_r *denotes the Malliavin Derivative at time* r. *Then for* $\phi \in$

$C^1(\mathbb{R})$, $\mathbb{R}$-valued differentiable functions, $F \in L^2_{\mathcal{F}_t}(\Omega, \mathbb{R})$, $\mathcal{F}_t$-adapted $\mathbb{R}^n$-valued square integrable random variable, $u \in L^2_{\mathbb{F}}(0, T; \mathbb{R})$, $0 \leq r < t < s$,

$$
\begin{aligned}
D_t(\phi(F)) &= \phi'(F) D_t F \\
D_s F &= 0 \\
D_r(\int_0^T u(t)dt) &= \int_r^T D_r u(t)dt \\
D_r(\int_0^T u(t)dB(t)) &= \int_r^T D_r u(t)dB(t) + u(r),
\end{aligned}
$$

Proof. For example, refer to [49]. □

So we have for all $r < t$

$$
D_r Y(t) = \int_r^t D_r \left[\alpha_1(s)Y(s) + \alpha_2(s)\bar{Y}(s) + \beta_1 Z(s) + \beta_2 \bar{Z}(s)\right] ds + \int_r^t D_r Z(s)dB(s) + Z(r).
$$

Letting $r \to t^-$, we get $Z(t) = D_t Y(t)$. Thus to find $Z(t)$ we only need to compute $D_t Y(t)$. Using the expression (5.14),the identity

$$
D_t \mathbb{E}[F|\mathcal{F}_t] = \mathbb{E}[D_t F|\mathcal{F}_t], \tag{5.17}
$$

as well as the fact that $D_t \Gamma(t, T) = \Gamma(t, T)\beta_1(t)$,

$$
Z(t) = \mathbb{E}\left[D_t \xi \Gamma(t, T) + \xi \Gamma(t, T)\beta_1(t) + \int_t^T \Gamma(t, s)\beta_1(t)\{\alpha_2(s)\bar{Y}(s) + \beta_2(s)\bar{Z}(s)\}ds|\mathcal{F}_t\right]. \tag{5.18}
$$

Taking expectation, we have

$$
\bar{Z}(t) = \mathbb{E}\left[D_t \xi \Gamma(t, T) + \xi \Gamma(t, T)\beta_1(t) + \int_t^T \Gamma(t, s)\beta_1(t)\{\alpha_2(s)\bar{Y}(s) + \beta_2(s)\bar{Z}(s)\}ds\right]. \tag{5.19}
$$

Remark *However, note that* $\mathbb{E}[\Gamma(t,s)|\mathcal{F}_t] = \mathbb{E}[\Gamma(t,s)] = e^{\int_t^s \alpha_1 dr}$*. Then*

$$\begin{cases} D_t\mathbb{E}[\Gamma(t,s)|\mathcal{F}_t] = D_t(e^{\int_t^s \alpha_1 dr}) = 0 \\ \mathbb{E}[D_t\Gamma(t,s)|\mathcal{F}_t] = \beta_1(t)\mathbb{E}[\Gamma(t,s)|\mathcal{F}_t] = \beta_1(t)e^{\int_t^s \alpha_1 dr}, \end{cases}$$

which contradicts that claim (5.17)*. This means that the identity property used in the theorem above is incorrect. The identity is probably a variation to Prop. 1.2.8 from [49]:*

$$D_t(\mathbb{E}[F|\mathcal{F}_A]) = \mathbb{E}[D_tF|\mathcal{F}_A]\mathbb{1}_A(t)$$

a.e. in $T \times \Omega$ *for any* A *in Borel-sigma-algebra over* $[0,T]$*. The main problem with the identity is that* $\mathbb{E}[\Gamma(t,s)|\mathcal{F}_t]$ *is* $\mathcal{F}_t$ *measurable and the Malliavin derivative* D_t *is also* t*-dependent while* A *in the equation above should be independent of* t*. As such, the conclusion after* (5.17) *should be invalid.*

Instead (5.14) *can be written as*

$$\begin{aligned} Y(t) &= \mathbb{E}\left[\xi\Gamma(t,T)|\mathcal{F}_t\right] + \int_t^T \mathbb{E}[\Gamma(t,s)|\mathcal{F}_t]\{\alpha_2(s)\mathbb{E}[Y(s)] + \beta_2(s)\mathbb{E}[Z(s)]\}ds \\ &= \mathbb{E}\left[\xi\Gamma(t,T)|\mathcal{F}_t\right] + \int_t^T \mathbb{E}[\Gamma(t,s)]\{\alpha_2(s)\mathbb{E}[Y(s)] + \beta_2(s)\mathbb{E}[Z(s)]\}ds \\ &= \mathbb{E}\left[\xi\Gamma(t,T)|\mathcal{F}_t\right] + \int_t^T e^{\int_t^s \alpha_1 ds}\{\alpha_2(s)\mathbb{E}[Y(s)] + \beta_2(s)\mathbb{E}[Z(s)]\}ds \end{aligned}$$

and as such

$$Z(t) = D_tY(t) = D_t\mathbb{E}\left[\xi\Gamma(t,T)|\mathcal{F}_t\right] \tag{5.20}$$

To show (5.20) is true instead of what is given in Theorem 5.3.1, we use the same example as set in Example 4.3.9.

Example 5.3.3 *With the same setting as Example 4.3.9, the results from Theorem 5.3.1 contradicts the result from Example 5.2.2.*

Proof. Using the result from Theorem 5.3.1, by substituting the followings:

$$\begin{cases} Y & = Y_2 \\ Z & = Z_2 \\ \alpha_1 & = A_2 \\ \alpha_2 & = \bar{A}_2 \\ \beta_1 & = C_2 \\ \beta_2 & = \bar{C}_2 \\ \xi & = Y_2(T) = G_{T2}X_2(T) + \bar{G}_{T2}\mathbb{E}[X_2(T)] + \hat{g}_{T2} \\ B(t) & = W(t) \end{cases} \tag{5.21}$$

(5.14) becomes

$$\begin{aligned} Y_2(t) &= \mathbb{E}\left[Y_2(T)\Gamma(t,T) + \int_t^T \Gamma(t,s)(\bar{A}_2\mathbb{E}[Y_2(s)] + \bar{C}_2\mathbb{E}[Z_2(s)])ds|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[Y_2(T)\Gamma(t,T)|\mathcal{F}_t\right] + \int_t^T \mathbb{E}[\Gamma(t,s)|\mathcal{F}_t](\bar{A}_2\mathbb{E}[Y_2(s)] + \bar{C}_2\mathbb{E}[Z_2(s)])ds \\ &= \mathbb{E}\left[Y_2(T)\Gamma(t,T)|\mathcal{F}_t\right] + \int_t^T \mathbb{E}[\Gamma(t,s)](\bar{A}_2\mathbb{E}[Y_2(s)] + \bar{C}_2\mathbb{E}[Z_2(s)])ds \qquad (5.22) \end{aligned}$$

and (5.18) becomes

$$\begin{aligned} & Z_2(t) \\ &= \mathbb{E}\left[D_t(Y_2(T))\Gamma(t,T) + Y_2(T)\Gamma(t,T)C_2(t) + \int_t^T \Gamma(t,s)C_2(t)[\bar{A}_2\mathbb{E}[Y_2(s)] + \bar{C}_2\mathbb{E}[Z_2(s)]]ds|\mathcal{F}_t\right] \\ &= C_2(t)Y_2(t) + \mathbb{E}[D_t(Y_2(T))\Gamma(t,T)|\mathcal{F}_t]. \qquad (5.23) \end{aligned}$$

Also, (5.16) and (5.19) become

$$\begin{cases} \mathbb{E}[Y_2(t)] & = \mathbb{E}\Big[Y_2(T)\Gamma(t,T) + \int_t^T \Gamma(t,s)(\bar{A}_2\mathbb{E}[Y_2(s)] + \bar{C}_2\mathbb{E}[Z_2(s)]ds\Big] \\ \mathbb{E}[Z_2(t)] & = \mathbb{E}\Big[D_t(Y_2(T))\Gamma(t,T) + Y_2(T)\Gamma(t,T)C_2(t) \\ & \quad + \int_t^T \Gamma(t,s)C_2(t)[\bar{A}_2\mathbb{E}[Y_2(s)] + \bar{C}_2\mathbb{E}[Z_2(s)]]ds\Big] \\ & = C_2(t)\mathbb{E}[Y_2(t)] + \mathbb{E}[D_t(Y_2(T))\Gamma(t,T)], \end{cases}$$

Since at the optimality, $Y_2(T) = G_{T2}X_2(T) + \bar{G}_{T2}\mathbb{E}[X_2(T)] + \hat{g}_{T2}$, and SDE for $X_2(t)$ is known, $D_t(Y_2(T))$ can be represented by a function of $D_t(X_2(T))$:

$$D_t(Y_2(T)) = D_t(G_{T2}X_2(T) + \bar{G}_{T2}\mathbb{E}[X_2(T)] + \hat{g}_{T2}) = G_{T2}D_tX_2(T)$$

Apply Malliavin's derivative to $X_2(s)$ where $s > t$, we have

$$\begin{aligned} D_tX_2(s) &= D_t(X_2(0) + \int_0^s dX_2(r)) \\ \text{for } t > 0 &= D_t(\int_0^s (A_2X_2 + \bar{A}_2\mathbb{E}[X_2])dr + \int_0^s (C_2X_2 + \bar{C}_2\mathbb{E}[X_2])dW(r)) \\ &= \int_t^s A_2D_tX_2dr + \int_t^s C_2D_tX_2dW(r) + (C_2(t)X_2(t) + \bar{C}_2\mathbb{E}[X_2(t)]). \end{aligned}$$

So

$$\begin{cases} d(D_tX_2(s)) & = A_2(s)D_tX_2(s)ds + C_2(s)D_tX_2(s)dW(s) \\ D_tX_2(t^+) & = C_2(t^+)X_2(t^+) + \bar{C}_2(t^+)\mathbb{E}[X_2(t^+)] \end{cases}$$

By uniqueness of strong solution to SDE and continuity,

$$D_tX_2(s) = \Gamma(t,s)(C_2(t)X_2(t) + \bar{C}_2(t)\mathbb{E}[X_2(t)]), \tag{5.24}$$

and so

$$D_tY_2(T) = G_{T2}\Gamma(t,T)(C_2(t)Y(t) + \bar{C}_2(t)\mathbb{E}[Y(t)]).$$

To verify the result of (5.24), from (4.32) and (4.33), we can compute $D_tY(T)$

explicitly:

$$
\begin{aligned}
& D_t X_2(T) \\
&= D_t\{\chi e^{\int_0^T (A_2-\frac{C_2^2}{2})ds+\int_0^T C_2 dW(s)}\}\Big[1+\int_0^T [e^{-\int_0^s (A_2-\frac{C_2^2}{2})dr-\int_0^s C_2 dW(r)}(\bar{A}_2 - C_2\bar{C}_2)e^{\int_0^s \hat{A}_2 dr}]ds \\
&\quad + \int_0^T [e^{-\int_0^s (A_2-\frac{C_2^2}{2})dr-\int_0^s C_2 dW(r)}\bar{C}_2 e^{\int_0^s \hat{A}_2 dr}]dW(s)\Big] \\
&\quad + \chi e^{\int_0^T (A_2-\frac{C_2^2}{2})ds+\int_0^t C_2 dW(s)} D_t\{\int_0^T [e^{-\int_0^s (A_2-\frac{C_2^2}{2})dr-\int_0^s C_2 dW(r)}(\bar{A}_2 - C_2\bar{C}_2)e^{\int_0^s \hat{A}_2 dr}]ds\} \\
&\quad + \chi e^{\int_0^T (A_2-\frac{C_2^2}{2})ds+\int_0^t C_2 dW(s)} D_t\{\int_0^T [e^{-\int_0^s (A_2-\frac{C_2^2}{2})dr-\int_0^s C_2 dW(r)}\bar{C}_2 e^{\int_0^s \hat{A}_2 dr}]dW(s)\} \\
&= C_2(t)X_2(T) + \chi e^{\int_0^T (A_2-\frac{C_2^2}{2})ds+\int_0^t C_2 dW(s)}(-C_2(t))\int_t^T [e^{-\int_0^s (A_2-\frac{C_2^2}{2})dr-\int_0^s C_2 dW(r)}\bar{C}_2 e^{\int_0^s \hat{A}_2 dr}]dW(s) \\
&\quad + \chi e^{\int_0^T (A_2-\frac{C_2^2}{2})ds+\int_0^t C_2 dW(s)}(-C_2(t))\int_t^T [e^{-\int_0^s (A_2-\frac{C_2^2}{2})dr-\int_0^s C_2 dW(r)}(\bar{A}_2 - C_2\bar{C}_2)e^{\int_0^s \hat{A}_2 dr}]ds \\
&\quad + \chi e^{\int_0^T (A_2-\frac{C_2^2}{2})ds+\int_0^t C_2 dW(s)} e^{-\int_0^t (A_2-\frac{C_2^2}{2})dr-\int_0^t C_2 dW(r)}\bar{C}_2(t)e^{\int_0^t \hat{A}_2 dr} \\
&= C_2(t)X_2(T) + \bar{C}_2(t)\mathbb{E}[X_2(t)]e^{\int_t^T (A_2+\frac{C_2^2}{2})ds-\int_0^t C_2 dW(s)} \\
&\quad - C_2(t)\chi e^{\int_0^T (A_2-\frac{C_2^2}{2})ds+\int_0^t C_2 dW(s)} X_2(T)\chi^{-1}e^{-\int_0^T (A_2-\frac{C_2^2}{2})ds-\int_0^T C_2 dW(s)} \\
&\quad + C_2(t)\chi e^{\int_0^T (A_2-\frac{C_2^2}{2})ds+\int_0^t C_2 dW(s)} X_2(t)\chi^{-1}e^{-\int_0^t (A_2-\frac{C_2^2}{2})ds-\int_0^t C_2 dW(s)} \\
&= C_2(t)X_2(T) + \bar{C}_2(t)\mathbb{E}[X_2(t)]e^{\int_t^T (A_2+\frac{C_2^2}{2})ds-\int_0^t C_2 dW(s)} - C_2(t)X_2(T) \\
&\quad + C_2(t)X_2(t)e^{\int_t^T (A_2+\frac{C_2^2}{2})ds-\int_0^t C_2 dW(s)} \\
&= (C_2(t)X_2(t) + \bar{C}_2(t)\mathbb{E}[X_2(t)])\Gamma(t,T),
\end{aligned}
$$

which is indeed match (5.24).

Next to compute

$$
\mathbb{E}[D_t(Y_2(T))\Gamma(t,T)|\mathcal{F}_t] = G_{T2}C_2(t)\mathbb{E}[\Gamma^2(t,T)X_2(t)|\mathcal{F}_t] + G_{T2}\bar{C}_2(t)\mathbb{E}[\Gamma^2(t,T)|\mathcal{F}_t]\mathbb{E}[X_2(t)].
$$

Since $\mathbb{E}[\Gamma^2(t,T)|\mathcal{F}_t] = \mathbb{E}[\Gamma^2(t,T)] = e^{\int_t^T 2A_2 + C_2^2 dr}$, the only term that remains to be found in $\mathbb{E}[D_t(Y_2(T))\Gamma(t,T)|\mathcal{F}_t]$ is $\mathbb{E}[X_2(t)\Gamma^2(t,T)|\mathcal{F}_t]$.

$$\begin{aligned}\mathbb{E}[X_2(t)\Gamma^2(t,T)|\mathcal{F}_t] &= X_2(t)\mathbb{E}[\Gamma^2(t,T)|\mathcal{F}_t]\\ &= X_2(t)e^{\int_t^T 2A_2+C_2^2dr},\end{aligned} \tag{5.25}$$

where $X_2(t)$ can be expressed as in (4.32). As such

$$\mathbb{E}[D_t(Y_2(T))\Gamma(t,T)|\mathcal{F}_t] = G_{T2}C_2(t)e^{\int_t^T 2A_2+C_2^2dr}X_2(t) + G_{T2}\bar{C}_2(t)e^{\int_t^T 2A_2+C_2^2dr}\mathbb{E}[X_2(t)] \tag{5.26}$$

$$G_{T2}e^{\int_t^T 2A_2+C_2^2dr}(C_2(t)X_2(t) + \bar{C}_2(t)\mathbb{E}[X_2(t)]).$$

From (5.20) and (5.11),

$$\begin{aligned}Z_2(t) &= G_{T2}e^{\int_t^T 2A_2+C_2^2dr}(C_2(t)Y(t) + \bar{C}_2(t)\mathbb{E}[Y(t)])\\ &= P_2(t)(C_2(t)Y(t) + \bar{C}_2(t)\mathbb{E}[Y(t)]),\end{aligned}$$

which coincides with (5.14). This shows that (5.20) agrees with the results from [59]. □

Notice that $\mathbb{E}\left[\xi\Gamma(t,T)|\mathcal{F}_t\right]$ from (5.20) also seems to be in the form of a solution to the state process in a linear non-Mean-Field BSDE.

Theorem 5.3.4 *With the same linear mean-field BSDE as above, consider another normal BSDE pair $(\check{Y},\check{Z})$ that is the solution to*

$$\begin{cases} d\check{Y}(t) &= -[\alpha_1(t)\check{Y}(t) + \beta_1(t)\check{Z}(t)]dt + \check{Z}(t)dB(t), t \in [0,T],\\ \check{Y}(T) &= \xi, \end{cases} \tag{5.27}$$

then $Z = \check{Z}$.

Proof. To solve (5.27), consider the same multiplying factor $\Gamma(t,s)$ as in (5.15).

Then by Ito's Lemma,

$$
\begin{aligned}
& d(\check{Y}(s)\Gamma(t,s)) \\
&= (d\check{Y}(s))\Gamma(t,s) + \check{Y}(s)(d\Gamma(t,s)) + d\langle \check{Y}, \Gamma(t,.)\rangle_s \\
&= \{-[\alpha_1(s)\check{Y}(s) + \beta_1(s)\check{Z}(s)]\Gamma(t,s) + \check{Y}(s)\Gamma(t,s)\alpha_1(s) + \check{Z}(s)\Gamma(t,s)\beta_1(s)\}ds + E_3 dB(t) \\
&= E_3 dB(t)
\end{aligned}
$$

for some $E_3 \in L^2_{\mathbb{F}}(t,T;\mathbb{R})$. Therefore,

$$
\check{Y}(t) = \Gamma(t,t)^{-1}\mathbb{E}[\check{Y}(T)\Gamma(t,T)|\mathcal{F}_t] = \mathbb{E}[\xi\Gamma(t,T)|\mathcal{F}_t].
$$

Similarly as in the approach of Theorem 5.3.1, express $\check{Y}$ in the form of forward SDE:

$$
\check{Y}(t) = \check{Y}(0) + \int_0^t [\alpha_1(s)\check{Y}(s) + \beta_1(s)\check{Z}(s)]ds + \int_0^t \check{Z}(s)dB(s), \; t \in [0,T],
$$

for some deterministic initial value $\check{Y}(0)$. Applying the Malliavin derivative with $r < t$, we have

$$
D_r\check{Y}(t) = \int_r^t D_r[\alpha_1(s)\check{Y}(s) + \beta_1(s)\check{Z}(s)]ds + \int_r^t D_r\check{Z}(s)dB(s) + \check{Z}(r).
$$

Letting $r \to t^-$, we have $\check{Z}(t) = D_t\check{Y}(t)$ which means

$$
\check{Z}(t) = D_t\check{Y}(t) = D_t\mathbb{E}[\xi\Gamma(t,T)|\mathcal{F}_t] = Z(t). \tag{5.28}
$$

□

5.4 Machine Learning and solution to Mean-Field FBSDEs

Divide the time interval $[0,T]$ evenly into $0 = t_0 < t_1 < ... < t_N = T$ where $\Delta t_i = t_{i+1} - t_i = \frac{T}{n}$ for $0 \le i \le N-1$. Let $\Delta W_i = W(t_{i+1}) - W(t_i)$ for $0 \le i \le N-1$, then ΔW_i has the same distribution as $\sqrt{\Delta t_i} n_i$ where $n_i \sim N(0,1)$, the standard normal distribution. Denote $x_i = X(t_i), y_i = Y(t_i)$ and $z_i = Z(t_i)$ for $0 \le i \le N$.

5.4.1 Forward SDE

Suppose the SDE equation follows

$$\begin{cases} dX(t) = b(t, X(t))dt + \sigma(t, X(t))dW(t), \\ X(0) = x_0. \end{cases} \tag{5.29}$$

Theorem 5.4.1 (Euler-Maruyama) *The SDE can be disretised into*

$$x_{i+1} = x_i + b(t_i, x_i)\Delta t_i + \sigma(t_i, x_i)\Delta W_i.$$

The SDE can be realised by continuously simulating ΔW_i in each time step.

5.4.2 Mean-Field Forward SDE

Suppose the SDE now follows

$$\begin{cases} dX(t) = b(t, X(t), \mathbb{E}[X(t)])dt + \sigma(t, X(t), \mathbb{E}[X(t)])dW(t), \\ X(0) = x_0. \end{cases}$$

Similar methods as in Subsection 5.4.1 can still be adopted.

Theorem 5.4.2 (Euler-Maruyama) *The SDE can be disretised into*

$$\begin{cases} \bar{x}_0 = x_0 \\ x_{i+1} = x_i + b(t_i, x_i, \bar{x}_i)\Delta t_i + \sigma(t_i, x_i, \bar{x}_i)\Delta W_i \\ \bar{x}_{i+1} = \mathbb{E}[x_{i+1}] = \bar{x}_i + \mathbb{E}[b(t_i, x_i, \bar{x}_i)]\Delta t_i, \end{cases}$$

where $\mathbb{E}[b(t_i, x_i, \bar{x}_i)]$ *can be approximated by Monte Carlo simulation when* b *is complicated.*

5.4.3 Backward SDE

Suppose the BSDE equation follows

$$\begin{cases} dY(t) = -f(t, Y(t), Z(t))dt + ZdW(t) \\ Y(T) = g(X(T)) \end{cases}$$

where $X(t)$ is the state process following the same SDE as in (5.29).

Theorem 5.4.3 (Forward discretisation of BSDE) *Suppose that the BSDE has a unique solution; then we can assume* $Y(t) = u(t, X(t))$ *for some function* u *with* $u(T, \cdot) = g(\cdot)$. *Then either by Ito's Lemma or taking the Malliavin's derivative,* $Z(t) = \frac{\partial u}{\partial x}(t, X(t)) \cdot \sigma(t, X(t))$.
As such the discretised BSDE is

$$\begin{cases} y_0 = u(t_0, x_0) \\ z_i = \frac{\partial u}{\partial x}(t_i, x_i) \cdot \sigma(t_i, x_i) \\ y_{i+1} = y_i - f(t_i, y_i, z_i)\Delta t_i + z_i \Delta W_i. \end{cases}$$

The selection of u *is trained by minimising the function* $\mathbb{E}[\left|Y(T) - g(X(T))\right|^2]$

5.4.4 Fully coupled FBSDE

Suppose that the state process X and the adjoint processes (Y, Z) follow the following equations:

$$\begin{cases} dX(t) = b(t, X(t), Y(t), Z(t))dt + \sigma(t, X(t), Y(t), Z(t))dW(t) \\ dY(t) = -f(t, X(t), Y(t), Z(t))dt + Z(t)dW(t) \\ X(0) = x_0 \\ Y(T) = g(X(T)). \end{cases}$$

Theorem 5.4.4 (Discretisation of fully coupled FBSDE) *Note that the diffusion term in the state process σ now depends on (Y, Z). Z appears as both input and output, which is not desirable in deep learning. Therefore, we introduce a parallel network that approximates $Z(t) = v(X(t))$. The discretisation then becomes:*

$$\begin{cases} u_i = u(x_i) \\ z_i = v(x_i) \\ x_{i+1} = x_i + b(t_i, x_i, u_i, z_i)\Delta t_i + \sigma(t_i, x_i, u_i, z_i)\Delta W_i \\ y_{i+1} = y_i - f(t_i, x_i, y_i, z_i)\Delta t_i + z_i \Delta W_i, \end{cases}$$

with minimizing function

$$\mathbb{E}[\big|g(X(T)) - Y(T)\big|^2 + \int_0^T \big|Y(t) - u_t\big|^2 dt].$$

5.4.5 Fully coupled Mean-Field FBSDE with Diffusion Dependent on the State Variable only

First when the diffusion term of the state process does not depend on Z:

$$\begin{cases} dX(t) = b(t, X(t), \mathbb{E}[X(t)], Y(t), \mathbb{E}[Y(t)], Z(t), \mathbb{E}[Z(t)])dt + \sigma(t, X(t), \mathbb{E}[X(t)])dW(t) \\ dY(t) = -f(t, X(t), \mathbb{E}[X(t)], Y(t), \mathbb{E}[Y(t)], Z(t), \mathbb{E}[Z(t)])dt + Z(t)dW(t) \\ X(0) = x_0 \\ Y(T) = g(X(T), \mathbb{E}[X(T)]). \end{cases}$$

Again, assuming the uniqueness of the system, we can approximate $Y(t) = u(t, X(t), \mathbb{E}[X(t)])$ and $Z(t) = v(t, X(t), \mathbb{E}[X(t)]) = \frac{\partial u}{\partial x}(t, X(t), \mathbb{E}[X(t)]) \cdot \sigma(t, X(t), \mathbb{E}[X(t)])$. Note that since $\mathbb{E}[X(t)]$ would be a deterministic function on t when $X(t)$ is known, the problem can be discretised using the following method:

$$\begin{cases} \bar{x}_i, \bar{y}_i, \bar{z}_i = \mathbb{E}[x_i], \mathbb{E}[y_i], \mathbb{E}[z_i] \\ y_0 = \tilde{u}(t_0, x_0) \\ z_i = \frac{\partial \tilde{u}}{\partial x}(t_i, x_i) \cdot \sigma(t_i, x_i) \\ x_{i+1} = x_i + b(t_i, x_i, y_i, z_i, \bar{x}_i, \bar{y}_i, \bar{z}_i)\Delta t_{i+1} + \sigma(t_i, x_i, \bar{x}_i)\Delta W_{i+1} \\ y_{i+1} = y_i - f(t_i, x_i, y_i, z_i, \bar{x}_i, \bar{y}_i, \bar{z}_i)\Delta t_{i+1} + z_i \Delta W_{i+1}, \end{cases}$$

with minimizing function

$$\mathbb{E}\left[\left|Y(T) - g(X(T), \mathbb{E}[X(T)]\right|^2\right].$$

5.4.6 Fully Coupled Mean-Field Linear Quadratic FBSDEs

With the same settings of coefficients as Section 5.2, we can write both the primal and dual problems in Section 4.2 into

$$
\begin{cases}
dX_i(t) &= (A_i\tilde{X}_i + \hat{A}_i\bar{X}_i + \mathcal{B}_i\tilde{u}_i + \hat{\mathcal{B}}_i\bar{u}_i)dt \\
&\qquad + (C_i\tilde{X}_i + \hat{C}_i\bar{X}_i + \mathcal{D}_i\tilde{u}_i + \hat{\mathcal{D}}_i\bar{u}_i)dW(t) \\
dY_i(t) &= (A_i^\intercal\tilde{Y}_i + \hat{A}_i^\intercal\bar{Y}_i + C_i^\intercal\tilde{Z}_i + \hat{C}_i^\intercal\bar{Z}_i + Q_i\tilde{X}_i + \hat{Q}_i\bar{X}_i + S_i^\intercal\tilde{u}_i + \hat{S}_i^\intercal\bar{u}_i)dt + Z_i dW(t),
\end{cases}
$$

with SMP equation:

$$
\mathcal{B}_i^\intercal\tilde{Y}_i + \hat{\mathcal{B}}_i^\intercal\bar{Y}_i + \mathcal{D}_i^\intercal\tilde{Z}_i + \hat{\mathcal{D}}_i^\intercal\bar{Z}_i + S_i\tilde{X}_i + \hat{S}_i\bar{X}_i + R_i\tilde{u}_i + \hat{R}_i\bar{u}_i = 0,
$$

for $i \in \{1, 2\}$.

From the SMP equation above, we can express controls in terms of X_i, Y_i and Z_i:

$$
\begin{cases}
\tilde{u}_i &= -(R_i^{-1}\mathcal{B}_i^\intercal\tilde{Y}_i + R_i^{-1}\mathcal{D}_i^\intercal\tilde{Z}_i + R_i^{-1}S_i\tilde{X}_i) \\
\bar{u}_i &= -(\hat{R}_i^{-1}\hat{\mathcal{B}}_i^\intercal\bar{Y}_i + \hat{R}_i^{-1}\hat{\mathcal{D}}_i^\intercal\bar{Z}_i + \hat{R}_i^{-1}\hat{S}_i\bar{X}_i)
\end{cases}
$$

Then the FBSDE can be rewritten without controls:

$$
\begin{cases}
dX_i &= (\mathbf{A}_i\tilde{X}_i + \hat{\mathbf{A}}_i\bar{X}_i + \mathbf{B}_i\tilde{Y}_i + \hat{\mathbf{B}}_i\bar{Y}_i + \mathbf{C}_i\tilde{Z}_i + \hat{\mathbf{C}}_i\bar{Z}_i)dt \\
&\qquad + (\mathbf{D}_i\tilde{X}_i + \hat{\mathbf{D}}_i\bar{X}_i + \mathbf{E}_i\tilde{Y}_i + \hat{\mathbf{E}}_i\bar{Y}_i + \mathbf{F}_i\tilde{Z}_i + \hat{\mathbf{F}}_i\bar{Z}_i)dW \\
dY_i &= -(\mathbf{J}_i\tilde{X}_i + \hat{\mathbf{J}}_i\bar{X}_i + \mathbf{K}_i\tilde{Y}_i + \hat{\mathbf{K}}_i\bar{Y}_i + \mathbf{L}_i\tilde{Z}_i + \hat{\mathbf{L}}_i\bar{Z}_i)dt + Z_i dW \\
Y_i(T) &= \mathbf{G}_{Ti}\tilde{X}_i(T) + \hat{\mathbf{G}}_{T]}\bar{X}_i(T) + \hat{\mathbf{g}}_{Ti}.
\end{cases}
\tag{5.30}
$$

$$
\begin{cases}
\mathbf{A}_i = A_i - \mathcal{B}_i R_i^{-1} S_i \\
\mathbf{B}_i = -\mathcal{B}_i R_i^{-1} \mathcal{B}_i^\intercal \\
\mathbf{C}_i = -\mathcal{B}_i R_i^{-1} \mathcal{D}_i^\intercal \\
\mathbf{D}_i = C_i - \mathcal{D}_i R_i^{-1} S_i \\
\mathbf{E}_i = -\mathcal{D}_i R_i^{-1} \mathcal{B}_i^\intercal \\
\mathbf{F}_i = -\mathcal{D}_i R_i^{-1} \mathcal{D}_i^\intercal \\
\mathbf{J}_i = Q_i - S_i^\intercal R_i^{-1} S_i \\
\mathbf{K}_i = A_i^\intercal - S_i^\intercal R_i^{-1} \mathcal{B}_i^\intercal \\
\mathbf{L}_i = C_i^\intercal - S_i^\intercal R_i^{-1} \mathcal{D}_i^\intercal
\end{cases}
\qquad
\begin{cases}
\hat{\mathbf{A}}_i = \hat{A}_i - \hat{\mathcal{B}}_i \hat{R}_i^{-1} \hat{S}_i \\
\hat{\mathbf{B}}_i = -\hat{\mathcal{B}}_i \hat{R}_i^{-1} \hat{\mathcal{B}}_i^\intercal \\
\hat{\mathbf{C}}_i = -\hat{\mathcal{B}}_i \hat{R}_i^{-1} \hat{\mathcal{D}}_i^\intercal \\
\hat{\mathbf{D}}_i = \hat{C}_i - \hat{\mathcal{D}}_i \hat{R}_i^{-1} \hat{S}_i \\
\hat{\mathbf{E}}_i = -\hat{\mathcal{D}}_i \hat{R}_i^{-1} \hat{\mathcal{B}}_i^\intercal \\
\hat{\mathbf{F}}_i = -\hat{\mathcal{D}}_i \hat{R}_i^{-1} \hat{\mathcal{D}}_i^\intercal \\
\hat{\mathbf{J}}_i = \hat{Q}_i - \hat{S}_i^\intercal \hat{R}_i^{-1} \hat{S}_i \\
\hat{\mathbf{K}}_i = \hat{A}_i^\intercal - \hat{S}_i^\intercal \hat{R}_i^{-1} \hat{\mathcal{B}}_i^\intercal \\
\hat{\mathbf{L}}_i = \hat{C}_i^\intercal - \hat{S}_i^\intercal \hat{R}_i^{-1} \hat{\mathcal{D}}_i^\intercal .
\end{cases}
$$

As such we can discretise the Mean-Field FBSDE and try to use Machine Learning to solve it:

Theorem 5.4.5 *For the primal problem, $X_1(0) = x_1$, treat $Y_1(0)$ as a control, while for the dual problem, $Y_2(0) = -x_1$, treat $X_2(0)$ as a control. Then approximate $Z_j(t) = v(X_j(t), \bar{X}_j(t))$ for $j \in \{1, 2\}$. The discretisation then becomes:*

$$
\begin{cases}
\bar{x}_i, \bar{y}_i, \bar{z}_i = \mathbb{E}[x_i], \mathbb{E}[y_i], \mathbb{E}[z_i] \\
\tilde{x}_i, \tilde{y}_i, \tilde{z}_i = x_i - \bar{x}_i, y_i - \bar{y}_i, z_i - \bar{z}_i \\
z_i = v(x_i, \bar{x}_i) \\
x_{i+1} = x_i + \mathbf{b}_j(t_i, \tilde{x}_i, \bar{x}_i, \tilde{y}_i, \bar{y}_i, \tilde{z}_i, \bar{z}_i)\Delta t_i + \boldsymbol{\sigma}_j(t_i, \tilde{x}_i, \bar{x}_i, \tilde{y}_i, \bar{y}_i, \tilde{z}_i, \bar{z}_i)\Delta W_i \\
y_{i+1} = y_i - \mathbf{f}_j(t_i, \tilde{x}_i, \bar{x}_i, \tilde{y}_i, \bar{y}_i, \tilde{z}_i, \bar{z}_i)\Delta t_i + z_i \Delta W_i,
\end{cases}
$$

where

$$\begin{cases} \mathbf{b}_j(t_i, \tilde{x}_i, \bar{x}_i, \tilde{y}_i, \bar{y}_i, \tilde{z}_i, \bar{z}_i) = \mathbf{A}_j\tilde{x}_i + \hat{\mathbf{A}}_j\bar{x}_i + \mathbf{B}_j\tilde{y}_i + \hat{\mathbf{B}}_j\bar{y}_i + \mathbf{C}_j\tilde{z}_i + \hat{\mathbf{C}}_j\bar{z}_i \\ \boldsymbol{\sigma}_j(t_i, \tilde{x}_i, \bar{x}_i, \tilde{y}_i, \bar{y}_i, \tilde{z}_i, \bar{z}_i) = \mathbf{D}_j\tilde{x}_i + \hat{\mathbf{D}}_j\bar{x}_i + \mathbf{E}_j\tilde{y}_i + \hat{\mathbf{E}}_j\bar{y}_i + \mathbf{F}_j\tilde{z}_i + \hat{\mathbf{F}}_j\bar{z}_i \\ \mathbf{f}_j(t_i, \tilde{x}_i, \bar{x}_i, \tilde{y}_i, \bar{y}_i, \tilde{z}_i, \bar{z}_i) = \mathbf{J}_j\tilde{x}_i + \hat{\mathbf{J}}_j\bar{x}_i + \mathbf{K}_j\tilde{y}_i + \hat{\mathbf{K}}_j\bar{y}_i + \mathbf{L}_j\tilde{z}_i + \hat{\mathbf{L}}_j\bar{z}_i \\ \mathbf{g}_j(x, \bar{x}) = \mathbf{G}_{Tj}x + \bar{\mathbf{G}}_{Tj}\bar{x} + \hat{\mathbf{g}}_{Tj} \end{cases}$$

with minimizing function

$$\mathbb{E}\left[\left|\mathbf{g}_1(X_j(T), \mathbb{E}[X_j(T)]) - Y_j(T)\right|^2\right].$$

The strategy would be to apply stochastic gradient descent method to minimize the cost function with respect to controls $Y_1(0)$ and v for the primal problem and $X_2(0)$ and v for the dual problem.

From Section 5.2, we can use the solution to Riccati equations to recreate solutions and use them as a benchmark to measure the accuracy of results from Machine Learning:

$$\begin{cases} P_i\tilde{X}_i = \tilde{Y}_i \\ \Pi_i\bar{X}_i + \bar{\eta}_i = \bar{Y}_i \\ Z_i = D_tY = P_i(C_i\tilde{X}_i + \mathcal{D}_i\tilde{u}_i + \hat{C}_i\bar{X}_i + \hat{\mathcal{D}}_i\bar{u}_i) \\ \tilde{u}_i = \Theta_i\tilde{X}_i \\ \bar{u}_i = \Theta_i\bar{X}_i + \bar{\varphi}_i \end{cases} \tag{5.31}$$

5.5 Deep Learning on Mean-Field Linear Quadratic FBSDEs

Remark ***In this section, we assume*** P_2 ***and*** Π_2 ***in*** (5.31) ***to be invertible on*** $[0, T]$***. This assumption is only for computation and numerical results and is not included for theoretical analysis in other sections.***

Example 5.5.1 *For the first example, the following coefficients are used.*

$$
\begin{cases}
A_1 = \begin{pmatrix} 1 & 0.5 \\ 0.4 & 1 \end{pmatrix} \\
B_1 = \begin{pmatrix} 1 & 0.2 \\ 0.1 & 1 \end{pmatrix} \\
C_1 = \begin{pmatrix} 1 & 1 \\ 0.2 & 1 \end{pmatrix} \\
D_1 = \begin{pmatrix} 0.5 & 0.4 \\ 0.3 & 0.5 \end{pmatrix} \\
Q_1 = \begin{pmatrix} 1 & 0.5 \\ 0.4 & 1 \end{pmatrix}^2 \\
S_1 = \begin{pmatrix} 0.2 & 0.2 \\ 0.1 & 0.4 \end{pmatrix} \\
R_1 = \begin{pmatrix} 1 & 0.2 \\ 0.1 & 1 \end{pmatrix}^2 \\
G_1 = \begin{pmatrix} 1 & 0.2 \\ 0.3 & 1 \end{pmatrix}^2 \\
g_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
x_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}
\end{cases}
\qquad
\begin{cases}
\hat{A}_1 = \begin{pmatrix} 1.5 & 0.8 \\ 1.0 & 1.5 \end{pmatrix} \\
\hat{B}_1 = \begin{pmatrix} 2 & 0.5 \\ 0.3 & 2 \end{pmatrix} \\
\hat{C}_1 = \begin{pmatrix} 1.2 & 1.5 \\ 0.5 & 1.2 \end{pmatrix} \\
\hat{D}_1 = \begin{pmatrix} 1 & 0.9 \\ 0.6 & 1 \end{pmatrix} \\
\hat{Q}_1 = \begin{pmatrix} 1.5 & 0.8 \\ 0.8 & 1.5 \end{pmatrix}^2 \\
\hat{S}_1 = \begin{pmatrix} 0.5 & 0.4 \\ 0.3 & 0.8 \end{pmatrix} \\
\hat{R}_1 = \begin{pmatrix} 1.2 & 0.5 \\ 0.4 & 1.2 \end{pmatrix}^2 \\
\hat{G}_1 = \begin{pmatrix} 1.5 & 0.5 \\ 0.4 & 1.5 \end{pmatrix}^2 \\
\hat{g}_1 = \begin{pmatrix} 0.1 \\ 0.2 \end{pmatrix}
\end{cases}
$$

The processes are run using Adam algorithm as optimizer. The loss function is calculated on the basis of 1024 BM paths while each epoch consists of 256 paths. There are 2 hidden layers, each with a dimension of 4. The activation function is set to be sigmoid. The total time is set to 0.3. $100k$ iterations are run for each timestep at a learning rate of $5e-4$. Table 5.1 and Table 5.2 show the relative errors for each variables and each time steps. The values in the table refer to the percentile difference between variables time series generated from deep learning and the time series generated from solutions to Riccati equations.

$$\text{Relative Error} = \frac{1}{2 \times T \times N} \sum_{t=1}^{T} \sum_{j=1}^{N} \sum_{i=1}^{2} \left| ((X_j)_i^{DL}(t_i) - X_i^R(t_i))/X_i^R(t_i) \right|,$$

where $(X_j)_i^{DL}(t)$ refers to the values i-th entry of j-th X in the batch at time t from deep learning, while $X_i^R(t)$ refers to the values i-th entry X at time t from time series generated from Riccati solutions.

Var\Timestep	5	10	15	20	25
X	12.96%	5.07%	5.20%	5.62%	4.65%
Y	2.88%	1.48%	1.23%	0.99%	0.91%
Z	4.11%	2.27%	1.97%	1.62%	1.48%

Table 5.1: Primal Problem Analysis (Adam)

Var\Timestep	5	10	15	20	25
X	3.41%	2.51%	2.27%	2.21%	2.11%
Y	58.48%	10.49%	12.95%	29.98%	18.60%
Z	3.57%	3.15%	3.07%	3.19%	3.22%

Table 5.2: Dual Problem Analysis (Adam)

The graphs below show hoe the log relative errors of X, Y and Z change as the number of iteration increases for different time steps. Graphs on the left are from Primal problems while graphs on the right are from Dual problems.

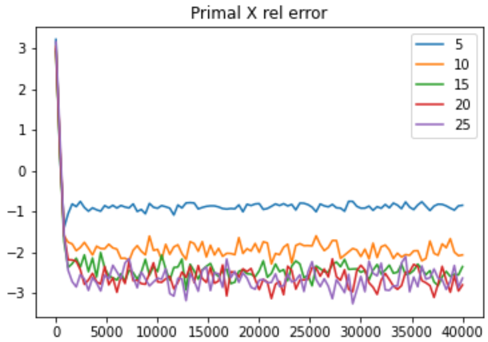

Figure 5.1: Relative errors of X_1 (ADAM)

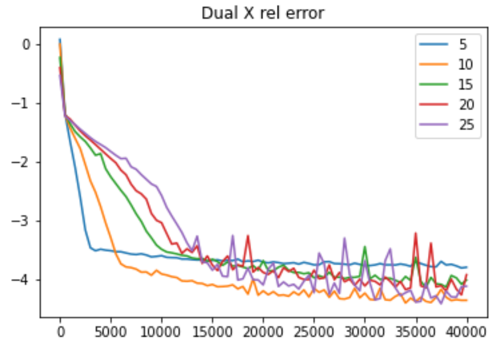

Figure 5.2: Relative errors of X_2 (ADAM)

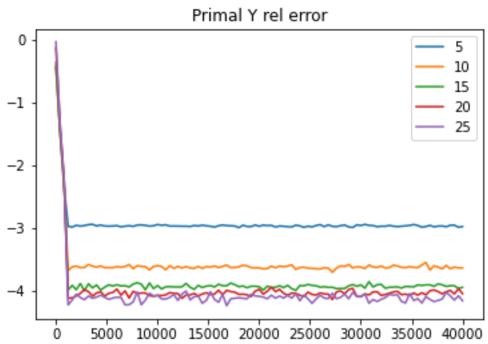

Figure 5.3: Relative errors of Y_1 (ADAM)

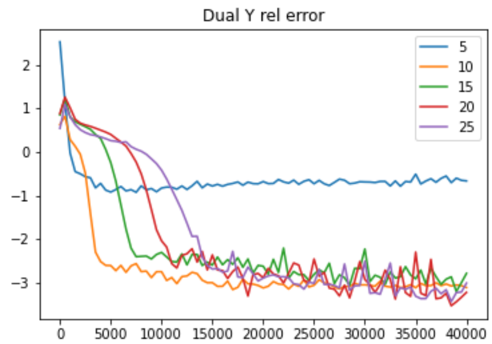

Figure 5.4: Relative errors of Y_2 (ADAM)

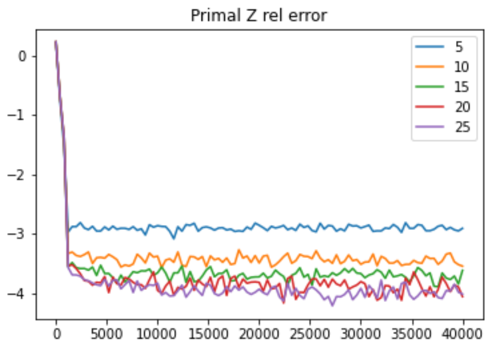

Figure 5.5: Relative errors of Z_1 (ADAM)

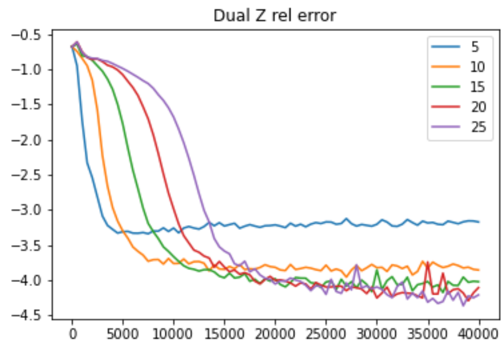

Figure 5.6: Relative errors of Z_2 (ADAM)

Example 5.5.2 *Due to the structure of the primal and dual problems, we compare the precision between X_1 and Y_2, as well as between Y_1 and X_2. As the number of iterations increases, the relative errors in primal problem decrease significantly* *faster. However, when the number of iterations increases further, the dual relative errors catch up. Moreover, the dual relative error is usually still on the downward trend, which means that it can decrease further. An example would be the relative error of Y_1 and X_2 with time step 5 of Figure 5.3 and Figure 5.2. The relative error of Y_1 appears to fluctuate around e^{-3}, while at the end of the iteration the relative error of X_2 approaches e^{-4}. Therefore, for the same level of learning rate, if a faster result is preferred, the primal problem would be the one to choose, whereas if a more accurate result is desired, the dual problem would usually be a better choice of the two. The second example uses a non-square D_1. The updated coefficients are as follows:*

$$
\begin{cases}
B_1 = \begin{pmatrix} 1 \\ \\ 0.1 \end{pmatrix} \\
D_1 = \begin{pmatrix} 0.5 \\ \\ 0.3 \end{pmatrix} \\
S_1 = \begin{pmatrix} 0.2 & 0.2 \end{pmatrix} \\
R_1 = \begin{pmatrix} 1^2 \end{pmatrix}
\end{cases}
\qquad
\begin{cases}
\hat{B}_1 = \begin{pmatrix} 2 \\ \\ 0.3 \end{pmatrix} \\
\hat{D}_1 = \begin{pmatrix} 0.5 \\ \\ 0.3 \end{pmatrix} \\
\hat{S}_1 = \begin{pmatrix} 0.5 & 0.4 \end{pmatrix} \\
\hat{R}_1 = \begin{pmatrix} 1.2^2 \end{pmatrix}
\end{cases}
$$

The other coefficients remain the same as in Example 5.5.1. The primal and dual tables consisting of relative errors are shown below.

For the second example, all the parameters are kept the same except for the total number of iteration is reduced to $50k$ instead as relatively steady loss errors were observed within a small number of iterations for both primal and dual problem. The tables and graphs are constructed in the same way as the last example.

Var\Timestep	5	10	15	20	25
X	4.67%	4.60%	3.19%	2.05%	2.01%
Y	5.45%	2.14%	1.71%	1.69%	1.08%
Z	15.29%	30.16%	17.55%	27.44%	21.92%

Table 5.3: Primal Problem Relative Error with varied D (Adam)

Var\Timestep	5	10	15	20	25
X	4.21%	1.60%	0.81%	0.61%	0.51%
Y	2.61%	4.01%	2.24%	1.50%	0.97%
Z	88.29%	46.67%	521.82%	38.48%	24.79%

Table 5.4: Dual Problem Relative Error with varied D (Adam)

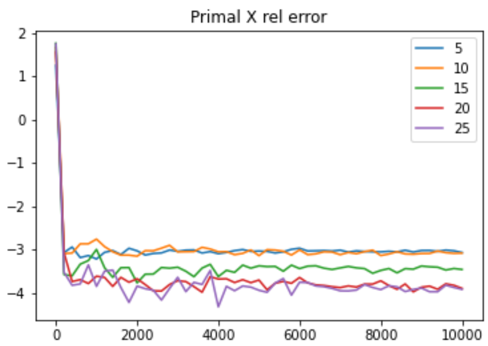

Figure 5.7: Relative errors of X_1 varied D (ADAM)

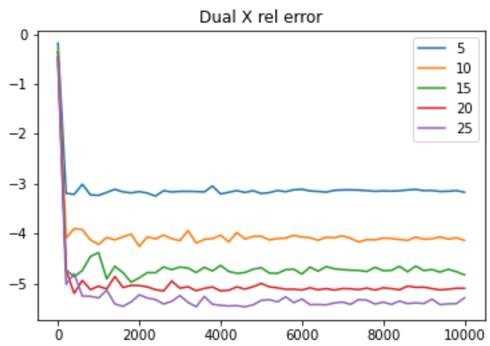

Figure 5.8: Relative errors of X_2 varied D (ADAM)

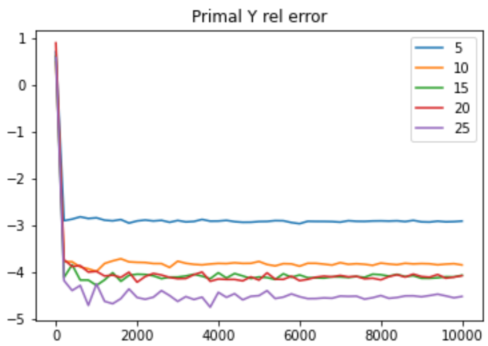

Figure 5.9: Relative errors of Y_1 varied D (ADAM)

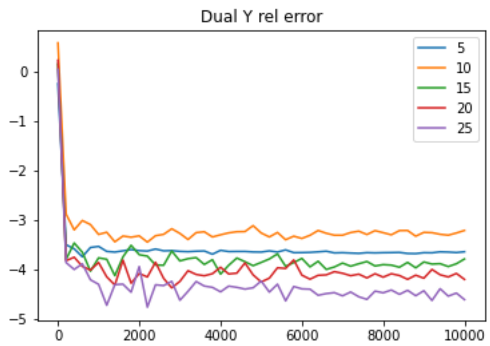

Figure 5.10: Relative errors of Y_2 varied D (ADAM)

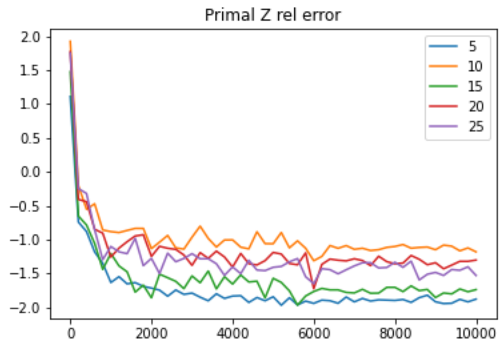

Figure 5.11: Relative errors of Z_1 varied D (ADAM)

Figure 5.12: Relative errors of Z_2 varied D (ADAM)

As can be seen from both examples, with higher time steps, the results are closer to the true values. From Table 5.4, it seems strange at first that while the relative errors of X and Y are low, the relative error of Z is very high. This is, in fact, caused by the fact that D_1 and D_2 are not invertible. Since they are not of full row rank, their nullities are greater than 0 which means there exists $v_1 \neq v_2$ such that $D_1 v_1 = D_1 v_2$. But as the problem is only interested in the value functions, an close approximation on X and Y is good enough.

5.6 Conclusion

In the chapter, we compare the solutions of 4 with the results of [59] to show that the derivation of the dual problem is indeed correct. We also point out the error in [3] and give a correct representation of the solution to the MF-BSDEs. Lastly, with help of Machine Learning, we are able to find numerical solutions that are close to solutions derived from Riccati solutions in [59].

One possible further research is to use machine learning to find solutions to constrained Mean-Field Stochastic control problem. The main obstacle would be to identify a problem that has an explicit solution. Although one can always use the loss function to measure accuracy, it is still good to measure how accurate the results from machine learning will be by comparing them to the true values.

6

Conclusion

Combining results in Chapter 2 with those in Chapter 4, we can find the dual problem to the constrained Mean-Field Stochastic Optimal Control. However, the conditions required in the proofs limit the method's application. It remains an interesting area to see if the results can be extended outside the Linear-Quadratic setting. While we treat X (or $\tilde{X}$) and $\bar{X}$ as independent variables when deriving the dual running cost and terminal cost, it is unlikely possible to repeat such method for more complicated problem since the stochastic and the law terms would be coupled.

Even in a Linear Quadratic setting, it will still be interesting to see what would happen when D_1 and $\hat{D}_1$ are not of full column rank, i.e., what would happen if $\tilde{N}$ and $\bar{N}$ from (4.11) cannot be represented by dual variables? Although we can still express the dual problem as it is now, what would be the relationship between it and the primal problem?

Lastly, while the current algorithm shows that the dual problem generally produces a more accurate result than its primal counterpart with the same learning rate, it requires a significantly greater number of iteration steps to be achieved. A possible further research would be to look into possible improvements to fasten the converging rate of the dual problem while keeping the accuracy level more or less unchanged.

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A

Proofs for Chapter 4 Section 3

By considering the expectation of the equations and equations subtracting their expectations, we can divide the FBSDEs and the SMP conditions into 2 parts.

Primal FBSDEs:

$$\begin{cases} d\tilde{X}_1(t) &= (A_1\tilde{X}_1 + B_1\tilde{\pi})dt + (C_1\tilde{X}_1 + \hat{C}_1\bar{X}_1 + D_1\tilde{\pi} + \hat{D}_1\bar{\pi})dW(t), \\ d\tilde{Y}_1(t) &= -(A_1^\intercal\tilde{Y}_1 + C_1^\intercal\tilde{Z}_1 + Q_1\tilde{X}_1 + S^\intercal\tilde{\pi})dt + Z_1 dW(t), \\ \tilde{Y}_1(T) &= G_{T1}\tilde{X}_1(T) \end{cases} \tag{A.1}$$

and

$$\begin{cases} d\bar{X}_1(t) &= (\hat{A}_1\bar{X}_1 + \hat{B}_1\bar{\pi})dt, \\ d\bar{Y}_1(t) &= -(\hat{A}_1^\intercal\bar{Y}_1 + \hat{C}_1^\intercal\bar{Z}_1 + \hat{Q}_1\bar{X}_1 + \hat{S}_1^\intercal\bar{\pi})dt, \\ \bar{Y}_1(T) &= \hat{G}_{T1}\bar{X}_1(T) + \hat{g}_{T1}. \end{cases} \tag{A.2}$$

Primal SMP:

$$B_1^\intercal \tilde{Y}_1 + D_1^\intercal \tilde{Z}_1 + S_1 \tilde{X}_1 + R_1 \tilde{\pi} = 0 \tag{A.3}$$

and

$$\hat{B}_1^\intercal \bar{Y}_1 + \hat{D}_1^\intercal \bar{Z}_1 + \hat{S}_1 \bar{X}_1 + \hat{R}_1 \bar{\pi} = 0. \tag{A.4}$$

Dual FBSDEs:

$$\begin{cases} d\tilde{X}_2(t) &= (A_2 \tilde{X}_2 + \tilde{\alpha} + B_2 \tilde{\beta})dt + (C_2 \tilde{X}_2 + \hat{C}_2 \bar{X}_2 + D_2 \tilde{\beta} + \hat{D}_2 \bar{\beta})dW(t) \\ d\tilde{Y}_2(t) &= -(A_2^\intercal \tilde{Y}_2 + C_2^\intercal \tilde{Z}_2)dt + Z_2 dW(t) \\ \tilde{Y}_2(T) &= G_{T2} \tilde{X}_2(T) \end{cases} \tag{A.5}$$

and

$$\begin{cases} d\bar{X}_2(t) &= (\hat{A}_2 \bar{X}_2 + \bar{\alpha} + \hat{B}_2 \bar{\beta})dt \\ d\bar{Y}_2(t) &= -(\hat{A}_2^\intercal \bar{Y}_2 + \hat{C}_2^\intercal \bar{Z}_2)dt \\ \bar{Y}_2(T) &= \hat{G}_{T2} \bar{X}_2(T) + \hat{g}_{T2}. \end{cases} \tag{A.6}$$

Dual SMP:

$$\begin{pmatrix} \tilde{Y}_2 \\ B_2^\intercal \tilde{Y}_2 \end{pmatrix} + \begin{pmatrix} 0 \\ D_2^\intercal \tilde{Z}_2 \end{pmatrix} + R_2 \begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{pmatrix} \tag{A.7}$$

and

$$\begin{pmatrix} \bar{Y}_2 \\ \hat{B}_2^\intercal \bar{Y}_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \hat{D}_2^\intercal \bar{Z}_2 \end{pmatrix} + \hat{R}_2 \begin{pmatrix} \bar{\alpha} \\ \bar{\beta} \end{pmatrix} \tag{A.8}$$

Proofs to Theorem 4.3.4:

Proof. Be definitions from above:

$$
\begin{cases}
\tilde{X}_2 = -\tilde{Y}_1 \\
\bar{X}_2 = -\bar{Y}_1 \\
\tilde{Y}_2 = -\tilde{X}_1 \\
\bar{Y}_2 = -\bar{X}_1 \\
\tilde{Z}_2 = -C_1\tilde{X}_1 - D_1\tilde{\pi} \\
\bar{Z}_2 = -\hat{C}_1\bar{X}_1 - \hat{D}_1\bar{\pi} \\
\tilde{\alpha} = Q_1\tilde{X}_1 + S_1^\intercal\tilde{\pi} \\
\bar{\alpha} = \hat{Q}_1\bar{X}_1 + \hat{S}_1^\intercal\bar{\pi} \\
\tilde{\beta} = -D_1^\intercal\tilde{Z}_1 - B_1^\intercal\tilde{Y}_1 \\
\bar{\beta} = -\hat{D}_1^\intercal\bar{Z}_1 - \hat{B}_1^\intercal\bar{Y}_1.
\end{cases}
\tag{A.9}
$$

As such

$$
\begin{aligned}
d\tilde{X}_2 &= -d\tilde{Y}_1 \\
(\text{A.1}) &= (A_1^\intercal\tilde{Y}_1 + C_1^\intercal\tilde{Z}_1 + Q_1\tilde{X}_1 + S_1^\intercal\tilde{\pi})dt - Z_1 dW(t) \\
(\text{A.9}) &= [(A_1 - B_1D_1^\dagger C_1)^\intercal\tilde{Y}_1 + (B_1D_1^\dagger C_1)^\intercal\tilde{Y}_1 + C_1^\intercal\tilde{Z}_1 + \tilde{\alpha}]dt \\
&\quad - [(B_1D_1^\dagger - B_1D_1^\dagger)^\intercal\tilde{Y}_1 + \tilde{Z}_1 + (\hat{B}_1\hat{D}_1^\dagger - \hat{B}_1\hat{D}_1^\dagger)^\intercal\bar{Y}_1 + \bar{Z}_1]dW(t) \\
(4.23) &= [(A_1 - B_1D_1^\dagger C_1)^\intercal\tilde{Y}_1 + (D_1^\dagger C_1)^\intercal B_1^\intercal\tilde{Y}_1 + C_1^\intercal(D_1D_1^\dagger)^\intercal\tilde{Z}_1 + \tilde{\alpha}]dt \\
&\quad - [(B_1D_1^\dagger - B_1D_1^\dagger)^\intercal\tilde{Y}_1 + (D_1D_1^\dagger)^\intercal\tilde{Z}_1 + (\hat{B}_1\hat{D}_1^\dagger - \hat{B}_1\hat{D}_1^\dagger)^\intercal\bar{Y}_1 + (\hat{D}_1\hat{D}_1^\dagger)^\intercal\bar{Z}_1]dW(t) \\
&= [(A_1 - B_1D_1^\dagger C_1)^\intercal\tilde{Y}_1 + (D_1^\dagger C_1)^\intercal(B_1^\intercal\tilde{Y}_1 + D_1^\intercal\tilde{Z}_1) + \tilde{\alpha}]dt \\
&\quad - [-(B_1D_1^\dagger)^\intercal\tilde{Y}_1 + (D_1^\dagger)^\intercal(B_1^\intercal\tilde{Y}_1 + D_1^\intercal\tilde{Z}_1) - (\hat{B}_1\hat{D}_1^\dagger)^\intercal\bar{Y}_1 + (\hat{D}_1^\dagger)^\intercal(\hat{B}_1^\intercal\bar{Y}_1 + \hat{D}_1^\intercal\bar{Z}_1)]dW(t) \\
(\text{A.9}) &= [-(A_1 - B_1D_1^\dagger C_1)^\intercal\tilde{X}_2 - (D_1^\dagger C_1)^\intercal\tilde{\beta} + \tilde{\alpha}]dt \\
&\quad + [-(B_1D_1^\dagger)^\intercal\tilde{X}_2 + (D_1^\dagger)^\intercal\tilde{\beta} - (\hat{B}_1\hat{D}_1^\dagger)^\intercal\bar{Y}_1 + (\hat{D}_1^\dagger)^\intercal\bar{\beta}]dW(t) \\
(4.8) &= (A_2\tilde{X}_2 + \tilde{\alpha} + B_2\tilde{\beta})dt + (C_2\tilde{X}_2 + D_2\tilde{\beta} + \hat{C}_2\bar{X}_2 + \hat{D}_2\bar{\beta})dW(t),
\end{aligned}
$$

and

$$
\begin{aligned}
d\bar{X}_2 &= -d\bar{Y}_1 \\
\text{(A.2)} &= (\hat{A}_1^\intercal \bar{Y}_1 + \hat{C}_1^\intercal \bar{Z}_1 + \hat{Q}_1 \bar{X}_1 + \hat{S}_1^\intercal \bar{\pi})dt \\
\text{(A.9)} &= [(\hat{A}_1 - \hat{B}_1\hat{D}_1^\dagger \hat{C}_1)^\intercal \bar{Y}_1 + (\hat{B}_1\hat{D}_1^\dagger \hat{C}_1)^\intercal \bar{Y}_1 + \hat{C}_1^\intercal \bar{Z}_1 + \bar{\alpha}]dt \\
\text{(4.24)} &= [(\hat{A}_1 - \hat{B}_1\hat{D}_1^\dagger \hat{C}_1)^\intercal \bar{Y}_1 + (\hat{B}_1\hat{D}_1^\dagger \hat{C}_1)^\intercal \bar{Y}_1 + \hat{C}_1^\intercal (\hat{D}_1\hat{D}_1^\dagger)^\intercal \bar{Z}_1 + \bar{\alpha}]dt \\
&= [(\hat{A}_1 - \hat{B}_1\hat{D}_1^\dagger \hat{C}_1)^\intercal \bar{Y}_1 + (\hat{D}_1^\dagger \hat{C}_1)^\intercal (\hat{B}_1^\intercal \bar{Y}_1 + \hat{D}_1^\intercal \bar{Z}_1) + \bar{\alpha}]dt \\
\text{(A.9)} &= [-(\hat{A}_1 - \hat{B}_1\hat{D}_1^\dagger \hat{C}_1)^\intercal \bar{X}_2 + \bar{\alpha} - (\hat{D}_1^\dagger \hat{C}_1)^\intercal \bar{\beta}]dt \\
\text{(4.8)} &= (\hat{A}_2 \bar{X}_2 + \bar{\alpha} + \hat{B}_2\bar{\beta})dt.
\end{aligned}
$$

Adding the two equations to obtain dual forward equation.

As for the backward equation,

$$
\begin{aligned}
d\tilde{Y}_2 &= -d\tilde{X}_1 \\
\text{(A.1)} &= -(A_1\tilde{X}_1 + B_1\tilde{\pi})dt - (C_1\tilde{X}_1 + \hat{C}_1\bar{X}_1 + D_1\tilde{\pi} + \hat{D}_1\bar{\pi})dW(t) \\
\text{(4.26)} &= [(-A_1 + B_1D_1^\dagger C_1)\tilde{X}_1 - B_1D_1^\dagger C_1\tilde{X}_1 - B_1\tilde{\pi}]dt + Z_2 dW(t) \\
\text{((A6))} &= [(-A_1 + B_1D_1^\dagger C_1)\tilde{X}_1 - B_1D_1^\dagger C_1\tilde{X}_1 - B_1D_1^\dagger D_1\tilde{\pi}]dt + Z_2 dW(t) \\
&= [(-A_1 + B_1D_1^\dagger C_1)\tilde{X}_1 - B_1D_1^\dagger (C_1\tilde{X}_1 + D_1\tilde{\pi})]dt + Z_2 dW(t) \\
\text{(A.9)} &= [-(-A_1 + B_1D_1^\dagger C_1)\tilde{Y}_2 + B_1D_1^\dagger \tilde{Z}_2]dt + Z_2 dW(t) \\
\text{(4.8)} &= -(A_2^\intercal \tilde{Y}_2 + C_2^\intercal \tilde{Z}_2)dt + Z_2 dW(t),
\end{aligned}
$$

and

$$
\begin{aligned}
d\bar{Y}_2 &= -d\bar{X}_1 \\
\text{(A.2)} &= -(\hat{A}_1\bar{X}_1 + \hat{B}_1\bar{\pi})dt \\
\text{(4.26)} &= [(-\hat{A}_1 + \hat{B}_1\hat{D}_1^\dagger\hat{C}_1)\bar{X}_1 - \hat{B}_1\hat{D}_1^\dagger\hat{C}_1\bar{X}_1 - \hat{B}_1\bar{\pi}]dt \\
\text{((A6))} &= [(-\hat{A}_1 + \hat{B}_1\hat{D}_1^\dagger\hat{C}_1)\bar{X}_1 - \hat{B}_1\hat{D}_1^\dagger\hat{C}_1\bar{X}_1 - \hat{B}_1\hat{D}_1^\dagger\hat{D}_1\bar{\pi}]dt \\
&= [(-\hat{A}_1 + \hat{B}_1\hat{D}_1^\dagger\hat{C}_1)\bar{X}_1 - \hat{B}_1\hat{D}_1^\dagger(\hat{C}_1\bar{X}_1 + \hat{D}_1\bar{\pi})]dt \\
\text{(A.9)} &= [-(-\hat{A}_1 + \hat{B}_1\hat{D}_1^\dagger\hat{C}_1)\bar{Y}_2 + \hat{B}_1\hat{D}_1^\dagger\bar{Z}_2]dt \\
\text{(4.8)} &= -(A_2^\intercal\bar{Y}_2 + C_2^\intercal\bar{Z}_2)dt.
\end{aligned}
$$

Combining the two to produce the backward equation. Furthermore, the SMP condition (4.21) can be divided into (A.7) and (A.8):

$$
\begin{aligned}
&\begin{pmatrix} \tilde{Y}_2 \\ B_2^\intercal\tilde{Y}_2 \end{pmatrix} + \begin{pmatrix} 0 \\ D_2^\intercal\tilde{Z}_2 \end{pmatrix} + R_2\begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{pmatrix} \\
\text{(A.9)} &= \begin{pmatrix} -\tilde{X}_1 \\ -B_2^\intercal\tilde{X}_1 \end{pmatrix} + \begin{pmatrix} 0 \\ -D_2^\intercal(C_1\tilde{X}_1 + D_1\tilde{\pi}) \end{pmatrix} + R_2\begin{pmatrix} Q_1\tilde{X}_1 + S_1^\intercal\tilde{\pi} \\ -D_1^\intercal\tilde{Z}_1 - B_1^\intercal\tilde{Y}_1 \end{pmatrix} \\
\text{(4.8)} &= \begin{pmatrix} -\tilde{X}_1 \\ D_1^\dagger C_1\tilde{X}_1 - D_1^\dagger C_1\tilde{X}_1 - D_1^\dagger D_1\tilde{\pi} \end{pmatrix} + R_2\begin{pmatrix} Q_1\tilde{X}_1 + S_1^\intercal\tilde{\pi} \\ -D_1^\intercal\tilde{Z}_1 - B_1^\intercal\tilde{Y}_1 \end{pmatrix} \\
\text{((A6))} &= R_2(-\begin{pmatrix} Q_1 & S_1^\intercal \\ S_1 & R_1 \end{pmatrix}\begin{pmatrix} \tilde{X}_1 \\ \tilde{\pi} \end{pmatrix} + \begin{pmatrix} Q_1\tilde{X}_1 + S_1^\intercal\tilde{\pi} \\ -D_1^\intercal\tilde{Z}_1 - B_1^\intercal\tilde{Y}_1 \end{pmatrix}) \\
&= R_2\begin{pmatrix} -Q_1\tilde{X}_1 - S_1^\intercal\tilde{\pi} + Q_1\tilde{X}_1 + S_1^\intercal\tilde{\pi} \\ -S_1\tilde{X}_1 - R_1\tilde{\pi} - D_1^\intercal\tilde{Z}_1 - B_1^\intercal\tilde{Y}_1 \end{pmatrix} \\
\text{(A.3)} &= 0
\end{aligned}
$$

and

$$
\begin{aligned}
&\begin{pmatrix} \bar{Y}_2 \\ \hat{B}_2^\intercal \bar{Y}_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \hat{D}_2^\intercal \bar{Z}_2 \end{pmatrix} + \hat{R}_2 \begin{pmatrix} \bar{\alpha} \\ \bar{\beta} \end{pmatrix} \\
(\text{A.9}) &= \begin{pmatrix} -\bar{X}_1 \\ -\hat{B}_2^\intercal \bar{X}_1 \end{pmatrix} + \begin{pmatrix} 0 \\ -\hat{D}_2^\intercal (\hat{C}_1 \bar{X}_1 + \hat{D}_1 \bar{\pi}) \end{pmatrix} + \hat{R}_2 \begin{pmatrix} \hat{Q}_1 \bar{X}_1 + \hat{S}_1^\intercal \bar{\pi} \\ -\hat{D}_1^\intercal \bar{Z}_1 - \hat{B}_1^\intercal \bar{Y}_1 \end{pmatrix} \\
(4.8) &= \begin{pmatrix} -\bar{X}_1 \\ \hat{D}_1^\dagger \hat{C}_1 \bar{X}_1 - \hat{D}_1^\dagger \hat{C}_1 \bar{X}_1 - \hat{D}_1^\dagger \hat{D}_1 \bar{\pi} \end{pmatrix} + \hat{R}_2 \begin{pmatrix} \hat{Q}_1 \bar{X}_1 + \hat{S}_1^\intercal \bar{\pi} \\ -\hat{D}_1^\intercal \bar{Z}_1 - \hat{B}_1^\intercal \bar{Y}_1 \end{pmatrix} \\
(4.8), ((\text{A6})) &= \hat{R}_2 (- \begin{pmatrix} \hat{Q}_1 & \hat{S}_1^\intercal \\ \hat{S}_1 & \hat{R}_1 \end{pmatrix} \begin{pmatrix} \bar{X}_1 \\ \bar{\pi} \end{pmatrix} + \begin{pmatrix} \hat{Q}_1 \bar{X}_1 + \hat{S}_1^\intercal \bar{\pi} \\ -\hat{D}_1^\intercal \bar{Z}_1 - \hat{B}_1^\intercal \bar{Y}_1 \end{pmatrix}) \\
&= \hat{R}_2 \begin{pmatrix} -\hat{Q}_1 \bar{X}_1 - \hat{S}_1^\intercal \bar{\pi} + \hat{Q}_1 \bar{X}_1 + \hat{S}_1^\intercal \bar{\pi} \\ -\hat{S}_1 \bar{X}_1 - \hat{R}_1 \bar{\pi} - \hat{D}_1^\intercal \bar{Z}_1 - \hat{B}_1^\intercal \bar{Y}_1 \end{pmatrix} \\
(\text{A.4}) &= 0.
\end{aligned}
$$

Hence (4.21) equals 0. Lastly, check the initial and terminal conditions:

$$
\begin{cases} \chi = X_2(0) = -Y_1(0) \\ Y_2(0) = -X_1(0) = -x_1, \end{cases}
$$

$$
\begin{aligned}
&\tilde{Y}_2(T) \\
(\text{A.9}) &= -\tilde{X}_1(T) \\
(\text{A.1}) &= -G_{T1}^{-1} \tilde{Y}_1(T) \\
(\text{A.9}) &= G_{T1}^{-1} \tilde{X}_2(T) \\
(4.8) &= G_{T2} \tilde{X}_2(T)
\end{aligned}
$$

and

$$
\begin{aligned}
&\bar{Y}_2(T) \\
(A.9) &= -\bar{X}_1(T) \\
(A.2) &= -\hat{G}_{T1}^{-1}(\bar{Y}_1(T) - \hat{g}_{T1}) \\
(A.9) &= \hat{G}_{T1}^{-1}\bar{X}_2(T) + \hat{G}_{T1}^{-1}\hat{g}_{T1} \\
(4.8) &= \hat{G}_{T2}\bar{X}_2(T) + \hat{g}_{T2}.
\end{aligned}
$$

We are able to obtain the initial and terminal conditions of the dual problem. □

Proofs to Theorem 4.3.5:

Proof. Be definitions from above:

$$
\begin{cases}
\tilde{X}_1 = -\tilde{Y}_2 \\
\bar{X}_1 = -\bar{Y}_2 \\
\tilde{Y}_1 = -\tilde{X}_2 \\
\bar{Y}_1 = -\bar{X}_2 \\
\tilde{Z}_1 = -C_2\tilde{X}_2 - D_2\tilde{\beta} \\
\bar{Z}_1 = -\hat{C}_2\bar{X}_2 - \hat{D}_2\bar{\beta} \\
\tilde{\pi} = -B_2^\intercal\tilde{Y}_2 - D_2^\intercal\tilde{Z}_2 \\
\bar{\pi} = -\hat{B}_2^\intercal\bar{Y}_2 - \hat{D}_2^\intercal\bar{Z}_2.
\end{cases}
\tag{A.10}
$$

As such

$$
\begin{aligned}
d\tilde{X}_1 &= -d\tilde{Y}_2 \\
\text{(A.5)} &= (A_2^\intercal \tilde{Y}_2 + C_2^\intercal \tilde{Z}_2)dt - Z_2 dW(t) \\
\text{(4.8), (4.27)} &= [(-A_1 + B_1 D_1^\dagger C_1)\tilde{Y}_2 - B_1 D_1^\dagger \tilde{Z}_2]dt \\
&\quad + [(-C_1 + D_1 D_1^\dagger C_1)\tilde{Y}_2 - D_1 D_1^\dagger \tilde{Z}_2 + (-\hat{C}_1 + \hat{D}_1 \hat{D}_1^\dagger \hat{C}_1)\bar{Y}_2 - \hat{D}_1 \hat{D}_1^\dagger \bar{Z}_2]dW(t) \\
&= [-A_1 \tilde{Y}_2 + B_1(D_1^\dagger C_1 \tilde{Y}_2 - D_1^\dagger \tilde{Z}_2)]dt \\
&\quad + [-C_1 \tilde{Y}_2 - D_1(-D_1^\dagger C_1 \tilde{Y}_2 + D_1^\dagger \tilde{Z}_2) - \hat{C}_1 \bar{Y}_2 - \hat{D}_1(-\hat{D}_1^\dagger \hat{C}_1 \bar{Y}_2 + \hat{D}_1^\dagger \bar{Z}_2)]dW(t) \\
\text{(4.8), (4.8)} &= [-A_1 \tilde{Y}_2 + B_1(-C_2^\intercal \tilde{Y}_2 - D_2^\intercal \tilde{Z}_2)]dt \\
&\quad + [-C_1 \tilde{Y}_2 - D_1(B_2^\intercal \tilde{Y}_2 + D_2^\intercal \tilde{Z}_2) - \hat{C}_1 \bar{Y}_2 - \hat{D}_1(\hat{B}_2^\intercal \bar{Y}_2 + \hat{D}_2^\intercal \bar{Z}_2)]dW(t) \\
\text{(A.10)} &= (A_1 \tilde{X}_1 + B_1 \tilde{\pi})dt + (C_1 \tilde{X}_1 + \hat{C}_1 \bar{X}_1 + D_1 \tilde{\pi} + \hat{D}_1 \bar{\pi})dW(t),
\end{aligned}
$$

and

$$
\begin{aligned}
d\bar{X}_1 &= -d\bar{Y}_2 \\
\text{(A.6)} &= (\hat{A}_2^\intercal \bar{Y}_2 + \hat{C}_2^\intercal \bar{Z}_2)dt \\
\text{(4.8)} &= [(-\hat{A}_1 + \hat{B}_1 \hat{D}_1^\dagger \hat{C}_1)\bar{Y}_2 - \hat{B}_1 \hat{D}_1^\dagger \bar{Z}_2]dt \\
&= [-\hat{A}_1 \bar{Y}_2 + \hat{B}_1(\hat{D}_1^\dagger \hat{C}_1 \bar{Y}_2 - \hat{D}_1^\dagger \bar{Z}_2)]dt \\
\text{(4.8)} &= [-\hat{A}_1 \bar{Y}_2 + \hat{B}_1(-\hat{C}_2^\intercal \bar{Y}_2 - \hat{D}_2^\intercal \bar{Z}_2)]dt \\
\text{(A.10)} &= (\hat{A}_1 \bar{X}_1 + \hat{B}_1 \bar{\pi})dt.
\end{aligned}
$$

Adding the two equations to obtain the primal forward equation. Observe $O_1 = R_2^{-1}$ times (A.7), we have

$$
0 = \begin{pmatrix} Q_1 & S_1^\intercal \\ S_1 & R_1 \end{pmatrix} \begin{pmatrix} \tilde{Y}_2 \\ B_2^\intercal \tilde{Y}_2 + D_2^\intercal \tilde{Z}_2 \end{pmatrix} + \begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \end{pmatrix} = \begin{pmatrix} Q_1 \tilde{Y}_2 + S_1^\intercal (B_2^\intercal \tilde{Y}_2 + D_2^\intercal \tilde{Z}_2) + \tilde{\alpha} \\ S_1 \tilde{Y}_2 + R_1(B_2^\intercal \tilde{Y}_2 + D_2^\intercal \tilde{Z}_2) + \tilde{\beta} \end{pmatrix}. \tag{A.11}
$$

Similarly, let $\hat{R}_2^{-1}$ time (A.8),

$$0 = \begin{pmatrix} \hat{Q}_1 & \hat{S}_1^\intercal \\ \hat{S}_1 & \hat{R}_1 \end{pmatrix} \begin{pmatrix} \bar{Y}_2 \\ \hat{B}_2^\intercal \bar{Y}_2 + \hat{D}_2^\intercal \bar{Z}_2 \end{pmatrix} + \begin{pmatrix} \bar{\alpha} \\ \bar{\beta} \end{pmatrix} = \begin{pmatrix} \hat{Q}_1 \bar{Y}_2 + \hat{S}_1^\intercal (\hat{B}_2^\intercal \bar{Y}_2 + \hat{D}_2^\intercal \bar{Z}_2) + \bar{\alpha} \\ \hat{S}_1 \bar{Y}_2 + \hat{R}_1 (\hat{B}_2^\intercal \bar{Y}_2 + \hat{D}_2^\intercal \bar{Z}_2) + \bar{\beta} \end{pmatrix}. \tag{A.12}$$

As for the backward equation,

$$\begin{aligned} d\tilde{Y}_1 &= -d\tilde{X}_2 \\ (\text{A.5}) &= -(A_2 \tilde{X}_2 + \tilde{\alpha} + B_2 \tilde{\beta})dt - (C_2 \tilde{X}_2 + \hat{C}_2 \bar{X}_2 + D_2 \tilde{\beta} + \hat{D}_2 \bar{\beta}) dW(t) \\ (4.8), (4.30) &= [(A_1 - B_1 D_1^\dagger C_1)^\intercal \tilde{X}_2 - \tilde{\alpha} + (D_1^\dagger C_1)^\intercal \tilde{\beta}]dt + Z_1 dW(t) \\ (4.8) &= [A_1^\intercal \tilde{X}_2 + C_1^\intercal (C_2 \tilde{X}_2 + D_2 \tilde{\beta}) - \tilde{\alpha}]dt + Z_1 dW(t) \\ (\text{A.11}) &= [A_1^\intercal \tilde{X}_2 + C_1^\intercal (C_2 \tilde{X}_2 + D_2 \tilde{\beta}) + Q_1 \tilde{Y}_2 + S_1^\intercal (B_2^\intercal \tilde{Y}_2 + D_2^\intercal \tilde{Z}_2)]dt + Z_1 dW(t) \\ (\text{A.10}) &= -(A_1^\intercal \tilde{Y}_1 + C_1^\intercal \tilde{\pi} + Q_1 \tilde{X}_1 + S_1^\intercal \tilde{\pi})dt + Z_1 dW(t), \end{aligned}$$

and

$$\begin{aligned} d\bar{Y}_1 &= -d\bar{X}_2 \\ (\text{A.5}) &= -(\hat{A}_2 \bar{X}_2 + \bar{\alpha} + \hat{B}_2 \bar{\beta})dt \\ (4.8) &= [(\hat{A}_1 - \hat{B}_1 \hat{D}_1^\dagger \hat{C}_1)^\intercal \bar{X}_2 - \bar{\alpha} + (\hat{D}_1^\dagger \hat{C}_1)^\intercal \bar{\beta}]dt \\ (4.8) &= [\hat{A}_1^\intercal \bar{X}_2 + \hat{C}_1^\intercal (\hat{C}_2 \bar{X}_2 + \hat{D}_2 \bar{\beta}) - \bar{\alpha}]dt \\ (\text{A.12}) &= [\hat{A}_1^\intercal \bar{X}_2 + \hat{C}_1^\intercal (\hat{C}_2 \bar{X}_2 + \hat{D}_2 \bar{\beta}) + \hat{Q}_1 \bar{Y}_2 + \hat{S}_1^\intercal (\hat{B}_2^\intercal \bar{Y}_2 + \hat{D}_2^\intercal \bar{Z}_2)]dt \\ (\text{A.10}) &= -(\hat{A}_1^\intercal \bar{Y}_1 + \hat{C}_1^\intercal \bar{\pi} + \hat{Q}_1 \bar{X}_1 + \hat{S}_1^\intercal \bar{\pi})dt. \end{aligned}$$

Combining the two to produce the backward equation. Furthermore, the SMP condition (4.19) can be divided into (A.3) and (A.4):

$$\begin{aligned}
& B_1^\intercal \tilde{Y}_1 + D_1^\intercal \tilde{Z}_1 + S_1 \tilde{X}_1 + R_1 \tilde{\pi} \\
\text{(A.10)} &= -B_1^\intercal \tilde{X}_2 - D_1^\intercal (C_2 \tilde{X}_2 + D_2 \tilde{\beta}) - S_1 \tilde{Y}_2 - R_1 (B_2^\intercal \tilde{Y}_2 + D_2^\intercal \tilde{Z}_2) \\
\text{(4.8)} &= -B_1^\intercal \tilde{X}_2 - D_1^\intercal (D_1^\dagger)^\intercal (\tilde{\beta} - B_1^\intercal \tilde{X}_2) - [S_1 \tilde{Y}_2 + R_1 (B_2^\intercal \tilde{Y}_2 + D_2^\intercal \tilde{Z}_2)] \\
\text{((A6))} &= -B_1^\intercal \tilde{X}_2 + B_1^\intercal \tilde{X}_2 - [S_1 \tilde{Y}_2 + R_1 (B_2^\intercal \tilde{Y}_2 + D_2^\intercal \tilde{Z}_2) + \tilde{\beta}] \\
\text{(A.11)} &= 0
\end{aligned}$$

and

$$\begin{aligned}
& \hat{B}_1^\intercal \bar{Y}_1 + \hat{D}_1^\intercal \bar{Z}_1 + \hat{S}_1 \bar{X}_1 + \hat{R}_1 \bar{\pi} \\
\text{(A.10)} &= -\hat{B}_1^\intercal \bar{X}_2 - \hat{D}_1^\intercal (\hat{C}_2 \bar{X}_2 + \hat{D}_2 \bar{\beta}) - \hat{S}_1 \bar{Y}_2 - \hat{R}_1 (\hat{B}_2^\intercal \bar{Y}_2 + \hat{D}_2^\intercal \bar{Z}_2) \\
\text{(4.8)} &= -\hat{B}_1^\intercal \bar{X}_2 - \hat{D}_1^\intercal (\hat{D}_1^\dagger)^\intercal (\bar{\beta} - \hat{B}_1^\intercal \bar{X}_2) - [\hat{S}_1 \bar{Y}_2 + \hat{R}_1 (\hat{B}_2^\intercal \bar{Y}_2 + \hat{D}_2^\intercal \bar{Z}_2)] \\
\text{((A6))} &= -\hat{B}_1^\intercal \bar{X}_2 + \hat{B}_1^\intercal \bar{X}_2 - [\hat{S}_1 \bar{Y}_2 + \hat{R}_1 (\hat{B}_2^\intercal \bar{Y}_2 + \hat{D}_2^\intercal \bar{Z}_2) + \bar{\beta}] \\
\text{(A.12)} &= 0
\end{aligned}$$

Hence (4.19) equals 0. Lastly, check the initial and terminal conditions:

$$X_1(0) = -Y_2(0) = X_1,$$

$$\begin{aligned}
& \tilde{Y}_1(T) \\
\text{(A.10)} &= -\tilde{X}_2(T) \\
\text{(A.5)} &= -G_{T2}^{-1} \tilde{Y}_2(T) \\
\text{(A.10)} &= G_{T2}^{-1} \tilde{X}_1(T) \\
\text{(4.8)} &= G_{T1} \tilde{X}_2(T)
\end{aligned}$$

and

$$
\begin{aligned}
&\bar{Y}_1(T) \\
(\text{A.10}) &= -\bar{X}_2(T) \\
(\text{A.6}) &= -\hat{G}_{T2}^{-1}(\bar{Y}_2(T) - \hat{g}_{T2}) \\
(\text{A.10}) &= \hat{G}_{T2}^{-1}\bar{X}_1(T) + \hat{G}_{T2}^{-1}\hat{g}_{T2} \\
(4.8) &= \hat{G}_{T1}\bar{X}_1(T) + \hat{g}_{T1}.
\end{aligned}
$$

We are able to obtain the initial and terminal conditions of the primal problem. □