Imperial College London
Department of Physics

Generalised Geometry of Supergravity

Charles Strickland-Constable

August 1, 2012

Supervised by Professor Daniel Waldram

Submitted in part fulfilment of the requirements for the degree of Doctor of Philosophy in Physics of Imperial College London and the Diploma of Imperial College London

Declaration

I herewith certify that, to the best of my knowledge, all of the material in this dissertation which is not my own work has been properly acknowledged. The research described has been done in collaboration with Daniel Waldram and André Coimbra, and the presentation in chapters 3-6 follows closely the papers [1, 2, 3].

Charles Strickland-Constable

Abstract

We reformulate type II supergravity and dimensional restrictions of eleven-dimensional supergravity as generalised geometrical analogues of Einstein gravity. The bosonic symmetries are generated by generalised vectors, while the bosonic fields are unified into a generalised metric. The generalised tangent space features a natural action of the relevant (continuous) duality group. Also, the analogues of orthonormal frames for the generalised metric are related by the well-known enhanced local symmetry groups, which provide the analogue of the local Lorentz symmetry in general relativity.

Generalised connections and torsion feature prominently in the construction, and we show that the analogue of the Levi-Civita connection is not uniquely determined by metric compatibility and vanishing torsion. However, connections of this type can be used to extract the derivative operators which appear in the supergravity equations, and the undetermined pieces of the connection cancel out from these, leaving the required unique expressions. We find that the bosonic action and equations of motion can be interpreted as generalised curvatures, while the derivative operators appearing in the supersymmetry variations and equations of motion for the fermions become very simple expressions in terms of the generalised connection.

In the final chapter, the construction is used to reformulate supersymmetric flux backgrounds as torsion-free generalised G-structures. This is the direct analogue of the special holonomy condition which arises for supersymmetric backgrounds without flux in ordinary Riemannian geometry.

In loving memory of
Polly Strickland-Constable

Acknowledgements

There are many people who have been of great help and support to me during my studies.

First and foremost, I would like to thank my supervisor, Daniel Waldram, for many enlightening meetings and great patience. It seems rare to encounter someone who has both amazing insight and the ability to share their ideas so coherently, and I feel extremely lucky to have had such a great mentor.

I would also like to thank my colleagues and friends at Imperial. You have made Imperial a great place to be during my time here. On the academic front, I should express special thanks to André Coimbra, with whom I have spent many enjoyable hours discussing the material in this thesis (as well as football and many other things). I have also benefitted greatly from collaboration and discussions with Tom Pugh, concerning many topics in theoretical physics. Thanks also to Kelly Stelle for collaboration and discussions, and to Daniel Thomas for being a great office-mate.

Finally, I cannot express enough gratitude to my parents Fred and Polly, whose unwavering support has always gone beyond the extra mile and made everything possible for me.

Contents

1. Introduction

In this thesis, we will present generalised geometrical descriptions of supergravity theories which arise in the study of superstring theory at low energy. These new formulations write supergravity with the same geometric structure as Einstein's theory of gravity and unify the bosonic degrees of freedom. Simultaneously, the hidden symmetries of supergravity appear in the construction. In this introductory chapter, we outline the historical development and motivations behind this research, and also discuss significant precursors and related works.

Quantum mechanics and gravity

The central problem of modern theoretical physics is to find a unified quantum theory which describes all observed physical interactions. The standard model of particle physics describes the microscopic quantum mechanical behaviour of the elementary particles seen in accelerator experiments to a staggering degree of precision. At its core are the quantisations of some particular gauge field theories in a fixed special relativistic spacetime background. However, it does not contain gravity, whose quantisation is non-renormalisable when viewed as a conventional field theory.

Currently, our best experimentally verified theory of gravity is Einstein's theory of general relativity, which explains how gravitational effects are due to the curvature of spacetime. This curvature is determined, via Einstein's field equation, from the configuration of matter and energy it contains, so spacetime itself becomes a dynamical element, in contrast to its role as a fixed background in the standard model. The equations governing this dynamical spacetime have a geometrical structure, which makes the theory particularly elegant. However, this is a purely classical deterministic theory and the scales on which gravitational effects are observable reflects this. Gravity is a very weak force compared to the others in the standard model, and there is a natural scale at which quantum gravity effects are expected to become important. This scale is constructed from the fundamental constants h (Planck's constant), c (speed of light) and G (Newton's constant) which characterise the theory. It can be expressed as the Planck length ($\sim 10^{-33}$cm) or the Planck mass ($\sim 10^{19}$GeV). As these scales are totally inaccessible to instruments made with existing technology, it is currently impossible to probe the quantum nature of gravity directly in experiments. However, there are situations in which such quantum effects would play an important role, thus a quantum theory is a requirement on more than just theoretical grounds. One is the microscopic description of quantum properties of black holes. A more fundamental problem is how to describe spacetime correctly in the very early universe and resolve the physics of the big-bang singularity predicted by general relativity.

The construction of a quantum theory of gravity (even in the absence of matter) has proved very troublesome. However, before the discovery of QCD, a model of hadrons was put forward which became known as the dual resonance model. The plethora of hadrons observed in experiments appeared to have masses following a Regge slope, with mass squared proportional to the spin, and it seemed conceivable that the tower of spins extended infinitely. In 1968, Veneziano [4] constructed an amplitude for a theory containing such an infinite set of spins, featuring an exact crossing symmetry. This seemed to fit reasonably well with the known experimental results, but the model was eventually superseded by QCD, which was favoured by later high energy data.

Prior to this failure, it had emerged that the Veneziano model could be derived by replacing point particles by one dimensional objects, or relativistic strings, whose vibrational modes gave rise to the tower of spins. A feature of this theory was that it automatically contained a particle of spin-2 in its spectrum, and this had been regarded as a difficulty. However, in time, it was realised that this spin-2 particle could be the elusive graviton and that this model might give rise to a unified quantum theory including gravity. The study of string theory [5, 6, 7, 8, 9, 10] as a theory of everything was born!

String theories

The earliest work concerned only bosonic strings in flat space, the classical action of a string being the area of the worldsheet in spacetime. On quantisation, cancellation of the Virasoro anomaly requires 26 spacetime dimensions and even then the spectrum always contains an unwanted tachyon and does not contain fermions. On the other hand, the first excitations of the string give massless particles corresponding to a spacetime metric, an anti-symmetric 2-form gauge field and a scalar field. Further, one can include background values for these fields as couplings in the two-dimensional worldsheet action for the string, which is commonly referred to as the string sigma-model. The vanishing of the corresponding one-loop beta-functions then leads to Einstein's equations for the spacetime metric coupled to the additional 2-form and scalar fields [11]. Thus, maintaining the Weyl invariance at the quantum level imposes Einstein's equations of gravity on the target spacetime. Since Weyl invariance is associated to tadpole cancellation, which in turn means that the vacuum is a classical solution, the vanishing of the beta-functions must correspond to a solution of the equations of motion, which are thus also Einstein's equations. Despite its many failings, the bosonic string does give hints of a quantum theory of gravity.

Remarkably, all of these shortcomings can be remedied by a single cure. One considers the addition of fermionic modes on the string such that the resulting worldsheet theory has $N = 1$ supersymmetry. In this case, the anomalies of the super-Virasoro algebra cancel in 10 spacetime dimensions. On the surface, the resulting spectrum appears to have some problems. Firstly, it still contains a tachyonic ground state and secondly, it contains a spin-$\frac{3}{2}$ particle, the gauge field of local supersymmetry, but cannot have spacetime supersymmetry as the numbers of fermionic and bosonic degrees of freedom do not match. Both of these problems are resolved by imposing the GSO projection on the spectrum, which removes even numbers of fermionic excitations in the Neveu-Schwarz (NS) sector, and odd numbers in the Rammond (R) sector. These two sectors correspond to taking odd or even boundary conditions for the worldsheet fermions.

The result is that the NS and R sectors have a massless vector and a massless Majorana-Weyl spinor, respectively as their ground states. The massless modes of open strings therefore form vector supermultiplets while those of closed strings are the supergravity multiplets. For example, the type IIA and type IIB supergravity multiplets which feature prominently in this thesis arise from the ground states of oriented closed superstrings, where the left- and right-moving R sector ground states have opposite chirality in IIA and the same chirality in IIB. The massless particle spectrum is thus made up of the tensor products of the left- and right-handed ground states. If we label the vector representation of the Lorentz group as $\mathbf{v}$ and the chiral spinor representations as $\mathbf{s}^{\pm}$, these are the representations $\mathbf{v} \otimes \mathbf{v}$ (NS-NS), $\mathbf{s}^{+} \otimes \mathbf{v}$ (R-NS), $\mathbf{v} \otimes \mathbf{s}^{\mp}$ (NS-R) and $\mathbf{s}^{+} \otimes \mathbf{s}^{\mp}$ (R-R), where the upper sign is for IIA and the lower for IIB. The NS-NS fields are the metric, 2-form $B_{(2)}$ and scalar dilaton ϕ, while the NS-R and R-NS sectors provide the two gravitini and the two dilatini. The tensor product for the RR sector can be decomposed into a sum of odd forms in IIA and even forms in IIB. Background vacuum expectation values (VEVs) for the various massless tensor field strengths, such as $H = \mathrm{d}B$ and $\mathrm{d}\phi$, are known as fluxes, and these have become important the study of backgrounds of string theory in recent times. Tree-level amplitudes for superstrings can be seen to give the tree-level amplitudes of supergravity, so the effective low-energy approximation gives precisely these theories.

In fact, by taking the various allowed types of superstrings it is possible to build five different consistent string theories. The type II theories have closed oriented superstrings. Type I theory has unoriented closed superstrings coupled to open superstrings with gauge group $SO(32)$ and can be thought of as an orientifold projection of type IIB[1]. Only one of the supersymmetries survives this projection, so this theory has $N = 1$ spacetime supersymmetry. Surprisingly, one can also consider theories of closed oriented strings which combine the right-moving modes of the superstring with the left-moving modes of the bosonic string. This give rise to two ten-dimensional $N = 1$ supersymmetric theories, with massless modes forming the supergravity multiplet and a vector supermultiplet in the adjoint representation of a gauge group. It was found that only two consistent choices of gauge group are $SO(32)$ and $E_8 \times E_8$. The latter option is attractive on phenomenological grounds as it naturally contains the standard model gauge group via the chain of embeddings $SU(3) \times SU(2) \times U(1) \subset SU(5) \subset$

[1]More precisely, one considers type IIB with a spacetime filling $O9^{-}$ plane and 16 $D9$ branes, which give rise to the open string sector with gauge group $SO(32)$.

$Spin(10) \subset E_6 \subset E_7 \subset E_8$, and has been the starting point of many attempts to build realistic models of physics from string theory.

The low-energy approximations of these string theories are given by ten-dimensional supergravities, which were shown to be anomaly-free by Green and Schwarz in a renound paper [12]. As for the bosonic string, the equations of motion of the supergravity ensure one-loop quantum consistency of the string theory in question. Many features of low-energy string theory can be seen in the supergravity limit and historically studying supergravity has been a fruitful approach to understanding them.

Another important development in string theory was the realisation that strings are not the only fundamental objects of the theory. D-branes are hypersurfaces in space on which open strings end. They were first discussed in [13], but their importance was only brought to light by Polchinski [14], with the realisation that they are stable BPS states carrying the charges of the RR fields in type II theories. D-branes thus act as sources of RR fields. Also, as the ends of open strings are point-like, they can carry the charges of a vector gauge-field living on the brane. D-branes therefore provide a construction of non-abelian gauge symmetry in string theory, as N coincident D-branes gives rise to an $U(N)$ gauge theory on their worldvolume.

In addition to D-branes, there are other types of brane with different properties. The string carries the electric charge of the NS-NS (Kalb-Rammond) 2-form $B_{(2)}$, as can be seen from the term $\sim \int_{\text{worldsheet}} B_{(2)}$ in the string sigma-model. The magnetic charge is carried by a different object, the NS5-brane [15]. There are also orientifold planes, which are present when a string theory is quotiented by the simultaneous action of a spacetime reflection of the coordinates on the plane and an orientation reversal of the string. These carry RR charges and can enhance the gauge symmetry of N coincident D-branes to $SO(2N)$ if they also coincide with the orientifold plane. This provides the construction of type I theory from type IIB in footnote 1. Branes can be seen in the supergravity approximations to string theories as solitonic solutions or solutions with delta function sources for the fields to which they couple.

Dualities

It would certainly be desirable for the fundamental theory of everything to be unique, so the existence of five different possibilities would seem to be a drawback. However, much further investigation reveals that the five superstring theories are not as distinct as they first appear, but are in fact related by dualities. They are therefore viewed as being different manifestations of a higher unified theory. This theory is often referred to as M theory, but in this thesis we will reserve this term for a more specific meaning: the eleven-dimensional limit of the theory.

This eleven-dimensional limit was first truly recognised in [16, 17]. One of the celebrated results of early studies of higher dimensional supergravity, was the discovery of a unique supergravity theory in eleven dimensions [18]. Counting supermultiplet degrees of freedom indicates that this is the highest dimension in which a supergravity can exist, as more than 32 supercharges necessitates fields of spin greater than 2. It was immediately recognised that dimensional reduction on a circle resulted in the non-chiral IIA supergravity. The exponential of the VEV of the ten-dimensional dilaton field is proportional to the radius of this circle, so as this grows large, the spacetime "decompactifies" back to having eleven dimensions. On the other hand, when one considers strings in background fields, the term which one adds to the string sigma-model for the dilaton field is $S_\phi \sim \int \phi \mathcal{R}^{(2)}$. Computing string amplitudes at constant dilaton, this term is topological contributing a factor $\exp(-S_\phi) \sim (\mathrm{e}^{2\phi})^{(g-1)}$ where g is the genus of the worldsheet. Each loop is thus accompanied by a factor of $\mathrm{e}^{2\phi}$, so that the string coupling constant is e^{ϕ}. Combining these two observations, one is led to suspect that the strong coupling limit of type IIA string theory is an eleven-dimensional theory whose low-energy limit is eleven-dimensional supergravity.

One can also see this relation by considering the brane solutions. Eleven-dimensional supergravity has two fundamental brane solutions, the M2 and M5 branes [19, 20], and these are conjectured to be the fundamental objects of M-theory. The dimensional reduction of the M2 brane solution, with one direction wrapping the circle, gives the fundamental string solution of type IIA. In fact, it was shown earlier that the classical world-volume actions for these two objects are also related in this way [21], so one can view the string as a wrapped M2 brane. Quantum mechanical considerations led to the full conjecture that the IIA string theory can arise from supermembranes on a circle [16]. Similar results have been found for the other objects in type IIA [22].

Horava and Witten found a similar picture of the $E_8 \times E_8$ heterotic theory as a compactification of the same eleven-dimensional theory [23]. In this case, the compactification is not on a circle, but on a line interval (also referred to as the orbifold $S^1/\mathbb{Z}_2$). On the ten-dimensional boundaries of the space, one adds E_8 gauge supermultiplets to cancel the resulting anomalies. Taking the size of the interval to zero, one recovers a ten-dimensional theory with $E_8 \times E_8$ gauge symmetry. Further, the supersymmetry is halved to $N = 1$ by the presence of the boundaries. In this picture, the heterotic string comes about as the zero-separation limit of a cylindrical M2 brane with each of its circular ends lying on one of the boundaries.

The first duality to be discovered was T-duality [24], a correspondance between string theories compactified on tori. This can be seen in bosonic string theory, but in superstring theory it is easiest to discuss as a symmetry between type IIA and type IIB. In the simplest case, one can see that type IIA compactified on a circle of radius R gives the same theory as type IIB on a circle of radius $\tilde{R} = \alpha'/R$. Considering the worldsheet oscillator expansions one can see that the nine-dimensional spectra are the same, with the transformation exchanging compact momenta with winding modes. From this it is apparent that this duality is a stringy effect with no field theory counterpart. In [25], it was shown to be a symmetry of the full perturbation expansion, order-by-order in the string coupling, provided one shifts the dilaton such that $\sqrt{g}\mathrm{e}^{-2\phi}$ is invariant.

This symmetry of backgrounds is self-inverse, and so defines the group $\mathbb{Z}_2 \simeq O(1,1;\mathbb{Z})$. It is also clear that, since we have considered no background fields, the direction around the circle is an isometry of the background. More generally, one can consider compactification on a d-dimensional torus, with similar $U(1)$ isometries around the various circles it contains. Each circle gives a $\mathbb{Z}_2$ transformation mapping between IIA and IIB.

However, the symmetry of the perturbative spectrum can in fact be extended to a larger group $O(d,d;\mathbb{Z})$[2]. This is easiest to see for the bosonic string. The allowed left- and right-moving momenta on the string form an

[2] This group structure was first discovered in toroidal compactifications of the heterotic string [26, 27].

even self-dual lattice, and any two such lattices are related by an $O(d,d;\mathbb{R})$ transformation [26]. The automorphism group of the lattice is $O(d,d;\mathbb{Z})$, and so this becomes a symmetry of the overall spectrum, though it mixes the different mass levels of the string. It can be generated by $SL(d,\mathbb{Z})$ transformations, corresponding to large diffeomorphisms of the torus, and the $\mathbb{Z}_2$ transformations corresponding to the radial inversion in each circle. For the type II superstring, one sees that determinant -1 transformations exchange IIA and IIB, while determinant $+1$ transformations relate different backgrounds of the same theory.

Including constant background fields into the toroidal setup above, one can derive the the action of $M \in O(d,d;\mathbb{Z})$ on their values. For the metric and B-field, these can be written concisely as

$$G' = M^{-T} G M^{-1} \qquad \text{for} \qquad G = \tfrac{1}{2}\begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix} \tag{1.1}$$

Taking M to correspond to radial inversion of the last direction in the torus, this gives the well-known Buscher rules [28]

$$\begin{aligned} g'_{ij} &= g_{ij} - \tfrac{g_{id}g_{jd} - B_{id}B_{jd}}{g_{dd}} & \qquad B'_{ij} &= B_{ij} + 2\tfrac{g_{d[i}B_{j]d}}{g_{dd}} \\ g'_{id} &= \tfrac{B_{id}}{g_{dd}} & B'_{id} &= \tfrac{g_{id}}{g_{dd}} \\ g'_{dd} &= \tfrac{1}{g_{dd}} \end{aligned} \tag{1.2}$$

where $i = 1, \ldots, d-1$. Note that these relations should not be interpreted as tensor equations, since they relate the components of tensors on different manifolds, and T-duality does not induce a mapping of points in one manifold onto points in the other. In fact g and B should be regarded as moduli for the compactification.

With the background fields included, another type of $O(d,d;\mathbb{Z})$ generator gains a natural interpretation. The $O(d,d;\mathbb{Z})$ element

$$M = \begin{pmatrix} \mathbb{1} & 0 \\ \Lambda & \mathbb{1} \end{pmatrix} \tag{1.3}$$

has the effect of shifting B by Λ, and is a discretised version of the gauge symmetry associated to B. If one also shifts the integer labels of the winding momenta by Λ contracted on their corresponding momentum label, one finds that each point in the lattice is invariant.

T-duality symmetry has a manifestation in the supergravity approximation, as type II supergravity compactified on a torus has a global continuous $O(d,d;\mathbb{R})$ symmetry. The VEVs of the NS-NS internal scalar fields parameterise the coset $O(d,d;\mathbb{R})/O(d;\mathbb{R})\times O(d;\mathbb{R})$, which is thus the moduli space of such compactifications with constant internal NS-NS fields.

Considering the spectrum of the string, we see that the possible even self-dual lattices of momenta are related by $O(d,d;\mathbb{R})$ transformations, while the mass shell condition depends on the squares of the left- and right-moving momenta separately and so is stabilised only by $O(d;\mathbb{R})\times O(d;\mathbb{R})$. The moduli space of toroidal string compactifications becomes

$$\frac{O(d,d;\mathbb{R})}{O(d;\mathbb{R})\times O(d;\mathbb{R})\times O(d,d;\mathbb{Z})}, \tag{1.4}$$

where the additional quotient is by the discrete T-duality symmetry[3].

This is an example of the action of a continuous group $(O(d,d;\mathbb{R}))$ moving the system around in moduli space while a discretised version $(O(d,d;\mathbb{Z}))$ forms an exact symmetry of the quantised spectrum (i.e. a symmetry of the moduli space). The continuous version of the group is visible in the supergravity approximation as a global symmetry. This is a pattern which will recur in the next discussions.

Type IIB supergravity can be written with a manifest global $SL(2,\mathbb{R})$ symmetry, by pairing the axion with the dilaton and the two 2-form potentials into doublets. One is led to wonder whether this too is the result of a string duality with quantised group[4] $SL(2,\mathbb{Z})$. Unlike T-duality, this duality must be non-perturbative as it has an action which flips the sign of the dilaton, thus inverting the string coupling. One can also see this by considering that it rotates the doublet of 2-forms into each other, one coupling to the string and the other to the $D1$-brane. Due to this non-perturbative nature, there is no way to prove its existence directly with current understanding. However, strong evidence for it has been found by considering results which are believed to be exact in perturbation theory, such as the invariance of the masses of certain BPS objects [30]. Simpler evidence comes from the

[3]Note that this coset is defined by identifying $\Lambda \sim \Lambda'\Lambda\Lambda''$ where $\Lambda \in O(d,d;\mathbb{R})$, $\Lambda' \in O(d;\mathbb{R})\times O(d;\mathbb{R})$ and $\Lambda'' \in O(d,d;\mathbb{Z})$, as when acting on a fixed reference lattice of momenta, two elements related in this way give the same physical spectrum.

[4]This type of duality was also first considered in the context of heterotic string theory [29, 30, 31]. The first discussion for type II strings appeared in [32].

observation that the tensions of the type IIB strings and branes match up if one inverts the string coupling.

In a landmark paper [32], Hull and Townsend combined the S- and T-dualities into a unified U-duality with group $E_{d+1(d+1)}(\mathbb{Z})$ for type II theories compactified on a d-dimensional torus. This contains the S and T-dualities in the subgroup $O(d, d; \mathbb{Z}) \times SL(2, \mathbb{Z})$. U-duality must also be non-perturbative, but was shown to pass the same tests as S-duality [32]. The continuous exceptional groups had been seen in eleven-dimensional supergravity by Cremmer and Julia [33] as early as 1978, and this global symmetry is now recognised as the U-duality analogue of the $O(d, d; \mathbb{R})$ symmetry connected to the discussion of NS-NS fields above, i.e. the continuous group which acts transitively on the full moduli space. However, it is important to remember that duality transformations mix the different types of modes in string theory (e.g. they can exchange strings and branes), so there is no hope that these dualities can be a true symmetry of the (point-particle) supergravity approximation.

As mentioned above, type I theory can be viewed as an orientifold projection of type IIB with D-branes added to cancel the total RR charge [13, 14]. It is therefore not so surprising to learn that T-duality of this leads to a type I′ theory, which is similarly related to type IIA [34]. S-duality on the type I theory results in the $SO(32)$ heterotic theory [17, 34], and the T-dual of this is the $E_8 \times E_8$ heterotic theory [26, 27, 35]. Thus we see that all five of the superstring theories are connected by a web of dualities.

We should stress that the list of dualities discussed above is in no way the full picture. Another very important duality was discovered by Maldacena in 1997 and is known as AdS/CFT duality [36] (see [37] for a review). This is conjectured to give an exact quantum equivalence between string theory on an AdS background and a dual superconformal gauge theory in Minkowski space of one dimension less. It can be viewed as a consequence of open/closed string duality [38], which intuitively relates a gauge theory with a gravitational theory. The duality also inverts the coupling constant, so that a strongly-coupled gauge theory is described by a weakly-coupled string theory and vice-versa. An enormous literature has emerged on this subject, not least because of its potential applications in describing the strongly-coupled systems which appear in more conventional particle physics and condensed matter systems.

Backgrounds: Geometric and Non-geometric

One of the apparent drawbacks of string theory is that it does not immediately give us a four-dimensional model that can be related to the real world. One way to remedy this is to compactify on a tiny internal manifold such that the effective physics appears four dimensional. The two length scales in the problem are then the compactification scale and the string scale, and if both are suitably small compared to an observable cut-off scale, then only the lowest modes associated to each survive into the effective theory.

For phenomenological reasons, it is desirable to preserve $N = 1$ supersymmetry in the four-dimensional effective theory. This requirement places strong constraints on the geometry of the internal space, which are discussed in more detail in chapter 7. Here we merely note that for heterotic strings with no warping, the constraints imply the vanishing of all internal fluxes and the background is a Calabi-Yau space [39]. This is an excellent example of the beauty of the mathematics of string theory.

Calabi-Yau spaces are also related by dualities [40, 41]. It was found that type IIA compactified on a Calabi-Yau manifold gives the same theory as type IIB compactified on a different Calabi-Yau. This relation, which is known as mirror symmetry, gives a mathematically surprising symmetry between apparently very different manifolds. For example, mirror symmetry exchanges the hodge numbers $h^{1,1}$ and $h^{2,1}$ thus relating manifolds of different topology. This further emphasises that dualities are not mappings of manifolds in any conventional sense. It has been argued that mirror symmetry can be thought of as a kind of T-duality [42], though Calabi-Yau manifolds have no continuous isometries, so the duality must act along non-isometry directions.

However, Calabi-Yau manifolds are not the whole story. The parameters which define the continuous deformations of the shape and size of a Calabi-Yau space (the moduli) have no potential in such a compactification and give rise to massless scalar fields. The VEVs of these fields are undetermined by the theory and this causes a loss of predictive power. For example, coupling constants can be set by the VEVs. This is the moduli space problem.

One therefore wishes to find a mechanism which stabilises the moduli dynamically. Considering a warped geometry with the addition of internal fluxes can generate a potential which gives masses to some of the moduli fields [43, 44, 45], thus fixing their values at the minimum of the potential. Fluxes can also provide a mechanism to break supersymmetry [44], and can even break all the supersymmetry, as required to recover the standard model at low energies. They can also generate large hierarchies [45].

Flux compactifications are also important in AdS/CFT duality, as the backgrounds involved generally have non-zero fluxes. For example, the classic $\mathrm{AdS}_5 \times S^5$ background generically has a non-zero RR 5-form flux and solutions with 3-form flux can provide the string theory duals of confining gauge theories [46]. Further, they often provide the string theory uplifts of lower dimensional gauged supergravities (see e.g. [47]).

Another curious feature of string dualities is that they can generate exotic types of background from ordinary supergravity backgrounds [48]. The simplest example one can consider is the bosonic string on a rectangular 3-torus with metric $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ and $B = x\mathrm{d}y \wedge \mathrm{d}z$. Applying the Buscher rules along the z-direction leads to a twisted torus with vanishing B-field. A further T-duality along the y-direction leads to a non-geometric background, with the monodromy in the periodic coordinate $x \sim x+1$ given by a T-duality transformation. In a sense, the source of the problem is that the local Killing vector of the twisted torus $\partial/\partial y$ is not globally defined. It is also worth considering that T-duality will always have a small circle on one side of the duality, hence the supergravity limit may not be a good approximation in these cases.

That the CFT description of strings need not have a conventional spacetime target space is made apparent by the construction of the heterotic string, where the left- and right-movers can be thought to see different target spaces. This idea is also used in the construction of asymmetric orbifolds in [49]. Early constructions of non-geometric backgrounds as backgrounds in their own right, motivated by stabilisation of moduli, can be found in [50, 51, 52].

One class of non-geometric backgrounds are made up of locally geometric solutions which are patched together by duality transformations. Examples are local torus fibrations patched together by T-dualities (as in the example above), which were dubbed T-folds by Hull [53], who also named backgrounds with S- and U-duality transition functions S-folds and U-folds respectively. In some ways it seems natural to extend the diffeomorphism and gauge patching to include the full set of transformations available in string theory. However, there are also backgrounds which are not even locally geometric [54], and conceptually these are harder to grasp. The precise mathematics of non-geometry is still yet to be fully understood.

Hidden Symmetries and Duality Covariant Formulations

Dualities of string theory and M theory can appear as global symmetries in the low energy supergravity. In fact, these "hidden symmetries" were observed in supergravity long before the discovery of dualities in string theory and there is a substantial history of efforts to reformulate the various theories in such a way that these symmetries become manifest. Here we will briefly mention the results and proposals of these works and other approaches to understanding these higher symmetries.

The hidden symmetries of eleven-dimensional supergravity first appeared in [33]. Here it was found that dimensional reduction of eleven-dimensional supergravity on a seven-dimensional torus resulted in a four-dimensional supergravity with global $E_{7(7)}$ and local $SU(8)$ symmetry. The authors note that this was the first time exceptional groups had appeared naturally in physics. It was established in [55] that these symmetries were present in the eleven-dimensional theory with no dimensional reduction, only requiring the mild assumption of a local product structure on the eleven-dimensional manifold. The analysis was extended to an eight-dimensional split with global $E_{8(8)}$ and local $SO(16)$ symmetries in [56]. This paper also makes the observation that the theory does not possess true $E_{d(d)}$ invariance, as the fermions do not transform in $E_{d(d)}$ representations and the explicit form of the "vielbein" written with indices of the larger symmetry group breaks the symmetry. These works commented on the mysterious origin of the $E_{d(d)}$ group and put forward the idea that these formulations may have an underlying geometry.

Another line of research was that pursued by Duff [57], who examined duality at the level of the string worldsheet theory. He constructed a first-order lagrangian for the sigma-model and also a dual lagrangian, written in terms of new coordinates dual to the string winding modes. The roles of equations of motion and Bianchi identities are exchanged for the dual version. He went on to write classical equations of motion for the string with a manifest action of the $O(d,d)$ group, which mixes the normal and winding coordinates, and also rotates the equations of motion and Bianchi identities into each other. Independently, Tseytlin [58] worked out a Lagrangian in two dimensions containing both a scalar field and its dual, and also considered the addition of interactions containing both fields. When applied to the string sigma-model on a torus, this Lagrangian was such that the dual fields could be identified with the dual target space coordinates and the duality appeared as a symmetry of the worldsheet theory.

Duff's analysis was also performed for the M2-brane worldvolume theory in [59][5]. The extra coordinates associated with the membrane have the index structure of a 2-form with respect to the ordinary spacetime coordinates and are dual to the winding of membranes.

The idea of coordinates dual to winding modes was also adopted by Siegel [61], who formulated the low-energy effective action and equations of motion of the NS-NS sector fields as a curvature on a doubled space. The dependence of fields on the doubled space was restricted to a conventional space section via the same conditions as in modern approaches to be discussed. This remarkable work effectively contains many of the results of more recent research, and we will comment on the precise connections to the present work in the conclusion.

Later works on the hidden symmetries of eleven-dimensional supergravity have also featured extra coordinates. In [62], a "generalised vielbein" parameterising the coset $E_{8(8)}/SO(16)$ is constructed and the authors argue that this encompasses the propagating degrees of freedom, like the vielbein in Einstein gravity. They also speculate that there may exist extra coordinates connected to a mixing of diffeomorphisms and gauge transformations. Further discussion of such extra coordinates can be found in [63].

Much later still, Hillmann [64] considered the problem of hidden $E_{7(7)}$ and $SU(8)$ symmetries from a new angle. He considered a $4 + 56$-dimensional spacetime, as proposed in [63], with a manifest $E_{7(7)}$ symmetry and constructed a Lagrangian for it such that eleven-dimensional supergravity equations are recovered by restricting to a $4+7$ dimensional spacetime. The Lagrangian is fixed by the requirement that the $GL(7,\mathbb{R})$ subgroup of $E_{7(7)}$ is promoted to full diffeomorphism symmetry on this reduction. His construction also includes the local $SU(8)$ covariant form of the fermionic sector.

[5]A nice review can be found in [60], where the formulation was used to study U-duality of toroidal compactifications of M-theory.

Hillmann also constructed a particular $SU(8)$ connection with similar properties to those introduced in this thesis, and demonstrated its relation with the supersymmetry variations.

More ambitious proposals in which eleven-dimensional supergravity, or even M theory itself, are claimed to have certain Kac-Moody symmetries have been put forward in recent times. The idea that the Kac-Moody algebras E_d for $d > 8$ could appear in supergravity goes back to [65]. The mildest extension of de Wit and Nicolai's construction to a $2 + 9$-dimensional split with $E_{9(9)}$ replacing $E_{7(7)}$ was examined in [66], and details of the bolder extension to E_{10} were first considered in [67].

A non-linear realisation of the Kac-Moody algebra E_{11} has been conjectured to be the symmetry underlying M theory by West [68]. In an earlier work [69], it was observed that eleven-dimensional supergravity can be written as a non-linear realisation of an algebra, and in [68] it was conjectured that this can be extended to a non-linear realisation of E_{11} for some appropriate reformulation of the supergravity. The author also argues that a discrete version the symmetry may carry over to M theory and provide an algebraic tool with which to investigate its fundamental nature. In [70], the $GL(11, \mathbb{R})$ decomposition of the first fundamental representation of E_{11} was shown to have the known central charges (TM, $\Lambda^2 T^*M$ and $\Lambda^5 T^*M$) of the supersymmetry algebra at low levels. At the same time, the remaining infinite tower of representations was conjectured to give the rest of the fundamental charges present in M theory, the next one along in the series corresponding to the dual graviton (or KK-monopole) of [71][6]. It was also proposed that spacetime must be extended to contain extra coordinates corresponding to each of these charges. There have been many subsequent papers exploring this construction [73, 74].

Damour, Henneaux and Nicolai [75] proposed that M theory compactified on all ten space dimensions can be formulated as a gradient expansion, with the spacial gradients of the fields filling out infinite dimensional representations of E_{10}. Since the theory is effectively considered at a point in space, which becomes causally disconnected from other point in the limit taken, spacetime is emergent in this construction. Fermions can be included as representations of the subalgebra KE_{10} [76], which builds on the earlier studies [77].

[6]The has been some debate over the relation of dual gravity and E_{11} [72].

Hidden symmetries have also surfaced in the study of supersymmetric backgrounds. In [78], the derivative appearing in the external gravitino variation is considered in the context of dimensional splits of eleven-dimensional supergravity, where the local symmetry is enhanced as in [55, 56]. It is conjectured that one can associate a generalised holonomy group to this derivative and that the number of supercharges preserved by a given solution will be the number of singlets in the decomposition of the eleven-dimensional spinor under this subgroup of the usual hidden symmetry group. A related and extended analysis with no dimensional split was given by Hull in [79]. The full generalised holonomy was shown to be $SL(32,\mathbb{R})$, which contains the groups considered in [78][7], and solutions with generalised holonomy falling outside the scope of [78] were exhibited. It was also argued that $SL(32,\mathbb{R})$ should be a hidden symmetry of M theory as one needs the local $SL(32,\mathbb{R})$ bundle in order to couple fermions, similarly to the coupling of spinors in general relativity.

Generalised Geometry and Doubled Formalisms

During the last decade, a new mathematical construction has appeared named generalised geometry [81, 82]. This is the study of structures on a vector bundle $E \simeq T \oplus T^*$ over a manifold. There is a natural $O(d,d)$ metric on this bundle and one can think of it as having an $O(d,d)$ structure. One can also define the analogue of a complex structure to obtain generalised complex geometry, which contains complex and symplectic geometries as limiting cases, and this unification was a large part of the original motivation for the construction.

In the mathematics literature, the basic notion of the generalised tangent space with an $O(d,d)$ metric and a suitable bracket is known as an exact Courant algebroid (see [83, 84] and references therein). Additional "generalised geometry" structures on such objects, specifically generalised complex structures and $O(d) \times O(d)$ generalised metrics, were introduced by Hitchin and Gualtieri [81, 82]. Connections on Courant algebroids were introduced in [85] (see also [84]) and again in [86, 87], together with a notion of torsion and compatibility with the generalised metric.

[7] However, some concerns as to the consistency of the embeddings of the lower dimensional hidden symmetry groups into $SL(32,\mathbb{R})$ have been noted in [80].

The relevance of generalised geometry for the NS-NS sector of string theory was observed soon after its birth. To date, this construction has largely been used to describe supersymmetric backgrounds of type II string theory, which has been a successful and fruitful program [88, 89, 90, 91]. It was also found [92] that the various potentials of the compactified theories could be written in terms of Hitchin functionals [93, 94, 81]. Further still, the $O(d,d)$ group appears and there are links with T-duality, mirror symmetry and non-geometric backgrounds [95, 96, 97, 98]. Generalised complex geometry has also been widely used in the study of sigma models and topological string theory [99].

The extension including RR fields in type II theories and M theory compactifications was found independently by Hull [100] and Pacheco and Waldram [101]. The $O(d,d)$ group of the original generalised geometry is replaced by an $E_{d(d)}$ group, reflecting U-duality. These exceptional generalised geometries have been used to find the superpotentials of M theory compactifications [101] and study supersymmetric backgrounds of type II theories with RR fluxes [102, 103]. They were shown to have the structure of Leibniz algebroids in [104]. This geometry, or in fact an extension of it containing an $\mathbb{R}^+$ factor in the structure group, and a similar extension of the original generalised geometry are the main subject of this thesis, and we will show that the low energy supergravity can be completely reformulated in this language.

Another important development in this area is doubled geometry [53]. In order to describe T-folds, Hull considered backgrounds with a T^n torus fibration structure and considered the enlargement of the fibres to T^{2n}. Taking the base of the fibration to be a circle, one can then consider the case where the transition function joining the ends of the circle is by an $O(n,n;\mathbb{Z})$ T-duality transformation, so that the background is non-geometric in nature. It was shown that the $O(n,n;\mathbb{Z}) \subset GL(2n,\mathbb{Z})$ transformation could be viewed as a large diffeomorphism of the doubled fibre T^{2n}, and the physical space (or "polarisation") as a slice of the doubled torus. The different polarisations in T^{2n} are then the T-duality-related backgrounds, and T-folds arise when there is no global choice of polariasation. The extra coordinates on T^{2n} are interpreted as the duals of the winding modes of the string, as in the older works [57, 58, 61]. If the $O(n,n;\mathbb{Z})$ transition function is not contained in the subgroup $GL(n,\mathbb{Z})$ of large diffeomorphisms of T^n then it mixes the momentum and winding modes of the string, and is thus a truly stringy effect.

This construction was extended to consider T-duality of the base circles as well as the T^n fibres [54]. This requires an extension to "generalised T-duality" (further discussed in [105, 106, 107]) which allows T-dualisation along non-isometry directions, such as the base circle in question. The coordinates of the base were then doubled in the same way as the fibre. T-duality around a base circle leads to a new type of non-geometric background which is not even locally geometric. This is because the geometry has a non-trivial dependence on the coordinate x around the circle, which under generalised T-duality is mapped to a dependence on the dual winding coordinate $\tilde{x}$. A nice summary of this is to say that a T^n fibration over T^m is geometric, a T^{2n} fibration over T^m is a T-fold and a T^{2n} fibration over T^{2m} is not even locally geometric.

The relation between doubled geometry and generalised geometry was neatly summarised in [108]. Essentially, generalised geometry has only transition functions which are true symmetries of supergravity (diffeomorphisms and gauge transformations), while doubled geometry includes the more exotic $O(d, d; \mathbb{Z})$ transition functions. However, generalised geometry can be defined on any manifold, while doubled geometry is only naturally associated with torus fibrations of the type discussed. Intuitively, this is due to the fact that string winding is only possible if the topology of the spacetime has the necessary non-trivial cycles.

Double field theory [109, 110, 111, 112] was introduced as a natural continuation of doubled geometry inspired by string field theory, studying the massless fields of closed strings on a doubled torus T^{2d}. The fields, in principle, are allowed to depend on both ordinary and winding coordinates. However, their dependence is restricted by implementing the level matching constraint of closed string theory by requiring the fields to be annihilated by the second order operator $\eta^{AB}\partial_A\partial_B$. In fact, gauge invariance of the resulting expressions requires the "strong constraint" which asserts that all products of fields must also be annihilated. In [111], this is shown to be equivalent to local dependence only on a d-dimensional isotropic slice of the doubled torus. Such a null subspace is always related, by an $O(d, d)$ transformation, to the usual physical subspace in which only the ordinary physical coordinates vary. Thus the level matching constraint appears locally to force recovery of a sensible d-dimensional physical space.

The natural generalised Lie derivative (or Dorfman derivative) and Courant bracket of generalised geometry naturally carry over to double field theory, when one switches on dependence on the winding coordinates, and the algebra of the former closes into the latter on imposing the strong constraint [112] (this is noted purely in terms of the gauge transformations in [110]). In [112], an action for the NS-NS fields is constructed from first-derivatives of the generalised metric (which is built from the metric and B-field as in generalised geometry), and is shown to be gauge invariant and equivalent to the standard action up to integration-by-parts on imposing the strong constraint. It is argued that this first-order expression should be related to a curvature scalar for the geometry, in the same way that the Einstein-Hilbert action can be integrated by parts to involve only first derivatives of the metric, though no geometrical construction of the expression is offered. This same action was also derived by West in [113].

The analogue of this construction has been worked through for dimensional restrictions of eleven-dimensional supergravity [114]. In this case, the extra coordinates introduced are those proposed in [63], which are said to be dual to the winding modes of the membrane and five-brane[8]. A similar action to that in [112] is found by brute force methods imposing diffeomorphism and gauge invariance. On restricting the dependence of fields to the usual physical coordinates the expected action is recovered after integration-by-parts. The analogue of the strong constraint in four dimensions is derived in [115] by requiring the algebra of generalised Lie derivatives to close. This agrees with the result given later in this thesis for the case $d = 4$.

In very recent times, there has been a flurry of activity in this area. Several papers [116, 117] have appeared which essentially relate the action of [112] to the much earlier work of Siegel [61]. A discussion of RR fields in double field theory was made in [118] (see also [119]), resulting in similar expressions to those presented in this thesis, and a no-go result concerning the construction of Riemann-squared terms from derivatives of the generalised metric appears in [120]. Considerations of heterotic theory in this context have appeared in [121, 122], again building on the ideas of Siegel [61]. As suggested in [106, 107, 123], the effective action of compactifications

[8]The final set in seven dimensions correspond to the Kaluza-Klein monopole as in [100, 101]

on doubled tori are considered in [124, 125] and the general gaugings[9] of $N = 4$ supergravity are recovered. Interestingly some indication is found that the strong constraint may not be necessary in these cases. This idea is also echoed in [128, 129]. Our results for the fermions have been re-cast into double field theory in [130, 131]. Other recent papers include [132, 133, 134].

The Structure of this Thesis

The main questions raised in the previous discussion concern the understanding of dualities and hidden symmetries in string theory and M theory. Dualities are relatively well understood for torus backgrounds, and mirror symmetry is well-established, but in other scenarios much less can be said using present techniques. In particular, the fact that duality can map a geometric background to a non-geometric configuration suggests that some new mathematics encompassing both classes needs to be found. While doubled geometry provides such a framework for torus backgrounds and their duals, little progress has been made for arbitrary solutions.

In this thesis, we use generalised geometry to reformulate the supergravity limits of type II theories and M theory. While the symmetry of the theory is still restricted to the geometric subgroup, the fields are organised into objects which transform under the (continuous) duality group. The dynamics becomes geometrical, as for Einstein gravity, answering the question posed in [55] as to the nature of the geometry of hidden symmetries. The formulation uncovers surprising new structure in these supergravities and the geometry of their supersymmetric solutions. The hope is that this new structure will help to shed light on dualities and non-geometric backgrounds, as well as providing new tools with which to study geometric ones.

Subjectively, the construction also has a very pleasing naturalness and elegance, which is usually a sign that one is looking at something in the right way!

We begin with a brief review of some of the geometrical ideas which will feature throughout the thesis. We examine the way that these appear in Einstein gravity and study the bosonic symmetries of the NS-NS sector of type II supergravity. Comparing the two provides motivation for the definition of the enlarged tangent space of generalised geometry.

[9] General gaugings of supergravity have been classified using the embedding tensor formalism [126, 127]. The relation to our results is discussed in [2]

In chapter 3, we briefly review some basic objects one can define on $T \oplus T^*$, and then proceed to construct $O(d,d) \times \mathbb{R}^+$ generalised geometry. Particular focus here is on generalised connections, torsion and curvatures, and we provide a thorough treatment of these. These results are used in chapter 4 to give a complete reformulation of type II supergravity theories as a generalised geometrical analogue of Einstein gravity, defined by an $O(9,1) \times O(1,9) \subset O(10,10) \times \mathbb{R}^+$ structure on the generalised tangent space. In this description the NS-NS fields are unified as a generalised metric, while the RR fields and fermions fall into representations of the enlarged symmetry groups and their supersymmetry variations and equations of motion are neatly expressed in terms of the generalised connection.

Chapters 5 and 6 present the $E_{d(d)} \times \mathbb{R}^+$ generalised geometry description of dimensional restrictions of eleven-dimensional supergravity. Chapter 5 runs through the definitions of the geometrical structures, which are exactly analogous to those in chapter 3, though considerably more complicated due to the exceptional groups involved. The equations are presented under the $GL(d,\mathbb{R})$ decomposition of $E_{d(d)} \times \mathbb{R}^+$ and the $O(d)$ decomposition its maximal compact subgroup. This allows us to write equations which hold in all dimensions d.

Chapter 6 begins by defining exactly what we mean by dimensional restrictions of eleven-dimensional supergravity. The next section provides some very abstract equations, which describe how to derive the supergravity equations from the geometry. In fact, these equations are equally true in the context of chapters 3 and 4. The realisation of these equations in $Spin(d)$ representations is then given, resulting in a similar reformulation of the supergravity to that in chapter 4. However, since in this case the geometry encompasses all of the bosonic fields, the entire bosonic action is given simply by the generalised scalar curvature.

Finally, in chapter 7, we apply the formalism to supersymmetric backgrounds. After reviewing the basics of G-structures, intrinsic torsion and supersymmetric backgrounds, we demonstrate that the Killing spinor equations translate into the analogue of special holonomy in generalised geometry. Concluding remarks follow in the last chapter.

The appendices describe our general conventions, the necessary Clifford algebras, the details of the exceptional groups, a group theoretical proof needed in section 5.2.3 and some spinor decompositions.

2. Differential Geometry and Gravity

To set the stage, we review some of the key constructions of ordinary differential geometry, in particular the notion of a G-structure. We then discuss the formulation of general relativity in this language, with a view to writing more complicated supergravity theories in the same elegant way. In the final subsection, we study some features of type II supergravities to motivate the introduction of generalised geometry.

2.1. Metric structures, torsion and the Levi–Civita connection

Let M be a d-dimensional manifold. We write $\{\hat{e}_a\}$ for a basis of the tangent space T_xM at $x \in M$ and $\{e^a\}$ for the dual basis of T^*_xM satisfying $i_{\hat{e}_a} e^b = \delta_a{}^b$. Recall that the frame bundle F is the bundle of all bases $\{\hat{e}^a\}$ over M,

$$F = \{(x, \{\hat{e}_a\}) : \ x \in M \text{ and } \{\hat{e}_a\} \text{ is a basis for } T_xM\} . \tag{2.1}$$

On each fibre of F there is an action of $A^a{}_b \in GL(d, \mathbb{R})$, given $v = v^a \hat{e}_a \in \Gamma(T_xM)$,

$$v^a \mapsto v'^a = A^a{}_b v^b, \qquad \hat{e}_a \mapsto \hat{e}'_a = \hat{e}_b (A^{-1})^b{}_a . \tag{2.2}$$

giving F the structure of a $GL(d, \mathbb{R})$ principal bundle[1].

The Lie derivative $\mathcal{L}_v$ encodes the effect of an infinitesimal diffeomorphism. On a vector field w it is equal to the Lie bracket

$$\mathcal{L}_v w = -\mathcal{L}_w v = [v, w] , \tag{2.3}$$

[1] We define a principal G-bundle as a fibre bundle $F \xrightarrow{\pi} M$ together with a continuous action $G \times F \to F$ which preserves the fibres of F and acts freely and transitively on them. This definition implies that the bundle is equivalent to a bundle with fibre G, the mapping arising from a choice of local sections to map to the identity in the fibre.

while on a general tensor field α one has, in coordinate indices,

$$\begin{aligned}\mathcal{L}_v \alpha^{\mu_1\dots\mu_p}_{\nu_1\dots\nu_q} &= v^\mu \partial_\mu \alpha^{\mu_1\dots\mu_p}_{\nu_1\dots\nu_q} \\ &\quad + (\partial_\mu v^{\mu_1})\, \alpha^{\mu\mu_2\dots\mu_p}_{\nu_1\dots\nu_q} + \dots + (\partial_\mu v^{\mu_p})\, \alpha^{\mu_1\dots\mu_{p-1}\mu}_{\nu_1\dots\nu_q} \\ &\quad - (\partial_{\nu_1} v^\mu)\, \alpha^{\mu_1\dots\mu_p}_{\mu\nu_2\dots\nu_q} - \dots - \left(\partial_{\nu_q} v^\mu\right) \alpha^{\mu_1\dots\mu_p}_{\nu_1\dots\nu_{q-1}\mu}.\end{aligned} \tag{2.4}$$

Note that the terms on the second and third lines can be viewed as the adjoint action of the $\mathfrak{gl}(d,\mathbb{R})$ matrix $a^\mu{}_\nu = \partial_\nu v^\mu$ on the particular tensor field α, i.e. we can write

$$\mathcal{L}_v \alpha = \partial_v \alpha - (\partial v) \cdot \alpha \tag{2.5}$$

This form will have an analogous expression when we come to generalised geometry.

Let $\nabla_\mu v^\nu = \partial_\mu v^\nu + \omega_\mu{}^\nu{}_\lambda v^\lambda$ be a general connection on TM. The torsion $T \in \Gamma(TM \otimes \Lambda^2 T^* M)$ of ∇ is defined by

$$T(v,w) = \nabla_v w - \nabla_w v - [v,w]\,. \tag{2.6}$$

or concretely, in coordinate indices,

$$T^\mu{}_{\nu\lambda} = \omega_\nu{}^\mu{}_\lambda - \omega_\lambda{}^\mu{}_\nu, \tag{2.7}$$

while, in a general basis where $\nabla_\mu v^a = \partial_\mu v^a + \omega_\mu{}^a{}_b v^b$, one has

$$T^a{}_{bc} = \omega_b{}^a{}_c - \omega_c{}^a{}_b + [\hat{e}_b, \hat{e}_c]^a\,. \tag{2.8}$$

Since it has a natural generalised geometric analogue, it is useful to define the torsion equivalently in terms of the Lie derivative. If $\mathcal{L}^\nabla_v \alpha$ is the analogue of the Lie derivative (2.4) but with ∂ replaced by ∇, and $(i_v T)^\mu{}_\nu = v^\lambda T^\mu{}_{\lambda\nu}$ then

$$(i_v T)\alpha = \mathcal{L}^\nabla_v \alpha - \mathcal{L}_v \alpha, \tag{2.9}$$

where we view $i_v T$ as a section of the $\mathfrak{gl}(d,\mathbb{R})$ adjoint bundle acting on the given tensor field α.

The curvature of a connection ∇ is given by the Riemann tensor $\mathcal{R} \in \Gamma(\Lambda^2 T^*M \otimes TM \otimes T^*M)$, defined by

$$\begin{aligned} \mathcal{R}(u,v)w &= [\nabla_u, \nabla_v]w - \nabla_{[u,v]}w, \\ \mathcal{R}_{\mu\nu}{}^{\lambda}{}_{\rho} w^{\rho} &= [\nabla_\mu, \nabla_\nu]w^\lambda - T^{\rho}{}_{\mu\nu}\nabla_\rho w^\lambda. \end{aligned} \tag{2.10}$$

The Ricci tensor is the trace of the Riemann curvature

$$\mathcal{R}_{\mu\nu} = \mathcal{R}_{\lambda\mu}{}^{\lambda}{}_{\nu}. \tag{2.11}$$

If the manifold admits a metric g then the Ricci tensor becomes symmetric in its indices and one can define the Ricci scalar curvature as

$$\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu}. \tag{2.12}$$

A G-structure is a principal sub-bundle $P \subset F$ with fibre G. In the case of the metric g, the $G = O(d)$ sub-bundle is formed by the set of orthonormal bases

$$P = \{(x, \{\hat{e}_a\}) \in F : g(\hat{e}_a, \hat{e}_b) = \delta_{ab}\}, \tag{2.13}$$

related by an $O(d) \subset GL(d,\mathbb{R})$ action. (A Lorentzian metric defines a $O(d-1,1)$-structure and δ_{ab} is replaced by η_{ab}.) At each point $x \in M$, the metric defines a point in the coset space

$$g|_x \in GL(d,\mathbb{R})/O(d). \tag{2.14}$$

In general the existence of a G-structure can impose topological conditions on the manifold, since it implies that the tangent space can be patched using only $G \subset GL(d,\mathbb{R})$ transition functions. For example, for even d, if $G = GL(d/2,\mathbb{C})$, the manifold must admit an almost complex structure, while for $G = SL(d,\mathbb{R})$ it must be orientable. However, for $G = O(d)$ there is no such restriction.

A connection ∇ is compatible with a G-structure $P \subset F$ if the corresponding connection on the principal bundle F reduces to a connection on P. This means that, given a basis $\{\hat{e}_a\}$, one has a set of connection one-forms $\omega^a{}_b$ taking values in the adjoint representation of G given by

$$\nabla_{\partial/\partial x^\mu}\hat{e}_a = \omega_\mu{}^b{}_a \hat{e}_b. \tag{2.15}$$

For a metric structure this is equivalent to the condition $\nabla g = 0$. If there exists a torsion-free compatible connection, the G-structure is said to be torsion-free or equivalently integrable (to first order). In general this can further restrict the structure, for instance in the case of $GL(d/2, \mathbb{C})$ it is equivalent to the existence of a complex structure (satisfying the Nijenhuis condition). However, for a metric structure no further conditions are implied, and furthermore the torsion-free, compatible connection, namely the Levi–Civita connection, is unique.

2.2. General Relativity

As a motivation for things to come, we will examine Einstein's theory of gravity written in terms of the geometry presented in the previous section. The eventual goal will be to find a formulation of more complicated supergravity theories which have the same geometrical structure.

In general relativity, spacetime is taken to be a d-dimensional differentiable manifold M equipped with a pseudo-Riemannian metric of signature $(- + \ldots +)$. The metric, which makes up the field content of the pure gravity theory, reduces the structure group of the tangent bundle TM to the Lorentz group[2] $SO(d-1, 1) \subset GL(d, \mathbb{R})$. As stated previously, there is no obstruction to the existence of a torsion-free connection which is compatible with this structure. The Levi-Civita connection is the unique torsion-free compatible connection and this gives rise to the usual measures of curvature (2.10), (2.11) and (2.12). The $SO(d-1, 1)$ structure also provides a volume element $\mathrm{vol}_g = \sqrt{-g}$, since these transformations have unit determinant.

The Einstein-Hilbert action for the theory is the integral of the scalar curvature with the volume element

$$S = \int \mathrm{vol}_g \, \mathcal{R}, \tag{2.16}$$

and varying with respect to the metric we find the vacuum Einstein equation

$$\mathcal{R}_{\mu\nu} = 0, \tag{2.17}$$

[2]We assume that we have an orientation on the manifold.

expressing the Ricci-flatness of the manifold.

We now review how the number of degrees of freedom of the theory matches up with these results. The metric can be encoded in a local veilbein frame $\{\hat{e}_a\}$ satisfying $g(\hat{e}_a, \hat{e}_b) = \eta_{ab}$, where $\eta_{ab} = \mathrm{diag}(-,+,\ldots,+)$ is the Minkowski metric. The components of the vielbein $(\hat{e}_a)^\mu$ form a $GL(d,\mathbb{R})$ transformation matrix with d^2 components, but any other frame which is related to this one by a Lorentz transformation encodes the same metric and thus the same $SO(d-1,1)$-structure. The structure therefore has

$$\dim GL(d,\mathbb{R}) - \dim SO(d-1,1) = d^2 - \tfrac{1}{2}d(d-1) = \tfrac{1}{2}d(d+1) \qquad (2.18)$$

degress of freedom. The metric is a symmetric matrix and thus (unsurprisingly) it has precisely this number of degrees of freedom, as does its equation of motion (2.17).

We can view this more abstractly as follows. If we denote the frame bundle of TM as F and the $SO(d-1,1)$ principle sub-bundle P, we have the associated adjoint bundles

$$\mathrm{ad}(F) \simeq TM \otimes T^*M \qquad\qquad \mathrm{ad}(P) \simeq \Lambda^2 T^*M. \qquad (2.19)$$

At each point p of the manifold the metric provides an element of the coset

$$g|_p \in \frac{GL(d,\mathbb{R})}{SO(d-1,1)} \qquad (2.20)$$

and so for a small fluctuation of the metric $\delta g_{\mu\nu}$, we have that the Lie algebra-valued tensor $g^{-1}\delta g$ is a section of the bundle

$$\mathrm{ad}(P)^\perp = \mathrm{ad}(F)/\,\mathrm{ad}(P) \simeq S^2 T^*M. \qquad (2.21)$$

Since $g^{-1}\delta g$ is merely a covariant index-raising of $\delta g_{\mu\nu}$ we can view δg itself as a section of $\mathrm{ad}(P)^\perp$. When we vary the action (2.16) with respect to the metric, the resulting equation of motion must live in the dual of this bundle, which in this case is equal to $\mathrm{ad}(P)^\perp$. Sure enough, we see that the Ricci tensor $R_{\mu\nu}$ is indeed a section of $\mathrm{ad}(P)^\perp \simeq S^2T^*M$.

Note that in (pseudo-)Riemannian geometry, one can define the Ricci tensor of a connection on F without the need for additional structure, but that the additional structure P constrains it to be a section of $\mathrm{ad}(P)^\perp \simeq$

S^2T^*M. This feature will not carry over to generalised geometry, where we will find that we require the extra structure to write down the analogue of the Ricci curvature.

We now point out the symmetries of the theory and the role that they play in the geometry. The equations are tensor in nature so it is clear that the theory has diffeomorphism symmetry. An infinitesimal diffeomorphism is parameterised by a vector $v \in TM$ and it acts on the fields via the Lie derivative

$$\delta_v g = \mathcal{L}_v g \tag{2.22}$$

It is important to note that the diffeomorphism symmetry and Lie derivative can be defined prior to the introduction of the metric, and so are present in the geometry before introducing the physical fields. This connection between the tangent space, the symmetry generators and the Lie derivative will be crucial in constructing generalised geometry.

There is also a local Lorentz symmetry relating the vielbein frames for the metric, which one must introduce if one wishes to couple fermions to the theory. This is essentially the reduced structure on the tangent bundle defined by the metric.

2.3. NS-NS Sector of Type II Supergravity

We now make some brief remarks about the NS-NS sector of type II supergravity in order to motivate the definitions made in the next section. The complete equations of type IIA and type IIB supergravity will be presented in chapter 4. For now we focus on the NS-NS bosonic fields, which are common to both type IIA and type IIB.

The NS-NS fields are comprised of a metric tensor $g_{\mu\nu}$, a two-form gauge potential $B_{\mu\nu}$ and the dilaton scalar ϕ. The potential B is only locally defined, so that, given an open cover $\{U_i\}$ of M, across coordinate patches $U_i \cap U_j$ it can be patched via

$$B_{(i)} = B_{(j)} - \mathrm{d}\Lambda_{(ij)}. \tag{2.23}$$

Furthermore the one-forms $\Lambda_{(ij)}$ satisfy

$$\Lambda_{(ij)} + \Lambda_{(jk)} + \Lambda_{(ki)} = \mathrm{d}\Lambda_{(ijk)}, \tag{2.24}$$

on $U_i \cap U_j \cap U_k$. This makes B a "connection structure on a gerbe" [136][3]. Thus for the NS-NS sector symmetry algebra we see that, in addition to diffeomorphism invariance, we have the local bosonic gauge symmetry

$$B'_{(i)} = B_{(i)} - \mathrm{d}\lambda_{(i)}, \tag{2.25}$$

where the choice of sign in the gauge transformation is to match the generalised geometry conventions that follow. Given the patching of B, the only requirement is $\mathrm{d}\lambda_{(i)} = \mathrm{d}\lambda_{(j)}$ on $U_i \cap U_j$. Thus globally $\lambda_{(i)}$ is equivalent to specifying a closed two-form. The set of gauge symmetries is then the Abelian group of closed two-forms under addition $\Omega^2_{\text{cl}}(M)$. The gauge transformations do not commute with the diffeomorphisms so the NS-NS bosonic symmetry group G_{NS} has a fibred structure

$$\Omega^2_{\text{cl}}(M) \longrightarrow G_{\text{NS}} \longrightarrow \text{Diff}(M), \tag{2.26}$$

sometimes written as the semi-direct product $\text{Diff}(M) \ltimes \Omega^2_{\text{cl}}(M)$.

One can see this structure infinitesimally by combining the diffeomorphism and gauge symmetries, given a vector v and one-form $\lambda_{(i)}$, into a general variation

$$\delta_{v+\lambda} g = \mathcal{L}_v g, \qquad \delta_{v+\lambda}\phi = \mathcal{L}_v \phi, \qquad \delta_{v+\lambda} B_{(i)} = \mathcal{L}_v B_{(i)} - \mathrm{d}\lambda_{(i)}, \tag{2.27}$$

where the patching (2.23) of B implies that

$$\mathrm{d}\lambda_{(i)} = \mathrm{d}\lambda_{(j)} - \mathcal{L}_v \mathrm{d}\Lambda_{(ij)}. \tag{2.28}$$

Recall that $\lambda_{(i)}$ and $\lambda_{(i)} + \mathrm{d}\phi_{(i)}$ define the same gauge transformation. One can use this ambiguity to integrate (2.28) and set

$$\lambda_{(i)} = \lambda_{(j)} - i_v \mathrm{d}\Lambda_{(ij)}, \tag{2.29}$$

on $U_i \cap U_j$.

In the NS-NS sector of type II theories, we therefore have two symmetry generators: a vector field $v \in TM$ and a collection of one-forms $\{\lambda_{(i)}\}$

[3] Note that technically the cocycle conditions for the gerbe structure actually only hold for quantised fluxes where H is suitably related to an integral cohomology class. This is not required by supergravity, but is necessary in string theory.

patched according to (2.29). In our discussion of general relativity we saw that the symmetry generator was a section of the tangent bundle, and that the geometry of this bundle (Lie derivatives, connections, torsion, metric and curvature) were the key ingredients of the geometrical formulation of the theory. The above leads us to wonder whether we can define a new tangent bundle which also includes the one-forms $\{\lambda_{(i)}\}$. This is precisely what we will construct in the next section.

3. $O(d,d) \times \mathbb{R}^+$ Generalised Geometry

We describe the construction of $O(d,d) \times \mathbb{R}^+$ generalised geometry closely following [1]. This is motivated by the bosonic symmetry algebra of NS-NS fields in type II supergravity as discussed in the previous section, but here we present it purely as a mathematical construction.

3.1. $T \oplus T^*$ Generalised Geometry

In this section, following [82], we review some linear algebra and differential structures on $T \oplus T^*$ which will feature extensively in the construction of the generalised tangent bundle.

3.1.1. Linear algebra of $F \oplus F^*$ and $SO(d,d)$

Let F be a d-dimensional vector space and consider the direct sum $F \oplus F^*$. Writing an element of $F \oplus F^*$ as $V = v + \lambda$ for $v \in F$, $\lambda \in F^*$, we find a natural symmetric inner product

$$\langle V, V' \rangle = \tfrac{1}{2}(\lambda(v') + \lambda'(v)) \tag{3.1}$$

The union of a basis of F and its dual basis for F^* form a basis for $F \oplus F^*$. In this basis the inner product can be written as a matrix

$$\eta = \frac{1}{2}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \tag{3.2}$$

The special orthogonal group which preserves this inner product is $SO(d,d)$ and its Lie algebra is

$$\{T : \langle TV, V' \rangle + \langle V, TV' \rangle = 0 \quad \forall\, V, V' \in F \oplus F^*\} \tag{3.3}$$

A general element can be expressed as a matrix

$$T = \begin{pmatrix} a & \beta \\ B & -a^T \end{pmatrix}, \tag{3.4}$$

where $a \in F \otimes F^*$, $\beta \in \Lambda^2 F$ and $B \in \Lambda^2 F^*$ provide the appropriate mappings. The exponentials of these can be evaluated individually and their actions form important types of $SO(d,d)$ transformations. Exponentiating the action of a, we recover the standard action of $A = \exp(a) \in GL^+(d,\mathbb{R})$ on $F \oplus F^*$. The exponential of the action of B gives the B-transformation[1]

$$e^{\hat{B}} = \begin{pmatrix} \mathbb{1} & 0 \\ B & \mathbb{1} \end{pmatrix}, \tag{3.5}$$

which sends $v + \lambda \mapsto v + \lambda - i_v B$. Similarly, the final part can be exponentiated to give the β-transformation

$$e^{\hat{\beta}} = \begin{pmatrix} \mathbb{1} & \beta \\ 0 & \mathbb{1} \end{pmatrix}. \tag{3.6}$$

which sends $v + \lambda \mapsto v + i_\beta \lambda + \lambda$. This transformation will not be as important to us as the others, as it does not correspond to a supergravity symmetry.

We now turn to the construction of spinors on $F \oplus F^*$. Consider a polyform

$$\Phi \in \Lambda^\bullet F^* = \sum_k \Lambda^k F^*. \tag{3.7}$$

We can define a natural action of the Clifford algebra on this object via

$$V \cdot \Phi = i_v \Phi + \lambda \wedge \Phi. \tag{3.8}$$

which satisfies $\{V\cdot, V'\cdot\} = 2\langle V, V'\rangle$ and we see that $\dim \Lambda^\bullet F^* = 2^d$ as one would expect for a spinor representation in $2d$ dimensions. We thus have that $S(F \oplus F^*) \simeq \Lambda^\bullet F^*$ and the positive and negative chirality spinors are encoded as the even and odd degree forms respectively. The action of the three basic transformations above on the spinor will be important later, so

[1]We use a hat to indicate the embedding of the relevant generator in the $O(d,d)$ algebra.

we present the details here. Via the Clifford action we have

$$\hat{B} \cdot \Phi = B \wedge \Phi \qquad\qquad \hat{\beta} \cdot \Phi = -i_\beta \Phi \tag{3.9}$$

for the B- and β-adjoint actions and these exponentiate to

$$\mathrm{e}^{\hat{B}} \cdot \Phi = \exp(B) \wedge \Phi \qquad\qquad \mathrm{e}^{\hat{\beta}} \cdot \Phi = \exp(-i_\beta)\Phi \tag{3.10}$$

The action of the $GL(d,\mathbb{R})$ subgroup of $Spin(d,d)$ on $\Phi \in \Lambda^\bullet F^*$ is not the standard one. For $a \in F \otimes F^*$ with $A = \exp(a)$, we have

$$\hat{a} \cdot \Phi = -a^* \Phi + \tfrac{1}{2}\,\mathrm{tr}(a)\Phi \qquad \Rightarrow \qquad \mathrm{e}^{\hat{a}} \cdot \Phi = (\det A)^{\frac{1}{2}} (A^{-1})^* \Phi. \tag{3.11}$$

This transformation law would be associated more naturally with the vector space $(\det F)^{\frac{1}{2}} \otimes \Lambda^\bullet F^*$.

This leads us to consider instead spinors of $Spin(d,d)\times\mathbb{R}^+$ and we examine a $GL(d,\mathbb{R})$ subgroup which embeds not only into the $Spin(d,d)$ factor, but also the $\mathbb{R}^+$ factor via the determinant. The $Spin(d,d) \times \mathbb{R}^+$ spinor with weight p under the $\mathbb{R}^+$ group then transforms as an element of

$$(\det F)^{\frac{1}{2}-p} \otimes \Lambda^\bullet F^* \tag{3.12}$$

under this $GL(d,\mathbb{R})$ subgroup. For $p = \frac{1}{2}$, the $GL(d,\mathbb{R})$ transformation of the spinor is given by the usual action on $\Lambda^\bullet F^*$. Later, it will be desirable to have a spinor transforming as a true polyform, as it will allow us to define an exterior derivative when we consider differential structures.

Finally, one can write a bilinear form on two such spinors (for general p), the Mukai pairing,

$$\left\langle \Psi, \Psi' \right\rangle = \sum_n (-)^{[(n+1)/2]} \Psi^{(d-n)} \wedge \Psi'^{(n)}. \tag{3.13}$$

This transforms as $(\det F^*)^{2p}$ under the $GL(d,\mathbb{R})$ subgroup.

3.1.2. Differential structures on $T \oplus T^*$

We now consider the sum of the tangent and cotangent bundles $T \oplus T^*$ over a d-dimensional manifold M. The linear algebra results of the previous section can be implemented here by sending $F \to T$. We are now wish to examine differential structures on $T \oplus T^*$.

One can define a natural analogue of the Lie derivative for $V, V' \in T \oplus T^*$, the Dorfman derivative, by

$$L_V V' = \mathcal{L}_v v' + \mathcal{L}_v \lambda' - i_{v'} \mathrm{d}\lambda \tag{3.14}$$

where $V = v + \lambda$ and $V' = v' + \lambda'$. As this definition involves only the ordinary Lie derivative and exterior derivative, this is automatically diffeomorphism covariant. However, one finds that it is also covariant under closed B-transformations, i.e. for $B \in \Omega^2_{\mathrm{cl}}(M)$ we have

$$L_{(\mathrm{e}^{\hat{B}} \cdot V)}(\mathrm{e}^{\hat{B}} \cdot V') = \mathrm{e}^{\hat{B}}(L_V V') \tag{3.15}$$

In fact diffeomorphisms and B-transformations are precisely the symmetries of the Dorfman derivative [82], and these form the group G_{NS} in (2.26). For a general two-form $B \in \Lambda^2 T^*$ we have

$$L_{(\mathrm{e}^{\hat{B}} \cdot V)}(\mathrm{e}^{\hat{B}} \cdot V') = \mathrm{e}^{\hat{B}}(L_V V' + i_v i_{v'} H) \tag{3.16}$$

where $H = \mathrm{d}B$.

A second differential structure one can naturally write down is the Dirac operator on the polyform representation of spinors. The exterior derivative is the obvious candidate as it is first order and maps even forms to odd forms and vice-versa, thus mixing the two chiralities as required. However, for spinors of $Spin(d, d)$ we are faced with the problematic $GL(d, \mathbb{R})$ transformation law (3.11), which means that the exterior derivative does not give a covariant definition. Again, we see that it is beneficial to consider spinors transforming under $Spin(d, d) \times \mathbb{R}^+$ with the appropriate weight $p = \frac{1}{2}$ in (3.12). The $GL(d, \mathbb{R})$ transformation is then returned to the conventional transformation of forms, so we have the usual action of the exterior derivative $\mathrm{d} : \Lambda^\bullet T^* \to \Lambda^\bullet T^*$, which is diffeomorphism covariant.

For any value of the weight p, we can examine the behaviour of the exterior derivative under a B-transformation. We have

$$\mathrm{d}(\mathrm{e}^{\hat{B}} \cdot \Phi) = \mathrm{e}^{\hat{B}} \cdot (\mathrm{d}\Phi + H \wedge \Phi) \tag{3.17}$$

where $H = \mathrm{d}B$, so for a closed 2-form B, the form of our Dirac operator is unchanged. Again, the closed B-transformations are a symmetry. For the case $p = \frac{1}{2}$, where we also have the diffeomorphism symmetry, the full set of symmetries form the group G_{NS} from (2.26)

In this section, we have seen the appearance of the symmetry group G_{NS}, which was the bosonic symmetry group of the NS-NS sector of type II supergravity, in natural structures on $T \oplus T^*$. For the Dorfman derivative, this was immediately present, whereas for the spinors we had to enlarge the structure group to $Spin(d, d) \times \mathbb{R}^+$. In the following sections, we will write down in more detail exactly how to define a geometry with structure group $Spin(d, d) \times \mathbb{R}^+$ which has the properties we desire. This $\mathbb{R}^+$ factor has previous appeared in generalised geometry in [98, 113, 135]

3.2. $O(d, d) \times \mathbb{R}^+$ Generalised Geometry

3.2.1. Generalised structure bundle

We start by recalling the generalised tangent space and defining what we will call the "generalised structure" which is the analogue of the frame bundle F in conventional geometry.

Let M be a d-dimensional spin manifold. In line with the patching of the transformation parameters (2.29), one starts by defining the generalised tangent space E. It is defined as an extension of the tangent space by the cotangent space

$$0 \longrightarrow T^*M \longrightarrow E \longrightarrow TM \longrightarrow 0, \tag{3.18}$$

which depends on the patching one-forms $\Lambda_{(ij)}$. If $v_{(i)} \in \Gamma(TU_i)$ and $\lambda_{(i)} \in \Gamma(T^*U_i)$, so $V_{(i)} = v_{(i)} + \lambda_{(i)}$ is a section of E over the patch U_i, then

$$v_{(i)} + \lambda_{(i)} = v_{(j)} + \big(\lambda_{(j)} - i_{v_{(j)}} \mathrm{d}\Lambda_{(ij)}\big), \tag{3.19}$$

on the overlap $U_i \cap U_j$. Hence as defined, while the $v_{(i)}$ globally are equivalent to a choice of vector, the $\lambda_{(i)}$ do not globally define a one-form. The patching of the $\lambda_{(i)}$ is the same as that of the symmetry generators in (2.29). Hence, as promised at the end of section 2.3, our definition has resulted in a new bundle, sections of which are precisely the generators of G_{NS} from (2.26).

E is in fact isomorphic to $TM \oplus T^*M$ though there is no canonical isomorphism. Instead one must choose a splitting of the sequence (3.18) as will be discussed in section 3.2.2. Crucially the definition of E is consistent with an $O(d,d)$ metric given by, for $V = v + \lambda$

$$\langle V, V \rangle = i_v \lambda, \tag{3.20}$$

since $i_{v_{(i)}} \lambda_{(i)} = i_{v_{(j)}} \lambda_{(j)}$ on $U_i \cap U_j$.

In order to describe the dilaton correctly we will actually need to consider a slight generalisation of E. We define the bundle $\tilde{E}$ weighted by an $\mathbb{R}^+$-bundle $\tilde{L}$ so that

$$\tilde{E} = \tilde{L} \otimes E. \tag{3.21}$$

The point is that, given the metric (3.20), one can now define a natural principal bundle with fibre $O(d,d) \times \mathbb{R}^+$ in terms of bases of $\tilde{E}$. We define a *conformal basis* $\{\hat{E}_A\}$ with $A = 1, \dots 2d$ on $\tilde{E}_x$ as one satisfying

$$\langle \hat{E}_A, \hat{E}_B \rangle = \Phi^2 \eta_{AB} \quad \text{where} \quad \eta = \frac{1}{2} \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \tag{3.22}$$

That is $\{\hat{E}_A\}$ is orthonormal up to a frame-dependent conformal factor $\Phi \in \Gamma(\tilde{L})$. We then define the *generalised structure bundle*

$$\tilde{F} = \big\{ (x, \{\hat{E}_A\}) : x \in M, \text{ and } \{\hat{E}_A\} \text{ is a conformal basis of } \tilde{E}_x \big\}. \tag{3.23}$$

By construction, this is a principal bundle with fibre $O(d,d) \times \mathbb{R}^+$. One can make a change of basis

$$V^A \mapsto V'^A = M^A{}_B V^B, \qquad \hat{E}_A \mapsto \hat{E}'_A = \hat{E}_B (M^{-1})^B{}_A, \tag{3.24}$$

where $M \in O(d,d) \times \mathbb{R}^+$ so that $(M^{-1})^C{}_A (M^{-1})^D{}_B \eta_{CD} = \sigma^2 \eta_{AB}$ for some σ. The topology of $\tilde{F}$ encodes both the topology of the tangent bundle TM and of the B-field gerbe.

Given the definition (3.18) there is one natural conformal basis defined by the choice of coordinates on M, namely $\{\hat{E}_A\} = \{\partial/\partial x^\mu\} \cup \{\mathrm{d}x^\mu\}$. Given $V \in \Gamma(E)$ over the patch U_i, we have $V = v^\mu (\partial/\partial x^\mu) + \lambda_\mu \mathrm{d}x^\mu$. We will sometimes denote the components of V in this frame by an index M such that

$$V^M = \begin{cases} v^\mu & \text{for } M = \mu \\ \lambda_\mu & \text{for } M = \mu + d \end{cases}. \tag{3.25}$$

Suppose now that we have different coordinates on two patches U_i and U_j. The transition functions (3.19) can be written acting explicitly on the components of the vector and 1-form parts as

$$\begin{aligned} v^\mu_{(i)} &= M^\mu{}_\nu v^\nu_{(j)}, \\ \lambda_{(i)\mu} &= (M^{-1})^\nu{}_\mu(\lambda_{(j)\nu} - v^\lambda_{(j)}(\mathrm{d}\Lambda_{(ij)})_{\lambda\nu}), \end{aligned} \tag{3.26}$$

for $M^\mu{}_\nu = \partial x^\mu_{(i)}/\partial x^\nu_{(j)} \in GL(d,\mathbb{R})$.

We now describe the details of the $\mathbb{R}^+$-bundle $\tilde{L}$. The transition functions (3.26) lie in the group $GL(d,\mathbb{R}) \ltimes \mathbb{R}^{d(d-1)/2}$. We define $\tilde{L}$ such that, between the same patches U_i and U_j considered above, the transition functions acting on the corresponding components of a section of $\tilde{E}$ are

$$\begin{aligned} v^\mu_{(i)} &= (\det M)^{-1} M^\mu{}_\nu v^\nu_{(j)}, \\ \lambda_{(i)\mu} &= (\det M)^{-1}(M^{-1})^\nu{}_\mu(\lambda_{(j)\nu} - v^\lambda_{(j)}(\mathrm{d}\Lambda_{(ij)})_{\lambda\nu}). \end{aligned} \tag{3.27}$$

These transition functions lie in a $GL(d,\mathbb{R}) \ltimes \mathbb{R}^{d(d-1)/2}$ subgroup of $O(d,d) \times \mathbb{R}^+$ acting on the components V^M, and the embedding is such that, under this subgroup, we can identify $\tilde{L} \simeq \det T^*M$. A section of $\tilde{L}$ is thus equivalent to a section of $\det T^*M$.

3.2.2. Generalised tensors and spinors and split frames

Generalised tensors are simply sections of vector bundles constructed from different representations of $O(d,d) \times \mathbb{R}^+$, that is representations of $O(d,d)$ of definite weight under $\mathbb{R}^+$. Since the $O(d,d)$ metric gives an isomorphism between E and E^*, one has the bundle

$$E^{\otimes n}_{(p)} = \tilde{L}^p \otimes E \otimes \cdots \otimes E, \tag{3.28}$$

for a general tensor of weight p.

One can also consider $Spin(d,d)$ spinor representations [82]. The $O(d,d)$ Clifford algebra

$$\{\Gamma_A, \Gamma_B\} = 2\eta_{AB}, \tag{3.29}$$

can be realised on each coordinate patch U_i by identifying spinors with weighted sums of forms $\Psi_{(i)} \in \Gamma((\det T^*U_i)^{1/2} \otimes \Lambda^\bullet T^*U_i)$, with the Clifford action

$$V^A\Gamma_A\Psi_{(i)} = i_v\Psi_{(i)} + \lambda_{(i)} \wedge \Psi_{(i)}. \tag{3.30}$$

The patching (3.19) then implies

$$\Psi_{(i)} = \mathrm{e}^{\mathrm{d}\Lambda_{(ij)}} \wedge \Psi_{(j)}. \tag{3.31}$$

Projecting onto the chiral spinors then defines two $Spin(d,d)$ spinor bundles isomorphic to weighted sums of odd or even forms $S^{\pm}(E) \simeq (\det T^*M)^{-1/2} \otimes \Lambda^{\text{even/odd}}T^*M$, where again specifying the isomorphism requires a choice of splitting.

More generally one defines $Spin(d,d) \times \mathbb{R}^+$ spinors of weight p as sections of

$$S^{\pm}_{(p)} = \tilde{L}^p \otimes S^{\pm}(E). \tag{3.32}$$

Note that there is a natural $Spin(d,d)$ invariant bilinear on these spinor spaces given by the Mukai pairing [81, 82]. For $\Psi, \Psi' \in \Gamma(S^{\pm}_{(p)})$ one has

$$\left\langle \Psi, \Psi' \right\rangle = \sum_n (-)^{[(n+1)/2]}\Psi^{(d-n)} \wedge \Psi'^{(n)} \in \Gamma(\tilde{L}^{2p}), \tag{3.33}$$

where $\Psi^{(n)}$ and $\Psi'^{(n)}$ are the local weighted n-form components.

A special class of conformal frames are those defined by a splitting of the generalised tangent space E. A splitting is a map $TM \to E$. It is equivalent to specifying a local two-form B patched as in (2.23) and defines an isomorphism $E \simeq TM \oplus T^*M$. If $\{\hat{e}_a\}$ is a generic basis for TM and $\{e^a\}$ be the dual basis on T^*M, one can then define what we call a *split frame* $\{\hat{E}_A\}$ for $\tilde{E}$ by

$$\hat{E}_A = \begin{cases} \hat{E}_a = (\det e)\,(\hat{e}_a + i_{\hat{e}_a}B) & \text{for } A = a \\ E^a = (\det e)e^a & \text{for } A = a + d \end{cases}. \tag{3.34}$$

We immediately see that

$$\left\langle \hat{E}_A, \hat{E}_B \right\rangle = (\det e)^2\eta_{AB}, \tag{3.35}$$

and hence the basis is conformal. Writing $V = v^a\hat{E}_a + \lambda_a E^a \in \Gamma(\tilde{E})$ we have

$$\begin{aligned} V^{(B)} &= v^a(\det e)\hat{e}_a + \lambda_a(\det e)e^a \\ &= v_{(i)} + \lambda_{(i)} - i_{v_{(i)}}B_{(i)}, \end{aligned} \tag{3.36}$$

demonstrating that the splitting defines an isomorphism $\tilde{E} \simeq (\det T^*M) \otimes (TM \oplus T^*M)$ since $\lambda_{(i)} - i_{v_{(i)}}B_{(i)} = \lambda_{(j)} - i_{v_{(j)}}B_{(j)}$.

The class of split frames defines a sub-bundle of $\tilde{F}$. Such frames are related by transformations (3.24) where M takes the form

$$M = (\det A)^{-1} \begin{pmatrix} \mathbb{1} & 0 \\ \omega & \mathbb{1} \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & (A^{-1})^T \end{pmatrix}, \tag{3.37}$$

where $A \in GL(d, \mathbb{R})$ is the matrix transforming $\hat{e}_a \mapsto \hat{e}_b(A^{-1})^b{}_a$ while $\omega = \frac{1}{2}\omega_{ab}e^a \wedge e^b$ transforms $B \mapsto B' = B + \omega$, where ω must be closed for B' to be a splitting. This defines a parabolic subgroup $G_{\text{split}} = GL(d, \mathbb{R}) \ltimes \mathbb{R}^{d(d-1)/2} \subset O(d,d) \times \mathbb{R}^+$ and hence the set of all frames of the form (5.13) defines a G_{split} principal sub-bundle of $\tilde{F}$, that is a G_{split}-structure. This reflects the fact that the patching elements in the definition of $\tilde{E}$ lie only in this subgroup of $O(d,d) \times \mathbb{R}^+$.

In what follows it will be useful to also define a class of *conformal split frames* given by the set of split bases conformally rescaled by a function ϕ so that

$$\hat{E}_A = \begin{cases} \hat{E}_a = \mathrm{e}^{-2\phi}(\det e)\,(\hat{e}_a + i_{\hat{e}_a}B) & \text{for } A = a \\ E^a = \mathrm{e}^{-2\phi}(\det e)e^a & \text{for } A = a + d \end{cases}, \tag{3.38}$$

thus defining a $G_{\text{split}} \times \mathbb{R}^+$ sub-bundle of $\tilde{F}$. In complete analogy with the split case, the components of $V \in \Gamma(\tilde{E})$ in the conformally split frame are related to those in the coordinate basis by

$$V^{(B,\phi)} = \mathrm{e}^{2\phi}\big(v_{(i)} + \lambda_{(i)} - i_{v_{(i)}}B_{(i)}\big). \tag{3.39}$$

We can similarly write the components of generalised spinors in different frames. The relation between the coordinate and split frames implies that if $\Psi^{(B)}_{a_1\dots a_n}$ are the polyform components of $\Psi \in \Gamma(S^{\pm}_{(p)})$ in the split frame then

$$\Psi^{(B)} = \sum_n \tfrac{1}{n!}\Psi^{(B)}_{a_1\dots a_n}e^{a_1} \wedge \cdots \wedge e^{a_n} = \mathrm{e}^{B_{(i)}} \wedge \Psi_{(i)}, \tag{3.40}$$

demonstrating the isomorphism $S^{\pm}_{(p)} \simeq (\det T^*M)^{p-1/2} \otimes \Lambda^{\text{even/odd}}T^*M$, since $\mathrm{e}^{B_{(i)}} \wedge \Psi_{(i)} = \mathrm{e}^{B_{(j)}} \wedge \Psi_{(j)}$. In the conformal split frame one similarly has

$$\Psi^{(B,\phi)} = \mathrm{e}^{2p\phi}\mathrm{e}^{B_{(i)}} \wedge \Psi_{(i)}. \tag{3.41}$$

3.2.3. The Dorfman derivative, Courant bracket and exterior derivative

We now demonstrate that the generalised tangent space admits a generalisation of the Lie derivative which encodes the bosonic symmetries of the NS-NS sector of type II supergravity, as we hoped in section 2.3. Given $V = v + \lambda \in \Gamma(E)$, one can define an operator L_V acting on any generalised tensor, which combines the action of an infinitesimal diffeomorphisms generated by v and a B-field gauge transformations generated by λ.

Acting on $W = w + \zeta \in E_{(p)}$, we define the *Dorfman derivative*[2] or "generalised Lie derivative" as [98]

$$L_V W = \mathcal{L}_v w + \mathcal{L}_v \zeta - i_w \mathrm{d}\lambda, \tag{3.42}$$

where, since w and ζ are weighted tensors, the action of the Lie derivative is

$$\begin{aligned} \mathcal{L}_v w^\mu &= v^\nu \partial_\nu w^\mu - w^\nu \partial_\nu v^\mu + p(\partial_\nu v^\nu) w^\mu, \\ \mathcal{L}_v \zeta_\mu &= v^\nu \partial_\nu \zeta_\mu + (\partial_\mu v^\nu)\zeta_\nu + p(\partial_\nu v^\nu)\zeta^\mu. \end{aligned} \tag{3.43}$$

Defining the action on a function f as simply $L_V f = \mathcal{L}_v f$, one can then extend the notion of Dorfman derivative to any $O(d,d) \times \mathbb{R}^+$ tensor using the Leibniz property.

To see this explicitly it is useful to note that we can rewrite (3.42) in a more $O(d,d) \times \mathbb{R}^+$ covariant way, in analogy with (2.4). First note that one can embed the action of the partial derivative operator into generalised geometry using the map $T^*M \to E$. In coordinate indices, as viewed as mapping to a section of E^*, one defines

$$\partial_M = \begin{cases} \partial_\mu & \text{for } M = \mu \\ 0 & \text{for } M = \mu + d \end{cases}. \tag{3.44}$$

[2]If $p = 0$ then $L_V W$ is none other than the Dorfman bracket [137]. Since it extends to a derivation on the tensor algebra of generalised tensors, it is natural in our context to call it the "Dorfman derivative".

One can then rewrite (5.23) in terms of generalised objects (as in [61, 112])

$$L_V W^M = V^N \partial_N W^M + \left(\partial^M V^N - \partial^N V^M\right) W_N + p \left(\partial_N V^N\right) W^M, \quad (3.45)$$

where indices are contracted using the $O(d,d)$ metric (3.20), which, by definition, is constant with respect to ∂. Note that this form is exactly analogous to the conventional Lie derivative (2.4), though now with the adjoint action in $\mathfrak{o}(d,d) \oplus \mathbb{R}$ rather than $\mathfrak{gl}(d)$. Specifically the second and third terms are (minus) the action of an $\mathfrak{o}(d,d) \oplus \mathbb{R}$ element m, given by

$$m \cdot W = \begin{pmatrix} a & 0 \\ -\omega & -a^T \end{pmatrix} \begin{pmatrix} w \\ \zeta \end{pmatrix} - p \operatorname{tr} a \begin{pmatrix} w \\ \zeta \end{pmatrix}, \quad (3.46)$$

where $a^\mu{}_\nu = \partial_\nu v^\mu$ and $\omega_{\mu\nu} = \partial_\mu \lambda_\nu - \partial_\nu \lambda_\mu$. Comparing with (3.37), we see that m in fact acts in the Lie algebra of the G_{split} subgroup of $O(d,d) \times \mathbb{R}^+$.

This form can then be naturally extended to an arbitrary $O(d,d) \times \mathbb{R}^+$ tensor $\alpha \in \Gamma(E_{(p)}^{\otimes n})$ as

$$\begin{aligned} L_V \alpha^{M_1 \dots M_n} &= V^N \partial_N \alpha^{M_1 \dots M_n} + \left(\partial^{M_1} V^N - \partial^N V^{M_1}\right) \alpha_N{}^{M_2 \dots M_n} \\ &\quad + \dots + \left(\partial^{M_n} V^N - \partial^N V^{M_n}\right) \alpha^{M_1 \dots M_{n-1}}{}_N + p \left(\partial_N V^N\right) W^M, \end{aligned} \quad (3.47)$$

again in analogy with (2.4). It similarly extends to generalised spinors $\Psi \in \Gamma(S_{(p)}^{\pm})$ as (see also [118])

$$L_V \Psi = V^N \partial_N \Psi + \tfrac{1}{4} \left(\partial_M V_N - \partial_N V_M\right) \Gamma^{MN} \Psi + p(\partial_M V^M) \Psi, \quad (3.48)$$

where $\Gamma_{MN} = \frac{1}{2}\left(\Gamma_M \Gamma_N - \Gamma_N \Gamma_M\right)$.

Note that when $W \in \Gamma(E)$ one can also define the antisymmetrisation of the Dorfman derivative

$$\begin{aligned} [\![V, W]\!] &= \tfrac{1}{2}\left(L_V W - L_W V\right) \\ &= [v, w] + \mathcal{L}_v \zeta - \mathcal{L}_w \lambda - \tfrac{1}{2} \mathrm{d}\left(i_v \zeta - i_w \lambda\right), \end{aligned} \quad (3.49)$$

which is know as the Courant bracket [138]. It can be rewritten in an $O(d,d)$ covariant form as

$$[\![U, V]\!]^M = U^N \partial_N V^M - V^N \partial_N U^M - \tfrac{1}{2}\left(U_N \partial^M V^N - V_N \partial^M U^N\right). \quad (3.50)$$

which follows directly from (3.45).

Finally note that since $S^{\pm}_{(1/2)} \simeq \Lambda^{\text{even/odd}} T^* M$ the Clifford action of ∂_M on $\Psi \in \Gamma(S^{\pm}_{(1/2)})$ defines a natural action of the exterior derivative. On U_i one defines $d : \Gamma(S^{\pm}_{(1/2)}) \to \Gamma(S^{\mp}_{(1/2)})$ by

$$(\mathrm{d}\Psi)_{(i)} = \tfrac{1}{2}\Gamma^M \partial_M \Psi_{(i)} = \mathrm{d}\Psi_{(i)}, \tag{3.51}$$

that is, it is simply the exterior derivative of the component p-forms. The Dorfman derivative and Courant bracket can then be regarded as derived brackets for this exterior derivative [139].

3.2.4. Generalised $O(d,d) \times \mathbb{R}^+$ connections and torsion

We now turn to the definitions of generalised connections, torsion and the possibility of defining a generalised curvature. The notion of connection on a Courant algebroid was first introduced by Alekseev and Xu [85, 84] and Gualtieri [86] (see also Ellwood [87]). At least locally, it is also essentially equivalent to the connection defined by Siegel [61] and discussed in doubled field theory [116]. It is also related to the differential operator introduced in the "stringy differential geometry" of [117].

Our definitions will follow closely those in [85, 86] though, in connecting to supergravity, it is important to extend the definitions to include the $\mathbb{R}^+$ factor in the generalised structure bundle.

Generalised connections

Here we will specifically be interested in those generalised connections that are compatible with the $O(d,d) \times \mathbb{R}^+$ structure. Following [85, 86] we can define a first-order linear differential operator D, such that, given $W \in \Gamma(\tilde{E})$, in frame indices,

$$D_M W^A = \partial_M W^A + \tilde{\Omega}_M{}^A{}_B W^B. \tag{3.52}$$

Compatibility with the $O(d,d) \times \mathbb{R}^+$ structure implies

$$\tilde{\Omega}_M{}^A{}_B = \Omega_M{}^A{}_B - \Lambda_M \delta^A{}_B, \tag{3.53}$$

where Λ is the $\mathbb{R}^+$ part of the connection and Ω the $O(d,d)$ part, so that we have

$$\Omega_M{}^{AB} = -\Omega_M{}^{BA}. \tag{3.54}$$

The action of D then extends naturally to any generalised tensor. In particular, if $\alpha \in \Gamma(E_{(p)}^{\otimes n})$ we have

$$\begin{aligned} D_M \alpha^{A_1 \dots A_n} = \partial_M \alpha^{A_1 \dots A_n} &+ \Omega_M{}^{A_1}{}_B \alpha^{B A_2 \dots A_n} \\ &+ \dots + \Omega_M{}^{A_n}{}_B \alpha^{A_1 \dots A_{n-1} B} - p \Lambda_M \alpha^{A_1 \dots A_n}. \end{aligned} \tag{3.55}$$

Similarly, if $\Psi \in \Gamma(S^{\pm}_{(p)})$ then

$$D_M \Psi = \left(\partial_M + \tfrac{1}{4} \Omega_M{}^{AB} \Gamma_{AB} - p \Lambda_M \right) \Psi. \tag{3.56}$$

Given a conventional connection ∇ and a conformal split frame of the form (3.38), one can construct the corresponding generalised connection as follows. Writing a generalised vector $W \in \Gamma(\tilde{E})$ as

$$W = W^A \hat{E}_A = w^a \hat{E}_a + \zeta_a E^a, \tag{3.57}$$

by construction $w = w^a (\det e) \hat{e}_a \in \Gamma((\det T^*M) \otimes TM)$ and $\zeta = \zeta_a (\det e) e^a \in \Gamma((\det T^*M) \otimes T^*M)$ and so we can define $\nabla_\mu w^a$ and $\nabla_\mu \zeta_a$. The generalised connection defined by ∇ lifted to an action on $\tilde{E}$ by the conformal split frame is then simply

$$(D^\nabla_M W^A) \hat{E}_A = \begin{cases} (\nabla_\mu w^a) \hat{E}_a + (\nabla_\mu \zeta_a) E^a & \text{for } M = \mu \\ 0 & \text{for } M = \mu + d \end{cases}. \tag{3.58}$$

Generalised torsion

We define the *generalised torsion* T of a generalised connection D in direct analogy to the conventional definition (2.9). Let α be any generalised tensor and $L^D_V \alpha$ be the Dorfman derivative (3.47) with ∂ replaced by D. The generalised torsion is a linear map $T : \Gamma(E) \to \Gamma(\text{ad}(\tilde{F}))$ where $\text{ad}(\tilde{F}) \simeq \Lambda^2 E \oplus \mathbb{R}$ is the $\mathfrak{o}(d,d) \oplus \mathbb{R}$ adjoint representation bundle associated to $\tilde{F}$. It is defined by

$$T(V) \cdot \alpha = L^D_V \alpha - L_V \alpha, \tag{3.59}$$

for any $V \in \Gamma(E)$ and where $T(V)$ acts via the adjoint representation on α. This definition is close to that of [86], except for the additional $\mathbb{R}^+$ action in the definition of L.

Viewed as a tensor $T \in \Gamma(E \otimes \operatorname{ad} \tilde{F})$, with indices such that $T(V)^M{}_N = V^P T^M{}_{PN}$, we can derive an explicit expression for T. Let $\{\hat{E}_A\}$ be a general conformal basis with $\langle \hat{E}_A, \hat{E}_B \rangle = \Phi^2 \eta_{AB}$. Then $\{\Phi^{-1}\hat{E}_A\}$ is an orthonormal basis for E. Given the connection $D_M W^A = \partial_M W^A + \tilde{\Omega}_M{}^A{}_B W^B$, we have

$$T_{ABC} = -3\tilde{\Omega}_{[ABC]} + \tilde{\Omega}_D{}^D{}_B \eta_{AC} - \Phi^{-2}\langle \hat{E}_A, L_{\Phi^{-1}\hat{E}_B} \hat{E}_C \rangle, \tag{3.60}$$

where indices are lowered with η_{AB}.

Naively one might expect that $T \in \Gamma((E \otimes \Lambda^2 E) \oplus E)$. However the form of the Dorfman derivative means that fewer components of $\tilde{\Omega}$ actually enter the torsion and

$$T \in \Gamma(\Lambda^3 E \oplus E). \tag{3.61}$$

This can be seen most easily in the coordinate basis where the two components are

$$T^M{}_{PN} = (T_1)^M{}_{PN} - (T_2)_P\, \delta^M{}_N, \tag{3.62}$$

with

$$\begin{aligned} (T_1)_{MNP} &= -3\tilde{\Omega}_{[MNP]} = -3\Omega_{[MNP]}, \\ (T_2)_M &= -\tilde{\Omega}_Q{}^Q{}_M = \Lambda_M - \Omega_Q{}^Q{}_M. \end{aligned} \tag{3.63}$$

An immediate consequence of this definition is that for $\Psi \in \Gamma(S^{\pm}_{(1/2)})$ the Dirac operator $\Gamma^M D_M \Psi$ is determined by the torsion of the connection [85]

$$\begin{aligned} \Gamma^M D_M \Psi &= \Gamma^M (\partial_M \Psi + \tfrac{1}{4}\Omega_{MNP}\Gamma^{NP}\Psi - \tfrac{1}{2}\Lambda_M \Psi) \\ &= \Gamma^M \partial_M \Psi + \tfrac{1}{4}\Omega_{[MNP]}\Gamma^{MNP}\Psi - \tfrac{1}{2}(\Lambda_M - \Omega_N{}^N{}_M)\Gamma^M \Psi \\ &= 2\mathrm{d}\Psi - \tfrac{1}{12}(T_1)_{[MNP]}\Gamma^{MNP}\Psi - \tfrac{1}{2}(T_2)_M \Gamma^M \Psi. \end{aligned} \tag{3.64}$$

This equation could equally well be used as a definition of the torsion of a generalised connection. Note in particular that if the connection is torsion-free we see that the Dirac operator becomes equal to the exterior derivative

$$\Gamma^M D_M \Psi = 2\mathrm{d}\Psi. \tag{3.65}$$

As an example, we can calculate the torsion for the generalised connection

D^{∇} defined in (3.58). In general we have

$$L_{\Phi^{-1}\hat{E}_A}\hat{E}_B = \left(L_{\Phi^{-1}\hat{E}_A}\Phi\right)\Phi^{-1}\hat{E}_B + \Phi\big(L_{\Phi^{-1}\hat{E}_A}(\Phi^{-1}\hat{E}_B)\big), \tag{3.66}$$

where here

$$L_{\Phi^{-1}\hat{E}_A}\Phi = \begin{cases} -\mathrm{e}^{-2\phi}(\det e)\left(i_{\hat{e}_a}i_{\hat{e}_b}\mathrm{d}e^b + 2i_{\hat{e}_a}\mathrm{d}\phi\right) & \text{for } A = a \\ 0 & \text{for } A = a+d \end{cases}, \tag{3.67}$$

and

$$L_{\Phi^{-1}\hat{E}_A}\Phi^{-1}\hat{E}_B = \begin{pmatrix} [\hat{e}_a,\hat{e}_b] + i_{[\hat{e}_a,\hat{e}_b]}B - i_{\hat{e}_a}i_{\hat{e}_b}H & \mathcal{L}_{\hat{e}_a}e^b \\ -\mathcal{L}_{\hat{e}_b}e^a & 0 \end{pmatrix}_{AB}, \tag{3.68}$$

where $H = \mathrm{d}B$. If the conventional connection ∇ is torsion-free, the corresponding generalised torsion is given by

$$T_1 = -4H, \qquad T_2 = -4\,\mathrm{d}\phi, \tag{3.69}$$

where we are using the embedding[3] $T^*M \to E$ (and the corresponding $T^*M \to \Lambda^3 E$) to write the expressions in terms of forms. This result is most easily seen by taking $\hat{e}_a$ to be the coordinate frame, so that all but the H and $\mathrm{d}\phi$ terms in (3.67) and (3.68) vanish.

The absence of generalised curvature

Having defined torsion it is natural to ask if one can also introduce a notion of generalised curvature in analogy to the usual definition (2.10), as the commutator of two generalised connections but now using the Courant bracket (3.49) rather than the Lie bracket

$$\mathrm{R}\,(U,V,W) = [D_U, D_V]\,W - D_{[\![U,V]\!]}W. \tag{3.70}$$

However, this object is non-tensorial [86]. We can check for linearity in the arguments explicitly. Taking $U \to fU$, $V \to gV$ and $W \to hW$ for some

[3]Note that with our definitions we have $(\partial^A\phi)\Phi^{-1}\hat{E}_A = 2\mathrm{d}\phi$ due to the factor $\frac{1}{2}$ in η_{AB}

scalar functions f, g, h, we obtain

$$\begin{aligned}
&[D_{fU}, D_{gV}]\, hW - D_{[\![fU,gV]\!]}hW \\
&\qquad = fgh\left([D_U, D_V]\, W - D_{[\![U,V]\!]}W\right) - \tfrac{1}{2}h\langle U, V\rangle D_{(f\mathrm{d}g - g\mathrm{d}f)}W,
\end{aligned} \tag{3.71}$$

and so the curvature is not linear in U and V.

Nonetheless, if there is additional structure, as will be relevant for supergravity, we are able to define other tensorial objects that are measures of generalised curvature. In particular, let $C_1 \subset E$ and $C_2 \subset E$ be subspaces such that $\langle U, V\rangle = 0$ for all $U \in \Gamma(C_1)$ and $V \in \Gamma(C_2)$. For such a U and V the final term in (3.71) vanishes, and so $\mathrm{R} \in \Gamma((C_1 \otimes C_2) \otimes \mathfrak{o}(d, d))$ is a tensor. A special example of this is when $C_1 = C_2$ is a null subspace of E.

The condition $\langle U, V\rangle = 0$ here is reminiscent of the section condition of double field theory. We will discuss this issue more fully in section 5.1.4.

3.3. $O(p, q) \times O(q, p)$ structures and torsion-free connections

We now turn to constructing the generalised analogue of the Levi–Civita connection. The latter is the unique torsion-free connection that preserves the $O(d) \subset GL(d, \mathbb{R})$ structure defined by a metric g. Here we will be interested in generalised connections that preserve an $O(p, q) \times O(q, p) \subset O(d, d) \times \mathbb{R}^+$ structure on $\tilde{F}$, where $p + q = d$. We will find that, in analogy to the Levi–Civita connection, it is always possible to construct torsion-free connections of this type but there is no unique choice. Locally this is same construction that appears in Siegel [61] and closely related to that of [117].

3.3.1. $O(p, q) \times O(q, p)$ structures and the generalised metric

Following closely the standard definition of the generalised metric [82], consider an $O(p, q) \times O(q, p)$ principal sub-bundle P of the generalised structure bundle $\tilde{F}$. As discussed below, this is equivalent to specifying a conventional metric g of signature (p, q), a B-field patched as in (2.23) and a dilaton ϕ. As such it clearly gives the appropriate generalised structure to capture the NS-NS supergravity fields.

Geometrically, an $O(p, q) \times O(q, p)$ structure does two things. First it fixes a nowhere vanishing section $\Phi \in \Gamma(\tilde{L})$, giving an isomorphism between weighted and unweighted generalised tangent space $\tilde{E}$ and E. Second it defines a splitting of E into two d-dimensional sub-bundles

$$E = C_+ \oplus C_- , \tag{3.72}$$

such that the $O(d,d)$ metric (3.20) restricts to a separate metric of signature (p,q) on C_+ and a metric of signature (q,p) on C_-. (Each sub-bundle is also isomorphic to TM using the map $E \to TM$.)

In terms of $\tilde{F}$ we can identify a special set of frames defining a $O(p,q) \times O(p,q)$ sub-bundle. We define a frame $\{\hat{E}^+_a\} \cup \{\hat{E}^-_{\bar{a}}\}$ such that $\{\hat{E}^+_a\}$ form an orthonormal basis for C_+ and $\{\hat{E}^-_{\bar{a}}\}$ form an orthonormal basis for C_-. This means they satisfy

$$\begin{aligned} \langle \hat{E}^+_a, \hat{E}^+_b \rangle &= \Phi^2 \eta_{ab}, \\ \langle \hat{E}^-_{\bar{a}}, \hat{E}^-_{\bar{b}} \rangle &= -\Phi^2 \eta_{\bar{a}\bar{b}}, \\ \langle \hat{E}^+_a, \hat{E}^-_{\bar{a}} \rangle &= 0, \end{aligned} \tag{3.73}$$

where $\Phi \in \Gamma(\tilde{L})$ is now some fixed density (independent of the particular frame element) and η_{ab} and $\eta_{\bar{a}\bar{b}}$ are flat metrics with signature (p,q). There is thus a manifest $O(p,q) \times O(q,p)$ symmetry with the first factor acting on $\hat{E}^+_a$ and the second on $\hat{E}^-_{\bar{a}}$.

Note that the natural conformal frame

$$\hat{E}_A = \begin{cases} \hat{E}^+_a & \text{for } A = a \\ \hat{E}^-_{\bar{a}} & \text{for } A = \bar{a} + d \end{cases}, \tag{3.74}$$

satisfies

$$\langle \hat{E}_A, \hat{E}_B \rangle = \Phi^2 \eta_{AB}, \quad \text{where} \quad \eta_{AB} = \begin{pmatrix} \eta_{ab} & 0 \\ 0 & -\eta_{\bar{a}\bar{b}} \end{pmatrix}, \tag{3.75}$$

where the form of η_{AB} differs from that used in (3.22). In this section, we will use this form of the metric η_{AB} throughout. It is also important to note that we will adopt the convention that we will always raise and lower the C_+ indices $a, b, c, \dots$ with η_{ab} and the C_- indices $\bar{a}, \bar{b}, \bar{c}, \dots$ with $\eta_{\bar{a}\bar{b}}$, while we continue to raise and lower $2d$ dimensional indices $A, B, C, \dots$ with the

$O(d,d)$ metric η_{AB}. Thus, for example we have

$$\hat{E}^A = \begin{cases} \hat{E}^{+a} & \text{for } A = a \\ -\hat{E}^{-\bar{a}} & \text{for } A = \bar{a} + d \end{cases}, \tag{3.76}$$

when we raise the A index on the frame.

One can write a generic $O(p,q) \times O(q,p)$ structure explicitly as

$$\begin{aligned} \hat{E}^+_a &= \mathrm{e}^{-2\phi}\sqrt{-g}\left(\hat{e}^+_a + e^+_a + i_{\hat{e}^+_a} B\right), \\ \hat{E}^-_{\bar{a}} &= \mathrm{e}^{-2\phi}\sqrt{-g}\left(\hat{e}^-_{\bar{a}} - e^-_{\bar{a}} + i_{\hat{e}^-_{\bar{a}}} B\right), \end{aligned} \tag{3.77}$$

where the fixed conformal factor in (3.73) is given by

$$\Phi = \mathrm{e}^{-2\phi}\sqrt{-g}, \tag{3.78}$$

and where $\{\hat{e}^+_a\}$ and $\{\hat{e}^-_{\bar{a}}\}$, and their duals $\{e^{+a}\}$ and $\{e^{-\bar{a}}\}$, are two independent orthonormal frames for the metric g, so that

$$\begin{aligned} g = \eta_{ab} e^{+a} \otimes e^{+b} &= \eta_{\bar{a}\bar{b}} e^{-\bar{a}} \otimes e^{-\bar{b}}, \\ g(\hat{e}^+_a, \hat{e}^+_b) = \eta_{ab}, &\qquad g(\hat{e}^-_{\bar{a}}, \hat{e}^-_{\bar{b}}) = \eta_{\bar{a}\bar{b}}. \end{aligned} \tag{3.79}$$

By this explicit construction we see that there is no topological obstruction to the existence of $O(p,q) \times O(q,p)$ structures.

In addition to the $O(p,q) \times O(q,p)$ invariant density (3.78) one can also construct the invariant *generalised metric* G [82]. It has the form

$$G = \Phi^{-2}\big(\eta^{ab}\hat{E}^+_a \otimes \hat{E}^+_b + \eta^{\bar{a}\bar{b}}\hat{E}^-_{\bar{a}} \otimes \hat{E}^-_{\bar{b}}\big). \tag{3.80}$$

In the coordinate frame we have the familiar expression

$$G_{MN} = \frac{1}{2}\begin{pmatrix} g - Bg^{-1}B & -Bg^{-1} \\ g^{-1}B & g^{-1} \end{pmatrix}_{MN}. \tag{3.81}$$

By construction, the pair (G, Φ) parametrise the coset $(O(d,d)\times\mathbb{R}^+)/O(p,q)\times O(q,p)$ where $p+q=d$.

Finally the $O(p,q) \times O(q,p)$ structure provides two additional chirality operators $\Gamma^{\pm}$ on $Spin(d,d)\times\mathbb{R}^+$ spinors which one can define as [98, 118, 140]

$$\Gamma^{(+)} = \tfrac{1}{d!}\epsilon^{a_1\dots a_d}\Gamma_{a_1}\dots\Gamma_{a_d}, \qquad \Gamma^{(-)} = \tfrac{1}{d!}\epsilon^{\bar{a}_1\dots\bar{a}_d}\Gamma_{\bar{a}_1}\dots\Gamma_{\bar{a}_d}. \tag{3.82}$$

Using that, in the split frame, the Clifford action takes the form

$$\Gamma_a\cdot\Psi^{(B)} = i_{\hat{e}^+_a}\Psi^{(B)} + e^+_a\wedge\Psi^{(B)}, \qquad \Gamma_{\bar{a}}\cdot\Psi^{(B)} = i_{\hat{e}^-_a}\Psi^{(B)} - e^-_a\wedge\Psi^{(B)}, \tag{3.83}$$

these can be evaluated on the weighted n-form components of Ψ as

$$\Gamma^{(+)}\Psi^{(B)}_{(n)} = (-)^{[n/2]} * \Psi^{(B)}_{(n)}, \qquad \Gamma^{(-)}\Psi^{(B)}_{(n)} = (-)^{d}(-)^{[n+1/2]} * \Psi^{(B)}_{(n)}, \tag{3.84}$$

and thus we have a generalisation of the Hodge dual on $Spin(d,d)\times\mathbb{R}^+$ spinors.

Since $G^T\eta G = \eta$, the generalised metric $G^A{}_B$ is an element of $O(d,d)$ and one can easily check that $G^2 = \mathbb{1}$. Connecting to the discussion of [118], for even dimensions d, one has $G \in SO(d,d)$ and $\Gamma^{(-)}$ is an element of $Spin(d,d)$ satisfying

$$\Gamma^{(-)}\Gamma^A\Gamma^{(-)-1} = G^A{}_B\Gamma^B, \tag{3.85}$$

so that $\Gamma^{(-)}$ is a preimage of G in the double covering map $Spin(d,d) \to SO(d,d)$. In odd dimensions d, $\Gamma^{(+)}$ is an element of $Pin(d,d)$ which maps to $G \in O(d,d)$ under the double cover $Pin(d,d) \to O(d,d)$.

3.3.2. Torsion-free, compatible connections

A generalised connection D is compatible with the $O(p,q)\times O(q,p)$ structure $P \subset \tilde{F}$ if

$$DG = 0, \qquad D\Phi = 0, \tag{3.86}$$

or equivalently, if the derivative acts only in the $O(p,q)\times O(q,p)$ sub-bundle so that for $W \in \Gamma(\tilde{E})$ given by

$$W = w^a_+\hat{E}^+_a + w^{\bar{a}}_-\hat{E}^-_{\bar{a}}, \tag{3.87}$$

we have

$$D_M W^A = \begin{cases} \partial_M w^a_+ + \Omega_M{}^a{}_b w^b_+ & \text{for } A = a \\ \partial_M w^{\bar{a}}_- + \Omega_M{}^{\bar{a}}{}_{\bar{b}} w^{\bar{b}}_- & \text{for } A = \bar{a} \end{cases}, \tag{3.88}$$

with

$$\Omega_{Mab} = -\Omega_{Mba}, \qquad \Omega_{M\bar{a}\bar{b}} = -\Omega_{M\bar{b}\bar{a}}. \tag{3.89}$$

In this subsection we will show, in analogy to the construction of the Levi–Civita connection, that

> *Given an $O(p,q) \times O(q,p)$ structure $P \subset \tilde{F}$ there always exists a torsion-free, compatible generalised connection D. However, it is not unique.*

We can construct a compatible connection as follows. Let ∇ be the Levi–Civita connection for the metric g. In terms of the two orthonormal bases we get two gauge equivalent spin-connections, so that if $v = v^a \hat{e}_a^+ = v^{\bar{a}} \hat{e}_{\bar{a}}^- \in \Gamma(TM)$ we have

$$\nabla_\mu v^\nu = \left(\partial_\mu v^a + \omega_\mu^{+a}{}_b v^b\right)(\hat{e}_a^+)^\nu = \left(\partial_\mu v^{\bar{a}} + \omega_\mu^{-\bar{a}}{}_{\bar{b}} v^{\bar{b}}\right)(\hat{e}_{\bar{a}}^-)^\nu. \tag{3.90}$$

We can then define, as in (3.58)

$$D^\nabla_M W^a = \begin{cases} \nabla_\mu w_+^a & \text{for } M = \mu \\ 0 & \text{for } M = \mu + d \end{cases}, \qquad D^\nabla_M W^{\bar{a}} = \begin{cases} \nabla_\mu w_-^{\bar{a}} & \text{for } M = \mu \\ 0 & \text{for } M = \mu + d \end{cases}. \tag{3.91}$$

Since $\omega^+_{\mu ab} = -\omega^+_{\mu ba}$ and $\omega^-_{\mu\bar{a}\bar{b}} = -\omega^-_{\mu\bar{b}\bar{a}}$, by construction, this generalised connection is compatible with the $O(p,q) \times O(q,p)$ structure.

However D^∇ is not torsion-free. To see this we note that, comparing with (3.38), when we choose the two orthonormal frames to be aligned so $e_a^+ = e_a^- = e_a$ we have

$$W = w_+^a \hat{E}_a^+ + w_-^{\bar{a}} \hat{E}_{\bar{a}}^- = \left(w_+^a + w_-^a\right) \hat{E}_a + \left(w_{+a} - w_{-a}\right) E^a, \tag{3.92}$$

and the two definitions of D^∇ in (3.58) and (3.91) agree. Hence from (3.69) we have the non-zero torsion components

$$T_1 = -4H, \qquad T_2 = -4\mathrm{d}\phi. \tag{3.93}$$

To construct a torsion-free compatible connection we simply modify D^∇. A generic generalised connection D can be always be written as

$$D_M W^A = D^\nabla_M W^A + \Sigma_M{}^A{}_B W^B. \tag{3.94}$$

If D is compatible with the $O(p,q)\times O(q,p)$ structure then we have $\Sigma_M{}^a{}_{\bar{b}} = \Sigma_M{}^{\bar{a}}{}_b = 0$ and

$$\Sigma_{Mab} = -\Sigma_{Mba}, \qquad \Sigma_{M\bar{a}\bar{b}} = -\Sigma_{M\bar{b}\bar{a}}. \tag{3.95}$$

By definition, the generalised torsion components of D are then given by

$$(T_1)_{ABC} = -4H_{ABC} - 3\Sigma_{[ABC]}, \qquad (T_2)_A = -4\mathrm{d}\phi_A - \Sigma_C{}^C{}_A. \tag{3.96}$$

The components H^{ABC} and $\mathrm{d}\phi^A$ are the components in frame indices of the corresponding forms under the embeddings $T^*M \hookrightarrow E$ and $\Lambda^3 T^*M \hookrightarrow \Lambda^3 E$. Given

$$\mathrm{d}x^\mu = \tfrac{1}{2}\Phi^{-1}\left(\hat{e}^{+\mu}_a \hat{E}^{+a} - \hat{e}^{-}_{\bar{a}}{}^{\mu}\hat{E}^{-\bar{a}}\right), \tag{3.97}$$

we have, for instance,

$$\mathrm{d}\phi = \tfrac{1}{2}\partial_a\phi\left(\Phi^{-1}\hat{E}^{+a}\right) - \tfrac{1}{2}\partial_{\bar{a}}\phi\left(\Phi^{-1}\hat{E}^{-\bar{a}}\right). \tag{3.98}$$

where there is a similar decomposition of H under

$$\Lambda^3 T^*M \hookrightarrow \Lambda^3 E \simeq \Lambda^3 C_+ \oplus (\Lambda^2 C_+ \otimes C_-) \oplus (C_+ \otimes \Lambda^2 C_-) \oplus \Lambda^3 C_-, \tag{3.99}$$

Note also that the middle index on $\Sigma_{[ABC]}$ in equation (3.96) has also been lowered with this η_{AB} which introduces some signs. The result is that the components are

$$\mathrm{d}\phi_A = \begin{cases} \frac{1}{2}\partial_a\phi & A = a \\ \frac{1}{2}\partial_{\bar{a}}\phi & A = \bar{a}+d \end{cases}, \qquad H_{ABC} = \begin{cases} \frac{1}{8}H_{abc} & (A,B,C) = (a,b,c) \\ \frac{1}{8}H_{ab\bar{c}} & (A,B,C) = (a,b,\bar{c}+d) \\ \frac{1}{8}H_{a\bar{b}\bar{c}} & (A,B,C) = (a,\bar{b}+d,\bar{c}+d) \\ \frac{1}{8}H_{\bar{a}\bar{b}\bar{c}} & (A,B,C) = (\bar{a}+d,\bar{b}+d,\bar{c}+d) \end{cases}, \tag{3.100}$$

and that setting the torsion of D to zero is equivalent to

$$\begin{aligned} \Sigma_{[abc]} &= -\tfrac{1}{6}H_{abc}, \quad \Sigma_{\bar{a}bc} = -\tfrac{1}{2}H_{\bar{a}bc}, \quad \Sigma_a{}^a{}_b = -2\partial_b\phi, \\ \Sigma_{[\bar{a}\bar{b}\bar{c}]} &= +\tfrac{1}{6}H_{\bar{a}\bar{b}\bar{c}}, \quad \Sigma_{a\bar{b}\bar{c}} = +\tfrac{1}{2}H_{a\bar{b}\bar{c}}, \quad \Sigma_{\bar{a}}{}^{\bar{a}}{}_{\bar{b}} = -2\partial_{\bar{b}}\phi. \end{aligned} \tag{3.101}$$

Thus we can always find a torsion-free compatible connection but clearly these conditions do not determine D uniquely. Specifically, one finds

$$
\begin{aligned}
D_a w_+^b &= \nabla_a w_+^b - \tfrac{1}{6} H_a{}^b{}_c w_+^c - \tfrac{2}{9}\big(\delta_a{}^b \partial_c \phi - \eta_{ac} \partial^b \phi\big) w_+^c + A^{+}_a{}^b{}_c w_+^c, \\
D_{\bar a} w_+^b &= \nabla_{\bar a} w_+^b - \tfrac{1}{2} H_{\bar a}{}^b{}_c w_+^c, \\
D_a w_-^{\bar b} &= \nabla_a w_-^{\bar b} + \tfrac{1}{2} H_a{}^{\bar b}{}_{\bar c} w_-^{\bar c}, \\
D_{\bar a} w_-^{\bar b} &= \nabla_{\bar a} w_-^{\bar b} + \tfrac{1}{6} H_{\bar a}{}^{\bar b}{}_{\bar c} w_-^{\bar c} - \tfrac{2}{9}\big(\delta_{\bar a}{}^{\bar b} \partial_{\bar c} \phi - \eta_{\bar a \bar c} \partial^{\bar b} \phi\big) w_-^{\bar c} + A^{-}_{\bar a}{}^{\bar b}{}_{\bar c} w_-^{\bar c},
\end{aligned}
\tag{3.102}
$$

where the undetermined tensors $A^{\pm}$ satisfy

$$
\begin{aligned}
A^{+}_{abc} &= -A^{+}_{acb}, \quad A^{+}_{[abc]} = 0, \quad A^{+}_a{}^a{}_b = 0, \\
A^{-}_{\bar a \bar b \bar c} &= -A^{-}_{\bar a \bar c \bar b}, \quad A^{-}_{[\bar a \bar b \bar c]} = 0, \quad A^{-}_{\bar a}{}^{\bar a}{}_{\bar b} = 0,
\end{aligned}
\tag{3.103}
$$

and hence do not contribute to the torsion.

3.3.3. Unique operators and generalised $O(p,q) \times O(q,p)$ curvatures

The fact that the $O(p,q) \times O(q,p)$ structure and torsion conditions are not sufficient to specify a unique generalised connection might raise ambiguities which could pose a problem for the applications to supergravity we are ultimately interested in. However, we will now show that it is still possible to find differential expressions that are independent of the chosen D, by forming $O(p,q) \times O(q,p)$ covariant operators which do not depend on the undetermined components $A^{\pm}$. For example, by examining (3.102) we already see that

$$
\begin{aligned}
D_{\bar a} w_+^b &= \nabla_{\bar a} w_+^b - \tfrac{1}{2} H_{\bar a}{}^b{}_c w_+^c, \\
D_a w_-^{\bar b} &= \nabla_a w_-^{\bar b} + \tfrac{1}{2} H_a{}^{\bar b}{}_{\bar c} w_-^{\bar c},
\end{aligned}
\tag{3.104}
$$

have no dependence on $A^{\pm}$ and so are unique. We find that this is also true for

$$
\begin{aligned}
D_a w_+^a &= \nabla_a w_+^a - 2(\partial_a \phi) w_+^a, \\
D_{\bar a} w_-^{\bar a} &= \nabla_{\bar a} w_-^{\bar a} - 2(\partial_{\bar a} \phi) w_-^{\bar a}.
\end{aligned}
\tag{3.105}
$$

Anticipating our application to supergravity, we will be especially interested in writing formulae for $Spin(p,q)$ spinors, so let us now assume that we have a $Spin(p,q) \times Spin(q,p)$ structure. If $S(C_\pm)$ are then the spinor bundles associated to the sub-bundles $C_\pm$, γ^a and $\gamma^{\bar a}$ the corresponding gamma matrices and $\epsilon^{\pm} \in \Gamma(S(C_{\pm}))$, we have that by definition a generalised connection acts as

$$\begin{aligned} D_M \epsilon^+ &= \partial_M \epsilon^+ + \tfrac{1}{4}\Omega_M{}^{ab}\gamma_{ab}\epsilon^+, \\ D_M \epsilon^- &= \partial_M \epsilon^- + \tfrac{1}{4}\Omega_M{}^{\bar{a}\bar{b}}\gamma_{\bar{a}\bar{b}}\epsilon^-. \end{aligned} \tag{3.106}$$

There are four operators which can be built out of these derivatives that are uniquely determined

$$\begin{aligned} D_{\bar{a}}\epsilon^+ &= \left(\nabla_{\bar{a}} - \tfrac{1}{8}H_{\bar{a}bc}\gamma^{bc}\right)\epsilon^+, \\ D_a\epsilon^- &= \left(\nabla_a + \tfrac{1}{8}H_{a\bar{b}\bar{c}}\gamma^{\bar{b}\bar{c}}\right)\epsilon^-, \\ \gamma^a D_a\epsilon^+ &= \left(\gamma^a\nabla_a - \tfrac{1}{24}H_{abc}\gamma^{abc} - \gamma^a\partial_a\phi\right)\epsilon^+, \\ \gamma^{\bar{a}} D_{\bar{a}}\epsilon^- &= \left(\gamma^{\bar{a}}\nabla_{\bar{a}} + \tfrac{1}{24}H_{\bar{a}\bar{b}\bar{c}}\gamma^{\bar{a}\bar{b}\bar{c}} - \gamma^{\bar{a}}\partial_{\bar{a}}\phi\right)\epsilon^-. \end{aligned} \tag{3.107}$$

The first two expressions follow directly from (3.104). In the final two expressions, there is an elegant cancellation from $\gamma^a\gamma^{bc} = \gamma^{abc} + \eta^{ab}\gamma^c - \eta^{ac}\gamma^b$ which removes the terms involving $A^{\pm}$.

The restriction that expressions involving generalised connections be determined unambiguously, irrespective of the particular D, now serves as a selection criteria for constructing new generalised objects. In particular, when defining a generalised notion of curvature, we find that even though we can actually build a tensorial $O(p,q) \times O(q,p)$ generalised Riemann curvature – by following the example in section 3.2.4 and taking $C_1 = C_{\pm}$ and $C_2 = C_{\mp}$ so that the index structure would be $\left(\mathrm{R}_{a\bar{b}}{}^c{}_d, \mathrm{R}_{a\bar{b}}{}^{\bar{c}}{}_{\bar{d}}\right)$ and $\left(\mathrm{R}_{\bar{a}b}{}^c{}_d, \mathrm{R}_{\bar{a}b}{}^{\bar{c}}{}_{\bar{d}}\right)$ – it would not result in a uniquely determined object. However, we can use combinations of (3.104) and (3.105) to define the corresponding *generalised Ricci tensor* as

$$R^0_{a\bar{b}}\, w^a_+ = [D_a, D_{\bar{b}}]\, w^a_+, \tag{3.108}$$

or as[4]

$$R^0_{\bar{a}b}\, w^{\bar{a}}_- = [D_{\bar{a}}, D_b]\, w^{\bar{a}}_-. \tag{3.109}$$

Note that the index contractions are precisely what is needed to guarantee uniqueness.

[4]Note that naively one might expect these definitions to give distinct tensors. However one can check that compatibility with the $O(p,q) \times O(q,p)$ structure means that the two agree.

It is not possible to contract the remaining two indices in the generalised Ricci. Nonetheless, there does exist a notion of generalised scalar curvature, but to define it we need the help of spinors and the operators in (3.107). We can obtain the generalised Ricci again from either

$$\begin{aligned} \tfrac{1}{2} R^0_{a\bar{b}} \gamma^a \epsilon^+ &= [\gamma^a D_a, D_{\bar{b}}]\, \epsilon^+, \\ \tfrac{1}{2} R^0_{\bar{a}b} \gamma^{\bar{a}} \epsilon^- &= \left[\gamma^{\bar{a}} D_{\bar{a}}, D_b\right] \epsilon^-. \end{aligned} \tag{3.110}$$

However, now we also find a *generalised curvature scalar*

$$-\tfrac{1}{4} R \epsilon^+ = \left(\gamma^a D_a \gamma^b D_b - D^{\bar{a}} D_{\bar{a}}\right) \epsilon^+, \tag{3.111}$$

or alternatively,

$$-\tfrac{1}{4} R \epsilon^- = \left(\gamma^{\bar{a}} D_{\bar{a}} \gamma^{\bar{b}} D_{\bar{b}} - D^a D_a\right) \epsilon^-. \tag{3.112}$$

Again, note the need to use the correct combinations of the operators in these definitions so that all the undetermined components drop out.

The fact that R is indeed a scalar and not itself an operator might not be immediately apparent, so it is useful to work out the explicit form of these curvatures. This can be done by again choosing the two orthogonal frames to be aligned, $e^+_a = e^-_a$, to find

$$R^0_{ab} = \mathcal{R}_{ab} - \tfrac{1}{4} H_{acd} H_b{}^{cd} + 2\nabla_a \nabla_b \phi + \tfrac{1}{2} \mathrm{e}^{2\phi} \nabla^c (\mathrm{e}^{-2\phi} H_{cab}), \tag{3.113}$$

and for the scalar

$$R = \mathcal{R} + 4\nabla^2 \phi - 4(\partial\phi)^2 - \tfrac{1}{12} H^2. \tag{3.114}$$

From these expressions it is clear that we have obtained genuine tensors which are uniquely determined by the torsion conditions, as desired. Furthermore, comparing with [61] we see that locally these are the same tensors that appear in Siegel's formulation. The expressions (3.113) and (3.114) also appear in the discussion of [117].

4. Type II Theories as $O(10, 10) \times \mathbb{R}^+$ Generalised Geometry

In this chapter, we will use the new geometry we have constructed to re-write the equations of type II supergravity. The NS-NS sector fields are packaged into the generalised metric, while the fermions fall into $Spin(9,1)\times Spin(1,9)$ representations. The RR sector fields form a chiral spinor of $Spin(d,d)\times\mathbb{R}^+$, which is coupled similarly to matter fields in general relativity. The bosonic action and equations of motion are written as generalised curvatures, while the supersymmetry variations and fermion equations have neat expressions in terms of the generalised connection. The presentation again follows [1] closely.

4.1. Type II supergravity

Let us briefly recall the structure of $d = 10$ type II supergravity. We essentially follow the conventions of the democratic formalism [141], as summarised in appendix A, and consider only the leading-order fermionic terms. We introduce a slightly unconventional notation in a few places in order to match more naturally with the underlying generalised geometry. It is also helpful to considerably rewrite the fermionic sector, introducing a particular linear combination of dilatini and gravitini, to match more closely what follows.

The type II fields are denoted

$$\{g_{\mu\nu}, B_{\mu\nu}, \phi, A^{(n)}_{\mu_1\dots\mu_n}, \psi^{\pm}_{\mu}, \lambda^{\pm}\}, \tag{4.1}$$

where $g_{\mu\nu}$ is the metric, $B_{\mu\nu}$ the two-form potential, ϕ is the dilaton and $A^{(n)}_{\mu_1\dots\mu_n}$ are the RR potentials in the democratic formalism, with n odd for type IIA and n even for type IIB. In each theory there is also a pair of chiral gravitini $\psi^\pm_\mu$ and a pair chiral dilatini $\lambda^\pm$. Here our notation is that $\pm$ does *not* refer to the chirality of the spinor but, as we will see, denote generalised geometrical subspaces. Specifically, in the notation of [141], for type IIA they are the chiral components of the gravitino and dilatino

$$\begin{aligned} \psi_\mu &= \psi^+_\mu + \psi^-_\mu && \text{where} \quad \gamma^{(10)}\psi^\pm_\mu = \mp\psi^\pm_\mu \\ \lambda &= \lambda^+ + \lambda^- && \text{where} \quad \gamma^{(10)}\lambda^\pm = \pm\lambda^\pm . \end{aligned} \tag{4.2}$$

(Note that ψ^+_μ and λ^+, and similarly ψ^-_μ and λ^-, have *opposite* chiralities.) For type IIB, in the notation of [141] one has two component objects

$$\begin{aligned} \psi_\mu &= \begin{pmatrix} \psi^+_\mu \\ \psi^-_\mu \end{pmatrix} && \text{where} \quad \gamma^{(10)}\psi^\pm_\mu = \psi^\pm_\mu \\ \lambda &= \begin{pmatrix} \lambda^+ \\ \lambda^- \end{pmatrix} && \text{where} \quad \gamma^{(10)}\lambda^\pm = -\lambda^\pm . \end{aligned} \tag{4.3}$$

and again the gravitini and dilatini have opposite chiralities.

In what follows, it will be very useful to consider the quantities

$$\rho^\pm := \gamma^\mu\psi^\pm_\mu - \lambda^\pm, \tag{4.4}$$

instead of $\lambda^\pm$. These are the natural combinations that appear in generalised geometry and from now on we will use $\rho^\pm$ rather than $\lambda^\pm$.

The bosonic "pseudo-action" takes the form

$$S_{\mathrm{B}} = \frac{1}{2\kappa^2}\int\sqrt{-g}\left[\mathrm{e}^{-2\phi}\left(\mathcal{R} + 4(\partial\phi)^2 - \tfrac{1}{12}H^2\right) - \tfrac{1}{4}\sum_n \tfrac{1}{n!}(F^{(B)}_{(n)})^2\right], \tag{4.5}$$

where $H = \mathrm{d}B$ and $F^{(B)}_{(n)}$ is the n-form RR field strength. Here we will use the "A-basis", where the field strengths, as sums of even or odd forms, take the form[1]

$$F^{(B)} = \sum_n F^{(B)}_{(n)} = \sum_n \mathrm{e}^B \wedge \mathrm{d}A_{(n-1)}, \tag{4.6}$$

where $\mathrm{e}^B = 1 + B + \frac{1}{2}B\wedge B + \dots$. This is a "pseudo-action" because the

[1]Note that in type IIA one cannot write a potential for the zero-form field strength, which must instead be added by hand in (4.6). Note also that in [141] these field strengths are denoted G.

RR fields satisfy a self-duality relation that does not follow from varying the action, namely,

$$F^{(B)}_{(n)} = (-)^{[n/2]} * F^{(B)}_{(10-n)}, \tag{4.7}$$

where $[n]$ denotes the integer part and $*\,\omega$ denotes the Hodge dual of ω. The fermionic action, keeping only terms quadratic in the fermions, can be written after some manipulation as

$$\begin{aligned} S_{\mathrm{F}} = -\frac{1}{2\kappa^2}\int \sqrt{-g}\Big[&\mathrm{e}^{-2\phi}\Big(2\bar{\psi}^{+\mu}\gamma^\nu\nabla_\nu\psi^+_\mu - 4\bar{\psi}^{+\mu}\nabla_\mu\rho^+ - 2\bar{\rho}^+\nabla\!\!\!/\rho^+ \\ &\qquad - \tfrac{1}{2}\bar{\psi}^{+\mu}H\!\!\!/\psi^+_\mu - \bar{\psi}^+_\mu H^{\mu\nu\lambda}\gamma_\nu\psi^+_\lambda - \tfrac{1}{2}\bar{\rho}^+H^{\mu\nu\lambda}\gamma_{\mu\nu}\psi^+_\lambda + \tfrac{1}{2}\bar{\rho}^+H\!\!\!/\rho^+\Big) \\ &+ \mathrm{e}^{-2\phi}\Big(2\bar{\psi}^{-\mu}\gamma^\nu\nabla_\nu\psi^-_\mu - 4\bar{\psi}^{-\mu}\nabla_\mu\rho^- - 2\bar{\rho}^-\nabla\!\!\!/\rho^- \\ &\qquad + \tfrac{1}{2}\bar{\psi}^{-\mu}H\!\!\!/\psi^-_\mu + \bar{\psi}^-_\mu H^{\mu\nu\lambda}\gamma_\nu\psi^-_\lambda + \tfrac{1}{2}\bar{\rho}^-H^{\mu\nu\lambda}\gamma_{\mu\nu}\psi^-_\lambda - \tfrac{1}{2}\bar{\rho}^-H\!\!\!/\rho^-\Big) \\ &- \tfrac{1}{4}\mathrm{e}^{-\phi}\Big(\bar{\psi}^+_\mu\gamma^\nu F\!\!\!/^{(B)}\gamma^\mu\psi^-_\nu + \bar{\rho}^+F\!\!\!/^{(B)}\rho^-\Big)\Big]. \end{aligned} \tag{4.8}$$

where ∇ is the Levi–Civita connection.

To match what follows it is useful to rewrite the standard equations of motion in a particular form. For the bosonic fields, with the fermions set to zero, one takes the combinations that naturally arise from the string β-functions, namely

$$\begin{aligned} \mathcal{R}_{\mu\nu} - \tfrac{1}{4}H_{\mu\lambda\rho}H_\nu{}^{\lambda\rho} + 2\nabla_\mu\nabla_\nu\phi - \tfrac{1}{4}\mathrm{e}^{2\phi}\sum_n \tfrac{1}{(n-1)!}F^{(B)}_{\mu\lambda_1\ldots\lambda_{n-1}}F_\nu^{(B)\lambda_1\ldots\lambda_{n-1}} &= 0, \\ \nabla^\mu\left(\mathrm{e}^{-2\phi}H_{\mu\nu\lambda}\right) - \tfrac{1}{2}\sum_n \tfrac{1}{(n-2)!}F^{(B)}_{\mu\nu\lambda_1\ldots\lambda_{n-2}}F^{(B)\lambda_1\ldots\lambda_{n-2}} &= 0, \\ \nabla^2\phi - (\nabla\phi)^2 + \tfrac{1}{4}\mathcal{R} - \tfrac{1}{48}H^2 &= 0, \\ \mathrm{d}F^{(B)} - H\wedge F^{(B)} &= 0, \end{aligned} \tag{4.9}$$

where the final Bianchi identity for F follows from the definition (4.6). Keeping only terms linear in the fermions, the fermionic equations of motion read

$$
\begin{aligned}
\gamma^\nu \Big[\Big(\nabla_\nu \mp \tfrac{1}{24} H_{\nu\lambda\rho}\gamma^{\lambda\rho} - \partial_\nu\phi\Big)\psi^\pm_\mu \pm \tfrac{1}{2} H_{\nu\mu}{}^\lambda \psi^\pm_\lambda\Big] - \Big(\nabla_\mu \mp \tfrac{1}{8} H_{\mu\nu\lambda}\gamma^{\nu\lambda}\Big)\rho^\pm& \\
= \tfrac{1}{16}\mathrm{e}^\phi \sum_n (\pm)^{[(n+1)/2]}\gamma^\nu \not{F}^{(B)}_{(n)}\gamma_\mu\psi^\mp_\nu,& \\
\Big(\nabla_\mu \mp \tfrac{1}{8} H_{\mu\nu\lambda}\gamma^{\nu\lambda} - 2\partial_\mu\phi\Big)\psi^{\mu\pm} - \gamma^\mu \Big(\nabla_\mu \mp \tfrac{1}{24} H_{\mu\nu\lambda}\gamma^{\nu\lambda} - \partial_\mu\phi\Big)\rho^\pm& \\
= \tfrac{1}{16}\mathrm{e}^\phi \sum_n (\pm)^{[(n+1)/2]} \not{F}^{(B)}_{(n)}\rho^\mp,&
\end{aligned}
\tag{4.10}
$$

The supersymmetry variations are parametrised by are pair of chiral spinors $\epsilon^\pm$ where, again, in the notation of [141], for type IIA, we have

$$
\epsilon = \epsilon^+ + \epsilon^- \qquad \text{where} \quad \gamma^{(10)}\epsilon^\pm = \mp\epsilon^\pm, \tag{4.11}
$$

while for type IIB we have the doublet

$$
\epsilon = \begin{pmatrix} \epsilon^+ \\ \epsilon^- \end{pmatrix} \qquad \text{where} \quad \gamma^{(10)}\epsilon^\pm = \epsilon^\pm. \tag{4.12}
$$

Again keeping only linear terms in the fermions field, the supersymmetry transformations for the bosons read

$$
\begin{aligned}
\delta e^a_\mu &= \bar\epsilon^+\gamma^a\psi^+_\mu + \bar\epsilon^-\gamma^a\psi^-_\mu, \\
\delta B_{\mu\nu} &= 2\bar\epsilon^+\gamma_{[\mu}\psi^+_{\nu]} - 2\bar\epsilon^-\gamma_{[\mu}\psi^-_{\nu]}, \\
\delta\phi - \tfrac{1}{4}\delta\log(-g) &= -\tfrac{1}{2}\bar\epsilon^+\rho^+ - \tfrac{1}{2}\bar\epsilon^-\rho^-, \\
\big(\mathrm{e}^B \wedge \delta A\big)^{(n)}_{\mu_1\dots\mu_n} &= \tfrac{1}{2}\Big(\mathrm{e}^{-\phi}\bar\psi^+_\nu\gamma_{\mu_1\dots\mu_n}\gamma^\nu\epsilon^- - \mathrm{e}^{-\phi}\bar\epsilon^+\gamma_{\mu_1\dots\mu_n}\rho^-\Big) \\
&\qquad \mp \tfrac{1}{2}\Big(\mathrm{e}^{-\phi}\bar\epsilon^+\gamma^\nu\gamma_{\mu_1\dots\mu_n}\psi^-_\nu + \mathrm{e}^{-\phi}\bar\rho^+\gamma_{\mu_1\dots\mu_n}\epsilon^-\Big),
\end{aligned}
\tag{4.13}
$$

where e_μ is an orthonormal frame for $g_{\mu\nu}$ and in the last equation the upper sign refers to type IIA and the lower to type IIB. For the fermions one has

$$
\begin{aligned}
\delta\psi^\pm_\mu &= \Big(\nabla_\mu \mp \tfrac{1}{8} H_{\mu\nu\lambda}\gamma^{\nu\lambda}\Big)\epsilon^\pm + \tfrac{1}{16}\mathrm{e}^\phi \sum_n (\pm)^{[(n+1)/2]} \not{F}^{(B)}_{(n)}\gamma_\mu\epsilon^\mp, \\
\delta\rho^\pm &= \gamma^\mu \Big(\nabla_\mu \mp \tfrac{1}{24} H_{\mu\nu\lambda}\gamma^{\nu\lambda} - \partial_\mu\phi\Big)\epsilon^\pm.
\end{aligned}
\tag{4.14}
$$

4.2. Type II supergravity as $O(9,1) \times O(1,9)$ generalised gravity

Let us now show how the dynamics and supersymmetry transformations of type II supergravity theories are encoded by an $O(9,1) \times O(1,9)$ structure with a compatible, torsion-free generalised connection. An outcome of this will be a formulation of type II supergravity with manifest local $O(9,1) \times O(1,9)$ symmetry.

In the following we will consider the full ten-dimensional supergraviy theory so that the relevant generalised structure is $O(10,10) \times \mathbb{R}^+$. However, one can equally well consider compactifications of theory of the form $\mathbb{R}^{9-d,1} \times M$

$$\mathrm{d}s^2_{10} = \mathrm{d}s^2(\mathbb{R}^{9-d,1}) + \mathrm{d}s^2_d, \tag{4.15}$$

where $\mathrm{d}s^2(\mathbb{R}^{9-d,1})$ is the flat metric on $\mathbb{R}^{9-d,1}$ and $\mathrm{d}s^2_d$ is a general metric on the d-dimensional manifold M. The relevant structure is then the $O(d) \times O(d) \subset O(d,d) \times \mathbb{R}^+$ generalised geometry on M. Below we will focus on the $O(10,10) \times \mathbb{R}^+$ case. The compactification case follows essentially identically, and the supersymmetry of such configurations will be examined for $d = 6$ in chapter 7.

4.2.1. NS-NS and fermionic supergravity fields

From the discussion of section 3.3.1 we see that an $O(9,1) \times O(1,9) \subset O(10,10) \times \mathbb{R}^+$ generalised structure is parametrised by a metric g of signature $(9,1)$, a two-form B patched as in (2.23) and a dilaton ϕ, that is, at each point $x \in M$

$$\{g, B, \phi\} \in \frac{O(10,10)}{O(9,1) \times O(1,9)} \times \mathbb{R}^+. \tag{4.16}$$

Thus it precisely captures the NS-NS bosonic fields of type II theories by packaging them into the generalised metric and conformal factor (G, Φ). As in [98], the infinitesimal bosonic symmetry transformation (2.27) is naturally encoded as the Dorfman derivative by $V = v + \lambda$

$$\delta_V G = L_V G, \qquad \delta_V \Phi = L_V \Phi \tag{4.17}$$

and the algebra of these transformations is given by the Courant bracket. The two parts of the generalised tangent space can be identified with the momentum and the electric charge for the B-field, and these are the generators of the bosonic symmetries. This relation is more obvious for the $E_{d(d)} \times \mathbb{R}^+$ generalised geometry of chapters 5 and 6.

The type II fermionic degrees of freedom fall into spinor and vector-spinor representations of $Spin(9,1) \times Spin(1,9)$[2]. Let $S(C_+)$ and $S(C_-)$ denote the $Spin(9,1)$ spinor bundles associated to the sub-bundles $C_\pm$ write γ^a and $\gamma^{\bar{a}}$ for the corresponding gamma matrices. Since we are in ten dimensions, we can further decompose into spinor bundles $S^\pm(C_+)$ and $S^\pm(C_-)$ of definite chirality under $\gamma^{(10)}$.

The gravitino degrees of freedom then correspond to

$$\psi^+_{\bar{a}} \in \Gamma(C_- \otimes S^\mp(C_+)), \qquad \psi^-_a \in \Gamma(C_+ \otimes S^+(C_-)), \tag{4.18}$$

where the upper sign on the chirality refers to type IIA and the lower to type IIB. Note that the vector and spinor parts of the gravitinos transform under different $Spin(9,1)$ groups. For the dilatino degrees of freedom one has

$$\rho^+ \in \Gamma(S^\pm(C_+)), \qquad \rho^- \in \Gamma(S^+(C_-)), \tag{4.19}$$

where again the upper and lower signs refer to IIA and IIB respectively. Similarly the supersymmetry parameters are sections

$$\epsilon^+ \in \Gamma(S^\mp(C_+)), \qquad \epsilon^- \in \Gamma(S^+(C_-)). \tag{4.20}$$

In terms of the string spectrum these gravitino and dilatino representations just correspond to the explicit left- and right-moving fermionic states of the superstring and, in a supergravity context were discussed, for example, in [142].

[2]Since the underlying manifold M is assumed to possess a spin structure, we are free to promote $O(9,1) \times O(1,9)$ to $Spin(9,1) \times Spin(1,9)$. Here will ignore more complicated extended spin structures that can arise in generalised geometry as described in [100].

4.2.2. RR fields

As is known from studying the action of T-duality, the RR field strengths transform as $Spin(10,10)$ spinors [32, 142, 143, 144]. Here, the patching[3]

$$A_{(i)} = \mathrm{e}^{\Lambda_{(ij)}} \wedge A_{(j)} + \mathrm{d}\hat{\Lambda}_{(ij)} \tag{4.21}$$

of $A_{(i)}$ on $U_i \cap U_j$ implies that the polyform $F_{(i)} = \mathrm{d}A_{(i)}$ is patched as in (3.31), and hence, as generalised spinors,

$$F \in \Gamma(S^{\pm}_{(1/2)}), \tag{4.22}$$

where the upper sign is for type IIA and the lower for type IIB. Furthermore, we see that the RR field strengths $F^{(B)}_{(n)}$ that appear in the supergravity (4.6) are simply F expressed in a split frame as in (3.40)

$$F^{(B)} = \mathrm{e}^{B_{(i)}} \wedge F_{(i)} = \mathrm{e}^{B_{(i)}} \wedge \sum_n \mathrm{d}A^{(n-1)}_{(i)}. \tag{4.23}$$

Note that the additional gauge transformations $\mathrm{d}\hat{\Lambda}$ in (4.21) imply that $A_{(i)}$ does not globally define a section of $S^{\pm}_{(1/2)}$. This additional gauge symmetry can be "geometrised" using $E_{d(d)}$ generalised geometry, which described in chapters 5 and 6. Since $A_{(i)}$ is still locally a generalised spinor on the patch U_i we can perform the same operations on it as we do on F in the remainder of this subsection.

Given the generalised metric structure, we can also write F in terms of $Spin(9,1)\times Spin(1,9)$ representations. One has the decomposition $\mathrm{Cliff}(10,10;\mathbb{R}) \simeq \mathrm{Cliff}(9,1;\mathbb{R}) \otimes \mathrm{Cliff}(1,9;\mathbb{R})$ with

$$\Gamma^A = \begin{cases} \gamma^a \otimes \mathbb{1} & \text{for } A = a \\ \gamma^{(10)} \otimes \gamma^{\bar{a}}\gamma^{(10)} & \text{for } A = \bar{a} + d \end{cases}. \tag{4.24}$$

and hence we can identify[4]

$$S_{(1/2)} \simeq S(C_+) \otimes S(C_-). \tag{4.25}$$

Using the spinor norm on $S(C_-)$ we can equally well view $F \in \Gamma(S_{(1/2)})$ as

[3] Here $\hat{\Lambda}_{(ij)}$ are a sum of even (odd) forms in type IIA (IIB).

[4] In fact $S_{(p)} \simeq S(C_+) \otimes S(C_-)$ for any p, but here we focus on the case of interest $p = \frac{1}{2}$

a map from sections of $S(C_-)$ to sections of $S(C_+)$. We denote the image under this isomorphism as

$$F_\# : S(C_-) \to S(C_+). \tag{4.26}$$

We have that $F \in \Gamma(S(C_+) \otimes S(C_-))$ naturally has spin indices $F^{\alpha\bar{\alpha}}$, while $F_\#$ naturally has indices $F^\alpha{}_{\bar{\alpha}}$. The isomorphism simply corresponds to lowering an index with the Cliff$(9,1;\mathbb{R})$ intertwiner $\tilde{C}_{\bar{\alpha}\bar{\beta}}$. The conjugate map, $F_\#^T : S(C_+) \to S(C_-)$, is given by

$$F_\#^T = (\tilde{C} F_\# \tilde{C}^{-1})^T, \tag{4.27}$$

which corresponds to lowering the other index on $F^{\alpha\bar{\alpha}}$ and taking the transpose.

We now give the relations between the components of the $Spin(d,d) \times \mathbb{R}^+$ spinor in all relevant frames. Note first that if the bases are aligned so that $e^+ = e^- = e$ then the $Spin(9,1) \times Spin(1,9)$ basis (3.77) is a split conformal basis and we have a $Spin(9,1) \subset Spin(9,1) \times Spin(1,9)$ structure. We can then use the isomorphism Cliff$(9,1;\mathbb{R}) \simeq \Lambda^\bullet T^*M$ to write $F^{(B,\phi)}$ as a spinor bilinear

$$\not{F}^{(B,\phi)} = \sum_n \tfrac{1}{n!} F^{(B,\phi)}_{a_1\dots a_n} \gamma^{a_1\dots a_n}. \tag{4.28}$$

More generally if the frames are related by Lorentz transformations $e^\pm_a = \Lambda^{\pm b}_a e_a$ and we write $\Lambda^\pm$ for the corresponding $Spin(9,1)$ transformations then we can define $F_\#$ explicitly as

$$F_\# = \Lambda^+ \not{F}^{(B,\phi)} (\Lambda^-)^{-1}, \tag{4.29}$$

which concretely realises the isomorphism between $F^{(B,\phi)}$ and $F_\#$.

This map can easily be inverted and so we can write the components of $F \in \Gamma(S_{(1/2)})$ in the coordinate frame as

$$\begin{aligned} F_{(i)} &= \mathrm{e}^{-B_{(i)}} \wedge F^{(B)} = \mathrm{e}^{-\phi} \mathrm{e}^{-B_{(i)}} \wedge F^{(B,\phi)} \\ &= \mathrm{e}^{-\phi} \mathrm{e}^{-B_{(i)}} \wedge \sum_n \Big[\frac{1}{32(n!)} (-)^{[n/2]} \operatorname{tr}\Big(\gamma_{(n)} (\Lambda^+)^{-1} F_\# \Lambda^- \Big) \Big]. \end{aligned} \tag{4.30}$$

This chain of equalities relates the components of F in all the frames we have discussed.

Finally, we note that the self-duality conditions satisfied by the RR field strengths $F \in \Gamma(S^{\pm}_{(1/2)})$ become a chirality condition under the operator $\Gamma^{(-)}$ defined in (3.82)

$$\Gamma^{(-)}F = -F, \tag{4.31}$$

as discussed in [118, 119].

4.2.3. Supersymmetry variations

We now show that the supersymmetry variations can be written in a simple, locally $Spin(9,1) \times Spin(1,9)$ covariant form using the torsion-free compatible connection D.

We start with the fermionic variations (4.14). Looking at the expressions (3.107), we see that the uniquely determined spinor operators allow us to write the supersymmetry variations compactly as

$$\begin{aligned} \delta\psi^{+}_{\bar{a}} &= D_{\bar{a}}\epsilon^{+} + \tfrac{1}{16}F_{\#}\gamma_{\bar{a}}\epsilon^{-}, \\ \delta\psi^{-}_{a} &= D_{a}\epsilon^{-} + \tfrac{1}{16}F^{T}_{\#}\gamma_{a}\epsilon^{+}, \\ \delta\rho^{+} &= \gamma^{a}D_{a}\epsilon^{+}, \\ \delta\rho^{-} &= \gamma^{\bar{a}}D_{\bar{a}}\epsilon^{-}, \end{aligned} \tag{4.32}$$

where we have also used the results from the previous section to add the RR field strengths to the gravitino variations.

For the bosonic fields, we need the variation of a generic $Spin(9,1) \times Spin(1,9)$ frame (3.77). Note that this means defining the variation of a pair of orthonormal bases $\{e^{+a}\}$ and $\{e^{-\bar{a}}\}$ whereas the conventional supersymmetry variations (4.13) are given in terms of a single basis $\{e^{a}\}$. The only possibility, compatible with the $Spin(9,1) \times Spin(1,9)$ representations of the fermions, is to take

$$\begin{aligned} \tilde{\delta}\hat{E}^{+}_{a} &= (\delta\log\Phi)\hat{E}^{+}_{a} + (\delta\Lambda^{+}_{a\bar{b}})\hat{E}^{-\bar{b}}, \\ \tilde{\delta}\hat{E}^{-}_{\bar{a}} &= (\delta\log\Phi)\hat{E}^{-}_{\bar{a}} + (\delta\Lambda^{-}_{\bar{a}b})\hat{E}^{+b}, \end{aligned} \tag{4.33}$$

where

$$\begin{aligned} \delta\Lambda^{+}_{a\bar{a}} &= \bar{\epsilon}^{+}\gamma_{a}\psi^{+}_{\bar{a}} + \bar{\epsilon}^{-}\gamma_{\bar{a}}\psi^{-}_{a}, \\ \delta\Lambda^{-}_{a\bar{a}} &= \bar{\epsilon}^{+}\gamma_{a}\psi^{+}_{\bar{a}} + \bar{\epsilon}^{-}\gamma_{\bar{a}}\psi^{-}_{a}, \end{aligned} \tag{4.34}$$

and

$$\delta \log \Phi = -2\delta\phi + \tfrac{1}{2}\delta \log(-g) = \bar{\epsilon}^+ \rho^+ + \bar{\epsilon}^- \rho^- . \tag{4.35}$$

Note that the variation of the basis (4.33) is by construction orthogonal to the $Spin(9,1) \times Spin(1,9)$ action. This is because it is impossible to construct an $Spin(9,1) \times Spin(1,9)$ tensor linear in $\psi^+_{\bar{a}}$ and ψ^-_a with two indices of the same type, that is L^+_{ab} or $L^-_{\bar{a}\bar{b}}$.

The corresponding variations of the frames $\hat{e}^{\pm}$ are

$$\begin{aligned} \tilde{\delta} e_\mu^{+a} &= \bar{\epsilon}^+ \gamma_\mu \psi^{+a} + \bar{\epsilon}^- \gamma^a \psi^-_\mu , \\ \tilde{\delta} e_\mu^{-\bar{a}} &= \bar{\epsilon}^+ \gamma^{\bar{a}} \psi^+_\mu + \bar{\epsilon}^- \gamma_\mu \psi^{-\bar{a}} , \end{aligned} \tag{4.36}$$

which both give

$$\tilde{\delta} g_{\mu\nu} = 2\bar{\epsilon}^+ \gamma_{(\mu} \psi^+_{\nu)} + 2\bar{\epsilon}^- \gamma_{(\mu} \psi^-_{\nu)} , \tag{4.37}$$

as required, but, when setting the frames equal so $e^{+a} = e^a$ and $e^{-\bar{a}} = e^{\bar{a}}$, differ by Lorentz transformations from the standard form (4.13)

$$\begin{aligned} \tilde{\delta} e_\mu^{+a} &= \delta e_\mu^{+a} - \left(\bar{\epsilon}^+ \gamma^a \psi^{+b} - \bar{\epsilon}^+ \gamma^b \psi^{+a} \right) e^+_{\mu b} , \\ \tilde{\delta} e_\mu^{-\bar{a}} &= \delta e_\mu^{+\bar{a}} - \left(\bar{\epsilon}^- \gamma^{\bar{a}} \psi^{-\bar{b}} - \bar{\epsilon}^- \gamma^{\bar{b}} \psi^{-\bar{a}} \right) e^-_{\mu\bar{b}} . \end{aligned} \tag{4.38}$$

This can also be expressed in terms of the generalised metric G_{AB} as

$$\delta G_{a\bar{a}} = \delta G_{\bar{a}a} = 2 \left(\bar{\epsilon}^+ \gamma_a \psi^+_{\bar{a}} + \bar{\epsilon}^- \gamma_{\bar{a}} \psi^-_a \right) . \tag{4.39}$$

The variation of the RR potential A can be written as a bispinor

$$\tfrac{1}{16} (\delta A_\#) = \left(\gamma^a \epsilon^+ \bar{\psi}^-_a - \rho^+ \bar{\epsilon}^- \right) \mp \left(\psi^+_{\bar{a}} \bar{\epsilon}^- \gamma^{\bar{a}} + \epsilon^+ \bar{\rho}^- \right) , \tag{4.40}$$

where the upper sign is for type IIA and the lower for type IIB.

4.2.4. Equations of motion

Finally, we rewrite the supergravity equations of motion (4.9) and (4.10) with local $Spin(9,1) \times Spin(1,9)$ covariance, using the generalised notions of curvature obtained in section 3.3.3.

From the generalised Ricci tensor (3.113), we find that the equations of motion for g and B can be written as

$$R^{0}_{a\bar{b}} + \tfrac{1}{16}\Phi^{-1}\langle F, \Gamma_{a\bar{b}} F\rangle = 0, \tag{4.41}$$

where we have made use of the Mukai pairing defined in (3.33)[5] to introduce the RR fields in a $Spin(9,1) \times Spin(1,9)$ covariant manner.

The equation of motion for ϕ does not involve the RR fields, so it is simply given by the generalised scalar curvature (3.114)

$$R = 0. \tag{4.42}$$

Using definition (3.51) and equation (3.65) we can write the equation of motion for the RR fields in the familiar form

$$\tfrac{1}{2}\Gamma^{A} D_{A} F = \mathrm{d}F = 0, \tag{4.43}$$

where the first equality serves as a reminder that this definition of the exterior derivative is fully covariant under $Spin(d,d) \times \mathbb{R}^{+}$.

We also have the bosonic pseudo-action (4.5) which takes the simple form[6]

$$S_B = \frac{1}{2\kappa^2} \int \Big(\Phi\, R + \tfrac{1}{4}\langle F, \Gamma^{(-)} F\rangle \Big), \tag{4.44}$$

using the density Φ. Note that the Mukai pairing is a top-form which can be directly integrated.

The fermionic action (4.8) is given by

$$\begin{aligned} S_F = -\frac{1}{2\kappa^2} \int 2\Phi \Big[& \bar{\psi}^{+\bar{a}} \gamma^{b} D_{b} \psi^{+}_{\bar{a}} + \bar{\psi}^{-a} \gamma^{\bar{b}} D_{\bar{b}} \psi^{-}_{a} \\ & + 2\bar{\rho}^{+} D_{\bar{a}} \psi^{+\bar{a}} + 2\bar{\rho}^{-} D_{a} \psi^{-a} \\ & - \bar{\rho}^{+} \gamma^{a} D_{a} \rho^{+} - \bar{\rho}^{-} \gamma^{\bar{a}} D_{\bar{a}} \rho^{-} \\ & - \tfrac{1}{8} \Big(\bar{\rho}^{+} F_{\#} \rho^{-} + \bar{\psi}^{+}_{\bar{a}} \gamma^{a} F_{\#} \gamma^{\bar{a}} \psi^{-}_{a} \Big) \Big]. \end{aligned} \tag{4.45}$$

Varying this with respect to the fermionic fields leads to the generalised

[5] Note that $\langle F, \Gamma_{a\bar{b}} F\rangle \in \Gamma(\tilde{L} \otimes C_{+} \otimes C_{-})$ so $\Phi^{-1}\langle F, \Gamma_{a\bar{b}} F\rangle \in \Gamma(C_{+} \otimes C_{-})$

[6] Up to integration by parts of the $\nabla^2\phi$ term

geometry version of (4.10)

$$
\begin{aligned}
\gamma^b D_b \psi^+_{\bar{a}} - D_{\bar{a}} \rho^+ &= +\tfrac{1}{16} \gamma^b F_\# \gamma_{\bar{a}} \psi^-_b , \\
\gamma^{\bar{b}} D_{\bar{b}} \psi^-_a - D_a \rho^- &= +\tfrac{1}{16} \gamma^{\bar{b}} F^T_\# \gamma_a \psi^+_{\bar{b}} , \\
\gamma^a D_a \rho^+ - D^{\bar{a}} \psi^+_{\bar{a}} &= -\tfrac{1}{16} F_\# \rho^- , \\
\gamma^{\bar{a}} D_{\bar{a}} \rho^- - D^a \psi^-_a &= -\tfrac{1}{16} F^T_\# \rho^+ ,
\end{aligned}
\tag{4.46}
$$

and it is straightforward to verify that by applying a supersymmetry variation (4.32) we recover the bosonic equations of motion (4.41)-(4.43).

We have thus rewritten all the supergravity equations from section 4.1 in terms of torsion free generalised connections and these expressions are therefore manifestly covariant under local $Spin(9,1) \times Spin(1,9)$ transformations.

5. $E_{d(d)} \times \mathbb{R}^+$ Generalised Geometry

We would now like to extend the generalised geometry developed thus far to a geometry for eleven-dimensional supergravity, which will also include the RR fields of type II theories when studied over a manifold of one dimension less. In the previous chapters we saw that the generalised geometry for the NS-NS sector of type II theories had an $O(d, d)$ factor in its structure group. This was reminiscent of the $O(d, d)$ action on the moduli space of NS-NS fields in compactifications on tori. The corresponding groups for eleven-dimensional supergravity reductions are the exceptional groups $E_{d(d)}$. We are therefore led to consider the structure group $E_{d(d)} \times \mathbb{R}^+$ as the natural replacement for $O(d, d) \times \mathbb{R}^+$ in this new geometry. As the exceptional groups are well-understood only in dimensional splits of eleven-dimensional supergravity, we aim to construct a geometry related to the internal sectors of these, rather than taking on the full theory.

The $E_{d(d)}$ generalised tangent space was first developed in [100] and independently in [101], where the exceptional Courant bracket was given for the first time. Our discussion follows closely that in [2], which also includes the $\mathbb{R}^+$ factor known as the "trombone symmetry" [145]. This allows one to specify the isomorphism between the generalised tangent space and a sum of vectors and forms. Physically, it is related to the "warp factor" of warped supergravity reductions. The need for this extra factor in the context of $E_{7(7)}$ geometries has previously been identified in [64, 104, 114].

5.1. $E_{d(d)} \times \mathbb{R}^+$ generalised tangent space

Following closely the construction given in section 3.2, here we introduce the generalised geometry versions of the tangent space, frame bundle, Lie derivative, connections and torsion, now in the more subtle context of an $E_{d(d)} \times \mathbb{R}^+$ structure.

5.1.1. Generalised bundles and frames

Generalised tangent space

We start by recalling the definition of the generalised tangent space for $E_{d(d)} \times \mathbb{R}^+$ generalised geometry [100, 101] and defining what is meant by the "generalised structure".

Let M be a d-dimensional spin manifold with $d \leq 7$. The generalised tangent space is isomorphic to a sum of tensor bundles

$$E \simeq TM \oplus \Lambda^2 T^*M \oplus \Lambda^5 T^*M \oplus (T^*M \otimes \Lambda^7 T^*M), \tag{5.1}$$

where for $d < 7$ some of these terms will of course be absent. The isomorphism is not unique. The bundle is actually described using a specific patching. If we write

$$\begin{aligned} V_{(i)} &= v_{(i)} + \omega_{(i)} + \sigma_{(i)} + \tau_{(i)} \\ &\quad \in \Gamma(TU_i \oplus \Lambda^2 T^*U_i \oplus \Lambda^5 T^*U_i \oplus (T^*U_i \otimes \Lambda^7 T^*U_i)), \end{aligned} \tag{5.2}$$

for a section of E over the patch U_i, then

$$V_{(i)} = \mathrm{e}^{\mathrm{d}\Lambda_{(ij)} + \mathrm{d}\tilde{\Lambda}_{(ij)}} V_{(j)}, \tag{5.3}$$

on the overlap $U_i \cap U_j$ where $\Lambda_{(ij)}$ and $\tilde{\Lambda}_{(ij)}$ are locally two- and five-forms respectively. The exponentiated action is given by

$$\begin{aligned} v_{(i)} &= v_{(j)}, \\ \omega_{(i)} &= \omega_{(j)} + i_{v_{(j)}} \mathrm{d}\Lambda_{(ij)}, \\ \sigma_{(i)} &= \sigma_{(j)} + \mathrm{d}\Lambda_{(ij)} \wedge \omega_{(j)} + \tfrac{1}{2} \mathrm{d}\Lambda_{(ij)} \wedge i_{v_{(j)}} \mathrm{d}\Lambda_{(ij)} + i_{v_{(j)}} \mathrm{d}\tilde{\Lambda}_{(ij)}, \\ \tau_{(i)} &= \tau_{(j)} + j\mathrm{d}\Lambda_{(ij)} \wedge \sigma_{(j)} - j\mathrm{d}\tilde{\Lambda}_{(ij)} \wedge \omega_{(j)} + j\mathrm{d}\Lambda_{(ij)} \wedge i_{v_{(j)}} \mathrm{d}\tilde{\Lambda}_{(ij)} \\ &\qquad + \tfrac{1}{2} j\mathrm{d}\Lambda_{(ij)} \wedge \mathrm{d}\Lambda_{(ij)} \wedge \omega_{(j)} + \tfrac{1}{6} j\mathrm{d}\Lambda_{(ij)} \wedge \mathrm{d}\Lambda_{(ij)} \wedge i_{v_{(j)}} \mathrm{d}\Lambda_{(ij)}, \end{aligned} \tag{5.4}$$

where we are using the notation of (C.8). Technically this defines E as a result of a series of extensions

$$\begin{aligned} 0 \longrightarrow \Lambda^2 T^*M \longrightarrow E'' \longrightarrow TM \longrightarrow 0, \\ 0 \longrightarrow \Lambda^5 T^*M \longrightarrow E' \longrightarrow E'' \longrightarrow 0, \\ 0 \longrightarrow T^*M \otimes \Lambda^7 T^*M \longrightarrow E \longrightarrow E' \longrightarrow 0. \end{aligned} \tag{5.5}$$

Note that while the $v_{(i)}$ are globally equivalent to a choice of vector, the $\omega_{(i)}$, $\sigma_{(i)}$ and $\tau_{(i)}$ are not globally tensors.

Note that globally $\Lambda_{(ij)}$ and $\tilde{\Lambda}_{(ij)}$ formally define[1] "connective structures on gerbes" (for a review see, for example, [136]). This essentially means there is a hierarchy of successive gauge transformations. For $\Lambda_{(ij)}$ on the multiple intersections we have

$$\begin{aligned} \Lambda_{(ij)} - \Lambda_{(ik)} + \Lambda_{(jk)} &= \mathrm{d}\Lambda_{(ijk)} \qquad \text{on } U_i \cap U_j \cap U_k, \\ \Lambda_{(ijk)} - \Lambda_{(ijl)} + \Lambda_{(ikl)} - \Lambda_{(jkl)} &= \mathrm{d}\Lambda_{(ijkl)} \qquad \text{on } U_i \cap U_j \cap U_k \cap U_l. \end{aligned} \tag{5.6}$$

For $\tilde{\Lambda}$, there is a similar set of structures with analogous relations to (5.6) going down to a septuple intersection $U_{i_1} \cap \cdots \cap U_{i_7}$.

The bundle E encodes all the topological information of the supergravity background: the twisting of the tangent space TM as well as that of the gerbes, which encode the topology of the supergravity form-field potentials.

Generalised $E_{d(d)} \times \mathbb{R}^+$ structure bundle and split frames

In all dimensions[2] $d \leq 7$ the fibre E_x of the generalised vector bundle at $x \in M$ forms a representation space of $E_{d(d)} \times \mathbb{R}^+$. These are listed in table 5.1. As we discuss below, the explicit action is defined using the $GL(d, \mathbb{R})$ subgroup that acts on the component spaces T_xM, $\Lambda^2 T_x^*M$, $\Lambda^5 T_x^*M$ and $T_x^*M \otimes \Lambda^7 T_x^*M$. Note that without the additional $\mathbb{R}^+$ action, sections of E would transform as tensors weighted by a power of $\det T^*M$. Thus it is key to extend the action to $E_{d(d)} \times \mathbb{R}^+$ in order to define E directly as the extension (5.5).

Crucially, the patching defined in (5.3) is compatible with this $E_{d(d)} \times \mathbb{R}^+$ action. This means that one can define a generalised structure bundle as a sub-bundle of the frame bundle F for E. Let $\{\hat{E}_A\}$ be a basis for E_x, where the label A runs over the dimension n of the generalised tangent space as listed in table 5.1. The frame bundle F formed from all such bases is, by construction, a $GL(n, \mathbb{R})$ principle bundle. We can then define the generalised structure bundle as the natural $E_{d(d)} \times \mathbb{R}^+$ principle sub-bundle of F compatible with the patching (5.3) as follows.

[1] Note that again the gerbe structure actually requires quantised fluxes which are suitably related to an integral cohomology classes (see e.g. [101]).

[2] In fact the $d \leq 2$ cases essentially reduce to normal Riemannian geometry, so in what follows we will always take $d \geq 3$.

$E_{d(d)}$ group	$E_{d(d)} \times\mathbb{R}^+$ rep.
$E_{7(7)}$	$\mathbf{56_1}$
$E_{6(6)}$	$\mathbf{27'_1}$
$E_{5(5)} \simeq Spin(5,5)$	$\mathbf{16^c_1}$
$E_{4(4)} \simeq SL(5,\mathbb{R})$	$\mathbf{10'_1}$
$E_{3(3)} \simeq SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$	$(\mathbf{3'},\mathbf{2})_\mathbf{1}$

Table 5.1.: Generalised tangent space representations where the subscript denotes the $\mathbb{R}^+$ weight

Let $\hat{e}_a$ be a basis for T_xM and e^a the dual basis for T^*_xM. We can use these to construct an explicit basis of E_x as

$$\{\hat{E}_A\} = \{\hat{e}_a\} \cup \{e^{ab}\} \cup \{e^{a_1\dots a_5}\} \cup \{e^{a,a_1\dots a_7}\}, \tag{5.7}$$

where $e^{a_1\dots a_p} = e^{a_1} \wedge \cdots \wedge e^{a_p}$ and $e^{a,a_1\dots a_7} = e^a \otimes e^{a_1} \wedge \cdots \wedge e^{a_7}$. A generic section of E at $x \in U_i$ takes the form

$$V = V^A\hat{E}_A = v^a\hat{e}_a + \tfrac{1}{2}\omega_{ab}e^{ab} + \tfrac{1}{5!}\sigma_{a_1\dots a_5}e^{a_1\dots a_5} + \tfrac{1}{7!}\tau_{a,a_1\dots a_7}e^{a,a_1\dots a_7}. \tag{5.8}$$

As usual, a choice of coordinates on U_i defines a particular such basis where $\{\hat{E}_A\} = \{\partial/\partial x^m\} \cup \{\mathrm{d}x^m \wedge \mathrm{d}x^n\} + \dots$. We will denote the components of V in such a coordinate frame by an index M, namely $V^M = (v^m, \omega_{mn}, \sigma_{m_1\dots m_5}, \tau_{m,m_1\dots m_7})$.

We then define a $E_{d(d)} \times\mathbb{R}^+$ *basis* as one related to (5.7) by an $E_{d(d)} \times\mathbb{R}^+$ transformation

$$V^A \mapsto V'^A = M^A{}_BV^B, \qquad \hat{E}_A \mapsto \hat{E}'_A = \hat{E}_B(M^{-1})^B{}_A, \tag{5.9}$$

where the explicit action of M is defined in appendix C.1. The action has a $GL(d,\mathbb{R})$ subgroup that acts in a conventional way on the bases $\hat{e}_a$, e^{ab} etc, and includes the patching transformation (5.3)[3].

The fact that the definition of the $E_{d(d)} \times\mathbb{R}^+$ action is compatible with the

[3]In analogy to the definitions for $O(d,d) \times \mathbb{R}^+$ generalised geometry from chapter 3, we could equivalently define an $E_{d(d)} \times\mathbb{R}^+$ basis using invariants constructed from sections of E. For example, in $d = 7$ there is a natural symplectic pairing and symmetric quartic invariant that can be used to define $E_{7(7)}$ (in the context of generalised geometry see [101]). However, these invariants differ in different dimension d so it is more useful here to define $E_{d(d)}$ by an explicit action.

patching means that we can then define the *generalised* $E_{d(d)} \times \mathbb{R}^+$ *structure bundle* $\tilde{F}$ as a sub-bundle of the frame bundle for E given by

$$\tilde{F} = \big\{(x, \{\hat{E}_A\}) : x \in M, \text{ and } \{\hat{E}_A\} \text{ is an } E_{d(d)} \times \mathbb{R}^+ \text{ basis of } E_x\big\}. \quad (5.10)$$

By construction, this is a principle bundle with fibre $E_{d(d)} \times \mathbb{R}^+$. The bundle $\tilde{F}$ is the direct analogue of the frame bundle of conventional differential geometry, with $E_{d(d)} \times \mathbb{R}^+$ playing the role of $GL(d, \mathbb{R})$.

A special class of $E_{d(d)} \times \mathbb{R}^+$ frames are those defined by a splitting of the generalised tangent space E, that is, an isomorphism of the form (5.1). Let A and $\tilde{A}$ be three- and six-form (gerbe) connections patched on $U_i \cap U_j$ by

$$\begin{aligned} A_{(i)} &= A_{(j)} + \mathrm{d}\Lambda_{(ij)}, \\ \tilde{A}_{(i)} &= \tilde{A}_{(j)} + \mathrm{d}\tilde{\Lambda}_{(ij)} - \tfrac{1}{2}\mathrm{d}\Lambda_{(ij)} \wedge A_{(j)}. \end{aligned} \quad (5.11)$$

Note that from these one can construct the globally defined field strengths

$$\begin{aligned} F &= \mathrm{d}A_{(i)}, \\ \tilde{F} &= \mathrm{d}\tilde{A}_{(i)} - \tfrac{1}{2}A_{(i)} \wedge F. \end{aligned} \quad (5.12)$$

Given a generic basis $\{\hat{e}_a\}$ for TM with $\{e^a\}$ the dual basis on T^*M and a scalar function Δ, we define a *conformal split frame* $\{\hat{E}_A\}$ for E by

$$\begin{aligned} \hat{E}_a &= \mathrm{e}^{\Delta}\Big(\hat{e}_a + i_{\hat{e}_a}A + i_{\hat{e}_a}\tilde{A} + \tfrac{1}{2}A \wedge i_{\hat{e}_a}A \\ &\qquad\qquad + jA \wedge i_{\hat{e}_a}\tilde{A} + \tfrac{1}{6}jA \wedge A \wedge i_{\hat{e}_a}A\Big), \\ \hat{E}^{ab} &= \mathrm{e}^{\Delta}\left(e^{ab} + A \wedge e^{ab} - j\tilde{A} \wedge e^{ab} + \tfrac{1}{2}jA \wedge A \wedge e^{ab}\right), \\ \hat{E}^{a_1\dots a_5} &= \mathrm{e}^{\Delta}\left(e^{a_1\dots a_5} + jA \wedge e^{a_1\dots a_5}\right), \\ \hat{E}^{a,a_1\dots a_7} &= \mathrm{e}^{\Delta}e^{a,a_1\dots a_7}, \end{aligned} \quad (5.13)$$

while a *split frame* has the same form but with $\Delta = 0$. To see that A and $\tilde{A}$ define an isomorphism (5.1) note that, in the conformal split frame,

$$\begin{aligned} V^{(A,\tilde{A},\Delta)} &= \mathrm{e}^{-\Delta}\mathrm{e}^{-A_{(i)}-\tilde{A}_{(i)}}V_{(i)} \\ &= v^a\hat{e}_a + \tfrac{1}{2}\omega_{ab}e^{ab} + \tfrac{1}{5!}\sigma_{a_1\dots a_5}e^{a_1\dots a_5} + \tfrac{1}{7!}\tau_{a,a_1\dots a_7}e^{a,a_1\dots a_7} \\ &\in \Gamma(TM \oplus \Lambda^2T^*M \oplus \Lambda^5T^*M \oplus (T^*M \otimes \Lambda^7T^*M)), \end{aligned} \quad (5.14)$$

since the patching implies $e^{-A_{(i)}-\tilde{A}_{(i)}}V_{(i)} = e^{-A_{(j)}-\tilde{A}_{(j)}}V_{(j)}$ on $U_i \cap U_j$.

The class of split frames defines a sub-bundle of $\tilde{F}$

$$P_{\text{split}} = \big\{(x, \{\hat{E}_A\}) : x \in M, \text{ and } \{\hat{E}_A\} \text{ is split frame}\big\} \subset \tilde{F}. \tag{5.15}$$

Split frames are related by transformations (5.9) where M takes the form $M = e^{a+\tilde{a}}m$ with $m \in GL(d,\mathbb{R})$. The action of $a + \tilde{a}$ shifts $A \mapsto A + a$ and $\tilde{A} \mapsto \tilde{A} + \tilde{a}$. This forms a parabolic subgroup $G_{\text{split}} = GL(d,\mathbb{R}) \ltimes (a + \tilde{a})\text{-shifts} \subset E_{d(d)} \times \mathbb{R}^+$ where $(a + \tilde{a})$-shifts is the nilpotent group of order two formed of elements $M = e^{a+\tilde{a}}$. Hence P_{split} is a G_{split} principle sub-bundle of $\tilde{F}$, that is a G_{split}-structure. This reflects the fact that the patching elements in the definition of E lie only in this subgroup of $E_{d(d)} \times \mathbb{R}^+$.

Generalised tensors

Generalised tensors are simply sections of vector bundles constructed from the generalised structure bundle using different representations of $E_{d(d)} \times \mathbb{R}^+$. We have already discussed the generalised tangent space E. There are four other vector bundles which will be of particular importance in the following. The relevant representations are summarised in table 5.2.

dimension	E^*	$\text{ad}\,\tilde{F} \subset E \otimes E^*$	$N \subset S^2E$	$K \subset E^* \otimes \text{ad}\,\tilde{F}$
7	$\mathbf{56}_{-1}$	$\mathbf{133}_{\mathbf{0}} + \mathbf{1}_{\mathbf{0}}$	$\mathbf{133}_{+\mathbf{2}}$	$\mathbf{912}_{-\mathbf{1}}$
6	$\mathbf{27}_{-\mathbf{1}}$	$\mathbf{78}_{\mathbf{0}} + \mathbf{1}_{\mathbf{0}}$	$\mathbf{27}'_{+\mathbf{2}}$	$\mathbf{351}'_{-\mathbf{1}}$
5	$\mathbf{16}^c_{-\mathbf{1}}$	$\mathbf{45}_{\mathbf{0}} + \mathbf{1}_{\mathbf{0}}$	$\mathbf{10}_{+\mathbf{2}}$	$\mathbf{144}^c_{-\mathbf{1}}$
4	$\mathbf{10}_{-\mathbf{1}}$	$\mathbf{24}_{\mathbf{0}} + \mathbf{1}_{\mathbf{0}}$	$\mathbf{5}'_{+\mathbf{2}}$	$\mathbf{40}_{-\mathbf{1}} + \mathbf{15}'_{-\mathbf{1}}$
3	$(\mathbf{3},\mathbf{2})_{-\mathbf{1}}$	$(\mathbf{8},\mathbf{1})_{\mathbf{0}} + (\mathbf{1},\mathbf{3})_{\mathbf{0}} + \mathbf{1}_{\mathbf{0}}$	$(\mathbf{3}',\mathbf{1})_{+\mathbf{2}}$	$(\mathbf{3}',\mathbf{2})_{-\mathbf{1}} + (\mathbf{6},\mathbf{2})_{-\mathbf{1}}$

Table 5.2.: Some generalised tensor bundles

The first is the dual of the generalised tangent space

$$E^* \simeq T^*M \oplus \Lambda^2 TM \oplus \Lambda^5 TM \oplus (TM \otimes \Lambda^7 TM). \tag{5.16}$$

Given a basis $\{\hat{E}_A\}$ for E we have a dual basis $\{E^A\}$ on E^* and sections of E^* can be written as $U = U_A E^A$.

Next we then have the adjoint bundle $\text{ad}\,\tilde{F}$ associated with the $E_{d(d)} \times \mathbb{R}^+$

principle bundle $\tilde{F}$ (see (C.3))

$$\operatorname{ad}\tilde{F} \simeq \mathbb{R} \oplus (TM \otimes T^*M) \oplus \Lambda^3 T^*M \oplus \Lambda^6 T^*M \oplus \Lambda^3 TM \oplus \Lambda^6 TM. \quad (5.17)$$

By construction $\operatorname{ad}\tilde{F} \subset E \otimes E^*$ and hence we can write sections as $R = R^A{}_B \hat{E}_A \otimes E^B$. We write the projection on the adjoint representation as

$$\otimes_{\text{ad}} : E^* \otimes E \to \operatorname{ad}\tilde{F}. \quad (5.18)$$

It is given explicitly in (C.14).

We also consider the sub-bundle of the symmetric product of two generalised tangent bundles $N \subset S^2 E$,

$$\begin{aligned} N \simeq T^*M &\oplus \Lambda^4 T^*M \oplus (T^*M \otimes \Lambda^6 T^*M) \\ &\oplus (\Lambda^3 T^*M \otimes \Lambda^7 T^*M) \oplus (\Lambda^6 T^*M \otimes \Lambda^7 T^*M). \end{aligned} \quad (5.19)$$

We can write sections as $S = S^{AB} \hat{E}_A \otimes \hat{E}_B$ with the projection

$$\otimes_N : E \otimes E \to N. \quad (5.20)$$

It is given explicitly in (C.17).

Finally, we also need the higher dimensional representation $K \subset E^* \otimes \operatorname{ad}\tilde{F}$ listed in the last column of table 5.2. Decomposing under $GL(d, \mathbb{R})$ one has

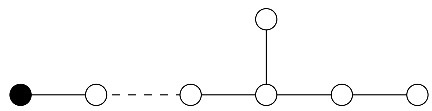
$$\begin{aligned} K \simeq T^*M &\oplus S^2 TM \oplus \Lambda^2 TM \oplus (\Lambda^2 T^*M \otimes TM)_0 \oplus (\Lambda^3 TM \otimes T^*M)_0 \\ &\oplus \Lambda^4 T^*M \oplus (\Lambda^4 TM \otimes TM)_0 \oplus \Lambda^5 TM \oplus (\Lambda^2 TM \otimes \Lambda^6 TM)_0 \\ &\oplus \Lambda^7 T^*M \oplus (TM \otimes \Lambda^7 TM) \oplus (\Lambda^7 TM \otimes \Lambda^7 TM) \\ &\oplus (S^2 T^*M \otimes \Lambda^7 TM) \oplus (\Lambda^4 TM \otimes \Lambda^7 TM), \end{aligned} \quad (5.21)$$

where, in fact, the $\Lambda^5 TM$ term is absent when $d = 5$. Note also that the zero subscripts are defined such that

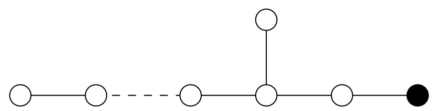
$$\begin{aligned} a_{mn}{}^n &= 0, && \text{if } a \in \Gamma((\Lambda^2 T^*M \otimes TM)_0), \\ a^{mnp}{}_p &= 0, && \text{if } a \in \Gamma((\Lambda^3 TM \otimes T^*M)_0), \\ a^{[m_1 m_2 m_3 m_4, m_5]} &= 0, && \text{if } a \in \Gamma((\Lambda^4 TM \otimes TM)_0), \\ a^{m[n_1, m_2 \dots, n_7]} &= 0, && \text{if } a \in \Gamma((\Lambda^2 TM \otimes \Lambda^6 TM)_0). \end{aligned} \quad (5.22)$$

Since $K \subset E^* \otimes \operatorname{ad} \tilde{F}$ we can write sections as $T = T_A{}^B{}_C E^A \otimes \hat{E}_B \otimes E^C$.

It is interesting to note that, up to symmetries of the E_d Dynkin diagram, the Dynkin labels of the representations E and N follow patterns as d varies. For each value of d, the Dynkin label for E can be represented on the Dynkin diagram as



while N has the label



5.1.2. The Dorfman derivative and Courant bracket

As in $O(d,d) \times \mathbb{R}^+$ generalised geometry, we find that our construction admits a generalisation of the Lie derivative which encodes the bosonic symmetries of the supergravity. Given $V = v + \omega + \sigma + \tau \in \Gamma(E)$, one can define an operator L_V acting on any generalised tensor, which combines the action of an infinitesimal diffeomorphism generated by v and A- and $\tilde{A}$-field gauge transformations generated by ω and σ. Formally this gives E the structure of a "Leibniz algebroid" [104].

Acting on $V' = v' + \omega' + \sigma' + \tau' \in \Gamma(E)$, one defines the *Dorfman derivative*[4] or "generalised Lie derivative"

$$\begin{aligned} L_V V' = \mathcal{L}_v v' &+ \left(\mathcal{L}_v \omega' - i_{v'} \mathrm{d}\omega\right) + \left(\mathcal{L}_v \sigma' - i_{v'} \mathrm{d}\sigma - \omega' \wedge \mathrm{d}\omega\right) \\ &+ \left(\mathcal{L}_v \tau' - j\sigma' \wedge \mathrm{d}\omega - j\omega' \wedge \mathrm{d}\sigma\right). \end{aligned} \tag{5.23}$$

Defining the action on a function f as simply $L_V f = \mathcal{L}_v f$, one can then extend the notion of Dorfman derivative to a derivative on the space of $E_{d(d)} \times \mathbb{R}^+$ tensors using the Leibniz property.

To see this, first note that we can rewrite (5.23) in a more $E_{d(d)} \times \mathbb{R}^+$ covariant way, in analogy with the corresponding expressions for the conventional Lie derivative (2.5) and the Dorfman derivative in $O(d,d) \times \mathbb{R}^+$

[4] We follow the nomenclature of $O(d,d) \times \mathbb{R}^+$ generalised geometry, as this directly corresponds to the definition given there.

generalised geometry (3.45). One can embed the action of the partial derivative operator via the map $T^*M \to E^*$ defined by the dual of the exact sequences (5.5). In coordinate indices M, as viewed as mapping to a section of E^*, one defines

$$\partial_M = \begin{cases} \partial_m & \text{for } M = m \\ 0 & \text{otherwise} \end{cases} . \tag{5.24}$$

Such an embedding has the property that under the projection onto N^* we have

$$\partial f \otimes_{N^*} \partial g = 0, \tag{5.25}$$

for arbitrary functions f, g. We will comment on this observation in section 5.1.4.

One can then rewrite (5.23) in terms of generalised objects as

$$L_V V'^M = V^N \partial_N V'^M - (\partial \otimes_{\text{ad}} V)^M{}_N V'^N, \tag{5.26}$$

where $\otimes_{\text{ad}}$ denotes the projection onto $\text{ad}\,\tilde{F}$ given in (5.18). Concretely, from (C.14) we have

$$\partial \otimes_{\text{ad}} V = r + a + \tilde{a}, \tag{5.27}$$

where $r^m{}_n = \partial_n v^m$, $a = \mathrm{d}\omega$ and $\tilde{a} = \mathrm{d}\sigma$. We see that the action actually lies in the adjoint of the $G_{\text{split}} \subset E_{d(d)} \times \mathbb{R}^+$ group. This form of the Dorfman derivative can then be naturally extended to an arbitrary $E_{d(d)} \times \mathbb{R}^+$ tensor by taking that appropriate adjoint action on the $E_{d(d)} \times \mathbb{R}^+$ representation.

Note that we can also define a bracket by taking the antisymmetrisation of the Dorfman derivative. This was originally given in [101] where it was called the "exceptional Courant bracket", and re-derived in [104]. It is given by

$$\begin{aligned} [\![V, V']\!] &= \tfrac{1}{2}\left(L_V V' - L_{V'} V\right) \\ &= [v, v'] + \mathcal{L}_v \omega' - \mathcal{L}_{v'} \omega - \tfrac{1}{2}\mathrm{d}\left(i_v \omega' - i_{v'} \omega\right) \\ &\quad + \mathcal{L}_v \sigma' - \mathcal{L}_{v'} \sigma - \tfrac{1}{2}\mathrm{d}\left(i_v \sigma' - i_{v'} \sigma\right) + \tfrac{1}{2}\omega \wedge \mathrm{d}\omega' - \tfrac{1}{2}\omega' \wedge \mathrm{d}\omega \\ &\quad + \tfrac{1}{2}\mathcal{L}_v \tau' - \tfrac{1}{2}\mathcal{L}_{v'} \tau + \tfrac{1}{2}\big(j\omega \wedge \mathrm{d}\sigma' - j\sigma' \wedge \mathrm{d}\omega\big) - \tfrac{1}{2}\big(j\omega' \wedge \mathrm{d}\sigma - j\sigma \wedge \mathrm{d}\omega'\big). \end{aligned} \tag{5.28}$$

Note that the group generated by closed A and $\tilde{A}$ shifts is a semi-direct product $\Omega^3_{\mathrm{cl}}(M) \ltimes \Omega^6_{\mathrm{cl}}(M)$ and corresponds to the symmetry group of gauge transformations in the supergravity. The full automorphism group of the exceptional Courant bracket is then the local symmetry group of the supergravity $G_{\mathrm{sugra}} = \mathrm{Diff}(M) \ltimes (\Omega^3_{\mathrm{cl}}(M) \ltimes \Omega^6_{\mathrm{cl}}(M))$.

For $U, V, W \in \Gamma(E)$, the Dorfman derivative also satisfies the Leibniz identity

$$L_U(L_V W) - L_V(L_U W) = L_{L_U V} W, \tag{5.29}$$

and hence E is a "Leibniz algebroid". On first inspection, one might expect that the bracket of $[\![U, V]\!]$ should appear on the RHS. However, the statement is correct since one can show that

$$L_{[\![U,V]\!]} W = L_{L_U V} W, \tag{5.30}$$

so that the RHS is automatically antisymmetric in U and V.

5.1.3. Generalised $E_{d(d)} \times \mathbb{R}^+$ connections and torsion

We now turn to the definitions of generalised connections and torsion. Definitions of derivative operators in $E_{7(7)}$ geometries of a similar type have been considered in [64, 103]. Here, for the $E_{d(d)} \times \mathbb{R}^+$ case, we follow the same procedure as in chapter 3.

Generalised connections

We first define generalised connections that are compatible with the $E_{d(d)} \times \mathbb{R}^+$ structure. These are first-order linear differential operators D, such that, given $W \in E$, in frame indices,

$$D_M W^A = \partial_M W^A + \Omega_M{}^A{}_B W^B. \tag{5.31}$$

where Ω is a section of E^* (denoted by the M index) taking values in $E_{d(d)} \times \mathbb{R}^+$ (denoted by the A and B frame indices), and as such, the action of D then extends naturally to any generalised $E_{d(d)} \times \mathbb{R}^+$ tensor.

Given a conventional connection ∇ and a conformal split frame of the form (5.13), one can construct the corresponding generalised connection as follows. Given the isomorphism (5.14), by construction $v^a \hat{e}_a \in \Gamma(TM)$, $\frac{1}{2}\omega_{ab} e^{ab} \in \Gamma(\Lambda^2 T^* M)$ etc and hence $\nabla_m v^a$ and $\nabla_m \omega_{ab}$ are well-defined.

The generalised connection defined by ∇ lifted to an action on E by the conformal split frame is then simply

$$D^{\nabla}_M V = \begin{cases} (\nabla_m v^a)\hat{E}_a + \frac{1}{2}(\nabla_m \omega_{ab})\hat{E}^{ab} \\ \quad + \frac{1}{5!}(\nabla_m \sigma_{a_1 \dots a_5})\hat{E}^{a_1 \dots a_5} + \frac{1}{7!}(\nabla_m \tau_{a,a_1 \dots a_7})\hat{E}^{a,a_1 \dots a_7} & \text{for } M = m, \\ 0 & \text{otherwise.} \end{cases} \tag{5.32}$$

Generalised torsion

We define the *generalised torsion* T of a generalised connection D in direct analogy to the conventional definition, as we did for the $O(d,d) \times \mathbb{R}^+$ case.

Let α be any generalised $E_{d(d)} \times \mathbb{R}^+$ tensor and let $L^D_V \alpha$ be the Dorfman derivative (5.26) with ∂ replaced by D. The generalised torsion is a linear map $T : \Gamma(E) \to \Gamma(\mathrm{ad}(\tilde{F}))$ defined by

$$T(V) \cdot \alpha = L^D_V \alpha - L_V \alpha, \tag{5.33}$$

for any $V \in \Gamma(E)$ and where $T(V)$ acts via the adjoint representation on α. Let $\{\hat{E}_A\}$ be an $E_{d(d)} \times \mathbb{R}^+$ frame for E and $\{E^A\}$ be the dual frame for E^* satisfying $E^A(\hat{E}_B) = \delta^A{}_B$. We then have the explicit expression

$$T(V) = V^C \left[\Omega_C{}^A{}_B - \Omega_B{}^A{}_C - E^A(L_{\hat{E}_C} \hat{E}_B) \right] \hat{E}_A \otimes_{\mathrm{ad}} E^B. \tag{5.34}$$

Note that we are projecting onto the adjoint representation on the A and B indices. Note also that in a coordinate frame the last term vanishes.

Viewed as a generalised $E_{d(d)} \times \mathbb{R}^+$ tensor we have $T \in \Gamma(E^* \otimes \mathrm{ad}\,\tilde{F})$. However, the form of the Dorfman derivative means that fewer components actually survive and we find

$$T \in \Gamma(K \oplus E^*), \tag{5.35}$$

where K was defined in table 5.2. Note that these representations are exactly the same ones that appear in the embedding tensor formulation of gauged supergravities [126], including gaugings [127] of the so-called "trombone" symmetry [145]. The relation between the embedding tensor and the generalised torsion can be made concrete by examining identity structures on the generalised tangent space [2], but we do not give further details here.

As an example, we can calculate the torsion of the generalised connection D^{∇} defined by a conventional connection ∇ and a conformal split frame as given in (5.32). Assuming ∇ is torsion-free we find

$$T(V) = \mathrm{e}^{\Delta}\left(-i_v \mathrm{d}\Delta + v \otimes \mathrm{d}\Delta - i_v F + \mathrm{d}\Delta \wedge \omega - i_v \tilde{F} + \omega \wedge F + \mathrm{d}\Delta \wedge \sigma\right), \tag{5.36}$$

where we are using the isomorphism (5.17), and F and $\tilde{F}$ are the (globally defined) field strengths of the potentials A and $\tilde{A}$ given by (5.12).

5.1.4. The "section condition", Jacobi identity and the absence of generalised curvature

Restricting our analysis to $d \leq 6$, we find that the bundle N given in (5.19) measures the failure of the generalised tangent bundle to satisfy the properties of a Lie algebroid. This follows from the observation that the difference between the Dorfman derivative and the exceptional Courant bracket (that is, the symmetric part of the Dorfman derivative), for $V, V' \in \Gamma(E)$, is precisely given by[5]

$$L_V V' - [\![V, V']\!] = \tfrac{1}{2}\mathrm{d}\left(i_v \omega' + i_{v'}\omega - i_v \sigma' - i_{v'}\sigma + \omega \wedge \omega'\right) = \partial \otimes_E \big(V \otimes_N V'\big), \tag{5.37}$$

where the last equality stresses the $E_{d(d)} \times \mathbb{R}^+$ covariant form of the exact term. Therefore, while the Dorfman derivative satisfies a sort of Jacobi identity via the Leibniz identity (5.29), the Jacobiator of the exceptional Courant bracket, like that of the $O(d,d)$ Courant bracket, does not vanish in general. In fact, it can be shown that

$$\mathrm{Jac}(U, V, W) = [\![[\![U, V]\!], W]\!] + \text{c.p.} = \tfrac{1}{3}\partial \otimes_E \left([\![U, V]\!] \otimes_N W + \text{c.p.}\right), \tag{5.38}$$

where $W \in \Gamma(E)$ and c.p. denotes cyclic permutations in U, V and W. We see that both the failure of the exceptional Courant bracket to be Jacobi and the Dorfman derivative to be antisymmetric is measured by an exact term given by the $\otimes_N$ projection. The proof is essentially the same as the one for the $O(d,d)$ case, see for example [82], section **3.2**[6].

[5]For $d \geq 7$ the RHS can no longer be written as a derivative of an object built from U and V in an $E_{d(d)} \times \mathbb{R}^+$ covariant way. Similar complications occur in the discussion of the curvature below. This is the reason for the restriction to $d \leq 6$ in this section.

[6]Note that $N \simeq \mathbb{R}$ in the $O(d,d)$ case.

Similarly, and as was the case with $O(d,d)\times\mathbb{R}^+$ generalised connections, one finds that the naive definition of generalised curvature $[D_U, D_V]\,W - D_{[\![U,V]\!]}W$ is not a tensor and its failure to be covariant is measured by the projection of the first two arguments to N. Explicitly, taking $U \to fU$, $V \to gV$ and $W \to hW$ for some scalar functions f, g, h, we obtain

$$\begin{aligned}&[D_{fU}, D_{gV}]\,hW - D_{[\![fU,gV]\!]}hW \\ &\quad = fgh\left([D_U, D_V]\,W - D_{[\![U,V]\!]}W\right) - \tfrac{1}{2}hD_{(f\partial g - g\partial f)\otimes_E(U\otimes_N V)}W.\end{aligned} \tag{5.39}$$

Note, however, that it is still possible to define analogues of the Ricci tensor and scalar when there is additional structure on the generalised tangent space, as we will see in section 5.2.3.

Finally, we note that from the point of view of "double field theory"-like geometries [61, 109, 112, 114, 115], the equation

$$\partial f \otimes_{N^*} \partial g = 0, \tag{5.40}$$

for any functions f and g acquires a special interpretation. In these theories, one starts by enlarging the spacetime manifold so that its dimension matches that of the generalised tangent space. The partial derivative $\partial_M f$ is then generically non-zero for all M. However, the corresponding Dorfman derivative does not then satisfy the Leibniz property, nor is the action for the generalised metric invariant. One must instead impose a "section condition" or "strong constraint". In the original $O(d,d)$ double field theory the condition takes the form $(\partial^A f)(\partial_A g) = 0$. It implies that, in fact, the fields only depend on half the coordinates [111]. For exceptional geometries, the $d = 4$ case was thoroughly analysed in [115], and is given by (5.40). Again it implies that the fields depend on only d of the coordinates.

It can be shown that satisfying (5.40) always implies the Leibniz property for the Dorfman derivative. Thus it gives the section condition in general dimension. In generalised geometry it is satisfied identically by taking ∂_M of the form (5.24). However given the $E_{d(d)}\times\mathbb{R}^+$ covariant form of the Dorfman derivative (5.26), any subspace of E^* in the same orbit under $E_{d(d)}\times\mathbb{R}^+$ will also satisfy the Leibniz condition. Note further that any such subspace, like T^*, is invariant under an action of the parabolic subgroup G_{split}.

5.2. H_d structures and torsion-free connections

We now turn to the construction of the analogue of the Levi–Civita connection by considering additional structure on the generalised tangent space. Again, this closely follows the constructions in $O(d,d) \times \mathbb{R}^+$ generalised geometry from chapter 3.

We consider H_d structures on E where H_d is the maximally compact-subgroup of $E_{d(d)}$. These, or rather their double covers[7] $\tilde{H}_d$ are listed in table 5.3. We will then be interested in generalised connections D that preserve the H_d structure. We find it is always possible to construct torsion-free connections of this type but they are not unique. Nonetheless we show that, using the H_d structure, one can construct unique projections of D, and that these can be used to define analogues of the Ricci tensor and scalar curvatures with a local H_d symmetry.

$E_{d(d)}$ group	$\tilde{H}_d$ group	$\operatorname{ad} P^{\perp} = \operatorname{ad} \tilde{F} / \operatorname{ad} P$
$E_{7(7)}$	$SU(8)$	$\mathbf{35} + \bar{\mathbf{35}} + \mathbf{1}$
$E_{6(6)}$	$Sp(8)$	$\mathbf{42} + \mathbf{1}$
$E_{5(5)} \simeq Spin(5,5)$	$Spin(5) \times Spin(5)$	$(\mathbf{5},\mathbf{5}) + (\mathbf{1},\mathbf{1})$
$E_{4(4)} \simeq SL(5,\mathbb{R})$	$Spin(5)$	$\mathbf{14} + \mathbf{1}$
$E_{3(3)} \simeq SL(3,\mathbb{R})) \times SL(2,\mathbb{R})$	$Spin(3) \times Spin(2)$	$(\mathbf{5},\mathbf{1}) + (\mathbf{1},\mathbf{2}) + (\mathbf{1},\mathbf{1})$

Table 5.3.: Double covers of the maximal compact subgroups of $E_{d(d)}$ and H_d representations of the coset bundle

5.2.1. H_d structures and the generalised metric

In $E_{d(d)} \times \mathbb{R}^+$ generalised geometry, the analogue of a metric structure is an H_d structure, i.e. a principal sub-bundle P, with fibre H_d, of the generalised structure bundle $\tilde{F}$. The choice of such a structure is parametrised, at each point on the manifold, by an element of the coset $(E_{d(d)} \times \mathbb{R}^+)/\, H_d$. The corresponding representations are listed in table 5.3. Note that there is always a singlet corresponding to the $\mathbb{R}^+$ factor.

[7] We give the double covers of the maximally compact group, since we will be interested in the analogues of spinor representations. A necessary and sufficient condition for the existence of the double cover is the vanishing of the 2nd Stiefel-Whitney class of the generalised tangent bundle [100]. As the underlying manifold is spin by assumption, this is automatically satisfied.

One can construct elements of P concretely, that is, identify the analogues of "orthonormal" frames, in the following way. Given an H_d structure, it is always possible to put the H_d frame in a conformal split form, namely,

$$\begin{aligned}
\hat{E}_a &= \mathrm{e}^{\Delta}\Big(\hat{e}_a + i_{\hat{e}_a}A + i_{\hat{e}_a}\tilde{A} + \tfrac{1}{2}A\wedge i_{\hat{e}_a}A \\
&\qquad\qquad + jA\wedge i_{\hat{e}_a}\tilde{A} + \tfrac{1}{6}jA\wedge A\wedge i_{\hat{e}_a}A\Big), \\
\hat{E}^{ab} &= \mathrm{e}^{\Delta}\left(e^{ab} + A\wedge e^{ab} - j\tilde{A}\wedge e^{ab} + \tfrac{1}{2}jA\wedge A\wedge e^{ab}\right), \\
\hat{E}^{a_1\dots a_5} &= \mathrm{e}^{\Delta}\left(e^{a_1\dots a_5} + jA\wedge e^{a_1\dots a_5}\right), \\
\hat{E}^{a,a_1\dots a_7} &= \mathrm{e}^{\Delta}e^{a,a_1\dots a_7}.
\end{aligned} \tag{5.41}$$

Any other frame is then related by an H_d transformation of the form given in appendix C.2. Concretely given $V = V^A\hat{E}_A \in \Gamma(E)$ expanded in such a frame, different frames are related by

$$V^A \mapsto V'^A = H^A{}_B V^B, \qquad \hat{E}_A \mapsto \hat{E}'_A = \hat{E}_B(H^{-1})^B{}_A, \tag{5.42}$$

where H is defined in (C.22). Note that the $O(d) \subset H_d$ action simply rotates the $\hat{e}_a$ basis, defining a set of orthonormal frames for a conventional metric g. It also keeps the frame in the conformal split form. Thus the set of conformal split H_d frames actually forms an $O(d)$ structure on E, that is

$$(P\cap P_{\text{split}}) \subset \tilde{F} \text{ with fibre } O(d). \tag{5.43}$$

One can also define the generalised metric acting on $V \in \Gamma(E)$ as

$$G(V,V) = |v|^2 + |\omega|^2 + |\sigma|^2 + |\tau|^2, \tag{5.44}$$

where $|v|^2 = v_a v^a$, $|\omega|^2 = \frac{1}{2!}\omega_{ab}\omega^{ab}$, $|\sigma|^2 = \frac{1}{5!}\sigma_{a_1\dots a_5}\sigma^{a_1\dots a_5}$ and $|\tau|^2 = \frac{1}{7!}\tau_{a,a_1\dots a_7}\tau^{a,a_1\dots a_7}$ evaluated in an H_d frame and indices are contracted using the flat frame metric δ_{ab} (as used to define the H_d subgroup in appendix C.2). Since, by definition, this is independent of the choice of H_d frame, it can be evaluated in the conformal split representative (5.41). Hence one sees explicitly that the metric is defined by the fields g, A, $\tilde{A}$ and Δ that determine the coset element.

Note that the H_d structure embeds as $H_d \subset E_{d(d)} \subset E_{d(d)} \times \mathbb{R}^+$. This mirrors the chain of embeddings in Riemannian geometry $SO(d) \subset SL(d,\mathbb{R}) \subset$

$GL(d,\mathbb{R})$ which allows one to define a $\det T^*M$ density that is $SO(d)$ invariant, $\sqrt{g}$. Likewise, here we can define a density that is H_d (and $E_{d(d)}$) invariant, corresponding to the choice of $\mathbb{R}^+$ factor which, in terms of the conformal split frame, is given by

$$\mathrm{vol}_G = \sqrt{g}\,\mathrm{e}^{(9-d)\Delta}, \tag{5.45}$$

as can be seen from appendix C.1.1. This can also be defined as the determinant of G to a suitable power.

5.2.2. Torsion-free, compatible connections

A generalised connection D is compatible with the H_d structure $P \subset \tilde{F}$ if

$$DG = 0, \tag{5.46}$$

or, equivalently, if the derivative acts only in the H_d sub-bundle. In this subsection we will show, in analogy to the construction of the Levi–Civita connection, that exactly as for the $O(d,d) \times \mathbb{R}^+$ geometry

> *Given an H_d structure $P \subset \tilde{F}$ there always exists a torsion-free, compatible generalised connection D. However, it is not unique.*

We construct the compatible connection explicitly by working in the conformal split H_d frame (5.41). However the connection is H_d covariant, so the form in any another frame simply follows from an H_d transformation.

Let ∇ be the Levi–Civita connection for the metric g. Via the conformal split frame, we can lift this connection to a generalised connection D^∇ as in (5.32). Since ∇ is compatible with the $O(d) \subset H_d$ subgroup, it necessarily gives rise to an H_d-compatible connection. However, the generalised torsion of D^∇ is given by equation (5.36), and thus D^∇ is not generically torsion-free.

To construct a torsion-free compatible connection we simply modify D^∇. A generic generalised connection D can always be written as

$$D_M W^A = D^\nabla_M W^A + \Sigma_M{}^A{}_B W^B. \tag{5.47}$$

If D is compatible with the H_d structure then

$$\Sigma \in \Gamma(E^* \otimes \operatorname{ad} P), \tag{5.48}$$

that is, it is a generalised covector taking values in the adjoint of H_d. The problem is then to find a suitable Σ such that the torsion of D vanishes. Fortunately, decomposing under H_d one finds that all the representations that appear in the torsion are already contained in Σ. Thus, as in the $O(d,d) \times \mathbb{R}^+$ case, a solution always exists, but is not unique[8]. The relevant representations are listed in table 5.4. As H_d tensor bundles one has

$$E^* \otimes \operatorname{ad} P \simeq (K \oplus E^*) \oplus U, \tag{5.49}$$

so that the torsion $T \in \Gamma(K \oplus E^*)$ and the unconstrained part of Σ is a section of U.

dimension	$K \oplus E^*$	$U \simeq (E^* \otimes \operatorname{ad} P)/(K \oplus E^*)$
7	$\mathbf{28} + \bar{\mathbf{28}} + \mathbf{36} + \bar{\mathbf{36}} + \mathbf{420} + \bar{\mathbf{420}}$	$\mathbf{1280} + \bar{\mathbf{1280}}$
6	$\mathbf{27} + \mathbf{36} + \mathbf{315}$	$\mathbf{594}$
5	$(\mathbf{4},\mathbf{4}) + (\mathbf{4},\mathbf{4}) + (\mathbf{16},\mathbf{4}) + (\mathbf{4},\mathbf{16})$	$(\mathbf{20},\mathbf{4}) + (\mathbf{4},\mathbf{20})$
4	$\mathbf{1} + \mathbf{5} + \mathbf{10} + \mathbf{14} + \mathbf{35}'$	$\mathbf{35}$
3	$(\mathbf{1},\mathbf{2}) + (\mathbf{3},\mathbf{2}) + (\mathbf{3},\mathbf{2}) + (\mathbf{5},\mathbf{2})$	-

Table 5.4.: Components of the connection Σ that are constrained by the torsion, T, and the unconstrained ones, U, as H_d representations

The solution for Σ can be written very explicitly as follows. Contracting with $V \in \Gamma(E)$ so $\Sigma(V) \in \operatorname{ad} P$ and using the basis for the adjoint of H_d given in (C.20) and (C.21) we have

$$\begin{aligned}
\Sigma(V)_{ab} &= \mathrm{e}^{\Delta}\left(2\left(\tfrac{7-d}{d-1}\right) v_{[a}\partial_{b]}\Delta + \tfrac{1}{4!}\omega_{cd}F^{cd}{}_{ab} + \tfrac{1}{7!}\sigma_{c_1\dots c_5}\tilde{F}^{c_1\dots c_5}{}_{ab} + C(V)_{ab}\right), \\
\Sigma(V)_{abc} &= \mathrm{e}^{\Delta}\left(\tfrac{6}{(d-1)(d-2)}(\mathrm{d}\Delta \wedge \omega)_{abc} + \tfrac{1}{4}v^d F_{dabc} + C(V)_{abc}\right), \\
\Sigma(V)_{a_1\dots a_6} &= \mathrm{e}^{\Delta}\left(\tfrac{1}{7}v^b \tilde{F}_{ba_1\dots a_6} + C(V)_{a_1\dots a_6}\right),
\end{aligned} \tag{5.50}$$

[8]In $d = 3$ all the components of Σ are contained in the torsion representations, $E^* \otimes \operatorname{ad} P \simeq K \oplus E^*$, and so, in that particular case, the generalised connection is in fact completely determined.

where the ambiguous part of the connection $C \in \Gamma(E^* \otimes \operatorname{ad} P)$ projects to zero under the map to the torsion representation $K \oplus E^*$, that is

$$C \in \Gamma(U). \tag{5.51}$$

Using the two possible embeddings of $\tilde{H}_d$ in $\mathrm{Cliff}(d, \mathbb{R})$ given in (C.26), we can thus write the full connection as

$$\begin{aligned}
D_a &= \mathrm{e}^{\Delta}\left(\nabla_a + \tfrac{1}{2}\left(\tfrac{7-d}{d-1}\right)(\partial_b\Delta)\gamma_a{}^b \pm \tfrac{1}{2}\tfrac{1}{4!}F_{ab_1b_2b_3}\gamma^{b_1b_2b_3} - \tfrac{1}{2}\tfrac{1}{7!}\tilde{F}_{ab_1\dots b_6}\gamma^{b_1\dots b_6} + \not{C}_a\right), \\
D^{a_1a_2} &= \mathrm{e}^{\Delta}\left(\tfrac{1}{4}\tfrac{2!}{4!}F^{a_1a_2}{}_{b_1b_2}\gamma^{b_1b_2} \pm \tfrac{3}{(d-1)(d-2)}(\partial_b\Delta)\gamma^{a_1a_2b} + \not{C}^{a_1a_2}\right), \\
D^{a_1\dots a_5} &= \mathrm{e}^{\Delta}\left(\tfrac{1}{4}\tfrac{5!}{7!}\tilde{F}^{a_1\dots a_5}{}_{b_1b_2}\gamma^{b_1b_2} + \not{C}^{a_1\dots a_5}\right), \\
D^{a,a_1\dots a_7} &= \mathrm{e}^{\Delta}\left(\not{C}^{a,a_1\dots a_7}\right),
\end{aligned} \tag{5.52}$$

where $\pm$ corresponds to the choice of embedding and

$$\begin{aligned}
\not{C}_m &= \tfrac{1}{2}\left(\tfrac{1}{2!}C_{m,ab}\gamma^{ab} \pm \tfrac{1}{3!}C_{m,a_1a_2a_3}\gamma^{a_1a_2a_3} - \tfrac{1}{6!}C_{m,a_1\dots a_6}\gamma^{a_1\dots a_6}\right), \\
\not{C}^{m_1m_2} &= \tfrac{1}{2}\left(\tfrac{1}{2!}C^{m_1m_2}{}_{ab}\gamma^{ab} \pm \tfrac{1}{3!}C^{m_1m_2}{}_{a_1a_2a_3}\gamma^{a_1a_2a_3} - \tfrac{1}{6!}C^{m_1m_2}{}_{a_1\dots a_6}\gamma^{a_1\dots a_6}\right), \\
&\quad \text{etc.}
\end{aligned} \tag{5.53}$$

is the embedding of the ambiguous part of the connection.

5.2.3. Unique operators and generalised H_d curvatures

We now turn to the construction of unique operators and curvatures from the torsion-free and $\tilde{H}_d$-compatible connection D constructed in the previous section. We give a fairly abstract discussion of the overall structure of these operators in this section. However, the entire construction can be made very concrete. In chapter 6 we will present explicit expressions for these unique operators and curvatures in terms of the $SO(d)$ decompositions of the $\tilde{H}_d$ representations involved.

Given a bundle X transforming as some representation of $\tilde{H}_d$, we define the map

$$\mathcal{Q}_X : U \otimes X \longrightarrow E^* \otimes X, \tag{5.54}$$

via the embedding $U \subset E^* \otimes \operatorname{ad} P$ and the adjoint action of $\operatorname{ad} P$ on X. We then have the projection

$$\mathcal{P}_X : E^* \otimes X \longrightarrow \frac{E^* \otimes X}{\mathrm{Im}\mathcal{Q}_X}. \tag{5.55}$$

Recall that the ambiguous part C of the connection D is a section of U, which acts on X via the map $\mathcal{Q}_X$. If $\alpha \in \Gamma(X)$, then, by construction, $\mathcal{P}_X(D \otimes \alpha)$ is uniquely defined, independent of C.

We can construct explicit examples of such operators as follows. Consider two real $\tilde{H}_d$ bundles S and J, which we refer to as the "spinor" bundle and the "gravitino" bundle respectively, since the supersymmetry parameter and the gravitino field in supergravity are sections of them. The relevant $\tilde{H}_d$ representations are listed in table 5.5. Note that the spinor representation

$\tilde{H}_d$	S	J
$SU(8)$	$\mathbf{8}+\bar{\mathbf{8}}$	$\mathbf{56}+\bar{\mathbf{56}}$
$USp(8)$	$\mathbf{8}$	$\mathbf{48}$
$USp(4)\times USp(4)$	$(\mathbf{4},\mathbf{1})+(\mathbf{1},\mathbf{4})$	$(\mathbf{4},\mathbf{5})+(\mathbf{5},\mathbf{4})$
$USp(4)$	$\mathbf{4}$	$\mathbf{16}$
$SU(2)\times U(1)$	$\mathbf{2}_{\mathbf{1}}+\mathbf{2}_{-\mathbf{1}}$	$\mathbf{4}_{\mathbf{1}}+\mathbf{4}_{-\mathbf{1}}+\mathbf{2}_{\mathbf{3}}+\mathbf{2}_{-\mathbf{3}}$

Table 5.5.: Spinor and gravitino representations in each dimension

is simply the $\mathrm{Cliff}(d,\mathbb{R})$ spinor representation using the embedding (C.26).

One finds that under the projection $\mathcal{P}_X$ we have[9]

$$\begin{aligned} \mathcal{P}_S(E^* \otimes S) &\simeq S \oplus J, \\ \mathcal{P}_J(E^* \otimes J) &\simeq S \oplus J. \end{aligned} \tag{5.56}$$

The details of the group-theoretical proof of this can be found in appendix D. Therefore, for any $\varepsilon \in \Gamma(S)$ and $\psi \in \Gamma(J)$, one has that the following are unique for any torsion-free connection

$$\begin{aligned} D \otimes_J \varepsilon, &\qquad D \otimes_S \varepsilon, \\ D \otimes_J \psi, &\qquad D \otimes_S \psi, \end{aligned} \tag{5.57}$$

where $\otimes_{X'}$ denotes the projection of $\mathcal{P}_X$ onto the X' bundle.

[9]Note that there is an exception for $d = 3$ since, as was previously mentioned, in that case the entire metric compatible, torsion-free connection is uniquely determined, and so $\mathcal{P}_X$ is just the identity map and $\mathcal{P}_X(E^* \otimes X) = E^* \otimes X$ for any bundle X.

One can show that the first two expressions encode the supersymmetry variation of the internal and external gravitino respectively, while the latter two are related to the gravitino equation of motion.

We would now like to define measures of generalised curvature. As was mentioned in section 5.1.4, the natural definition of a Riemann curvature does not result in a tensor. Nonetheless, for a torsion-free, $\tilde{H}_d$-compatible connection D there does exist a generalised Ricci tensor R_{AB}, and it is a section of the bundle

$$\operatorname{ad} P^{\perp} = \operatorname{ad} \tilde{F} / \operatorname{ad} P \subset E^* \otimes E^*, \tag{5.58}$$

where the last relation follows because, as representations of H_d, $E \simeq E^*$. It is not immediately apparent that we can make such a definition, but R_{AB} can in fact be constructed from compositions of the unique operators (5.57) as

$$\begin{aligned} D \otimes_J (D \otimes_J \varepsilon) + D \otimes_J (D \otimes_S \varepsilon) &= R^0 \cdot \varepsilon, \\ D \otimes_S (D \otimes_J \varepsilon) + D \otimes_S (D \otimes_S \varepsilon) &= R\, \varepsilon, \end{aligned} \tag{5.59}$$

where R and R^0_{AB} provide the scalar and non-scalar parts of R_{AB} respectively[10]. The existence of expressions of this type is a non-trivial statement. By computing in the split frame, it can be shown that the LHS is linear in ε, and since ε and the LHS are manifestly covariant, these expressions define a tensor. We will write the components explicitly in section 6.4, equation (6.49). This calculation further provides the non-trivial result that R_{AB} is restricted to be a section of $\operatorname{ad} P^{\perp}$, rather than a more general section of $(S \otimes J) \oplus \mathbb{R}$. In the context of supergravity, this calculation exactly corresponds to the closure of the supersymmetry algebra on the fermionic equations of motion. Finally, since it is built from unique operators, the generalised curvature is automatically unique for a torsion-free compatible connection.

The expressions (5.59) can be written with a different sequence of projections. This helps elucidate the nature of the curvature in terms of certain second-order differential operators. In conventional differential geometry the commutator of two connections $[\nabla_m, \nabla_n]$ has no second-derivative term simply because the partial derivatives commute. This is a neces-

[10]Note that $\operatorname{ad} P^{\perp} \subset (S \otimes J) \oplus \mathbb{R}$ and the $\tilde{H}_d$ structure gives an isomorphism $S \simeq S^*$ and $J \simeq J^*$. Thus, as in the first line of (5.59), we can also view R^0 as a map from S to J.

sary condition for the curvature to be tensorial. In $E_{d(d)}$ indices one can similarly write the commutator of two generalised derivatives formally as $(D \wedge D)_{AB} = [D_A, D_B]$. More precisely, acting on an $E_{d(d)} \times \mathbb{R}^+$ vector bundle X we have

$$(D \wedge D) : X \to \Lambda^2 E^* \otimes X. \tag{5.60}$$

Since again the partial derivatives commute, this operator contains no second-order derivative term, and so can potentially be used to construct a curvature tensor. However, in $E_{d(d)} \times \mathbb{R}^+$ generalised geometry, we also have $\partial f \otimes_{N^*} \partial g = 0$ for any f and g, and so we can take the projection to the bundle N^* defined earlier, giving a similar operator

$$(D \otimes_{N^*} D) : X \to N^* \otimes X, \tag{5.61}$$

which will again contain no second-order derivatives. One thus expects that these two operators, which can be defined for an arbitrary $E_{d(d)} \times \mathbb{R}^+$ connection, should appear in any definition of generalised curvature. Given an $\tilde{H}_d$ structure and a torsion-free compatible connection D, they indeed enter the definition of R_{AB}. Using $\tilde{H}_d$ covariant projections one finds

$$\begin{aligned} (D \wedge D) \otimes_J \varepsilon + (D \otimes_{N^*} D) \otimes_J \varepsilon &= R^0 \cdot \varepsilon, \\ (D \wedge D) \otimes_S \varepsilon + (D \otimes_{N^*} D) \otimes_S \varepsilon &= R\varepsilon. \end{aligned} \tag{5.62}$$

This structure suggests there will be similar definitions of curvature in terms of the operators $(D \wedge D)$ and $(D \otimes_{N^*} D)$ independent of the representation on which they act, and potentially without the need for additional structure.

6. Dimensional restrictions of $D = 11$ Supergravity as $E_{d(d)} \times \mathbb{R}^+$ Generalised Geometry

In this chapter, we show how to formulate dimensional restrictions of eleven-dimensional supergravity in terms of the geometry described in chapter 5. After reviewing the equations of eleven-dimensional supergravity, we begin the main discussion by giving a general set of relations which recover these equations elegantly from generalised geometry. In fact, these relations also apply in the case of $O(10,10) \times \mathbb{R}^+$ generalised geometry from chapter 4, if one views type II supergravity fields in terms of the corresponding representation structure. In the case of restricted eleven-dimensional supergravity, these abstract equations can be realised concretely using the decomposition of H_d under $SO(d)$, detailed in appendix C.2, and by writing the action on the spinors using an embedding into $\mathrm{Cliff}(10,1;\mathbb{R})$. This results in a set of equations with $\tilde{H}_d$ symmetry which hold for any chosen reduced dimension $d \leq 7$.

6.1. Dimensional restrictions of eleven-dimensional supergravity

6.1.1. Eleven dimensional supergravity

Let us start by reviewing the action, equations of motion and supersymmetry variations of eleven-dimensional supergravity, to leading order in the fermions, following the conventions of [146].

The fields are simply

$$\{g_{\mu\nu}, \mathcal{A}_{\mu\nu\rho}, \psi_\mu\}, \tag{6.1}$$

where $g_{\mu\nu}$ is the metric, $\mathcal{A}_{\mu\nu\rho}$ the three-form potential and ψ_μ is the gravitino. The bosonic action is given by

$$S_{\mathrm{B}} = \frac{1}{2\kappa^2} \int \left(\mathrm{vol}_g \, \mathcal{R} - \tfrac{1}{2} \mathcal{F} \wedge * \mathcal{F} - \tfrac{1}{6} \mathcal{A} \wedge \mathcal{F} \wedge \mathcal{F} \right), \tag{6.2}$$

where $\mathcal{R}$ is the Ricci scalar and $\mathcal{F} = \mathrm{d}\mathcal{A}$. This leads to the equations of motion

$$\begin{aligned} \mathcal{R}_{\mu\nu} - \tfrac{1}{12} \left(\mathcal{F}_{\mu\rho_1\rho_2\rho_3} \mathcal{F}_{\nu}{}^{\rho_1\rho_2\rho_3} - \tfrac{1}{12} g_{\mu\nu} \mathcal{F}^2 \right) &= 0, \\ \mathrm{d} * \mathcal{F} + \tfrac{1}{2} \mathcal{F} \wedge \mathcal{F} &= 0, \end{aligned} \tag{6.3}$$

where $\mathcal{R}_{\mu\nu}$ is the Ricci tensor.

Taking Γ^μ to be the $\mathrm{Cliff}(10,1;\mathbb{R})$ gamma matrices, the fermionic action is

$$S_{\mathrm{F}} = \frac{1}{\kappa^2} \int \sqrt{-g} \Big(\bar{\psi}_\mu \Gamma^{\mu\nu\lambda} \nabla_\nu \psi_\lambda + \tfrac{1}{96} \bar{\psi}_\mu (\Gamma^{\mu\nu\lambda_1\dots\lambda_4} \mathcal{F}_{\lambda_1\dots\lambda_4} + 12 \mathcal{F}^{\mu\nu}{}_{\lambda_1\lambda_2} \Gamma^{\lambda_1\lambda_2}) \psi_\nu \Big), \tag{6.4}$$

so the gravitino equation of motion is

$$\Gamma^{\mu\nu\lambda} \nabla_\nu \psi_\lambda + \tfrac{1}{96} (\Gamma^{\mu\nu\lambda_1\dots\lambda_4} \mathcal{F}_{\lambda_1\dots\lambda_4} + 12 \mathcal{F}^{\mu\nu}{}_{\lambda_1\lambda_2} \Gamma^{\lambda_1\lambda_2}) \psi_\nu = 0\,. \tag{6.5}$$

The supersymmetry variations of the bosons are

$$\begin{aligned} \delta g_{\mu\nu} &= 2\bar{\varepsilon} \Gamma_{(\mu} \psi_{\nu)}, \\ \delta \mathcal{A}_{\mu\nu\lambda} &= -3 \bar{\varepsilon} \Gamma_{[\mu\nu} \psi_{\lambda]}, \end{aligned} \tag{6.6}$$

and the supersymmetry variation of the gravitino is

$$\delta \psi_\mu = \nabla_\mu \varepsilon + \tfrac{1}{288} (\Gamma_\mu{}^{\nu_1\dots\nu_4} - 8 \delta_\mu{}^{\nu_1} \Gamma^{\nu_2\nu_3\nu_4}) \mathcal{F}_{\nu_1\dots\nu_4} \varepsilon, \tag{6.7}$$

where ε is the supersymmetry parameter.

6.1.2. Restriction to d dimensions

We will be interested in "restrictions" of eleven-dimensional supergravity where the spacetime is assumed to be a product $\mathbb{R}^{10-d,1} \times M$ of Minkowski space with a d-dimensional spin manifold M, with $d \leq 7$. The metric is taken to have the form

$$\mathrm{d}s^2_{11} = \mathrm{e}^{2\Delta} \mathrm{d}s^2(\mathbb{R}^{10-d,1}) + \mathrm{d}s^2_d(M), \tag{6.8}$$

where $\mathrm{d}s^2(\mathbb{R}^{10-d,1})$ is the flat metric on $\mathbb{R}^{10-d,1}$ and $\mathrm{d}s_d^2(M)$ is a general metric on M. The warp factor Δ and all the other fields are assumed to be independent of the flat $\mathbb{R}^{10-d,1}$ space. In this sense we restrict the full eleven-dimensional theory to M. We will split the eleven-dimensional indices as external indices $\mu = 0, 1, \ldots, c-1$ and internal indices $m = 1, \ldots, d$ where $c + d = 11$.

6.1.3. Action, equations of motion and supersymmetry

In the restricted theory, the surviving fields include the obvious internal components of the eleven-dimensional fields (namely the metric g and three-form A) as well as the warp factor Δ. If $d = 7$, the eleven-dimensional Hodge dual of the 4-form F can have a purely internal 7-form component. This leads one to introduce in addition a dual six-form potential $\tilde{A}$ on M which is related to the seven-form field strength $\tilde{F}$ by

$$\tilde{F} = \mathrm{d}\tilde{A} - \tfrac{1}{2} A \wedge F, \tag{6.9}$$

The Bianchi identities satisfied by $F = \mathrm{d}A$ and $\tilde{F}$ are then

$$\begin{aligned} \mathrm{d}F &= 0, \\ \mathrm{d}\tilde{F} + \tfrac{1}{2} F \wedge F &= 0. \end{aligned} \tag{6.10}$$

With these definitions one can see that F and $\tilde{F}$ are related to the eleven dimensional 4-form field strength $\mathcal{F}$ by

$$F_{m_1 \ldots m_4} = \mathcal{F}_{m_1 \ldots m_4}\,, \qquad \tilde{F}_{m_1 \ldots m_7} = (*_{11}\mathcal{F})_{m_1 \ldots m_7}\,. \tag{6.11}$$

F and $\tilde{F}$ are invariant under the gauge transformations of the potentials given by

$$\begin{aligned} A' &= A + \mathrm{d}\Lambda, \\ \tilde{A}' &= \tilde{A} + \mathrm{d}\tilde{\Lambda} - \tfrac{1}{2} \mathrm{d}\Lambda \wedge A, \end{aligned} \tag{6.12}$$

for some two-form Λ and five-form $\tilde{\Lambda}$.

In order to diagonalise the kinetic terms in the fermionic Lagrangian, one introduces the standard field redefinition of the external components of the gravitino

$$\psi'_\mu = \psi_\mu + \tfrac{1}{c-2} \Gamma_\mu \Gamma^m \psi_m. \tag{6.13}$$

We then denote its trace as

$$\rho = \tfrac{c-2}{c}\Gamma^\mu \psi'_\mu, \tag{6.14}$$

and allow this to be non-zero and dependant on the internal coordinates. The surviving degrees of freedom are thus

$$\{g_{mn}, A_{mnp}, \tilde{A}_{m_1\dots m_6}, \Delta, \psi_m, \rho\}, \tag{6.15}$$

One obtains the internal bosonic action

$$S_{\mathrm{B}} = \frac{1}{2\kappa^2}\int \sqrt{g}\, \mathrm{e}^{c\Delta}\left(\mathcal{R} + c(c-1)(\partial\Delta)^2 - \tfrac{1}{2}\tfrac{1}{4!}F^2 - \tfrac{1}{2}\tfrac{1}{7!}\tilde{F}^2\right), \tag{6.16}$$

by requiring that its associated equations of motion

$$\begin{aligned}
\mathcal{R}_{mn} - c\nabla_m\nabla_n\Delta - c(\partial_m\Delta)(\partial_n\Delta) - \tfrac{1}{2}\tfrac{1}{4!}\Big(4F_{mp_1p_2p_3}F_n{}^{p_1p_2p_3} - \tfrac{1}{3}g_{mn}F^2\Big)& \\
-\tfrac{1}{2}\tfrac{1}{7!}\left(7\tilde{F}_{mp_1\dots p_6}\tilde{F}_n{}^{p_1\dots p_6} - \tfrac{2}{3}g_{mn}\tilde{F}^2\right) &= 0, \\
\mathcal{R} - 2(c-1)\nabla^2\Delta - c(c-1)(\partial\Delta)^2 - \tfrac{1}{2}\tfrac{1}{4!}F^2 - \tfrac{1}{2}\tfrac{1}{7!}\tilde{F}^2 &= 0, \\
\mathrm{d}*(\mathrm{e}^{c\Delta}F) - \mathrm{e}^{c\Delta}(*\tilde{F})\wedge F &= 0, \\
\mathrm{d}*(\mathrm{e}^{c\Delta}\tilde{F}) &= 0.
\end{aligned} \tag{6.17}$$

are those obtained by substituting the field ansatz into (6.3). Similarly, to quadratic order in fermions, the action for the restricted fermion fields (for

$d \leq 7$) is

$$
\begin{aligned}
S_{\mathrm{F}} = - \frac{1}{\kappa^2 (c-2)^2} \int \sqrt{g}\, \mathrm{e}^{c\Delta} \Big(\\
& (c-4)\bar{\psi}_m \Gamma^{mnp} \nabla_n \psi_p - c(c-3)\bar{\psi}^m \Gamma^n \nabla_n \psi_m - c(\bar{\psi}^m \Gamma_n \nabla_m \psi^n + \bar{\psi}^m \Gamma_m \nabla_n \psi^n) \\
& + c(\bar{\psi}_m \Gamma^{mn} \nabla_n \rho - \bar{\rho} \Gamma^{mn} \nabla_m \psi_n) - c(c-2)\bar{\psi}_m \Gamma^{mn} (\partial_n \Delta)\rho \\
& + c(c-1)(\bar{\psi}^m \nabla_m \rho - \bar{\rho} \nabla^m \psi_m) - c(c-1)(c-2)\bar{\psi}^m (\partial_m \Delta)\rho \\
& + c(c-1)(\bar{\rho} \Gamma^m \nabla_m \rho + \tfrac{1}{4} \bar{\rho} \not{F} \rho - \tfrac{1}{4} \bar{\rho} \tilde{\not{F}} \rho) \\
& - \tfrac{1}{2} \tfrac{1}{4!} c \bar{\rho} \Gamma^m{}_{p_1 \dots p_4} F^{p_1 \dots p_4} \psi_m + \tfrac{1}{2} \tfrac{1}{3!} c(c-1) \bar{\rho} F^m{}_{pqr} \Gamma^{pqr} \psi_m \\
& + \tfrac{1}{4} \tfrac{1}{4!} (c-4) \bar{\psi}_m \Gamma^{mnp_1 \dots p_4} F_{p_1 \dots p_4} \psi_n + \tfrac{1}{4} c(c-3) \bar{\psi}_m \not{F} \psi^m \\
& + \tfrac{1}{2} \tfrac{1}{3!} c \bar{\psi}_m F^{(m}{}_{pqr} \Gamma^{n)pqr} \psi_n - \tfrac{1}{4} \tfrac{1}{2!} (2c^2 - 5c + 4) \bar{\psi}_m F^{mn}{}_{pq} \Gamma^{pq} \psi_n \\
& - \tfrac{1}{2} \tfrac{1}{6!} c(c-1) \bar{\psi}_m \tilde{F}^m{}_{p_1 \dots p_6} \Gamma^{p_1 \dots p_6} \rho + \tfrac{1}{4} \tfrac{1}{6!} c(c-1) \bar{\psi}_m \tilde{F}^{(m}{}_{p_1 \dots p_6} \Gamma^{n) p_1 \dots p_6} \psi_n \\
& - \tfrac{1}{4} \tfrac{1}{5!} (2c^2 - 5c + 4) \bar{\psi}_m \tilde{F}^{mn}{}_{p_1 \dots p_5} \Gamma^{p_1 \dots p_5} \psi_n \Big).
\end{aligned}
\tag{6.18}
$$

This action leads to the equation of motion for ψ_m,

$$
\begin{aligned}
& (c-4)\Gamma_m{}^{np} (\nabla_n + \tfrac{c}{2} \partial_n \Delta) \psi_p - c(c-3) \Gamma^n (\nabla_n + \tfrac{c}{2} \partial_n \Delta) \psi_m \\
& \qquad - c(\Gamma_n (\nabla_m + \tfrac{c}{2} \partial_m \Delta) \psi^n + \Gamma_m (\nabla_n + \tfrac{c}{2} \partial_n \Delta) \psi^n) \\
& + c \Gamma_m{}^n (\nabla_n + \partial_n \Delta) \rho + c(c-1)(\nabla_m + \partial_m \Delta) \rho \\
& + \tfrac{1}{4} c \tfrac{1}{4!} \Gamma_{mp_1 \dots p_4} F^{p_1 \dots p_4} \rho + \tfrac{1}{4} c(c-1) \tfrac{1}{3!} F_{mp_1 p_2 p_3} \Gamma^{p_1 p_2 p_3} \rho \\
& + \tfrac{1}{4} (c-4) \tfrac{1}{4!} \Gamma_{mn}{}^{p_1 \dots p_4} F_{p_1 \dots p_4} \psi^n + \tfrac{1}{4} c(c-3) \not{F} \psi_m \\
& + \tfrac{1}{2} c \tfrac{1}{3!} F_{(m}{}^{p_1 p_2 p_3} \Gamma_{n) p_1 p_2 p_3} \psi^n - \tfrac{1}{4} (2c^2 - 5c + 4) \tfrac{1}{2!} F_{mnpq} \Gamma^{pq} \psi^n \\
& - \tfrac{1}{4} c(c-1) \tfrac{1}{6!} \tilde{F}_{mn_1 \dots n_6} \Gamma^{n_1 \dots n_6} \rho + \tfrac{1}{4} c(c-1) \tfrac{1}{6!} \tilde{F}_{(m}{}^{p_1 \dots p_6} \Gamma_{n) p_1 \dots p_6} \psi^n \\
& - \tfrac{1}{4} (2c^2 - 5c + 4) \tfrac{1}{5!} \tilde{F}_{mnp_1 \dots p_5} \Gamma^{p_1 \dots p_5} \psi^n \\
& = 0,
\end{aligned}
\tag{6.19}
$$

and the equation of motion for ρ,

$$\begin{aligned}
&-(\nabla\!\!\!/ + \tfrac{c}{2}(\partial\!\!\!/\Delta) + \tfrac{1}{4}F\!\!\!/ - \tfrac{1}{4}\tilde{F}\!\!\!/)\rho \\
&+ (\nabla_m + (c-1)\partial_m\Delta)\psi^m + \tfrac{1}{c-1}\Gamma^{mn}(\nabla_m + (c-1)\partial_m\Delta)\psi_n \\
&+ \tfrac{1}{4}\tfrac{1}{c-1}\tfrac{1}{4!}\Gamma^m{}_{p_1\dots p_4}F^{p_1\dots p_4}\psi_m - \tfrac{1}{4}\tfrac{1}{3!}F^m{}_{p_1p_2p_3}\Gamma^{p_1p_2p_3}\psi_m \\
&- \tfrac{1}{4}\tfrac{1}{6!}\tilde{F}^m{}_{p_1\dots p_6}\Gamma^{p_1\dots p_6}\psi_m \\
&= 0.
\end{aligned} \tag{6.20}$$

The supersymmetry variations of the fermion fields are given by

$$\begin{aligned}
\delta\rho &= [\nabla\!\!\!/ - \tfrac{1}{4}F\!\!\!/ - \tfrac{1}{4}\tilde{F}\!\!\!/ + \tfrac{c-2}{2}(\partial\!\!\!/\Delta)]\varepsilon, \\
\delta\psi_m &= \nabla_m\varepsilon + \tfrac{1}{288}(\Gamma_m{}^{n_1\dots n_4} - 8\delta_m{}^{n_1}\Gamma^{n_2n_3n_4})F_{n_1\dots n_4}\varepsilon - \tfrac{1}{12}\tfrac{1}{6!}\tilde{F}_{mn_1\dots n_6}\Gamma^{n_1\dots n_6}\varepsilon,
\end{aligned} \tag{6.21}$$

and the variations of the bosons by

$$\begin{aligned}
\delta g_{mn} &= 2\bar{\varepsilon}\Gamma_{(m}\psi_{n)}, \\
(c-2)\delta\Delta + \delta\log\sqrt{g} &= \bar{\varepsilon}\rho, \\
\delta A_{mnp} &= -3\bar{\varepsilon}\Gamma_{[mn}\psi_{p]}, \\
\delta\tilde{A}_{m_1\dots m_6} &= 6\bar{\varepsilon}\Gamma_{[m_1\dots m_5}\psi_{m_6]}.
\end{aligned} \tag{6.22}$$

In what follows the fermionic fields will be reinterpreted as representations of larger symmetry groups. To mark that distinction, the fermions that have appeared in this section will be denoted by $\varepsilon^{\text{sugra}}$, ρ^{sugra} and ψ^{sugra}. In the absence of this label, the fields are to be viewed as "generalised" objects as shall be clarified in section 6.2.

6.2. Supergravity degrees of freedom and H_d structures

We now explain how one can view the supergravity fields as generalised geometry objects. Broadly speaking, the bosons form the generalised metric, which defines an H_d structure on the generalised tangent space E, while the fermions can be promoted to representations of the double covering group $\tilde{H}_d$. The general structure of what is said here applies equally well to the $O(d,d) \times \mathbb{R}^+$ generalised geometry for the NS-NS sector of type II supergravity.

Bosons

It is well known [147] that the bosonic fields of the reduced supergravity parametrise an $(E_{d(d)} \times \mathbb{R}^+)/H_d$ coset, that is, at each point $x \in M$,

$$\{g, A, \tilde{A}, \Delta\} \in \frac{E_{d(d)}}{H_d} \times \mathbb{R}^+. \tag{6.23}$$

Thus giving the bosonic fields is equivalent to specifying a generalised metric G. Furthermore, as in [98], the infinitesimal bosonic symmetry transformation is naturally encoded as the Dorfman derivative by $V \in \Gamma(E)$

$$\delta_V G = L_V G, \tag{6.24}$$

and the algebra of these transformations is given by the Leibniz property (5.29). Thus, we see that the $GL(d,\mathbb{R})$ representations which make up the generalised tangent space correspond to the fundamental charges of the theory. These are the momentum, the charges of the gauge fields A and $\tilde{A}$ and the Kaluza-Klein monopole (or dual graviton) charge, which generate the bosonic symmetries[1].

Fermions

The fermionic degrees of freedom fall into spinor representations of $\tilde{H}_d$, the double cover of H_d. Let S and J denote the representations of $\tilde{H}_d$ listed in table 5.5. The fermion fields ψ, ρ and the supersymmetry parameter ε can then be thought of as sections of these bundles

$$\psi \in \Gamma(J), \qquad \rho \in \Gamma(S), \qquad \varepsilon \in \Gamma(S). \tag{6.25}$$

However, in the supergravity equations of section 6.1 the fermion fields were viewed as $\mathrm{Cliff}(10,1;\mathbb{R})$ objects. It is therefore preferable to follow

[1]Note that the Kaluza-Klein monopole does not generate a gauge transformation for $d \leq 7$. This will be discussed further in the conclusion.

the chain of embeddings $\tilde{H}_d \subset \mathrm{Cliff}(d;\mathbb{R}) \subset \mathrm{Cliff}(10,1;\mathbb{R})$ explained in appendix C.2.2 and uplift ψ, ρ and ε to representations of $Spin(10-d,1) \times \tilde{H}_d$[2]. This will allow us to write expressions directly comparable to the ones in section 6.1. There exists a complication, in that there are actually two distinct ways of realising the action of $\tilde{H}_d$ on $\mathrm{Cliff}(d;\mathbb{R})$ spinors, related by a change of sign of the gamma matrices

$$N^{\pm} = \tfrac{1}{2}\left(\tfrac{1}{2!}n_{ab}\gamma^{ab} \pm \tfrac{1}{3!}b_{abc}\gamma^{abc} - \tfrac{1}{6!}\tilde{b}_{a_1\dots a_6}\gamma^{a_1\dots a_6}\right), \tag{6.26}$$

and so one finds that in general the spinor bundles split into $S^{\pm}$ and $J^{\pm}$. Putting it all together, in order to recover the original supergravity formulation we are led to consider the four $Spin(10-d,1) \times \tilde{H}_d$ bundles listed in table 6.1 (see also [78]).

d	$\hat{S}^-$	$\hat{S}^+$	$\hat{J}^-$	$\hat{J}^+$
7	$(\mathbf{2},\mathbf{8})+(\bar{\mathbf{2}},\bar{\mathbf{8}})$	$(\mathbf{2},\bar{\mathbf{8}})+(\bar{\mathbf{2}},\mathbf{8})$	$(\mathbf{2},\mathbf{56})+(\bar{\mathbf{2}},\bar{\mathbf{56}})$	$(\mathbf{2},\bar{\mathbf{56}})+(\bar{\mathbf{2}},\mathbf{56})$
6	$(\mathbf{4},\mathbf{8})$	$(\mathbf{4},\mathbf{8})$	$(\mathbf{4},\mathbf{48})$	$(\mathbf{4},\mathbf{48})$
5	$(\mathbf{4},\mathbf{4},\mathbf{1})+(\bar{\mathbf{4}},\mathbf{1},\mathbf{4})$	$(\mathbf{4},\mathbf{1},\mathbf{4})+(\bar{\mathbf{4}},\mathbf{4},\mathbf{1})$	$(\mathbf{4},\mathbf{4},\mathbf{5})+(\bar{\mathbf{4}},\mathbf{5},\mathbf{4})$	$(\mathbf{4},\mathbf{5},\mathbf{4})+(\bar{\mathbf{4}},\mathbf{4},\mathbf{5})$
4	$(\mathbf{8},\mathbf{4})$	$(\mathbf{8},\mathbf{4})$	$(\mathbf{8},\mathbf{16})$	$(\mathbf{8},\mathbf{16})$

Table 6.1.: Spinor and gravitino as $Spin(10-d,1)\times\tilde{H}_d$ representations. Note that when d is even the positive and negative representations are equivalent.

We then find that the supergravity fields can be identified as

$$\begin{aligned}
\hat{\varepsilon}^- &= \mathrm{e}^{-\Delta/2}\,\varepsilon^{\mathrm{sugra}} \in \Gamma(\hat{S}^-),\\
\hat{\rho}^+ &= \mathrm{e}^{\Delta/2}\,\rho^{\mathrm{sugra}} \in \Gamma(\hat{S}^+),\\
\hat{\psi}^-_a &= \mathrm{e}^{\Delta/2}\,\psi^{\mathrm{sugra}}_a \in \Gamma(\hat{J}^-).
\end{aligned} \tag{6.27}$$

Note that, due to the warping of the metric, the precise maps between the fermion fields as viewed in the geometry and in the supergravity description involve a conformal rescaling. This is of course purely conventional, since one could just as easily perform field redefinitions at the supergravity level. We choose, however, to maintain the conventions in section 6.1 as familiar as possible and so it is important to account for this subtlety.

[2]The alternative is to decompose the eleven-dimensional spinors which necessarily leads to dimension dependent expressions.

6.3. Supergravity equations from generalised geometry

We now present the main result, a complete rewriting of the supergravity equations in the language of generalised geometry, to leading order in fermions. In this section, we provide an abstract treatment describing the theory in complete generality.

The equations presented here are intended to be schematic, capturing the essence of the structure, but ignoring details such as numerical factors and the precise details of the representations involved. This presentation is easily seen to reproduce the NS-NS sector equations for type II theories given in chapter 4. In section 6.4, we apply the same prescription to $E_{d(d)} \times \mathbb{R}^+$ generalised geometry, using the $SO(d)$ decomposition of H_d described in appendix C.2. This will provide more explicit expressions for the restricted eleven-dimensional supergravity setup in this language.

We begin by looking at the supersymmetry algebra. Remarkably, the variations of the two fermion fields match precisely the two unique differential operators which act on spinors that were found in section 5.2.3. Therefore, the supergravity equations (6.21) can be written concisely as the $\tilde{H}_d$ covariant projections (5.57) of the torsion-free compatible connection acting on the supersymmetry parameter ε

$$\begin{aligned} \delta\psi &= D \otimes_J \varepsilon, \\ \delta\rho &= D \otimes_S \varepsilon. \end{aligned} \tag{6.28}$$

Since the bosons arrange themselves into the generalised metric, one expects that their supersymmetry variations (6.22) be given by the variation of G. Indeed, denoting projections to the bundle $\operatorname{ad} P^{\perp}$ (defined in table 5.3) by $\otimes_{\operatorname{ad} P^{\perp}}$, one finds that the equations can be rewritten in the $\tilde{H}_d$ covariant form

$$\delta G = (\psi \otimes_{\operatorname{ad} P^{\perp}} \varepsilon) + (\rho \otimes_{\operatorname{ad} P^{\perp}} \varepsilon). \tag{6.29}$$

In order to describe the dynamics, an important first step is to realise the fermionic equations of motion (6.19) and (6.20). Using the unique projections (5.57) once again, they become simply

$$\begin{aligned} (D \otimes_J \psi) + (D \otimes_J \rho) &= 0, \\ (D \otimes_S \psi) + (D \otimes_S \rho) &= 0. \end{aligned} \tag{6.30}$$

The bosonic equations of motion (6.17) are naturally given by the vanishing of the generalised Ricci curvature (5.59)

$$R_{AB} = 0. \tag{6.31}$$

Note again that the form of the generalised Ricci in (5.59) can be interpreted in a physical way – it reflects the closure of the supersymmetry algebra on (6.30) which, by virtue of (6.28), can be examined at an $\tilde{H}_d$ level.

The bosonic action (6.16) is given by the generalised curvature scalar, integrated with the volume form (5.45)

$$S_{\mathrm{B}} \propto \int \mathrm{vol}_G \, R. \tag{6.32}$$

Finally, the fermionic action (6.18) can be written using the natural invariant pairings of the terms in (6.30) with the fermionic fields

$$S_{\mathrm{F}} \propto \int \mathrm{vol}_G \Big(\langle \psi, (D \otimes_J \psi) \rangle + 2 \langle \rho, (D \otimes_S \psi) \rangle + \langle \rho, (D \otimes_S \rho) \rangle \Big). \tag{6.33}$$

6.4. Realisation in $SO(d)$ representations

We now use the expressions of appendix C.2.2 to write explicit versions of the abstract relations in the previous section. The resulting equations are written in terms of $\mathrm{Cliff}(10,1;\mathbb{R})$ spinors and match exactly those in section 6.1.3 when evaluated in the split frame.

Supersymmetry Algebra

The supersymmetry variations (6.21) of the two fermion fields take the form

$$\begin{aligned} \delta \hat{\psi}^- &= D \otimes_{\hat{J}^-} \hat{\varepsilon}^-, \\ \delta \hat{\rho}^+ &= D \otimes_{\hat{S}^+} \hat{\varepsilon}^-. \end{aligned} \tag{6.34}$$

For clarity we will demonstrate how to evaluate one of these expressions in the split frame. Using (C.34), we see that the projection can be written

$$(D \otimes_{\hat{S}^+} \hat{\varepsilon}^-) = \Gamma^a D_a \hat{\varepsilon}^- + \tfrac{1}{2!}\Gamma^{ab} D_{ab}\hat{\varepsilon}^- + \tfrac{1}{5!}\Gamma^{c_1\dots c_5} D_{c_1\dots c_5}\hat{\varepsilon}^- + \tfrac{1}{6!}\Gamma^{c_1\dots c_6} D^d{}_{,dc_1\dots c_6}\hat{\varepsilon}^- . \tag{6.35}$$

Substituting in the connection components from (5.50) and the expression $\hat{\varepsilon}^- = \mathrm{e}^{-\Delta/2}\varepsilon^{\text{sugra}}$ we have

$$(D \otimes_{\hat{S}^+} \hat{\varepsilon}^-) = \mathrm{e}^{\Delta/2}\Big(\nabla\!\!\!/ + \tfrac{9-d}{2}(\partial\!\!\!/\Delta) - \tfrac{1}{4}F\!\!\!/ - \tfrac{1}{4}\tilde{F}\!\!\!/\Big)\varepsilon^{\text{sugra}}, \tag{6.36}$$

which is the supersymmetry variation of $\hat{\rho}^+ = \mathrm{e}^{\Delta/2}\rho^{\text{sugra}}$. The other differential operators in this section are derived similarly. The result for the supersymmetry variation of the gravitino comes out as

$$\begin{aligned}(D \otimes_{\hat{J}^-} \hat{\varepsilon}^-)_a = \mathrm{e}^{\Delta/2}\Big(\nabla_a &+ \tfrac{1}{288}(\Gamma_a{}^{b_1\dots b_4} - 8\delta_a{}^{b_1}\Gamma^{b_2b_3b_4})F_{b_1\dots b_4} \\ &- \tfrac{1}{12}\tfrac{1}{6!}\tilde{F}_{ab_1\dots b_6}\Gamma^{b_1\dots b_6}\Big)\varepsilon^{\text{sugra}}.\end{aligned} \tag{6.37}$$

Since the bosons arrange themselves into the generalised metric, one expects that their supersymmetry variations (6.22) be given by the variation of G. In fact, the most convenient object to consider is $G^{-1}\delta G$ which is naturally a section of the bundle $\mathrm{ad}(P)^\perp$ listed in table 5.3. In this context, the isomorphism (C.23) becomes

$$\mathrm{ad}(P)^\perp \simeq \mathbb{R} \oplus S^2T^*M \oplus \Lambda^3T^*M \oplus \Lambda^6T^*M \tag{6.38}$$

and in this notation, the variation of the generalised metric can be written in the split frame as

$$\begin{aligned}(G^{-1}\delta G)_0 &= -2\delta\Delta, \\ (G^{-1}\delta G)_{ab} &= \delta g_{ab}, \\ (G^{-1}\delta G)_{abc} &= -\delta A_{abc}, \\ (G^{-1}\delta G)_{a_1\dots a_6} &= -\delta\tilde{A}_{a_1\dots a_6}.\end{aligned} \tag{6.39}$$

Comparing this with (C.38) and (C.39), one finds that the supersymmetry variations of the bosons (6.22) can be written in the $\tilde{H}_d$ covariant form

$$G^{-1}\delta G = (\hat{\psi}^- \otimes_{\mathrm{ad}\,P^\perp} \hat{\varepsilon}^-) + (\hat{\rho}^+ \otimes_{\mathrm{ad}\,P^\perp} \hat{\varepsilon}^-), \tag{6.40}$$

where $\otimes_{\mathrm{ad}\,P^\perp}$ denotes the projection to $\mathrm{ad}(P)^\perp$.

Generalised Curvatures and the Equations of Motion

Being more careful about numerical factors, the fermion equations of motion (6.19) and (6.20) become

$$\begin{aligned} -(D \otimes_{\hat{J}+} \hat{\psi}^-) - \tfrac{11-d}{9-d}(D \otimes_{\hat{J}+} \hat{\rho}^+) &= 0, \\ (D \otimes_{\hat{S}-} \hat{\psi}^-) + (D \otimes_{\hat{S}-} \hat{\rho}^+) &= 0. \end{aligned} \tag{6.41}$$

$\hat{\rho}^+$ is embedded with a different conformal factor to $\hat{\varepsilon}^-$, so the warp factor terms in these projections are different to those involving $\hat{\varepsilon}^-$. We have

$$(D \otimes_{\hat{S}-} \hat{\rho}^+) = -\mathrm{e}^{3\Delta/2}\Big(\nabla\!\!\!/ + \tfrac{11-d}{2}(\partial\!\!\!/\Delta) + \tfrac{1}{4}F\!\!\!/ - \tfrac{1}{4}\tilde{F}\!\!\!/\Big)\rho^{\mathrm{sugra}} \tag{6.42}$$

and

$$(D \otimes_{\hat{J}+} \hat{\rho}^+)_a = \mathrm{e}^{3\Delta/2}\Big[(\nabla_a + \partial_a\Delta) - \tfrac{1}{288}({\Gamma_a}^{b_1\dots b_4} - 8{\delta_a}^{b_1}\Gamma^{b_2b_3b_4})F_{b_1\dots b_4} - \tfrac{1}{12}\tfrac{1}{6!}\tilde{F}_{ab_1\dots b_6}\Gamma^{b_1\dots b_6}\Big]\rho^{\mathrm{sugra}} \tag{6.43}$$

The projections involving $\hat{\psi}^-$ are evaluated using (C.36) and (C.37). The result is

$$\begin{aligned}(D \otimes_{\hat{S}^-} \hat{\psi}^-) = \mathrm{e}^{3\Delta/2}\Big[&(\nabla^b + (10-d)\partial^b\Delta) + \tfrac{1}{10-d}\Gamma^{ab}(\nabla_a + (10-d)\partial_a\Delta) \\ &+ \tfrac{1}{4}\tfrac{1}{10-d}\tfrac{1}{4!}\Gamma^b{}_{c_1\dots c_4}F^{c_1\dots c_4} - \tfrac{1}{4}\tfrac{1}{3!}F^b{}_{c_1c_2c_3}\Gamma^{c_1c_2c_3} \\ &- \tfrac{1}{4}\tfrac{1}{6!}\tilde{F}^b{}_{c_1\dots c_6}\Gamma^{c_1\dots c_6}\Big]\psi_b^{\text{sugra}}, \\ (D \otimes_{\hat{J}^+} \hat{\psi}^-)_a = -\mathrm{e}^{3\Delta/2}\Big[&\Gamma^c(\nabla_c + \tfrac{11-d}{2}\partial_c\Delta)\delta_a{}^b + \tfrac{2}{9-d}\Gamma^b(\nabla_a + \tfrac{11-d}{2}\partial_a\Delta) \\ &- \tfrac{1}{12}(3 + \tfrac{2}{9-d})\not{F}\delta_a{}^b + \tfrac{1}{3}\tfrac{10-d}{9-d}\tfrac{1}{2!}F_a{}^b{}_{cd}\Gamma^{cd} \\ &- \tfrac{1}{3}\tfrac{1}{9-d}\tfrac{1}{3!}F_a{}^{c_1\dots c_3}\Gamma^b{}_{c_1\dots c_3} + \tfrac{1}{6}\tfrac{10-d}{9-d}\tfrac{1}{3!}F^{bc_1\dots c_3}\Gamma_{ac_1\dots c_3} \\ &- \tfrac{1}{6}\tfrac{1}{9-d}\tfrac{1}{4!}F_{c_1\dots c_4}\Gamma_a{}^{bc_1\dots c_4} + \tfrac{1}{4}\tfrac{1}{5!}\tilde{F}_a{}^b{}_{c_1\dots c_5}\Gamma^{c_1\dots c_5}\Big]\psi_b^{\text{sugra}}.\end{aligned} \tag{6.44}$$

The gravitino equation as written here does not exactly match that in section 6.1.3. This is due to the presence of additional gamma matrices in the inner product (C.31) which is used to write the fermion action. Using the expressions (C.31) and (C.32) for the spinor bilinears, we find that (6.18) can be rewritten as

$$\begin{aligned}S_{\text{F}} = \tfrac{1}{\kappa^2}\int \text{vol}_G \Big[&-\langle\hat{\psi}^-, D\otimes_{\hat{J}^+}\hat{\psi}^-\rangle - \tfrac{c}{c-2}\langle\hat{\psi}^-, D\otimes_{\hat{J}^+}\hat{\rho}^+\rangle \\ &+ \tfrac{c(c-1)}{(c-2)^2}\langle\hat{\rho}^+, D\otimes_{\hat{S}^-}\hat{\psi}^-\rangle + \tfrac{c(c-1)}{(c-2)^2}\langle\hat{\rho}^+, D\otimes_{\hat{S}^-}\hat{\rho}^+\rangle\Big].\end{aligned} \tag{6.45}$$

When this is varied with respect to $\hat{\psi}^-$, the equation of motion comes out in the form

$$\langle\delta\hat{\psi}^-, -(D\otimes_{\hat{J}^+}\hat{\psi}^-) - \tfrac{11-d}{9-d}(D\otimes_{\hat{J}^+}\hat{\rho}^+)\rangle = 0, \qquad \forall\delta\hat{\psi}^- \tag{6.46}$$

If one merely removes $\delta\hat{\psi}^-$ from the left side of the expression, the form of the inner product (C.31) gives the equation

$$(\delta_a{}^b + \tfrac{1}{9-d}\Gamma_a\Gamma^b)\Big[-(D\otimes_{\hat{J}^+}\hat{\psi}^-)_b - \tfrac{11-d}{9-d}(D\otimes_{\hat{J}^+}\hat{\rho}^+)_b\Big] = 0. \tag{6.47}$$

Some algebra reveals that this equation does exactly match equation (6.19), multiplied by an overall factor of $-(9-d)^{-2}$.

From the fermion equations of motion we can find explicit expressions for the generalised Ricci tensor R_{AB}, which is a section of the bundle $\mathrm{ad}\,P^{\perp} = \mathrm{ad}\,\tilde{F}/\,\mathrm{ad}\,P \subset E^{*} \otimes E^{*}$. Using the closure of the supersymmetry algebra on (6.41) which, by virtue of (6.34), can be examined at an $\tilde{H}_d$ level, we define

$$
\begin{aligned}
-D \otimes_{\hat{J}^{+}} (D \otimes_{\hat{J}^{-}} \hat{\varepsilon}^{-}) - \tfrac{11-d}{9-d} D \otimes_{\hat{J}^{+}} (D \otimes_{\hat{S}^{+}} \hat{\varepsilon}^{-}) &= R^{0} \cdot \hat{\varepsilon}^{-}, \\
D \otimes_{\hat{S}^{-}} (D \otimes_{\hat{J}^{-}} \hat{\varepsilon}^{-}) + D \otimes_{\hat{S}^{-}} (D \otimes_{\hat{S}^{+}} \hat{\varepsilon}^{-}) &= \tfrac{1}{4}\tfrac{9-d}{10-d} R\, \hat{\varepsilon}^{-},
\end{aligned}
\tag{6.48}
$$

for any $\hat{\varepsilon}^{-} \in \Gamma(\hat{S}^{-})$ and where R and R^{0}_{AB} provide the scalar and non-scalar of R_{AB} respectively. Explicitly, via the $\tilde{H}_d$ covariant projection (C.29), the action of the curvatures on the spinor are given by

$$
\begin{aligned}
(R^{0} \cdot \hat{\varepsilon}^{-})_a &= \Big(\tfrac{1}{2} R^{0}_{ab} \Gamma^{b} + \tfrac{1}{3}\tfrac{1}{2!} R^{0}_{abc} \Gamma^{bc} - \tfrac{1}{6}\tfrac{1}{3!} R^{0}_{c_1 \dots c_3} \Gamma_{a}{}^{c_1 \dots c_3} \\
&\qquad + \tfrac{1}{6}\tfrac{1}{5!} R^{0}_{a b_1 \dots b_5} \Gamma^{b_1 \dots b_5} - \tfrac{1}{3}\tfrac{1}{6!} R^{0}_{c_1 \dots c_6} \Gamma_{a}{}^{c_1 \dots c_6} \Big) \varepsilon^{\text{sugra}}, \\
R\, \hat{\varepsilon}^{-} &= \mathrm{e}^{2\Delta} \Big(\mathcal{R} - 2(c-1) \nabla^2 \Delta - c(c-1)(\partial \Delta)^2 - \tfrac{1}{2}\tfrac{1}{4!} F^2 - \tfrac{1}{2}\tfrac{1}{7!} \tilde{F}^2 \Big) \varepsilon^{\text{sugra}},
\end{aligned}
\tag{6.49}
$$

where $c = 11 - d$ and

$$
\begin{aligned}
R^{0}_{ab} &= \mathrm{e}^{2\Delta} \Big[\mathcal{R}_{mn} - c \nabla_m \nabla_n \Delta - c (\partial_m \Delta)(\partial_n \Delta) \\
&\qquad - \tfrac{1}{2}\tfrac{1}{4!} \Big(4 F_{m p_1 p_2 p_3} F_{n}{}^{p_1 p_2 p_3} - \tfrac{1}{3} g_{mn} F^2 \Big) \\
&\qquad - \tfrac{1}{2}\tfrac{1}{7!} \Big(7 \tilde{F}_{m p_1 \dots p_6} \tilde{F}_{n}{}^{p_1 \dots p_6} - \tfrac{2}{3} g_{mn} \tilde{F}^2 \Big) \Big], \\
R^{0}_{abc} &= \tfrac{1}{2} \mathrm{e}^{2\Delta} \Big[\mathrm{e}^{-c\Delta} * \mathrm{d} * (\mathrm{e}^{c\Delta} F) - *((*\tilde{F}) \wedge F) \Big]_{abc}, \\
R^{0}_{a_1 \dots a_6} &= \tfrac{1}{2} \mathrm{e}^{2\Delta} \Big[\mathrm{e}^{-c\Delta} * \mathrm{d} * (\mathrm{e}^{c\Delta} \tilde{F}) \Big]_{a_1 \dots a_6}.
\end{aligned}
\tag{6.50}
$$

The generalised Ricci tensor is manifestly uniquely determined and the bosonic equations of motion (6.17) become simply

$$
R_{AB} = 0. \tag{6.51}
$$

Finally, the bosonic action (6.16) is given by the generalised curvature scalar, integrated with the volume form (5.45)

$$S_{\mathrm{B}} = \frac{1}{2\kappa^2} \int \mathrm{vol}_G \, R. \tag{6.52}$$

We have now rewritten all of the supergravity equations from section 6.1.2 in terms of generalised geometry. Though we have chosen to write them out under an $SO(d)$ decomposition, the abstract equations have manifest $\tilde{H}_d$ symmetry. This formulation has the advantage that we were able to write formulae which hold true in any dimension $d \leq 7$.

6.5. Comments on type II theories

The $O(d,d) \times \mathbb{R}^+$ generalised geometry of chapter 3 was only able to incorporate the gauge symmetry of the NS-NS sector in the structure of the generalised tangent space. Here, we will briefly discuss how one may use $E_{d(d)} \times \mathbb{R}^+$ generalised geometry over a $(d-1)$-dimensional manifold to include also the RR gauge symmetry (4.21), an idea we have alluded to at various points. This has previously been described in [100, 102, 103, 148].

The $(E_{d(d)} \times \mathbb{R}^+)/\, H_d$ coset structure can equally well describe the fields type II theories in $d-1$ dimensions. Specifically

$$\{g, B, \tilde{B}, \phi, A^{\pm}, \Delta\} \in \frac{E_{d(d)}}{H_d} \times \mathbb{R}^+, \tag{6.53}$$

where B is the NS-NS two-form field, $\tilde{B}$ is the six-form potential dual to B, ϕ is the dilaton and $A^{\pm}$ are the RR potentials (in a democratic formalism) where A^- is a sum of odd-degree forms in type IIA and A^+ is a sum of even-degree forms in type IIB. All the fields now depend on a $d-1$ dimensional manifold M'.

The genearlised tangent space for the corresponding generalised geometry is twisted by the gauge transformations of all of these tensor gauge potentials. In particular the generalised tangent space takes the form [100, 102, 103, 148]

$$E \simeq TM' \oplus T^*M' \oplus \Lambda^5 T^*M' \oplus (T^*M' \otimes \Lambda^6 T^*M') \oplus \Lambda^{\mathrm{even/odd}} T^*M', \tag{6.54}$$

where “even” refers to type IIB and “odd” to IIA. The pieces of this can be identified as the charges for the momentum, fundamental string, NS5-brane, Kaluza-Klein monopole and D-branes.

The IIA case is given simply by the decomposition under the obvious $GL(d-1,\mathbb{R})$ subgroup of $GL(d,\mathbb{R})$ we have used so far (this was first discussed in [100]). One can see from the Dynkin diagram of $E_{d(d)}$ that there is another embedding of $GL(d-1,\mathbb{R})$, and the decomposition under this gives the corresponding result for type IIB. There are similarly two different realisations of H_d in terms of $Spin(d-1)$ which can be used to describe the $\tilde{H}_d$ structures and fermions.

We do not work through the details of these decompositions here (some are given in the appendix of [2]). However, we note that the construction of the torsion-free compatible connections, unique operators and curvatures will go through in exactly the same way. Also, the partial derivative operator as embedded in E^* will still satisfy the condition $\partial \otimes_{N^*} \partial = 0$, however, it no longer spans a maximal dimension subspace (which would be d dimensional).

7. Supersymmetric backgrounds as generalised G-structures

In this chapter, we discuss supersymmetric backgrounds of the theories we have considered thus far, in the language of generalised G-structures. First, we review the standard construction of G-structures in detail. In the next section we recall the basic equations of four dimensional Minkowski compactifications of type II theories and M-theory, and how these equations lead to the appearance of G-structures. In the presence of fluxes, we see that these G-structures have intrinsic torsion, so integrability is lost. Also, we see that for each level of preserved supersymmetry there are several different structure groups that can appear. A valuable review which we follow in part is [149].

Generalised geometry is able to combine these different classes of ordinary G-structure so that each level of preserved supersymmetry corresponds to a single generalised structure group. Furthermore, in the relevant cases, the Killing spinor equations become precisely the statement that this generalised G-structure is torsion-free. This is the main result of this section, and it is proposed as a major application of the technology. We prove these statements in some important cases, by direct computation of the representation structure of the intrinsic torsion.

7.1. Holonomy, G-structures and Intrinsic Torsion

In the very first section of chapter 2, we reviewed some differential geometry, introducing the notion of a G-structure on a manifold. We briefly mentioned that there might be some barrier to the existence of a torsion-free connection which was compatible with a given G-structure. We expand on this here, reviewing the concepts of holonomy and intrinsic torsion.

Holonomy

First, consider a vector bundle E, with structure group G, over a base manifold M. Suppose we have a connection D on E with connection one-form $\mathcal{A}_\mu$, so that if $\{\hat{e}_i\}$ is a local basis for E then

$$D_{\partial/\partial x^\mu}\hat{e}_j = \mathcal{A}_\mu{}^i{}_j \hat{e}_i. \tag{7.1}$$

Consider a curve $c : [0,1] \to M$, for convenience, assumed to lie within a single coordinate chart of M and a single local trivialisation of E, with tangent vector $v(t) \in TM$. A section $X \in \Gamma(E)$ is said to be parallel transported along c if

$$D_v X = 0 \tag{7.2}$$

Given an element X_0 of E at the point $c(0) \in M$, we can then find the parallel transport of X_0 along c as $X_1^i = (g_c)^i{}_j X_0^j$ where

$$(g_c) = \mathcal{P}\exp\Big(-\int_c \mathcal{A}_\mu \mathrm{d}x^\mu\Big) \in G, \tag{7.3}$$

where $\mathcal{P}$ is the path-ordering symbol. The new element X_1 is located above the point $c(1)$ in the manifold. Considering the case of a closed loop, $(g_c) \in G$ gives an endomorphism of the fibre of E at the point $p = c(0) = c(1) \in M$. The set

$$H_p = \{(g_c) \in G : c \text{ is a closed curve at } p\} \subset G \tag{7.4}$$

forms a group called the holonomy group at p, with the group operation given by concatenation of curves. Simple arguments show that for any $p, p' \in M$ one has $H_p \simeq H_{p'}$, so that the holonomy group is independent of the point p. In fact, we will take the holonomy group to be what is more precisely referred to as the restricted holonomy group, which includes only contractible curves.

If one takes E to be the tangent bundle of a Riemannian manifold (M, g) and ∇ to be the Levi-Civita connection, the resulting holonomy group is commonly referred to as the holonomy of M. If the holonomy of the manifold is a proper subgroup of $SO(d)$[1], the manifold is said to have special holonomy. The requirement of special holonomy turns out to be a very

[1] Parallel transport preserves the metric length of a vector, and the holonomy group can only have one connected component as it is built up from infinitesimal actions.

strong condition. Riemannian special holonomy manifolds were classified by Berger [150] and it turns out that only a short list of holonomy groups is possible. The details can be found in [151]. Similar results have also been obtained for Lorentzian manifolds ([152] discusses the status of this area of research as well as highlighting the distinction between full and restricted holonomy).

Intrinsic Torsion

Recall that a G-structure is a principal sub-bundle of the frame bundle on a manifold, that is

$$P = \{(x, \{\hat{e}_a\}) \in F : \text{different frames } \{\hat{e}_a\} \text{ related by } G\text{-transformations}\}. \tag{7.5}$$

Using these special frames, or equivalently the invariant tensors of G, all tensors on the manifold can then be decomposed into irreducible parts under $G \subset GL(d, \mathbb{R})$. A connection ∇ is compatible with a G-structure $P \subset F$ if the corresponding connection of the principal bundle F reduces to a connection on P. This means that, given a basis $\{\hat{e}_a\}$, one has a set of connection one-forms $\omega^a{}_b$ taking values in the adjoint representation of G given by

$$\nabla_{\partial/\partial x^\mu} \hat{e}_a = \omega_\mu{}^b{}_a \hat{e}_b. \tag{7.6}$$

Let $\mathrm{ad}(P)$ be the associated adjoint bundle, then a connection one-form ω_μ on P can locally be represented as a section

$$\omega \in T^*M \otimes \mathrm{ad}(P). \tag{7.7}$$

The torsion of ∇ will still be a section of the bundle $TM \otimes \Lambda^2 T^*M$, and in general both of these bundles can be decomposed into irreducible parts under G.

The intrinsic torsion of P can be defined as follows. Consider two such connections ∇ and ∇', both compatible with the structure P, and let $T(\nabla)$ and $T(\nabla')$ be their respective torsions. The difference of these $\Delta T = T(\nabla') - T(\nabla)$ is a section of $W := TM \otimes \Lambda^2 T^*M$. However, it can happen that, varying ∇' for fixed ∇, ΔT fills out only a subspace of the fibre of W at each point of M. Let $\Sigma \in T^*M \otimes \mathrm{ad}(P)$ be the difference of the two connections, which is a tensor such that for $v \in TM$

$$\Sigma_v = \nabla'_v - \nabla_v. \tag{7.8}$$

ΔT depends linearly on Σ. Therefore, if the dimension of $T^*M \otimes \mathrm{ad}(P)$ is less than the dimension of $TM \otimes \Lambda^2 T^*M$, it is clear that ΔT must be restricted to a subspace. Label the image of the torsion map on $T^*M \otimes \mathrm{ad}(P)$ as W_P, then we can define the bundle

$$W_I = \frac{W}{W_P}. \tag{7.9}$$

Now, given any compatible connection ∇ on P, its torsion defines an element of W_I, which is independent of which connection one chooses. This element of W_I is the intrinsic torsion of P, and if it is non-zero, then there does not exist a torsion-free connection which is compatible with P. G-structures with vanishing intrinsic torsion are said to be integrable.

In general, the vanishing of the intrinsic torsion is a first-order differential constraint on the structure. Suppose the structure is defined by a G-invariant tensor Φ, and let $\nabla' = \nabla + \Sigma$, where this time ∇ is torsion-free and ∇' is assumed to be torsion-free and compatible. This implies that

$$0 = \nabla'\Phi = \nabla\Phi + \Sigma \cdot \Phi \tag{7.10}$$

We must therefore be able to solve the equation $\nabla\Phi = -\Sigma \cdot \Phi$ for Σ, subject to the constraint that $T(\Sigma) = 0$, and in general this constrains which irreducible parts of $\nabla\Phi$ can be non-zero. Thus we have first-order differential constraints on the invariant tensor Φ which defines the structure.

For example, in the case of an almost complex structure J on a real manifold, the above condition becomes

$$\Sigma_m{}^n{}_q J^q{}_p - \Sigma_m{}^q{}_p J^n{}_q = -\nabla_m J^n{}_p, \tag{7.11}$$

where ∇ is an arbitrary torsion-free connection on TM. Using $\Sigma_{[m}{}^n{}_{p]} = 0$, we have

$$\Sigma_q{}^n{}_{[m} J^q{}_{p]} = -\nabla_{[m} J^n{}_{p]} \tag{7.12}$$

However, contracting (7.11) with J twice and then anti-symmetrising results in a different expression

$$\Sigma_q{}^n{}_{[m} J^q{}_{p]} = -J^n{}_r J^s{}_{[p} \nabla_{m]} J^r{}_s \tag{7.13}$$

Together these become

$$\nabla_{[m} J^n{}_{p]} - J^n{}_r J^s{}_{[p} \nabla_{m]} J^r{}_s = 0 \tag{7.14}$$

which is the well-known Nijenhuis condition for a complex manifold. Considering a torsion-free shift of Σ in the above argument reveals that the left-hand side is unique for a torsion-free connection, so that one can replace ∇ with ∂. This is generally true of such integrability conditions when written as a first order differential constraint.

We now connect the notions of intrinsic torsion and special holonomy. If one considers a Riemannian manifold M equipped with a metric g, this defines an $O(d)$ structure on M, given by the orthonormal frames of g. The holonomy of a connection compatible with this structure will be contained in $SO(d)$ (as the holonomy group is necessarily in the identity component of the structure group).

Now suppose we have a G-structure P on M, which is a sub-bundle of the orthonormal frame bundle of the metric, so that $G \subset O(d)$. A connection ∇ compatible with P will thus also be compatible with the metric, i.e. $\nabla g = 0$. If the intrinsic torsion vanishes, then the torsion-free connection will be the unique Levi-Civita connection on (M, g). The holonomy of the Levi-Civita connection is then contained in the group $G_0 \subset SO(d)$, where G_0 is the identity component of G, so the manifold has special holonomy. However, in the more general context of P being an arbitrary G-structure on the frame bundle of a differentiable manifold, there may be a family of torsion-free compatible connections.

7.2. Supersymmetric Backgrounds of String Theory and M Theory

An important class of backgrounds of string theory and M-theory are the solutions of the low-energy classical supergravity approximation. Such a solution is said to be supersymmetric if there exists a nowhere-vanishing choice of supersymmetry parameter on the manifold such that the supersymmetry variations of all of the background fields vanish. Since we are interested in classical solutions, the background fermionic fields are zero. The variations of the bosonic fields always have a fermionic factor, so these are automatically zero. Therefore, the non-trivial condition for supersymmetry is the vanishing of the variations of the fermionic fields, and we need only consider the lowest order terms in fermions.

A family of such solutions, used to attempt to construct physically realistic models, are manifolds of the form $M = M_4 \times M_{\text{int}}$ i.e. the product of a four-dimensional Minkowski space with a compact internal manifold M_{int}. Other popular choices are for the external space to be anti-de-Sitter, important for studying AdS/CFT, or de-Sitter, which has an observationally appealing positive cosmological constant. Together with Minkowski space, these are the maximally symmetric spaces.

We will focus on solutions with an external Minkowski factor. For the bosonic fields, one takes a metric ansatz of the form

$$\mathrm{d}s^2_{10} = \mathrm{e}^{2\Delta}\mathrm{d}s^2(\mathbb{R}^{3,1}) + \mathrm{d}s^2_d(M_{\text{int}}), \tag{7.15}$$

where the warp factor Δ depends only on the internal coordinates[2]. Otherwise, one can only keep those components of the fields which are scalars on the external space, as any other components would violate maximal symmetry. One also must take all background fields to depend only on the coordinates of M_{int} for the same reason.

Schematically, we take the higher-dimensional supersymmetry parameter to have a tensor product form $\hat{\varepsilon} = \eta \otimes \varepsilon$, where η is a covariantly constant spinor on the external Minkowski space and ε is an internal spinor field on M_{int}. The expressions for the vanishing of the higher-dimensional supersymmetry variations are known as Killing spinor equations, and these induce lower-dimensional Killing spinor equations on the internal spinor ϵ. The number of real supercharges which preserve the background is equal to the number of real degrees of freedom of the spinor η times the number of independent solution of these equations on M_{int}.

[2]With non-trivial warp factor, this ansatz can also describe solutions with an external anti-de-Sitter factor of one dimension higher, as in e.g. [153].

7.2.1. 4D Minkowski compactifications of type II theories

We consider Minkowski compactifications of type II theories on six-dimensional internal manifold M. We will set up our analysis for backgrounds preserving $N = 1$ supersymmetry in the resulting four dimensional theory, though we will see that the restricted cases we eventually consider will actually permit $N = 2$. For type IIA the two supersymmetry parameters have opposite chirality and we can use the ansatz

$$\begin{aligned} \varepsilon^+ &= \eta^- \otimes \epsilon_1^+ + \eta^+ \otimes \epsilon_1^- \\ \varepsilon^- &= \eta^+ \otimes \epsilon_2^+ + \eta^- \otimes \epsilon_2^- \end{aligned} \tag{7.16}$$

For type IIB the two supersymmetry parameters have the same chirality and so we take

$$\begin{aligned} \varepsilon^+ &= \eta^+ \otimes \epsilon_1^+ + \eta^- \otimes \epsilon_1^- \\ \varepsilon^- &= \eta^+ \otimes \epsilon_2^+ + \eta^- \otimes \epsilon_2^- \end{aligned} \tag{7.17}$$

In both of these decompositions η and ϵ_i are Majorana spinors in four and six dimensions respectively. On the right-hand sides, the superscripts $\pm$ indicate the chirality of the spinor as viewed in its respective dimension. On the left-hand sides they have they label the two ten dimensional supersymmetry parameters as in section 4.1. We take the Majorana representations of Cliff$(3, 1; \mathbb{R})$ and Cliff$(6; \mathbb{R})$ so that we have $\eta^- = (\eta^+)^*$ and $\epsilon_i^- = (\epsilon_i^+)^*$. (See appendix E for full details of conventions for the decomposition.) Each ϵ_i thus has eight real degrees of freedom overall, so the components of ϵ_i transform in the $\mathbf{4} + \bar{\mathbf{4}}$ representation of $SU(4) \simeq Spin(6)$ where the Majorana condition relates the two parts to be complex conjugate.

The background values of the bosonic fields are chosen such that only quantities which are scalars on the external spacetime are non-zero. In this case we have an internal metric g_{mn}, and internal fluxes H_{mnp} and $\partial_m \phi$. We set the warp factor to zero[3] and neglect RR fluxes for simplicity. The vanishing of the 10 dimensional supersymmetry variations, for either the

[3] We use the string-frame metric. If one took the Einstein frame metric, supersymmetry requires an overall warp factor equal to the dilaton, which essentially returns the discussion to the string-frame [154, 155].

IIA or IIB case, then imply the Killing spinor equations

$$\begin{aligned}
&\nabla_m\epsilon_1 - \tfrac{1}{8}H_{mnp}\gamma^{np}\epsilon_1 = 0 && \nabla\!\!\!/\epsilon_1 - \tfrac{1}{4}H\!\!\!/\epsilon_1 + (\partial\!\!\!/\phi)\epsilon_1 = 0 \\
&\nabla_m\epsilon_2 + \tfrac{1}{8}H_{mnp}\gamma^{np}\epsilon_2 = 0 && \nabla\!\!\!/\epsilon_2 + \tfrac{1}{4}H\!\!\!/\epsilon_2 + (\partial\!\!\!/\phi)\epsilon_2 = 0
\end{aligned} \tag{7.18}$$

for the spinors $\epsilon_{1,2}$ on the internal manifold M.

An important point to raise immediately is that such a configuration will not give an $N = 1$ compactification of a type II theory. This is because one can introduce a second four-dimensional supersymmetry parameter and use the spinor ansatz

$$\begin{aligned}
\varepsilon^+ &= \eta_1^- \otimes \epsilon_1^+ + \eta_1^+ \otimes \epsilon_1^- \\
\varepsilon^- &= \eta_2^+ \otimes \epsilon_2^+ + \eta_2^- \otimes \epsilon_2^-
\end{aligned} \tag{7.19}$$

for the IIA case and

$$\begin{aligned}
\varepsilon^+ &= \eta_1^+ \otimes \epsilon_1^+ + \eta_1^- \otimes \epsilon_1^- \\
\varepsilon^- &= \eta_2^+ \otimes \epsilon_2^+ + \eta_2^- \otimes \epsilon_2^-
\end{aligned} \tag{7.20}$$

for the IIB case. In the absence of RR fluxes, these will still solve the ten-dimensional Killing spinor equations on imposing (7.18). To obtain an $N = 1$ vacuum, one would need to have non-zero RR fluxes as these introduce terms mixing the spinors ε^+ and ε^-. This would then spontaneously break the $N = 2$ supersymmetry down to $N = 1$.

Our task is to determine the consequences of equations (7.18) for the properties of the internal manifold. The first obvious property is that it must possess two non-vanishing spinor fields $\epsilon_{1,2}$. This imposes a topological condition condition on the manifold, reducing the structure group to a subgroup of $SU(4)$. Firstly, the spin-frames in which $\epsilon_1^+ = (|\epsilon|, 0, 0, 0)$ (viewed as a 4-component Weyl spinor) form an $SU(3)$ structure. If the two spinors are parallel, then ϵ_2 provides no further reduction of the structure group. If they are non-parallel then, in similar fashion, the second spinor ϵ_2 reduces the structure group further to $SU(2)$. There are hybrid cases in which the spinors are parallel at some points of the manifold and non-parallel at others.

The second set of conditions on M_{int} come from the differential equations (7.18). These are far more complicated to analyse in general. One property which is easy to derive from them is that the norms of the Killing spinors are constant. The equations on the left of (7.18) relate $\nabla_m\epsilon_i$ with

$H_{mnp}\gamma^{np}\epsilon_i$. Since $H_{mnp}\gamma^{np}$ rotates the spinor by an $SU(4)$ algebra element, this will preserve any $SU(4)$ covariant norm, so we see that $\nabla_m(\bar{\epsilon}_i\epsilon_i) = 0$ for each $i = 1, 2$. In fact, using generalised geometry methods it is easy to see that all Killing spinors have constant norm in the systems we will study (see section 7.4).

With regards to further investigation of (7.18) at this stage, we will split our discussion in two, first considering the simplified case in which the fluxes vanish, and then commenting on how the fluxes spoil the nice properties obtained in that case.

The case without fluxes

In the absence of internal fluxes, the Killing spinor equations (7.18) become the statement that the spinors $\epsilon_{1,2}$ are covariantly constant. This means that the Levi-Civita connection is compatible with the reduced structure they define, which therefore has vanishing intrinsic torsion. The manifold is thus a special holonomy manifold.

Suppose M_{int} has a non-vanishing covariantly constant spinor ϵ. This defines an $SU(3)$ structure on M_{int} with vanishing intrinsic torsion. As reviewed below, this implies that the internal manifold is Calabi-Yau. Since the observation that compactifications of heterotic string theory on such manifolds could lead to phenomenologically viable models [39], these manifolds have received much attention in the literature.

Suppose ϵ is normalised so that $\bar{\epsilon}^+\epsilon^+ = \bar{\epsilon}^-\epsilon^- = 1$. The tensor $J_m{}^n = i\bar{\epsilon}^+\gamma_m{}^n\epsilon^+$ then defines an almost complex structure, since a Fierz identity reveals that $J_m{}^pJ_p{}^n = -\delta_m{}^n$. Furthermore it is integrable because $\nabla_m J_n{}^p = 0$ and so it's Nijenhuis tensor vanishes. This guarantees that one can find local complex coordinates $\{z^a\}$ on M_{int} such that $J_a{}^b = i\delta_a{}^b$ and $J_{\bar{a}}{}^{\bar{b}} = -i\delta_{\bar{a}}{}^{\bar{b}}$ are the only non-vanishing components. The complex coordinates also allow us to define the Dolbeault operators $\partial : \Lambda^{p,q}(M) \to \Lambda^{p+1,q}(M)$ and $\bar{\partial} : \Lambda^{p,q}(M) \to \Lambda^{p,q+1}(M)$ such that $\mathrm{d} = \partial + \bar{\partial}$. The metric is hermitian with respect to the complex structure, since $g_{mn} = J_m{}^pJ_n{}^qg_{pq}$, and the simple statement $J_{mn} = J_m{}^pg_{pn}$ leads immediately to $J_{a\bar{b}} = ig_{a\bar{b}}$.

Clearly $J_{mn} = J_{[mn]}$ defines a 2-form $J = ig_{a\bar{b}}\mathrm{d}z^a\wedge\mathrm{d}\bar{z}^{\bar{b}}$ with $\mathrm{d}J = 0$ (since $\nabla J = 0$ and ∇ is torsion-free), so the manifold is Kahler with Kahler form J.

One can also construct a holomorphic 3-form with non-vanishing components $\Omega_{abc} = \bar{\epsilon}^{+}\gamma_{abc}\epsilon^{-}$. Again, since $\nabla\epsilon = 0$ we have $\nabla_{\bar{a}}\Omega_{bcd} = 0$ so $\bar{\partial}\Omega = 0$ and $\mathrm{d}\Omega = (\partial + \bar{\partial})\Omega = 0$. Ω is thus a holomorphic $(3,0)$-form. However, Ω is not exact as

$$\Omega \wedge \bar{\Omega} = -\tfrac{i}{3!}||\Omega||^2 J \wedge J \wedge J = -i||\Omega||^2 \,\mathrm{vol}_g \tag{7.21}$$

is a constant multiple of the volume form, since

$$||\Omega||^2 = \tfrac{1}{3!} g^{a\bar{a}} g^{b\bar{b}} g^{c\bar{c}} \Omega_{abc} \bar{\Omega}_{\bar{a}\bar{b}\bar{c}} \tag{7.22}$$

is constant.

The existence of a non-vanishing holomorphic (3,0)-form implies that the Ricci form is exact. Recall that the Ricci form is defined as $\mathcal{R} = i\mathcal{R}_{a\bar{b}}\mathrm{d}z^a \wedge \mathrm{d}\bar{z}^{\bar{b}}$ for $\mathcal{R}_{a\bar{b}}$ the components of the Ricci tensor, and that $\mathcal{R} = i\partial\bar{\partial}\log\sqrt{g}$ for the Levi-Civita connection of a hermitian metric. Since Ω is holomorphic we must have $\Omega_{abc} = f(z)\epsilon_{abc}$ where ϵ_{abc} is the Levi-Civita symbol, and f is a non-vanishing holomorphic function. This leads to the relation

$$||\Omega||^2 = \tfrac{1}{3!}|f(z)|^2 \epsilon_{abc}\epsilon_{\bar{a}\bar{b}\bar{c}} g^{a\bar{a}} g^{b\bar{b}} g^{c\bar{c}} = |f(z)|^2 \det(g^{a\bar{b}}) \propto \frac{|f(z)|^2}{\sqrt{g}} \tag{7.23}$$

so that the Ricci form is

$$\mathcal{R} = i\partial\bar{\partial}\log\sqrt{g} = -i\partial\bar{\partial}\log||\Omega||^2 \tag{7.24}$$

This is exact since $||\Omega||^2$ is a globally defined (non-zero) scalar, so the manifold has vanishing first Cern class. Kahler manifolds with vanishing first Cern class are called Calabi-Yau manifolds. Using the further property that Ω is defined as above, we have that $||\Omega||^2$ is constant and so the Ricci form is zero. This illustrates the powerful existence theorem [156] stating that Calabi-Yau manifolds always admit a Ricci-flat metric.

The mathematically nice properties of Calabi-Yau spaces enable examples to be constructed and studied. The two crucial points are that the manifold is complex, allowing the use of algebraic geometry machinery, and the existence theorem for the metric, which guarantees that as long as the Ricci form is exact there exists a Ricci-flat metric. As a result, the physical consequences of many examples have been computed, especially in the context of heterotic string theory where Calabi-Yau compactifications directly give $N = 1$ supersymmetry (and physically reasonable examples have been found, e.g. [157, 158]). However, our goal is to shed light on the broader class of solutions which do include fluxes.

The case with fluxes

The inclusion of fluxes complicates the picture significantly. The globally non-vanishing spinor fields give rise to G-structures exactly as before, but these structures are no longer integrable, as the Killing spinors are not covariantly constant[4]. This is the general problem posed by the inclusion of fluxes in such compactifications.

In actual fact, it is possible to make some of the same statements in these cases. Focusing on one of the two Killing spinors, we can again build a complex structure and, after a slightly more involved calculation, it is integrable [39, 154]. However, the manifold is not Kahler, the obstruction being that $\mathrm{d}J$ is related to H. This also ignores the constraints on M_{int} imposed by the other set of Killing spinor equations, and so is not particularly useful here.

If one assumes the internal manifold has an $SU(3)$ structure, so that we have a decomposition of the form

$$\begin{aligned} \varepsilon^+ &= \eta^- \otimes b\epsilon^+ + \eta^+ \otimes \bar{b}\epsilon^- \\ \varepsilon^- &= \eta^+ \otimes a\epsilon^+ + \eta^- \otimes \bar{a}\epsilon^- \end{aligned} \tag{7.25}$$

for IIA or

$$\begin{aligned} \varepsilon^+ &= \eta^+ \otimes b\epsilon^+ + \eta^- \otimes \bar{b}\epsilon^- \\ \varepsilon^- &= \eta^+ \otimes a\epsilon^+ + \eta^- \otimes \bar{a}\epsilon^- \end{aligned} \tag{7.26}$$

for IIB, where a, b are complex functions on M_{int}, then it is possible to classify the solutions in terms of intrinsic torsion classes of the $SU(3)$ structure (see [149] for a review). This approach has assumed an $SU(3)$ structure from the outset, so it does not present the most general solution. If $\epsilon_{1,2}$ are nowhere parallel then we will have an $SU(2)$ structure, while if they are parallel at some points of M_{int} we obtain an interpolating $SU(2)$ and $SU(3)$ structure.

[4]They are preserved by a connection with skew-symmetric torsion [154, 155], but this connection (obviously) has non-zero torsion.

Therefore, it would seem that the inclusion of fluxes presents a departure from the nice mathematical properties of the fluxless case. In particular, the structures are no longer integrable, and we have the possibility of different local structures appearing at different points in the manifold. However, as we will see, when viewed as a generalised geometry, there is only one type of structure that can appear and the analogue of integrability continues to hold.

Before moving on, we must discuss a no-go theorem [159] (see also the earlier works [160]) concerning the existence of purely geometrical compact solutions with flux. The bosonic equations of motion imply that

$$\nabla^2(\mathrm{e}^{2\phi}) = \tfrac{1}{6}\mathrm{e}^{-2\phi}H^2 + \tfrac{1}{4}\sum_n \tfrac{1}{(n-1)!}F_{(n)}^2 \tag{7.27}$$

If M_{int} is compact without boundary, the integral over M_{int} of the LHS vanishes, while the terms on the RHS are positive semi-definite. Therefore, we deduce that H and F must vanish on M_{int}, implying that the LHS of the above equation must also vanish. This in turn gives $\nabla^2\phi = 2(\partial\phi)^2$ and a similar integration argument reveals that ϕ must be constant.

We have therefore deduced that no purely geometrical solution with compact M_{int} and non-vanishing fluxes exists. In type II string theory, this can be remedied by the addition of orientifold planes, which crucially have negative tension, thus circumventing the no-go theorem [14].

7.2.2. 4D warped Minkowski solutions of M-theory

We now examine solutions of eleven-dimensional supergravity with the same metric ansatz used in chapter 6

$$\mathrm{d}s^2_{11} = \mathrm{e}^{2\Delta}\mathrm{d}s^2(\mathbb{R}^{10-d,1}) + \mathrm{d}s^2_d(M), \tag{7.28}$$

and we keep exactly the same internal fluxes as we did there.

Again, we must discuss a no-go theorem. The bosonic equations of motion imply the equation

$$\nabla^2(\mathrm{e}^{c\Delta}) = c\mathrm{e}^{c\Delta}(\tfrac{1}{6}\tfrac{1}{4!}F^2 + \tfrac{1}{3}\tfrac{1}{7!}\tilde{F}^2) \tag{7.29}$$

If M is compact without boundary, the integral over M of the LHS vanishes, while both terms on the RHS are positive semi-definite. Therefore, we deduce that the fields F and $\tilde{F}$ must vanish on M. One way to proceed from this is to consider solutions with non-compact M and non-zero internal fluxes. Such solutions include those which can be viewed as an external AdS_5 factor times a compact internal space, which are important in AdS/CFT. However, it is still possible to have compact solutions with non-zero flux in M theory if one adds appropriate negative tension objects (which we do not discuss, see [161]), as for the type II case.

We employ the spinor decomposition of appendix E, first with only one external spinor so as to obtain an $N = 1$ vacuum

$$\varepsilon = \eta^{+} \otimes \epsilon + \eta^{-} \otimes \epsilon^{*} \tag{7.30}$$

taking the Majorana representations of $\mathrm{Cliff}(3,1;\mathbb{R})$ and $\mathrm{Cliff}(7;\mathbb{R})$. Note that the internal spinor ϵ is complex, which can be thought of as a pair of real $Spin(7)$ spinors $\epsilon = \Re(\epsilon) + i\Im(\epsilon)$

The relevant parts of the eleven-dimensional supersymmetry variations are precisely (6.21), which under the spinor decomposition become the Killing spinor equations

$$\begin{aligned}
0 &= [\nabla\!\!\!/ - \tfrac{1}{4}F\!\!\!/ - \tfrac{1}{4}\tilde{F}\!\!\!/ + (\partial\!\!\!/\Delta)]\epsilon, \\
0 &= \nabla_m \varepsilon + \tfrac{1}{288}(\gamma_m{}^{n_1 \dots n_4} - 8\delta_m{}^{n_1}\gamma^{n_2 n_3 n_4})F_{n_1 \dots n_4}\varepsilon - \tfrac{1}{12}\tfrac{1}{6!}\tilde{F}_{m n_1 \dots n_6}\gamma^{n_1 \dots n_6}\epsilon,
\end{aligned} \tag{7.31}$$

If we wish to extend to more supersymmetries, we merely insist on the existence of more internal non-vanishing spinors ϵ_i and gain multiple copies of the above equations.

The case of vanishing fluxes again forces M to have special holonomy as the Killing spinor equations become $\nabla_m \epsilon_i = 0$. Since Levi-Civita is real, the real and imaginary parts of the complex spinors ϵ_i are preserved separately. The stabiliser of the single internal complex spinor depends upon its form. Each of the real and imaginary parts is a real $Spin(7)$ spinor, which defines a G_2 structure. If both parts are non-zero and they are not parallel, the pair defines an $SU(3)$ structure. However, if they are parallel (or one of them is zero) we have a G_2 strcture on M. Again, the full class of cases includes the possibility that they are parallel at some points of M but not others.

For the case of a G_2 structure, the vacuum has $N = 1$ supersymmetry. The local existence of G_2 holonomy metrics was shown in [162] and non-compact examples were presented. The first construction of compact manifolds with G_2 holonomy was achieved in [163] (see [151] for a review). For the flux-less $SU(3)$ structure case, one can use each of the real and imaginary parts of ϵ to build different eleven-dimensional spinors, so one actually obtains an $N = 2$ vacuum.

As before, the inclusion of fluxes introduces great complications to the picture. Firstly, the flux terms mix the real and imaginary parts of the complex spinors in the equations, so that one can have different structure groups at each level of preserved supersymmetry. More importantly, they spoil the integrability of the G-structures defined by the Killing spinors. In the next section we will show how generalised geometry will restore this integrability and unify all of the different G-structures one could have for each value of N.

7.3. Generalised Complex Geometry and Supersymmetric Vacua

The first generalised geometry formulations of supersymmetric backgrounds were given in [88] for type II theories. These papers approached the problem from a different (but equally elegant) angle, which we briefly discuss and summarise, following [82, 149, 153].

We have included several structures familiar from differential geometry into generalised geometry. However, there are yet more such structures which carry over naturally. One is the almost complex structure [81, 82], if the manifold has even dimension d. In generalised geometry, one can define a generalised almost complex structure to be an $O(d, d)$ norm-preserving map $\mathcal{J} : E \to E$ with $\mathcal{J}^2 = -\mathbb{1}$. This defines a $U(d/2, d/2) = O(d, d) \cap GL(d, \mathbb{C})$ structure on E. The integrability condition on $\mathcal{J}$ is that the Courant bracket closes on its $\pm i$ eigenbundles in $E_{\mathbb{C}} = E \otimes \mathbb{C}$.

At the two extremes of generalised almost complex structures (on $T \oplus T^*$) are those built from either an ordinary complex structure or an ordinary sympletic form. The integrability condition then matches the normal integrability conditions for those ordinary structures. A generalised complex manifold therefore interpolates between complex manifolds and symplectic manifolds.

A generalised almost complex structure on E is equivalent to a complex pure spinor line bundle. A pure spinor is a definite chirality complex $O(d,d)$ spinor Φ such that the annihilator

$$L_\Phi = \{V \in \Gamma(E_{\mathbb{C}}) : V^A \Gamma_A \Phi = 0\} \tag{7.32}$$

is maximally isotropic. This annihilator is identified with the $+i$ eigenbundle of $\mathcal{J}$. Such a globally non-vanishing pure spinor defines an $SU(d/2, d/2)$ structure. In the coordinate frame, the integrability condition becomes $\mathrm{d}\Phi = V \cdot \Phi$ for some $V \in E$. If $\mathrm{d}\Phi = 0$, the manifold is called generalised Calabi-Yau [81].

A pair of generalised almost complex structures $\mathcal{J}_{1,2}$ is said to be compatible if they commute and their product $G = -\mathcal{J}_1\mathcal{J}_2$ defines a generalised metric. The pair then defines an $SU(d/2) \times SU(d/2)$ structure, which is integrable, or generalised Kahler, if both complex structures are integrable.

The same structure can be defined by a generalised metric and two $Spin(d)$ spinors. For the relevant case of $d = 6$, we can use two chiral spinors $\epsilon^+_{1,2}$ to build the two pure spinors (also in the coordinate frame)

$$\Phi^+ = \mathrm{e}^{-\phi}\mathrm{e}^{-B} \wedge (\epsilon^+_1 \otimes \bar{\epsilon}^+_2) \qquad\qquad \Phi^- = \mathrm{e}^{-\phi}\mathrm{e}^{-B} \wedge (\epsilon^+_1 \otimes \bar{\epsilon}^-_2) \tag{7.33}$$

where the tensor products are expanded in forms using the usual Clifford algebra isomorphism, in which the $\Lambda^k T^*M$ components of $\epsilon^+_1 \otimes \bar{\epsilon}^\pm_2$ are given by

$$(\epsilon^+_1 \otimes \bar{\epsilon}^\pm_2)_{m_1\dots m_k} = \tfrac{1}{4}(-1)^{[k+1/2]}(\bar{\epsilon}^\pm_2 \gamma_{m_1\dots m_k} \epsilon^+_1) \tag{7.34}$$

Schematically, the differential conditions for $N = 1$ supersymmetry become

$$\mathrm{d}\Phi_+ = 0 \qquad\qquad \mathrm{d}\Phi_- = F_{\mathrm{RR}} \tag{7.35}$$

where $\Phi_\pm = \Phi^\pm$ for IIA and $\Phi_\pm = \Phi^\mp$ for IIB. For vanishing RR flux, these conditions express the integrability of the $SU(3)\times SU(3)$ structure, or equivalently the integrability of both of the generalised complex structures. This result is essentially equivalent to the result we give using our generalised connections formalism in the next section.

7.4. Killing Spinor Equations and Generalised Holonomy

In generalised geometry language, we have seen that the vanishing of the supersymmetry variations of the fermions can be encoded concisely in the equations

$$D \otimes_J \epsilon = 0 \qquad\qquad D \otimes_S \epsilon = 0 \tag{7.36}$$

where D is a torsion-free generalised $\tilde{H}_d$ connection and the supersymmetry parameter ϵ is viewed as a section of the $\tilde{H}_d$ bundle S. In the present context, ϵ is a globally non-vanishing section, so its components are stabilised by transition functions in some subgroup $G^{(\epsilon)}$ of $\tilde{H}_d$. In other words, those $\tilde{H}_d$ frames in which the components of ϵ are fixed define a generalised $G^{(\epsilon)}$-structure. A generalised connection is compatible with this structure if $D\epsilon = 0$.

One can easily extend the above to higher supersymmetry. Here, one merely has several background spinors $\{\epsilon^i\}$, which are stabilised by a group $G^{\{\epsilon^i\}} \subset \tilde{H}_d$, and the compatibility condition becomes $D\epsilon^i = 0$ for each value of i.

The equations (7.36), which hold for any torsion-free $\tilde{H}_d$ compatible D, appear weaker than the compatibility condition as they constrain only two of the irreducible parts of $D\epsilon^i$. However, one can show that, for low numbers of supersymmetries, one can always construct some (more restricted) torsion-free connection $\hat{D}$ such that the full compatibility condition $\hat{D}\epsilon^i = 0$ is satisfied[5]. The generalised $G^{\{\epsilon^i\}}$-structure then has vanishing intrinsic torsion and we have the generalised analogue of special holonomy. The Killing spinor equations are thus equivalent to special generalised holonomy in these cases. We will demonstrate this statement for some important examples in the next sections.

7.4.1. $SU(3) \times SU(3)$ structures in type II theories

We will now explain how this works for the six-dimensional $SU(3) \times SU(3)$ compactifications of type II theories. As said in section 7.3, these have been studied extensively in the context of generalised complex geometry in [88].

[5]Note that this also provides a simple proof that the norm of a Killing spinor is constant in these setups

Here we explain how they are described equally well by our formalism, which can also incorporate more supersymmetries.

We consider an $O(6,6) \times \mathbb{R}^+$ generalised geometry on the internal manifold. The spin group associated to the generalised metric is then $Spin(6) \times Spin(6) \simeq SU(4) \times SU(4)$ and we can consider a generalised spinor bundle $S = S(C_+) \oplus S(C_-)$. Here the spinor bundles $S(C_\pm)$ are taken to include both chiralities such that the fibre of each is the $\mathbf{4} + \bar{\mathbf{4}}$ representation of $SU(4)$. For Majorana spinors the $\mathbf{4}$ and $\bar{\mathbf{4}}$ parts are related by complex conjugation. The fibre of S is thus the $(\mathbf{4} + \bar{\mathbf{4}}, \mathbf{1}) + (\mathbf{1}, \mathbf{4} + \bar{\mathbf{4}})$ representation of $SU(4) \times SU(4)$, and the direct sum of two spinors $\epsilon^\pm \in \Gamma(S(C_\pm))$ gives a section of S.

In six dimensions, our two supersymmetry parameters $\epsilon^+ = \epsilon_1$ and $\epsilon^- = \epsilon_2$ on M_{int} can therefore be promoted to sections of $S(C_\pm)$ and then combined into a single object $\epsilon \in \Gamma(S)$. The components of ϵ are stabilised by an $SU(3) \times SU(3)$ subgroup of $SU(4) \times SU(4)$, so ϵ defines an $SU(3) \times SU(3)$ structure. Note that we have not assumed that ϵ^+ and ϵ^- are non-parallel, only that they are non-vanishing. The generalised geometry sees them as transforming under different spin groups and thus combines all possible cases into one unified description.

In the absence of RR fluxes, the supersymmetry variations of the fermions can be equated to zero via the generalised geometry expressions

$$\begin{aligned} D_{\bar{a}}\epsilon^+ &= 0 & \qquad \gamma^a D_a \epsilon^+ &= 0 \\ D_a \epsilon^- &= 0 & \qquad \gamma^{\bar{a}} D_{\bar{a}} \epsilon^- &= 0 \end{aligned} \tag{7.37}$$

where D is any torsion-free $SU(4) \times SU(4)$ connection. These are the unique operators (3.107) for the $Spin(6) \times Spin(6)$ case, and so they are independent of the choice of torsion-free compatible D.

The equations (7.37) do indeed imply the existence of a torsion-free $SU(3) \times SU(3)$ connection $\hat{D}$ with $\hat{D}\epsilon = 0$. We will demonstrate this in the section 7.4.3 after discussing the relationship between these projected derivatives and the intrinsic torsion of the structure. An alternative proof comes from considering a connection of the form $\hat{D} = D + \Sigma$ where D is an arbitrary torsion-free $SU(4) \times SU(4)$ connection and Σ is an extra connection piece. One can then show that it is always possible to solve for the components of Σ in terms of $D\epsilon$ such that $\hat{D}$ is torsion-free and $\hat{D}\epsilon = 0$, and in the process one constructs a family of these connections.

We could also look at cases with more supersymmetry, simply by insisting that further $(\mathbf{4}+\bar{\mathbf{4}},\mathbf{1})+(\mathbf{1},\mathbf{4}+\bar{\mathbf{4}})$ spinors are annihilated by the above operators. We do not discuss this in detail, but it can be shown that the group theory works in the same way, so that there is again an integrable generalised G-structure.

Also, note that if we had non-zero RR fluxes on M_{int}, these would generate non-zero terms on the right had sides of equations (7.37). This would mean that even our generalised structure would not be integrable, and this is a consequence of not including the symmetries and charges of the RR fields in the generalised geometry. This is remedied in $E_{d(d)}\times\mathbb{R}^+$ generalised geometry, which, when decomposed under a $GL(d-1,\mathbb{R})$ subgroup as outlined in section 6.5, does include a "geometrisation" of the RR sector. The integrability of backgrounds with RR fluxes is then restored. Previously, $N=1$ and $N=2$ compactifications of type II theories with RR flux have been studied in exceptional generalised geometry in [102, 103].

At this point, it is also worth mentioning that the solution corresponding to an NS5-brane wrapped on a Kahler 2-cycle in a Calabi-Yau manifold [164], falls outside the classification of [88], as this is an $N=1$ background with vanishing RR fields. For this solution, one sets the second spinor ϵ_2 to zero, so the pure spinors vanish identically. However, in our formalism, where we do not take the tensor product of the spinors, this supersymmetry parameter still gives a non-vanishing section of $S\sim(\mathbf{4}+\bar{\mathbf{4}},\mathbf{1})+(\mathbf{1},\mathbf{4}+\bar{\mathbf{4}})$, describing an $SU(3)\times SU(4)$ structure. This is a case which we have ignored up until now, but it is included in the framework.

7.4.2. $SU(7)$ and $SU(6)$ structures in M-theory

Here we provide the same treatment for M-theory compactifications on seven-manifolds with $E_{d(d)}\times\mathbb{R}^+$ generalised geometry. We find that the logical structure of this case is the same as that in section 7.4.1. The corresponding results for type II theories with RR fluxes can be obtained by considering the results presented here over a six-dimensional manifold as outlined in section 6.5.

In this case, a single internal complex spinor ϵ is a section of the generalised spinor bundle S. The fibre of this bundle is the representation $\mathbf{8}+\bar{\mathbf{8}}$

of $SU(8)$, where the two parts are related by complex conjugation. Therefore ϵ is stabilised by $SU(7) \subset SU(8)$ and so defines an $SU(7)$ structure. This statement unifies all of the different subgroups of $Spin(7)$ which can stabilise both the real and imaginary parts of ϵ. The Killing spinor equations take the abstract form (7.36), which can be written out explicitly in $SU(8)$ indices if required. Again, we will find in the next section that these equations precisely correspond to the vanishing of the intrinsic torsion of the $SU(7)$ structure.

If we introduce a second complex spinor ϵ^2, which is not parallel to the other spinor ϵ^1, the pair define an $SU(6)$ structure and will lead to an $N = 2$ vacuum. The Killing spinor equations become two copies of (7.36), one for each spinor. We find that these equations express the vanishing of the intrinsic torsion of the $SU(6)$ structure, so that a torsion-free compatible connection exists.

We remark that the torsion-free compatible connections for these structures can be found by the constructive method outlined for $SU(3) \times SU(3)$ structures in the previous section. This provides an alternative proof of integrability.

7.4.3. Computation of the Intrinsic Torsion

One can show directly that the intrinsic torsion of the above structures is related to the Killing spinor derivatives. We calculate the representation in which the intrinsic torsion transforms and, assuming that certain maps are non-degenerate, we demonstrate that the Killing spinor equations annihilate precisely this representation. In this section, we study only the linear algebra involved at a single point in the manifold, so that we may discuss representations rather than bundles. In fact, we will use a slight abuse of notation in which we do not distinguish between the two.

Let D and D' be two $\tilde{H}_d$ compatible connections. Their difference $\Sigma = D' - D$ is then a section of $E^* \otimes \mathrm{ad}(P)$. At a point in the manifold, denote the vector space of such tensors by $\tilde{\Sigma} \sim E^* \otimes \mathrm{ad}(\tilde{H}_d)$, and the corresponding space for G-compatible connections by $\tilde{\Sigma}' \sim E^* \otimes \mathrm{ad}(G)$. These are (reducible) representations of $\tilde{H}_d$ and G respectively, and they split into two (also in general reducible) representations

$$\tilde{\Sigma} = \tilde{T} \oplus \tilde{U} \qquad\qquad \tilde{\Sigma}' = \tilde{T}' \oplus \tilde{U}' \tag{7.38}$$

where $\tilde{T}$ and $\tilde{T}'$ are the components constrained by the torsion, and $\tilde{U}$ and $\tilde{U}'$ are unconstrained by the torsion. Clearly $\tilde{\Sigma}' \subset \tilde{\Sigma}$, $\tilde{T}' \subset \tilde{T}$ and $\tilde{U}' \subset \tilde{U}$.

Under the decomposition of $\tilde{H}_d$ representations under G, we have

$$\tilde{T} = \tilde{T}' \oplus \tilde{T}_I \tag{7.39}$$

where $\tilde{T}_I \simeq \tilde{T}/\tilde{T}'$ is the (reducible) representation of G under which the intrinsic torsion transforms. This can be found as we know $\tilde{T}$ and $\tilde{T}' = \tilde{\Sigma}' \cap \tilde{T}$.

The Killing spinor equations transform under the representation $S \oplus J$ of $\tilde{H}_d$, and the projections that give rise to them from the generalised connection define a map

$$\begin{aligned} \mathcal{P} : \tilde{\Sigma} &\to S \oplus J \\ \Sigma &\mapsto (\Sigma \otimes_S \epsilon) \oplus (\Sigma \otimes_J \epsilon) \end{aligned} \tag{7.40}$$

Now, G is the stabiliser of ϵ so for $\Sigma \in \tilde{\Sigma}'$ we have $\Sigma \cdot \epsilon = 0$. Since we have the decomposition $\tilde{\Sigma} = \tilde{T}' \oplus \tilde{T}_I \oplus \tilde{U}$ and the projection depends only on torsion components of Σ, we can restrict to a map

$$\mathcal{P}|_{\tilde{T}_I} : \tilde{T}_I \to S \oplus J \tag{7.41}$$

If D is an $\tilde{H}_d$ compatible connection then, writing D as a G-compatible connection plus a connection piece Σ, we see that $(D \otimes_S \epsilon) \oplus (D \otimes_J \epsilon) \in S \oplus J$ is a linear function of the intrinsic torsion of the structure defined by ϵ.

In the cases considered, we will show that $\tilde{T}_I$ and $S \oplus J$ have the same decomposition into irreducible representations of G. We make the plausible assumption that the map $\mathcal{P}|_{\tilde{T}_I}$ is an isomorphism, and coupled with the previous statement this proves that the Killing spinor equations precisely set the intrinsic torsion to zero. Our assumption could easily be checked by explicit calculations coupled with Shur's lemma. It is also supported by the results in appendix D where it is seen that the projections appear to depend on all components of the torsion at $\tilde{H}_d$ level.

For $SU(3) \times SU(3)$ structures in type II compactifications without RR flux, we have the $SU(4) \times SU(4)$ representations

$$
\begin{aligned}
S \oplus J &= (\mathbf{4}+\bar{\mathbf{4}},\mathbf{1}) + (\mathbf{1},\mathbf{4}+\bar{\mathbf{4}}) + (\mathbf{6},\mathbf{4}+\bar{\mathbf{4}}) + (\mathbf{4}+\bar{\mathbf{4}},\mathbf{6}), \\
\tilde{T} &= (\mathbf{6},\mathbf{1}) + (\mathbf{1},\mathbf{6}) + (\mathbf{10}+\bar{\mathbf{10}},\mathbf{1}) + (\mathbf{1},\mathbf{10}+\bar{\mathbf{10}}) + (\mathbf{15},\mathbf{6}) + (\mathbf{6},\mathbf{15}),
\end{aligned}
\tag{7.42}
$$

and the $SU(3) \times SU(3)$ decompositions

$$
\begin{aligned}
S \oplus J &= 4 \times (\mathbf{1},\mathbf{1}) + 2 \times \big[(\mathbf{3}+\bar{\mathbf{3}},\mathbf{1}) + (\mathbf{1},\mathbf{3}+\bar{\mathbf{3}})\big] \\
&\quad + 2 \times (\mathbf{3},\mathbf{3}) + 2 \times \big[(\mathbf{3},\bar{\mathbf{3}}) + (\bar{\mathbf{3}},\mathbf{3})\big] + 2 \times (\bar{\mathbf{3}},\bar{\mathbf{3}}), \\
\tilde{T} &= \big(2 \times [\mathbf{1}] + 2 \times [\mathbf{3}+\bar{\mathbf{3}}] + [\mathbf{6}+\bar{\mathbf{6}}],\mathbf{1}\big) \\
&\quad + \big(\mathbf{1}, 2 \times [\mathbf{1}] + 2 \times [\mathbf{3}+\bar{\mathbf{3}}] + [\mathbf{6}+\bar{\mathbf{6}}]\big) \\
&\quad + (\mathbf{8},\mathbf{3}+\bar{\mathbf{3}}) + (\mathbf{3}+\bar{\mathbf{3}},\mathbf{8}) \\
\tilde{\Sigma}' &= E \otimes \mathrm{ad}(SU(3) \times SU(3)) \\
&= \big(\mathbf{3}+\bar{\mathbf{3}}+\mathbf{6}+\bar{\mathbf{6}}+\mathbf{15}+\bar{\mathbf{15}},\mathbf{1}\big) + \big(\mathbf{1},\mathbf{3}+\bar{\mathbf{3}}+\mathbf{6}+\bar{\mathbf{6}}+\mathbf{15}+\bar{\mathbf{15}}\big) \\
&\quad + \big(\mathbf{8},\mathbf{3}+\bar{\mathbf{3}}\big) + \big(\mathbf{3}+\bar{\mathbf{3}},\mathbf{8}\big).
\end{aligned}
\tag{7.43}
$$

After inspection of these we immediately write

$$
\tilde{T}' = \big([\mathbf{3}+\bar{\mathbf{3}}] + [\mathbf{6}+\bar{\mathbf{6}}],\mathbf{1}\big) + \big(\mathbf{1},[\mathbf{3}+\bar{\mathbf{3}}] + [\mathbf{6}+\bar{\mathbf{6}}]\big) + \big(\mathbf{8},\mathbf{3}+\bar{\mathbf{3}}\big) + \big(\mathbf{3}+\bar{\mathbf{3}},\mathbf{8}\big).
\tag{7.44}
$$

In fact, we cannot quite deduce this from what is written above. It is possible that the torsion map could vanish on some of the representations appearing in the $SU(3) \times SU(3)$ connection, which also appear in (7.44). To be fully watertight, one should check explicitly that this does not happen, which is easy to do, though we do not give details here. We then have

$$
\begin{aligned}
\tilde{T}_I &= 4 \times (\mathbf{1},\mathbf{1}) + 2 \times \big[(\mathbf{3}+\bar{\mathbf{3}},\mathbf{1}) + (\mathbf{1},\mathbf{3}+\bar{\mathbf{3}})\big] \\
&\quad + 2 \times (\mathbf{3},\mathbf{3}) + 2 \times \big[(\mathbf{3},\bar{\mathbf{3}}) + (\bar{\mathbf{3}},\mathbf{3})\big] + 2 \times (\bar{\mathbf{3}},\bar{\mathbf{3}}) \\
&= S \oplus J,
\end{aligned}
\tag{7.45}
$$

establishing that the desired relation.

We can go through the same procedure for the $SU(7)$ and $SU(6)$ structures in section 7.4.2. Here we have the $SU(8)$ decompositions

$$\begin{aligned} S \oplus J &= \mathbf{8} + \bar{\mathbf{8}} + \mathbf{56} + \overline{\mathbf{56}}, \\ \tilde{T} &= \mathbf{28} + \overline{\mathbf{28}} + \mathbf{36} + \overline{\mathbf{36}} + \mathbf{420} + \overline{\mathbf{420}}. \end{aligned} \tag{7.46}$$

The next step is to calculate the $SU(7)$ decompositions

$$\begin{aligned} S \oplus J &= 2 \times \mathbf{1} + \mathbf{7} + \bar{\mathbf{7}} + \mathbf{21} + \overline{\mathbf{21}} + \mathbf{35} + \overline{\mathbf{35}}, \\ \tilde{T} &= 2 \times \left[\mathbf{1} + \mathbf{7} + \bar{\mathbf{7}} + \mathbf{21} + \overline{\mathbf{21}}\right] + \left[\mathbf{28} + \overline{\mathbf{28}}\right] \\ &\quad + \left[\mathbf{35} + \overline{\mathbf{35}}\right] + \left[\mathbf{140} + \overline{\mathbf{140}}\right] + \left[\mathbf{224} + \overline{\mathbf{224}}\right], \\ \tilde{\Sigma}' &= E \otimes \mathrm{ad}(SU(7)) \\ &= \left[\mathbf{7} + \bar{\mathbf{7}}\right] + \left[\mathbf{21} + \overline{\mathbf{21}}\right] + \left[\mathbf{28} + \overline{\mathbf{28}}\right] + \left[\mathbf{140} + \overline{\mathbf{140}}\right] \\ &\quad + \left[\mathbf{189} + \overline{\mathbf{189}}\right] + \left[\mathbf{224} + \overline{\mathbf{224}}\right] + \left[\mathbf{735} + \overline{\mathbf{735}}\right]. \end{aligned} \tag{7.47}$$

from which we see that

$$\begin{aligned} \tilde{T}_I &= 2 \times \mathbf{1} + \mathbf{7} + \bar{\mathbf{7}} + \mathbf{21} + \overline{\mathbf{21}} + \mathbf{35} + \overline{\mathbf{35}} \\ &= S \oplus J. \end{aligned} \tag{7.48}$$

Again, technically we need to check that the torsion map is not zero on the relevant representations appearing in the $SU(7)$ connection in order to be sure of this.

Now let us examine $SU(6)$ case, where the relevant decompositions are

$$\begin{aligned} S \oplus J &= 4 \times \mathbf{1} + 2 \times \left[\mathbf{6} + \bar{\mathbf{6}} + \mathbf{15} + \overline{\mathbf{15}}\right] + \left[\mathbf{20} + \overline{\mathbf{20}}\right], \\ \tilde{T} &= 8 \times \mathbf{1} + 6 \times \left[\mathbf{6} + \bar{\mathbf{6}}\right] + 5 \times \left[\mathbf{15} + \overline{\mathbf{15}}\right] + \left[\mathbf{21} + \overline{\mathbf{21}}\right] \\ &\quad + 2 \times \left[\mathbf{20} + \overline{\mathbf{20}} + \mathbf{84} + \overline{\mathbf{84}}\right] + \left[\mathbf{35} + \overline{\mathbf{35}}\right] + \left[\mathbf{105} + \overline{\mathbf{105}}\right], \\ \tilde{\Sigma}' &= E \otimes \mathrm{ad}(SU(6)) \\ &= 2 \times \left[\mathbf{6} + \bar{\mathbf{6}} + \mathbf{84} + \overline{\mathbf{84}} + \mathbf{120} + \overline{\mathbf{120}}\right] \\ &\quad + \left[\mathbf{15} + \overline{\mathbf{15}}\right] + \left[\mathbf{21} + \overline{\mathbf{21}}\right] + \left[\mathbf{35} + \overline{\mathbf{35}}\right] \\ &\quad + \left[\mathbf{105} + \overline{\mathbf{105}}\right] + \left[\mathbf{384} + \overline{\mathbf{384}}\right], \end{aligned} \tag{7.49}$$

so, modulo the same checks as in the previous cases, we can write

$$\begin{aligned}\tilde{T}_I &= 8\times\mathbf{1}+4\times\big[\mathbf{6}+\bar{\mathbf{6}}+\mathbf{15}+\bar{\mathbf{15}}\big]+2\times\big[\mathbf{20}+\bar{\mathbf{20}}\big]\\ &= 2\times\big(S\oplus J\big).\end{aligned} \tag{7.50}$$

Thus, again we see that the Killing spinor equations precisely set to zero the irreducible representations corresponding to the intrinsic torsion.

8. Conclusion and Outlook

In this thesis, we have seen that generalised geometry is able to provide an extremely neat, geometrical formulation of the supergravities we have studied. The "geometrised" bosonic degrees of freedom are packaged as a generalised metric, which is equivalent to a G-structure on the generalised tangent bundle. The bosonic action and equations of motion are then given by curvatures of a torsion-free connection compatible with this structure. This is the exact analogue of Einstein gravity in the formulation of chapter 2. This formalism also realises the hidden symmetries of supergravity, so we can claim to have finally answered the question in [55] concerning the geometry underlying them. However, as mentioned previously, the only true symmetries are the geometric subgroup and the local group, which correspond to the diffeomorphism and local Lorentz symmetries of general relativity. The $E_{d(d)} \times \mathbb{R}^+$ group is only a symmetry in the sense that it relates different frames in the generalised frame bundle, in much the same way that tensors can be evaluated in any (not necessarily coordinate induced) frame in general relativity.

Our formalism has very naturally included the fermions and supersymmetry. The latter should come as a surprise, as we did not put this in by hand. In fact the torsion-free generalised connection D has precisely the properties necessary for the supersymmetry algebra to close. The commutator of two supersymmetries acting on the generalised metric is given by a Dorfman derivative with the connection D inserted. However, the torsion-free property is exactly the condition for this to be equal to the regular Dorfman derivative, which gives the infinitesimal action of the bosonic symmetries. Furthermore, the generalised curvatures we have constructed can be thought of as coming from the closure of the supersymmetry algebra on the fermionic equations of motion. These curvatures are unique and tensorial only if one takes a torsion-free connection. Thus there appears to be a strong relation between the vanishing of the generalised torsion and the closure of the supersymmetry algebra.

We should discuss the connections to other works noted in the introduction. Siegel [61], in particular, considered connections very similar to those of chapter 3. He proposed a separate conventional $GL(d, \mathbb{R})$ connection for each of the left- and right-moving sectors of the string, in contrast to our $O(p, q) \times O(q, p)$ connection. However, he goes on to impose compatibility with the $O(d, d)$ metric and the volume measure $\Phi = \sqrt{g}e^{-2\phi}$, which in fact imposes compatibility with $O(p, q) \times O(q, p)$ (the common subgroup of $GL(d, \mathbb{R}) \times GL(d, \mathbb{R})$ and $O(d, d) \times \mathbb{R}^+$). Therefore, Siegel's results for the generalised Ricci curvature and scalar curvature are in fact identical to ours (and also those in [116, 117]). Siegel also constructs a Riemann-like tensor, but has to add extra terms involving the connection components by hand in order to recover a tensor (again this is repeated in [116, 117]). This reflects that our construction (3.70) does not result in a tensor.

Another point we wish to highlight, is that the strong constraint imposed in "double field theory"-like geometries [61, 109, 112, 114, 115, 116, 117] always locally reduces the dependence of fields to the coordinates of a regular manifold. Thus, once the strong constraint is imposed, these geometries are locally equivalent to the corresponding generalised geometries. The results given here thus carry over directly to these setups, for example, providing the main results of [130, 131].

We also believe that the "section conditions" provided by the projection to the bundle N, as described in section 5.1.4, will provide the appropriate generalisation of the strong constraint for the exceptional geometries, though this has only explicitly been shown for $d = 4$ [115]. It guarantees the Leibniz identity for the Dorfman derivative, implying the closure of the algebra, and the condition $U \otimes_N V = 0$ forces linearity of the curvature (5.39), suggesting that it reduces it to an ordinary curvature. The $O(d, d)$ section condition has an interpretation as the level matching condition of the string [109]. It would be interesting to investigate whether some corresponding interpretation could be found for the M theory cases.

An obvious question to ask for $E_{d(d)} \times \mathbb{R}^+$ geometries is whether they can be extended to $d > 7$. Statements about Kac-Moody symmetries for $d > 8$ are mostly in the realm of speculation, but the construction of eleven-dimensional supergravity with $E_{8(8)}$ and $SO(16)$ appearing is well-understood [56]. There has also been success in considering these cases in the embedding tensor formalism (see [47] and references therein). One might therefore expect that it should be easy to extend our construction to an $E_{8(8)} \times \mathbb{R}^+$ generalised geometry. Indeed, in appendix D, we demonstrate that the algebraic properties of the torsion and projections go through exactly as one would hope. However, there is clear indication that the construction cannot work in exactly the same way for $d = 8$. The most immediate barrier is the absence of the Dorfman derivative in this case: the expression $\partial_V V' - (\partial \otimes_{\mathrm{ad}} V) \cdot V'$ is not diffeomorphism covariant. This is because there is no natural diffeomorphism covariant mapping

$$\partial : T^* \otimes \Lambda^7 T^* \to T^* \otimes \Lambda^8 T^* \tag{8.1}$$

constructed using only partial derivatives, which would be required in this definition. This would seem to be related to the lack of a non-linear theory of dual gravity, as this charge should be associated to the symmetries of the dual graviton [71].

As there is currently no known resolution of this issue, we do not dwell on the point. However, we do briefly note that in most approaches to the local symmetry in higher dimensions, the action of the three-form piece (the b term in (C.21)) generates the entire algebra by multiple commutators. Therefore, whatever expressions one may eventually be able to write down in higher dimensions, our formulae in $SO(d)$ representations look likely to provide at least the first few terms. The fact that several expressions in appendix C.2 agree precisely with the low-level decompositions of E_{10} is a consequence of this statement.

One might also wonder whether integrability of the generalised metric structure (which is guaranteed in the cases examined) could impose differential constraints on the fields in other possible cases. This is one idea of how the seemingly infinite number of fields present in Kac-Moody constructions could be truncated to the finite number of degrees of freedom in supergravity, within the framework of generalised geometry. This would fit well with the recent paper [165], which claims that all but a finite number of these fields are auxiliary.

A different line of thought would be to explore the possibility of constructing generalised geometries in this way for other supergravities. Some work has been done for the case of $N = 1$ $d = 4$ supergravity in the formulation of [166]. It appears that similar statements may be possible, though no final conclusion has yet been reached. It is noteworthy that we have an abstract prescription for the extraction of supergravity equations from generalised geometry, and one might hope that this was applicable in other cases which might exist.

Clearly the most concrete application of the technology is the description of supersymmetric backgrounds given in section 7. The unification of backgrounds with different (possibly interpolating) structure groups at each level of supersymmetry, and the equivalence of integrability and Killing spinor equations is no doubt a very pleasing result. One would hope that this could help in the ultimate goal of classification of all supersymmetric backgrounds, though obviously this could only provide a first step. However, special holonomy of a Riemannian manifold proved to be a very strong constraint, so one might hope that its generalisation would too. At the very least, one could hope to find new solutions similarly to those resulting from conventional G-structure analysis in e.g. [167]. It may also be interesting to understand the integrable $G \subset SU(8)$ structures for $d = 7$ in terms of extensions of the generalised complex structures and pure spinors used in [88] (much of the ground work for this has already been done in [102, 103]).

A less encouraging result arises from considering more supersymmetries. By the same calculations, one can see that our algebraic proof no longer holds for three or more Killing spinors, as the intrinsic torsion becomes larger than the representation content of the Killing spinor equations. For example, the existence of a torsion-free $SU(5) \subset SU(8)$ structure would appear to impose further constraints on the three Killing spinors and the fluxes. It would be easy to check this condition for known solutions, for example the pure (non-compact) M5 brane background, and it may hold in general, though this would be future work.

Probably the most interesting question comes back to possible links with doubled geometry and non-geometric backgrounds. Our formalism has discussed only supergravity, and makes no direct claims about dualities or non-geometry. However, unlike doubled geometry it is applicable to general manifolds, and features of supergravity can sometimes give us hints of stringy physics. Some recent papers [132, 133] have considered the evaluation of actions in different generalised frames. Roughly, they choose to use the β-transformation in $O(d, d)$ to construct their frame, instead of taking the globally defined B-transformed frame as we do. The evaluation of the curvature in this new frame gives a Lagrangian in which the so-called "non-geometric fluxes" appear. The frame is not globally defined on an ordinary manifold, but maybe it is somehow globally defined on a non-geometric space. Other intriguing works [168] have observed links between non-geometric backgrounds and non-commutative spaces, and the connection to generalised geometry and T-folds is discussed in [169]. It would be very interesting to see if there exists a generalisation of our results which moves away from conventional manifolds, but retains the same notions of connections and curvatures, in such a way as to include these exotic geometries.

A. General Conventions

Euclidean signature conventions in d dimensions

The d dimensional metric is positive definite. We use the indices $m, n, p, \ldots$ as the coordinate indices and $a, b, c \ldots$ for the tangent space indices. We take symmetrisation of indices with weight one. Our conventions for forms are

$$\begin{aligned}
\omega_{(k)} &= \tfrac{1}{k!}\omega_{m_1\ldots m_k}\mathrm{d}x^{m_1}\wedge\cdots\wedge\mathrm{d}x^{m_k}, \\
\omega_{(k)}\wedge\eta_{(l)} &= \tfrac{1}{(k+l)!}\left(\tfrac{(k+l)!}{k!\,l!}\omega_{[m_1\ldots m_k}\eta_{m_{k+1}\ldots m_{k+l}]}\right)\mathrm{d}x^{m_1}\wedge\cdots\wedge\mathrm{d}x^{m_{k+l}}, \\
*\omega_{(k)} &= \tfrac{1}{(d-k)!}\left(\tfrac{1}{k!}\sqrt{|g|}\epsilon_{m_1\ldots m_{d-k}n_1\ldots n_k}\omega^{n_1\ldots n_k}\right)\mathrm{d}x^{m_1}\wedge\cdots\wedge\mathrm{d}x^{m_{d-k}}, \\
\omega^2_{(k)} &= \omega_{m_1\ldots m_k}\omega^{m_1\ldots m_k},
\end{aligned} \tag{A.1}$$

where $\epsilon_{1\ldots d} = \epsilon^{1\ldots d} = +1$. We also use the j notation from [101, 2]

$$j\omega_{(p+1)}\wedge\eta_{(7-p)} = \frac{7!}{p!(7-p)!}\omega_{m[m_1\ldots m_p}\eta_{m_{p+1}\ldots m_7]}\mathrm{d}x^m\otimes\mathrm{d}x^{m_1}\wedge\cdots\wedge\mathrm{d}x^{m_7}. \tag{A.2}$$

Let $\nabla_m v^n = \partial_m v^n + \omega_m{}^n{}_p v^p$ be a general connection on TM. The torsion $T \in \Gamma(TM\otimes\Lambda^2T^*M)$ of ∇ is defined by

$$T(v,w) = \nabla_v w - \nabla_w v - [v,w]\,. \tag{A.3}$$

or concretely, in coordinate indices,

$$T^m{}_{np} = \omega_n{}^m{}_p - \omega_p{}^m{}_n, \tag{A.4}$$

while, in a general basis where $\nabla_m v^a = \partial_m v^a + \omega_m{}^a{}_b v^b$, one has

$$T^a{}_{bc} = \omega_b{}^a{}_c - \omega_c{}^a{}_b + [\hat{e}_b, \hat{e}_c]^a\,. \tag{A.5}$$

The curvature of a connection ∇ is given by the Riemann tensor $\mathcal{R} \in \Gamma(\Lambda^2 T^*M \otimes TM \otimes T^*M)$, defined by

$$\begin{aligned}\mathcal{R}(u,v)w &= [\nabla_u, \nabla_v]w - \nabla_{[u,v]}w, \\ \mathcal{R}_{mn}{}^{p}{}_{q}v^q &= [\nabla_m, \nabla_n]v^p - T^q{}_{mn}\nabla_q v^p.\end{aligned} \tag{A.6}$$

The Ricci tensor is the trace of the Riemann curvature

$$\mathcal{R}_{mn} = \mathcal{R}_{pm}{}^{p}{}_{n}. \tag{A.7}$$

If the manifold admits a metric g then the Ricci scalar is defined by

$$\mathcal{R} = g^{mn}\mathcal{R}_{mn}. \tag{A.8}$$

Lorentzian signature conventions in 10 dimensions

Our conventions largely follow [141] but we include a list for completeness. The only difference which is not purely notational is that we take the opposite sign for the Riemann tensor. The metric has the mostly plus signature $(- + + \cdots +)$. We use the indices $\mu, \nu, \lambda \ldots$ as the spacetime coordinate indices and $a, b, c \ldots$ for the tangent space indices. We take symmetrisation of indices with weight one. Our conventions for forms are

$$\begin{aligned}\omega_{(k)} &= \tfrac{1}{k!}\omega_{\mu_1\ldots\mu_k}\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_k}, \\ \omega_{(k)} \wedge \eta_{(l)} &= \tfrac{1}{(k+l)!}\left(\tfrac{(k+l)!}{k!\,l!}\omega_{[\mu_1\ldots\mu_k}\eta_{\mu_{k+1}\ldots\mu_{k+l}]}\right)\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_{k+l}}, \\ *\omega_{(k)} &= \tfrac{1}{(10-k)!}\left(\tfrac{1}{k!}\sqrt{-g}\,\epsilon_{\mu_1\ldots\mu_{10-k}\nu_1\ldots\nu_k}\omega^{\nu_1\ldots\nu_k}\right)\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_{10-k}}, \\ \omega^2_{(k)} &= \omega_{\mu_1\ldots\mu_k}\omega^{\mu_1\ldots\mu_k},\end{aligned} \tag{A.9}$$

where $\epsilon_{01\ldots9} = -\epsilon^{01\ldots9} = +1$. The formulae for connections, torsion and Riemann tensor are the same as for Euclidean signature in the previous section.

Lorentzian signature conventions in 11 dimensions

We follow the conventions of [146] precisely. This differs from the conventions used in ten dimensions in that here the square of a k-form is defined as $\omega^2_{(k)} = \frac{1}{k!}\omega_{\mu_1\ldots\mu_k}\omega^{\mu_1\ldots\mu_k}$.

B. Clifford Algebras

Conventions for all Clifford algebras

The following conventions are applied to all clifford algebras used in this thesis. Here the indices are taken as $m, n, p, \ldots$. but are intended to be replaced by any other set of indices as necessary.

The gamma matrices satisfy

$$\{\gamma^m, \gamma^n\} = 2g^{mn}, \qquad \gamma^{m_1 \ldots m_k} = \gamma^{[m_1} \ldots \gamma^{m_k]}. \tag{B.1}$$

The top gamma is defined as

$$\gamma^{(d)} = \gamma^0 \gamma^1 \ldots \gamma^d = \tfrac{1}{d!} \epsilon_{m_1 \ldots m_d} \gamma^{m_1 \ldots m_d}. \tag{B.2}$$

We use Dirac slash notation with weight one so that for $\omega \in \Gamma(\Lambda^k T^* M)$

$$\not{\omega} = \tfrac{1}{k!} \omega_{m_1 \ldots m_k} \gamma^{m_1 \ldots m_k}. \tag{B.3}$$

Where needed we will introduce $SU(2)$ indices $A, B, \ldots = 1, 2$ for symplectic Majorana spinors. The convention for raising and lowering these indices is taken as

$$\chi_A = \epsilon_{AB} \chi^B \qquad \chi^A = \epsilon^{AB} \chi_B. \tag{B.4}$$

The symplectic Majorana condition will always be taken as one of

$$\eta^A = \epsilon^{AB} (D \eta^B)^* \qquad \text{or} \qquad \eta^A = \epsilon^{AB} (\tilde{D} \eta^B)^*, \tag{B.5}$$

where D or $\tilde{D}$ is the chosen complex-conjugation intertwiner for the Clifford algebra in question.

Cliff$(9,1;\mathbb{R})$

Our conventions match [141]. We use the anti-symmetric transpose intertwiner

$$\tilde{C}\gamma^{\mu}\tilde{C}^{-1} = -(\gamma^{\mu})^{T}, \qquad \tilde{C}^{T} = -\tilde{C}, \tag{B.6}$$

to define the Majorana conjugate as $\bar{\epsilon} = \epsilon^{T}C$. This leads to the formulae

$$\begin{aligned} \tilde{C}\gamma^{\mu_1\dots\mu_k}\tilde{C}^{-1} &= (-)^{[(k+1)/2]}(\gamma^{\mu_1\dots\mu_k})^{T}, \\ \bar{\epsilon}\gamma^{\mu_1\dots\mu_k}\chi &= (-)^{[(k+1)/2]}\bar{\chi}\gamma^{\mu_1\dots\mu_k}\epsilon, \end{aligned} \tag{B.7}$$

where the spinors ϵ and χ are anti-commuting. We have

$$\gamma_{\mu_1\dots\mu_k}\gamma^{(10)} = (-)^{[k/2]}\tfrac{1}{(10-k)!}\sqrt{-g}\epsilon_{\mu_1\dots\mu_k\nu_1\dots\nu_{10-k}}\gamma^{\nu_1\dots\nu_{10-k}}, \tag{B.8}$$

which is also commonly written as

$$\gamma^{(k)}\gamma^{(10)} = (-)^{[k/2]} * \gamma^{(10-k)}. \tag{B.9}$$

We use Dirac slash notation with weight one so that for $\Psi \in \Gamma(\Lambda^{\bullet}T^{*}M)$

$$\not{\Psi} = \sum_{k} \tfrac{1}{k!}\Psi_{\mu_1\dots\mu_k}\gamma^{\mu_1\dots\mu_k}. \tag{B.10}$$

Cliff$(10,1;\mathbb{R})$

We use the transpose intertwiner $\tilde{C} = -\tilde{C}^{T}$ and a complex conjugation intertwiner D with $D^{*}D = 1$ and

$$\tilde{C}\Gamma^{M}\tilde{C}^{-1} = -(\Gamma^{M})^{T}, \qquad D\Gamma^{M}D^{-1} = (\Gamma^{M})^{*}. \tag{B.11}$$

The Majorana condition can be written as

$$\varepsilon = (D\varepsilon)^{*}, \tag{B.12}$$

and the Majorana conjugate is defined by

$$\bar{\varepsilon} = \varepsilon^{T}\tilde{C}. \tag{B.13}$$

This then satisfies

$$\overline{\Gamma^{M_1 \dots M_k} \varepsilon} = (-1)^{\lfloor \frac{k+1}{2} \rfloor} \bar{\varepsilon} \Gamma^{M_1 \dots M_k} \tag{B.14}$$

Following the conventions of [146] we take the representation with

$$\Gamma^{(11)} = \Gamma^0 \Gamma^1 \dots \Gamma^{10} = -1. \tag{B.15}$$

C. Details of $E_{d(d)} \times \mathbb{R}^+$ and H_d

C.1. $E_{d(d)} \times \mathbb{R}^+$ and $GL(d, \mathbb{R})$

C.1.1. Construction of $E_{d(d)} \times \mathbb{R}^+$ from $GL(d, \mathbb{R})$

In this section we give an explicit construction of $E_{d(d)} \times \mathbb{R}^+$ for $d \leq 7$ based on the $GL(d, \mathbb{R})$ subgroup.

We start with the Lie algebra. If $GL(d, \mathbb{R})$ acts linearly on the d-dimensional vector space F, consider first the space

$$W_1 = F \oplus \Lambda^2 F^* \oplus \Lambda^5 F^* \oplus \big(F^* \otimes \Lambda^7 F^*\big) . \tag{C.1}$$

We can write an element of $V \in W_1$ as

$$V = v + \omega + \sigma + \tau, \tag{C.2}$$

where $x \in F$ etc. If we write the index a for the fundamental $GL(d, \mathbb{R})$ representation note that τ has the index structure $\tau_{a,b_1 \ldots b_7}$, where a labels the F^* factor and $b_1 \ldots b_7$ the $\Lambda^7 F^*$ factor.

To define the Lie algebra we introduce

$$W_2 = \mathbb{R} \oplus (F \otimes F^*) \oplus \Lambda^3 F^* \oplus \Lambda^6 F^* \oplus \Lambda^3 F \oplus \Lambda^6 F, \tag{C.3}$$

with elements $R \in W_2$

$$R = c + r + a + \tilde{a} + \alpha + \tilde{\alpha}, \tag{C.4}$$

with $c \in \mathbb{R}$, $r \in F \otimes F^*$ etc. The Lie algebra of $E_{d(d)} \times \mathbb{R}^+$ can be defined by an action of $R \in W_2$ on $V \in W_1$ as follows. We take

$$\begin{aligned} R \cdot v &= cv + r \cdot v + \alpha \lrcorner \omega - \tilde{\alpha} \lrcorner \sigma, \\ R \cdot \omega &= c\omega + r \cdot \omega + v \lrcorner a + \alpha \lrcorner \sigma + \tilde{\alpha} \lrcorner \tau, \\ R \cdot \sigma &= c\sigma + r \cdot \sigma + v \lrcorner \tilde{a} + a \wedge \omega + \alpha \lrcorner \tau, \\ R \cdot \tau &= c\tau + r \cdot \tau + ja \wedge \sigma - j\tilde{a} \wedge \omega. \end{aligned} \tag{C.5}$$

Our notation here is that $r \cdot v$, etc. are the usual action of $gl(d, \mathbb{R})$ on the relevant tensor. Thus

$$(r \cdot v)^a = r^a{}_b v^b, \qquad (r \cdot \omega)_{ab} = -r^c{}_a \omega_{cb} - r^c{}_b \omega_{ac}, \qquad \text{etc.} \tag{C.6}$$

Note that the $E_{d(d)}$ sub-algebra is generated by setting $c = \frac{1}{(9-d)} r^a{}_a$.

For completeness, note that the contraction of forms and polyvectors in our conventions, given $w \in \Lambda^p F$. $\lambda \in \Lambda^q F^*$ and $\tau \in F^* \otimes \Lambda^7 F^*$, are given by

$$\begin{aligned} (w \lrcorner \lambda)_{a_1 \dots a_{q-p}} &:= \frac{1}{p!} w^{c_1 \dots c_p} \lambda_{c_1 \dots c_p a_1 \dots a_{q-p}} && \text{if } p \leq q, \\ (w \lrcorner \lambda)^{a_1 \dots a_{p-q}} &:= \frac{1}{q!} w^{a_1 \dots a_{p-q} c_1 \dots c_q} \lambda_{c_1 \dots c_q} && \text{if } p \geq q, \\ (w \lrcorner \tau)_{a_1 \dots a_{8-p}} &:= \frac{1}{(p-1)!} w^{c_1 \dots c_p} \tau_{c_1, c_2 \dots c_p a_1 \dots a_{8-p}}, \end{aligned} \tag{C.7}$$

while for $\lambda \in \Lambda^{p+1} F^*$ and $\mu \in \Lambda^{7-p} F^*$ we define $j\lambda \wedge \mu \in F^* \otimes \Lambda^7 F^*$ as

$$(j\lambda \wedge \mu)_{a, a_1 \dots a_7} := \frac{7!}{p!(7-p)!} \lambda_{a[a_1 \dots a_p} \mu_{a_{p+1} \dots a_7]}. \tag{C.8}$$

The $E_{d(d)} \times \mathbb{R}^+$ Lie group can then be constructed starting with $GL(d, \mathbb{R})$ and using the exponentiated action of a, $\tilde{a}$, α and $\tilde{\alpha}$. The $GL(d, \mathbb{R})$ action by an element m is standard so

$$(m \cdot v)^a = m^a{}_b v^b, \qquad (m \cdot \omega)_{ab} = (m^{-1})^c{}_a (m^{-1})^d{}_b \omega_{cd}, \qquad \text{etc.} \tag{C.9}$$

The action of a and $\tilde{a}$ form a nilpotent subgroup of nilpotency class two.

One has

$$
\begin{aligned}
\mathrm{e}^{a+\tilde{a}}V = v &+ (\omega + i_v a) \\
&+ \big(\sigma + a\wedge\omega + \tfrac{1}{2}a\wedge i_v a + i_v\tilde{a}\big) \\
&+ \big(\tau + ja\wedge\sigma - j\tilde{a}\wedge\omega + \tfrac{1}{2}ja\wedge a\wedge\omega \\
&\qquad + \tfrac{1}{2}ja\wedge i_v\tilde{a} - \tfrac{1}{2}j\tilde{a}\wedge i_v a + \tfrac{1}{6}ja\wedge a\wedge i_v a\big),
\end{aligned}
\tag{C.10}
$$

with no terms higher than cubic in the expansion. The action of α and $\tilde{\alpha}$ form a similar nilpotent subgroup of nilpotency class two with

$$
\begin{aligned}
\mathrm{e}^{\alpha+\tilde{\alpha}}V = &\big(v + \alpha\lrcorner\omega - \tilde{\alpha}\lrcorner\sigma + \tfrac{1}{2}\alpha\lrcorner\alpha\lrcorner\sigma \\
&\qquad + \tfrac{1}{2}\alpha\lrcorner\tilde{\alpha}\lrcorner\tau + \tfrac{1}{2}\tilde{\alpha}\lrcorner\alpha\lrcorner\tau + \tfrac{1}{6}\alpha\lrcorner\alpha\lrcorner\alpha\lrcorner\tau\big) \\
&+ (\omega + \alpha\lrcorner\sigma + \tilde{\alpha}\lrcorner\tau + \alpha\lrcorner\alpha\lrcorner\sigma) \\
&+ (\sigma + \alpha\lrcorner\tau) + \tau.
\end{aligned}
\tag{C.11}
$$

A general element of $E_{d(d)}\times\mathbb{R}^+$ then has the form

$$
M\cdot V = \mathrm{e}^{\lambda}\,\mathrm{e}^{\alpha+\tilde{\alpha}}\,\mathrm{e}^{a+\tilde{a}}\,m\cdot V, \tag{C.12}
$$

where e^{λ} with $\lambda\in\mathbb{R}$ is included to give a general $\mathbb{R}^+$ scaling.

C.1.2. Some tensor products

We can also define tensor products between representations in terms of the $GL(d,\mathbb{R})$ components. Given the dual space

$$
\begin{aligned}
W_1^* &= F^*\oplus\Lambda^2F\oplus\Lambda^5F\oplus\left(F\otimes\Lambda^7F\right), \\
Z &= \zeta + u + s + t \in W_1^*,
\end{aligned}
\tag{C.13}
$$

the map into the adjoint $W_1 \otimes W_1^* \to W_2$ is given by

$$\begin{aligned}
c &= -\tfrac{1}{3}u \lrcorner \omega - \tfrac{2}{3}s \lrcorner \sigma - t \lrcorner \tau, \\
r &= v \otimes \zeta - ju \lrcorner j\omega + \tfrac{1}{3}(u \lrcorner \omega)\mathbb{1} - js \lrcorner j\sigma + \tfrac{2}{3}(s \lrcorner \sigma)\mathbb{1} - jt \lrcorner j\tau, \\
a &= \zeta \wedge \omega + u \lrcorner \sigma + s \lrcorner \tau, \\
\alpha &= v \wedge u + s \lrcorner \omega + t \lrcorner \sigma, \\
\tilde{a} &= \zeta \wedge \sigma + u \lrcorner \tau, \\
\tilde{\alpha} &= -v \wedge s - t \lrcorner \omega,
\end{aligned} \tag{C.14}$$

where we are using the notation that, given $w \in \Lambda^p F$ and $\lambda \in \Lambda^p F^*$,

$$\begin{aligned}
(jw \lrcorner j\lambda)^a{}_b &:= \frac{1}{(p-1)!} w^{ac_1\dots c_{p-1}} \lambda_{bc_1\dots c_{p-1}}, \\
(jt \lrcorner j\tau)^a{}_b &:= \frac{1}{7!} t^{a,c_1\dots c_7} \tau_{b,c_1\dots c_7}, \\
(t \lrcorner \lambda)^{a_1\dots a_{8-p}} &:= \frac{1}{(p-1)!} t^{c_1,c_2\dots c_p a_1\dots a_{8-p}} \lambda_{c_1\dots c_p}, \\
(t \lrcorner \tau) &:= \frac{1}{7!} t^{a,b_1\dots b_7} \tau_{a,b_1\dots b_7}.
\end{aligned} \tag{C.15}$$

We can also consider the representations that appear in the bundle N as given in table 5.2. We consider

$$\begin{aligned}
W_3 &= F^* \oplus \Lambda^4 F^* \oplus (F^* \otimes \Lambda^6 F^*) \oplus (\Lambda^3 F^* \otimes \Lambda^7 F^*) \oplus (\Lambda^6 F^* \otimes \Lambda^7 F^*), \\
Y &= \lambda + \kappa + \mu + \nu + \pi.
\end{aligned} \tag{C.16}$$

The symmetric map $W_1 \otimes W_1 \to W_3$ is given by

$$\begin{aligned}
\lambda &= v \lrcorner \omega' + v' \lrcorner \omega, \\
\kappa &= v \lrcorner \sigma' + v' \lrcorner \sigma - \omega \wedge \omega', \\
\mu &= \big(j\omega \wedge \sigma' + j\omega' \wedge \sigma\big) - \tfrac{1}{4}\big(\sigma \wedge \omega' + \sigma' \wedge \omega\big) \\
&\qquad + (v \lrcorner j\tau) + (v \lrcorner j\tau') - \tfrac{1}{4}(v \lrcorner \tau' + v' \lrcorner \tau), \\
\nu &= j^3\omega \wedge \tau' + j^3\omega' \wedge \tau - j^3\sigma \wedge \sigma', \\
\pi &= j^6\sigma \wedge \tau' + j^6\sigma' \wedge \tau,
\end{aligned} \tag{C.17}$$

where, for $\omega \in \Lambda^p F^*$, σ, $\sigma' \in \Lambda^5 F^*$ and $\tau \in F^* \otimes \Lambda^7 F^*$,

$$\begin{aligned}
\left(j^{p+1}\omega \wedge \tau\right)_{a_1\dots a_{p+1},b_1\dots b_7} &:= (p+1)\omega_{[a_1\dots}\tau_{a_{p+1}],b_1\dots b_7}, \\
\left(j^3\sigma \wedge \sigma'\right)_{a_1\dots a_3,b_1\dots b_7} &:= \tfrac{7!}{5!\cdot 2!}\sigma_{a_1\dots a_3[b_1b_2}\sigma'_{\dots b_7]}, \\
(v \lrcorner j\tau)_{mn_1\dots n_6} &:= v^p\tau_{m,pn_1\dots n_6}.
\end{aligned} \tag{C.18}$$

C.2. H_d and $O(d)$

C.2.1. Construction of H_d from $SO(d)$

Given a positive definite metric g_{ab} on F, which for convenience we take to be in standard form δ_{ab}, we can define a metric on W_1 by

$$G(V,V) = |v|^2 + |\omega|^2 + |\sigma|^2 + |\tau|^2, \tag{C.19}$$

where $|v|^2 = v_a v^a$, $|\omega|^2 = \frac{1}{2!}\omega_{ab}\omega^{ab}$, $|\sigma|^2 = \frac{1}{5!}\sigma_{a_1\dots a_5}\sigma^{a_1\dots a_5}$ and $=|\tau|^2 = \frac{1}{7!}\tau_{a,a_1\dots a_7}\tau^{a,a_1\dots a_7}$. The subgroup of $E_{d(d)} \times \mathbb{R}^+$ that leaves this metric invariant is H_d, the maximal compact subgroup of $E_{d(d)}$ (see table 5.3). The corresponding Lie algebra is parametrised by

$$N = n + b + \tilde{b} \in \Lambda^2 F^* \oplus \Lambda^3 F^* \oplus \Lambda^6 F^*, \tag{C.20}$$

and embeds in W_2 as

$$\begin{aligned}
c &= 0, \\
m_{ab} &= n_{ab}, \\
a_{abc} = -\alpha_{abc} &= b_{abc}, \\
\tilde{a}_{a_1\dots a_6} = \tilde{\alpha}_{a_1\dots a_6} &= \tilde{b}_{a_1\dots a_6},
\end{aligned} \tag{C.21}$$

where indices are lowered with the metric g. Note that n_{ab} generates the $O(d) \subset GL(d,\mathbb{R})$ subgroup that preserves g. Concretely a general element can be written as

$$H \cdot V = \mathrm{e}^{\alpha+\tilde{\alpha}}\,\mathrm{e}^{a+\tilde{a}}\,h \cdot V, \tag{C.22}$$

where $h \in O(d)$ and a and α and $\tilde{a}$ and $\tilde{\alpha}$ are related as in (C.20).

An important representation of H_d is the complement of the adjoint of H_d in $E_{d(d)} \times \mathbb{R}^+$, which we denote as $H^\perp$. An element of $H^\perp$ be represented as

$$Q = c + h + q + \tilde{q} \in \mathbb{R} \oplus S^2F^* \oplus \Lambda^3F^* \oplus \Lambda^6F^*, \tag{C.23}$$

and it embeds in W_2 as

$$\begin{aligned} c &= c, \\ m_{ab} &= h_{ab}, \\ a_{abc} = \alpha_{abc} &= q_{abc}, \\ \tilde{a}_{a_1\dots a_6} = -\tilde{\alpha}_{a_1\dots a_6} &= \tilde{q}_{a_1\dots a_6}. \end{aligned} \tag{C.24}$$

The action of H_d on this representation is given by $E_{d(d)} \times \mathbb{R}^+$ Lie algebra. Writing $Q' = [N, Q]$ we have

$$\begin{aligned} c' &= -\tfrac{2}{3} b \lrcorner q - \tfrac{4}{3}\tilde{b} \lrcorner \tilde{q}, \\ h'_{ab} &= (n \cdot h)_{ab} - \tfrac{2}{2!} b_{(a}{}^{cd} q_{b)cd} - \tfrac{2}{5!}\tilde{b}_{(a}{}^{c_1\dots c_5}\tilde{q}_{b)c_1\dots c_5} + (\tfrac{2}{3} b \lrcorner q + \tfrac{4}{3}\tilde{b} \lrcorner \tilde{q})\delta_{ab}, \\ q' &= (n \cdot q) - (h \cdot b) + (b \lrcorner \tilde{q}) + (q \lrcorner \tilde{b}), \\ \tilde{q}' &= (n \cdot \tilde{q}) - (h \cdot \tilde{b}) - (b \wedge q), \end{aligned} \tag{C.25}$$

where we are using the $GL(d, \mathbb{R})$ adjoint action of $h_{(ab)}$ on Λ^3F^* and Λ^6F^*. The H_d invariant scalar part of Q is given by $c - \frac{1}{9-d} h^a{}_a$.

Finally we note that the double cover $\tilde{H}_d$ of H_d has a realisation in terms of the Clifford algebra $\mathrm{Cliff}(d; \mathbb{R})$. Consider the gamma matrices γ^a satisfying $\{\gamma^a, \gamma^b\} = 2g^{ab}$. The H_d Lie algebra can be realised on $\mathrm{Cliff}(d; \mathbb{R})$ spinors in two ways

$$N^{\pm} = \tfrac{1}{2}\left(\tfrac{1}{2!} n_{ab}\gamma^{ab} \pm \tfrac{1}{3!} b_{abc}\gamma^{abc} - \tfrac{1}{6!}\tilde{b}_{a_1\dots a_6}\gamma^{a_1\dots a_6}\right). \tag{C.26}$$

Again n_{ab} generates the $Spin(d)$ subgroup of $\tilde{H}_d$. The two representations are mapped into each other by $\gamma^a \to -\gamma^a$. As such, they are inequivalent in odd dimensions, while in even dimensions they are related by

$$\gamma^{(d)} N^+ \gamma^{(d)-1} = N^-. \tag{C.27}$$

We denote as $Z_1^{\pm}$ the spinor representation of H_d transforming under $N^{\pm}$. One also finds two different actions on the vector-spinor representations

$\varphi_a^\pm \in Z_2^\pm$ with[1]

$$N \cdot \varphi_a^\pm = N^\pm \varphi_a^\pm - r^b{}_a \varphi_b^\pm \mp \tfrac{2}{3} b_a{}^b{}_c \gamma^c \varphi_b^\pm \mp \tfrac{1}{3}\tfrac{1}{2!} b^b{}_{cd} \gamma_a{}^{cd} \varphi_b^\pm \\ + \tfrac{1}{3}\tfrac{1}{4!} \tilde{b}_a{}^b{}_{c_1\dots c_4} \gamma^{c_1\dots c_4} \varphi_b^\pm + \tfrac{2}{3}\tfrac{1}{5!} \tilde{b}^b{}_{c_1\dots c_5} \gamma_a{}^{c_1\dots c_5} \varphi_b^\pm . \tag{C.28}$$

We will also need the projections $H^\perp \otimes Z_1^\pm \to Z_2^\mp$, which, for $\chi^\pm \in Z_1^\pm$, is given by

$$(Q \otimes_{Z_2^\mp} \chi^\pm)_a = \tfrac{1}{2} h_{ab} \gamma^b \chi^\pm \mp \tfrac{1}{3}\tfrac{1}{2!} q_{abc} \gamma^{bc} \chi^\pm \pm \tfrac{1}{6}\tfrac{1}{3!} q_{c_1\dots c_3} \gamma_a{}^{c_1\dots c_3} \chi^\pm \\ + \tfrac{1}{6}\tfrac{1}{5!} \tilde{q}_{ab_1\dots b_5} \gamma^{b_1\dots b_5} \chi^\pm - \tfrac{1}{3}\tfrac{1}{6!} \tilde{q}_{c_1\dots c_6} \gamma_a{}^{c_1\dots c_6} \chi^\pm . \tag{C.29}$$

C.2.2. $\tilde{H}_d$ and Cliff$(10, 1; \mathbb{R})$

One can also find two embeddings of $\tilde{H}_d$ in Cliff$(10, 1; \mathbb{R})$ which are generated using the internal spacelike gamma matrices Γ^a for $a = 1, \dots, d$. Combined with the external spin generators $\Gamma^{\mu\nu}$, this gives us an action of $Spin(10 - d, 1) \times \tilde{H}_d$ on eleven-dimensional spinors. The adjoint of $\tilde{H}_d$ is embedded similarly as

$$\hat{N}^\pm = \tfrac{1}{2} \left(\tfrac{1}{2!} n_{ab} \Gamma^{ab} \pm \tfrac{1}{3!} b_{abc} \Gamma^{abc} - \tfrac{1}{6!} \tilde{b}_{a_1\dots a_6} \Gamma^{a_1\dots a_6} \right) . \tag{C.30}$$

Since the algebra of the $\{\Gamma^a\}$ is the same as Cliff$(d; \mathbb{R})$ all the equations of the previous section translate directly into this presentation of $\tilde{H}_d$. The advantage of the direct action on eleven-dimensional spinors is that it allows us to write $\tilde{H}_d$ covariant spinor bilinears in a dimension independent way.

Let $\hat{Z}_1^\pm$ be the eleven-dimensional spinors transforming under $\tilde{H}_d$ via $\hat{N}^\pm$. Let also $\hat{Z}_2^\pm$ be the representations with one eleven-dimensional spinor index and one internal vector index which transform as (C.28) (with Γ^a in place of γ^a). One can construct the singlet projections $\hat{Z}_2^\mp \otimes \hat{Z}_2^\pm \to \mathbf{1}$. For $\hat{\varphi}^\pm \in \hat{Z}_2^\pm$ these are given by[2]

$$\langle \hat{\varphi}^\mp, \hat{\varphi}^\pm \rangle = \bar{\hat{\varphi}}_a^\mp (\delta^{ab} + \tfrac{1}{9-d} \Gamma^a \Gamma^b) \hat{\varphi}_b^\pm . \tag{C.31}$$

The eleven-dimensional spinor conjugate provides the relevant inner prod-

[1] The formula given here matches those found in [76, 77] for levels 0, 1 and 2 of $K(E_{10})$. A similar formula also appears in the context of E_{11} in [74].
[2] Setting $d = 10$ in this reproduces the corresponding inner product in [76].

ucts $\hat{Z}_1^{\mp} \otimes \hat{Z}_1^{\pm} \to \mathbf{1}$ as

$$\langle \hat{\chi}^-, \hat{\chi}^+ \rangle = \bar{\hat{\chi}}^- \chi^+, \tag{C.32}$$

where $\hat{\chi}^{\pm} \in \hat{Z}_1^{\pm}$.

We now also give the decompositions of the H_d-covariant projections

$$\begin{aligned} &V \otimes_{\hat{Z}_1^{\mp}} \hat{\chi}^{\pm}, \qquad V \otimes_{\hat{Z}_2^{\pm}} \hat{\chi}^{\pm}, \\ &V \otimes_{\hat{Z}_1^{\pm}} \hat{\varphi}^{\pm}, \qquad V \otimes_{\hat{Z}_2^{\mp}} \hat{\varphi}^{\pm}, \end{aligned} \tag{C.33}$$

where $V \in W_1$, $\hat{\chi}^{\pm} \in \hat{Z}_1^{\pm}$ and $\hat{\varphi}_a^{\pm} \in \hat{Z}_2^{\pm}$. These are

$$(V \otimes_{\hat{Z}_1^{\mp}} \hat{\chi}^{\pm}) = \Big(\pm v^a \Gamma_a - \tfrac{1}{2!}\omega_{ab}\Gamma^{ab} \pm \tfrac{1}{5!}\sigma_{a_1 \dots a_5}\Gamma^{a_1 \dots a_5} - \tfrac{1}{6!}\tau^b{}_{ba_1 \dots a_6}\Gamma^{a_1 \dots a_6} \Big) \hat{\chi}^{\pm}, \tag{C.34}$$

$$\begin{aligned} (V \otimes_{\hat{Z}_2^{\pm}} \hat{\chi}^{\pm})_a = {} & v_a \hat{\chi}^{\pm} \pm \tfrac{2}{3}\Gamma^b \omega_{ab}\hat{\chi}^{\pm} \mp \tfrac{1}{3}\tfrac{1}{2!}\Gamma_a{}^{cd}\omega_{cd}\hat{\chi}^{\pm} - \tfrac{1}{3}\tfrac{1}{4!}\Gamma^{c_1 \dots c_4}\sigma_{ac_1 \dots c_4}\hat{\chi}^{\pm} \\ & + \tfrac{2}{3}\tfrac{1}{5!}\Gamma_a{}^{c_1 \dots c_5}\sigma_{c_1 \dots c_5}\hat{\chi}^{\pm} \pm \tfrac{1}{7!}\Gamma^{c_1 \dots c_7}\tau_{a, c_1 \dots c_7}\hat{\chi}^{\pm}, \end{aligned} \tag{C.35}$$

$$\begin{aligned} (V \otimes_{\hat{Z}_1^{\pm}} \hat{\varphi}^{\pm}) = {} & v^a \hat{\varphi}_a + \tfrac{1}{10-d} v_a \Gamma^{ab}\hat{\varphi}_b \pm \tfrac{1}{10-d}\tfrac{1}{2!}\omega_{bc}\Gamma^{abc}\hat{\varphi}_a^+ \pm \tfrac{8-d}{10-d}\omega^a{}_b \Gamma^b \hat{\varphi}_a^+ \\ & - \tfrac{1}{10-d}\tfrac{1}{5!}\sigma^{b_1 \dots b_5}\Gamma^a{}_{b_1 \dots b_5}\hat{\varphi}_a^+ - \tfrac{8-d}{10-d}\tfrac{1}{4!}\sigma^a{}_{b_1 \dots b_4}\Gamma^{b_1 \dots b_4}\hat{\varphi}_a^+ \\ & \mp \tfrac{1}{7!}\tau^a{}_{,b_1 \dots b_7}\Gamma^{b_1 \dots b_7}\hat{\varphi}_a^+ \mp \tfrac{1}{3}\tfrac{1}{5!}\tau^c{}_{,c}{}^a{}_{b_1 \dots b_5}\Gamma^{b_1 \dots b_5}\hat{\varphi}_a^+, \end{aligned} \tag{C.36}$$

$$\begin{aligned} (V \otimes_{\hat{Z}_2^{\mp}} \hat{\varphi}^{\pm})_a = {} & \pm v^c \Gamma_c \hat{\varphi}_a^{\pm} \pm \tfrac{2}{9-d}\Gamma^c v_a \hat{\varphi}_c^{\pm} - \tfrac{1}{2!}\omega_{cd}\Gamma^{cd}\hat{\varphi}_a^{\pm} + \tfrac{4}{3}\omega_a{}^b \hat{\varphi}_b^{\pm} \\ & - \tfrac{2}{3}\omega_{cd}\Gamma_a{}^c \hat{\varphi}^{\pm d} - \tfrac{4}{3}\tfrac{1}{9-d}\omega_{ab}\Gamma^b \Gamma^c \hat{\varphi}_c^{\pm} + \tfrac{2}{3}\tfrac{1}{9-d}\tfrac{1}{2!}\omega_{bc}\Gamma_a{}^{bc}\Gamma^d \hat{\varphi}_d^{\pm} \\ & \pm \tfrac{1}{5!}\sigma_{c_1 \dots c_5}\Gamma^{c_1 \dots c_5}\hat{\varphi}_a^{\pm} \mp \tfrac{2}{3}\tfrac{1}{3!}\sigma_a{}^b{}_{c_1 c_2 c_3}\Gamma^{c_1 c_2 c_3}\hat{\varphi}_b^{\pm} \mp \tfrac{4}{3}\tfrac{1}{4!}\sigma^b{}_{c_1 \dots c_4}\Gamma_a{}^{c_1 \dots c_4}\hat{\varphi}_b^{\pm} \\ & \mp \tfrac{2}{3}\tfrac{1}{9-d}\tfrac{1}{4!}\sigma_{ac_1 \dots c_4}\Gamma^{c_1 \dots c_4}\Gamma^d \hat{\varphi}_d^{\pm} \pm \tfrac{4}{3}\tfrac{1}{9-d}\tfrac{1}{5!}\sigma_{c_1 \dots c_5}\Gamma_a{}^{c_1 \dots c_5}\Gamma^d \hat{\varphi}_d^{\pm} \\ & + \tfrac{1}{7!}\tau_{c, d_1 \dots d_7}\Gamma^c \Gamma^{d_1 \dots d_7}\hat{\varphi}_a^{\pm} + \tfrac{1}{7!}\tau_{a, c_1 \dots c_7}\Gamma^{c_1 \dots c_7}\Gamma^d \hat{\varphi}_d^{\pm}. \end{aligned} \tag{C.37}$$

Another projection we will need is the projection $\hat{Z}_1^\pm \otimes \hat{Z}_2^\pm \to H^\perp$, which for $\hat{\chi}^\pm \in \hat{Z}_1^\pm$ and $\hat{\varphi}_a^\pm \in \hat{Z}_2^\pm$, is given by

$$\begin{aligned} c &= \tfrac{2}{9-d}\bar{\hat{\chi}}^\pm \Gamma^a \hat{\varphi}_a^\pm, \\ h_{ab} &= 2\bar{\hat{\chi}}^\pm \Gamma_{(a} \hat{\varphi}_{b)}^\pm \\ q_{abc} &= \mp 3\bar{\hat{\chi}}^\pm \Gamma_{[ab} \hat{\varphi}_{c]}^\pm, \\ \tilde{q}_{a_1 \dots a_6} &= -6\bar{\hat{\chi}}^\pm \Gamma_{[a_1 \dots a_5} \hat{\varphi}_{a_6]}^\pm, \end{aligned} \tag{C.38}$$

Note that the image of this projection does not include the $\tilde{H}_d$ scalar part of $H^\perp$, as $c - \frac{1}{9-d} h^a{}_a = 0$. We also define a projection $\hat{Z}_1^- \otimes \hat{Z}_1^+ \to H^\perp$ by

$$c = \tfrac{2}{9-d}\bar{\hat{\chi}}^- \hat{\chi}^+, \tag{C.39}$$

where $\hat{\chi}^\pm \in \hat{Z}_1^\pm$ and all other components of $H^\perp$ are set to zero. The image of this map is clearly the $\tilde{H}_d$ scalar part of $H^\perp$.

D. Group Theory Proof of Uniqueness of Operators

In this appendix we supply a group theoretical proof of the uniqueness of the operators (5.57). As in section 7.4 we need only consider linear algebra at a point in the manifold and so we do not distinguish between bundles and their fibres in this section. Since the proof we give here uses the particular details of the $\tilde{H}_d$ group in each dimension, we run through the dimensions in a case-by-case fashion. As we comment in the conclusion, this group theoretical structure would be the same for an $E_{8(8)} \times \mathbb{R}^+$ geometry in eight dimensions, so we include the details of the relevant representations here.

Of course, this is not the only way to prove this uniqueness. Indeed, one could demonstrate by explicit calculation that the relevant irreducible parts of the connection cancel looking at the $SO(d)$ decomposition of the generalised connection. This would be extremely long-winded. One could also do this using a decomposition under a larger subgroup. For example, in seven dimensions, it is lengthy but possible to do this looking at the $Spin(8)$ decomposition of the $SU(8)$ representations, though we do not give the details of the calculation here.

The main $E_{d(d)} \times \mathbb{R}^+$ generalised tensors we need are listed in table 5.2 and the representations appearing in the compatible connection under the $\tilde{H}_d$ subgroup are given in table 5.4. The representations for the spinor and gravitino are given in table 5.5. The general method to establish uniqueness is outlined in section 5.2.3, and here we run through this argument in each dimension.

4 dimensions

For the $Spin(5) \simeq Sp(4)$ case we have the representations

$$E \sim \mathbf{10} \qquad\qquad S \sim \mathbf{4} \qquad\qquad J \sim \mathbf{16}, \tag{D.1}$$

and the components of an $Sp(4)$ compatible connection lie in the space

$$E^* \times \mathrm{ad}(Sp(4)) \sim \mathbf{10} \times \mathbf{10} = \mathbf{1} + \mathbf{5} + \mathbf{10} + \mathbf{14} + \mathbf{35} + \mathbf{35'} \tag{D.2}$$

Here we have labeled the **35** to be that which has the index symmetries of the Young diagram $\yng(4)$ and the $\mathbf{35'}$ to have the symmetries $\yng(3,1)$.

For $\chi \in S$ we have

$$D \cdot \chi \in \mathbf{10} \times \mathbf{4} = \mathbf{4} + \mathbf{16} + \mathbf{20} \tag{D.3}$$

We are interested in the projection onto the S and J parts. The irreducible parts of the compatible connection which appear in these projections must therefore be those parts which can also be embedded in the tensor product $S \otimes S^*$ and $J \otimes S^*$ respectively. These tensor products decompose as

$$\begin{aligned} S \otimes S^* &= \mathbf{4} \times \mathbf{4} &&= \mathbf{1} + \mathbf{5} + \mathbf{10} \\ J \otimes S^* &= \mathbf{16} \times \mathbf{4} &&= \mathbf{5} + \mathbf{10} + \mathbf{14} + \mathbf{35'} \end{aligned} \tag{D.4}$$

Clearly, all of the representations appearing here are those in the torsion of the connection which has the decomposition

$$E^* \oplus K \sim \mathbf{1} + \mathbf{5} + \mathbf{10} + \mathbf{14} + \mathbf{35'} \tag{D.5}$$

and so the projection only depends on the torsion components of the compatible connection. If one considered the projection to the other possible representation, the **20**, one would examine

$$\mathbf{20} \otimes S^* = \mathbf{20} \times \mathbf{4} = \mathbf{10} + \mathbf{35'} + \mathbf{35} \tag{D.6}$$

which contains the non-torsion representation **35** of the connection. Therefore, this projection is not uniquely determined by the torsion of the connection. In this case, this is obvious since $D \cdot \chi$ does depend on all parts of the connection, so by process of elimination the final part must depend on the **35**.

Similarly $J \sim \mathbf{16}$ so for $\varphi \in J$

$$D \cdot \varphi \in \mathbf{10} \times \mathbf{16} = \mathbf{5} + 2 \times \mathbf{16} + \mathbf{20} + \mathbf{40} + \mathbf{64} \tag{D.7}$$

Looking to project onto the S and J parts we consider

$$\begin{aligned} S \otimes J^* &= \mathbf{4} \times \mathbf{16} = \mathbf{5} + \mathbf{10} + \mathbf{14} + \mathbf{35'} \\ J \otimes J^* &= \mathbf{16} \times \mathbf{16} \\ &= \mathbf{1} + \mathbf{5} + 2 \times \mathbf{10} + \mathbf{14} + \mathbf{35} + 2 \times \mathbf{35'} + \mathbf{30} + \mathbf{81} \end{aligned} \tag{D.8}$$

This case actually has a complication which does not appear in any of the remaining cases. The decomposition (D.7) contains two copies of J, while the tensor product $J \otimes J^*$ seems to contain the non-torsion **35** components of the connection. In this instance one therefore choses the unique linear combination of the projections onto the two J parts of $D \cdot \varphi$ such that this non-torsion component cancels. This also turns out to be the correct projection to recover the supergravity equations, and is the one we use in section 6.3.

5 dimensions

For the $H_5 \simeq Spin(5) \times Spin(5) \simeq Sp(4) \times Sp(4)$ case we have the representations

$$E \sim (\mathbf{4}, \mathbf{4}) \qquad S \sim (\mathbf{4}, \mathbf{1}) + (\mathbf{1}, \mathbf{4}) \qquad J \sim (\mathbf{4}, \mathbf{5}) + (\mathbf{5}, \mathbf{4}) \tag{D.9}$$

and the components of an $Sp(4) \times Sp(4)$ compatible connection lie in the space

$$\begin{aligned} E^* \times \mathrm{ad}(Sp(4) \times Sp(4)) &\sim (\mathbf{4}, \mathbf{4}) \times ((\mathbf{10}, \mathbf{1}) + (\mathbf{1}, \mathbf{10})) \\ &= (\mathbf{4}, \mathbf{4}) + (\mathbf{4}, \mathbf{4}) + (\mathbf{16}, \mathbf{4}) + (\mathbf{4}, \mathbf{16}) + (\mathbf{20}, \mathbf{4}) + (\mathbf{4}, \mathbf{20}) \end{aligned} \tag{D.10}$$

So for $\chi \in S$ we have

$$\begin{aligned} D \cdot \chi &\in (\mathbf{4}, \mathbf{4}) \times ((\mathbf{4}, \mathbf{1}) + (\mathbf{1}, \mathbf{4})) \\ &= (\mathbf{4}, \mathbf{1}) + (\mathbf{1}, \mathbf{4}) + (\mathbf{4}, \mathbf{5}) + (\mathbf{5}, \mathbf{4}) + (\mathbf{10}, \mathbf{4}) + (\mathbf{4}, \mathbf{10}) \end{aligned} \tag{D.11}$$

We are interested in the projections onto S and J, so again we look at the tensor products

$$\begin{aligned}
S \otimes S^* &= ((\mathbf{4},\mathbf{1}) + (\mathbf{1},\mathbf{4})) \times ((\mathbf{4},\mathbf{1}) + (\mathbf{1},\mathbf{4})) \\
&= (\mathbf{1}+\mathbf{5}+\mathbf{10},\mathbf{1}) + (\mathbf{1},\mathbf{1}+\mathbf{5}+\mathbf{10}) + 2\times(\mathbf{4},\mathbf{4}) \\
J \otimes S^* &= ((\mathbf{4},\mathbf{5}) + (\mathbf{5},\mathbf{4})) \times ((\mathbf{4},\mathbf{1}) + (\mathbf{1},\mathbf{4})) \\
&= (\mathbf{1}+\mathbf{5}+\mathbf{10},\mathbf{5}) + (\mathbf{5},\mathbf{1}+\mathbf{5}+\mathbf{10}) + 2\times(\mathbf{4},\mathbf{4}) + (\mathbf{16},\mathbf{4}) + (\mathbf{4},\mathbf{16})
\end{aligned} \tag{D.12}$$

Clearly, all of the representations appearing here which are common with the connection are those in the torsion of the connection which has the decomposition

$$E^* \oplus K \sim (\mathbf{4},\mathbf{4}) + (\mathbf{16},\mathbf{4}) + (\mathbf{4},\mathbf{16}) + (\mathbf{4},\mathbf{4}) \tag{D.13}$$

and so the projection only depends on the torsion components of the compatible connection.

Similarly for $\varphi \in J$

$$\begin{aligned}
D \cdot \varphi &\in (\mathbf{4},\mathbf{4}) \times ((\mathbf{4},\mathbf{5}) + (\mathbf{5},\mathbf{4})) \\
&= (\mathbf{1}+\mathbf{5}+\mathbf{10},\mathbf{4}+\mathbf{16}) + (\mathbf{4}+\mathbf{16},\mathbf{1}+\mathbf{5}+\mathbf{10})
\end{aligned} \tag{D.14}$$

Looking to project onto the S and J parts we consider

$$\begin{aligned}
S \otimes J^* &= ((\mathbf{4},\mathbf{1}) + (\mathbf{1},\mathbf{4})) \times ((\mathbf{4},\mathbf{5}) + (\mathbf{5},\mathbf{4})) \\
&= (\mathbf{1}+\mathbf{5}+\mathbf{10},\mathbf{5}) + (\mathbf{5},\mathbf{1}+\mathbf{5}+\mathbf{10}) + 2\times(\mathbf{4},\mathbf{4}) + (\mathbf{16},\mathbf{4}) + (\mathbf{4},\mathbf{16}) \\
J \otimes J^* &= ((\mathbf{4},\mathbf{5}) + (\mathbf{5},\mathbf{4})) \times ((\mathbf{4},\mathbf{5}) + (\mathbf{5},\mathbf{4})) \\
&= (\mathbf{1}+\mathbf{5}+\mathbf{10},\mathbf{1}+\mathbf{10}+\mathbf{14}) + (\mathbf{1}+\mathbf{10}+\mathbf{14},\mathbf{1}+\mathbf{5}+\mathbf{10}) \\
&\qquad + 2\times(\mathbf{4}+\mathbf{16},\mathbf{4}+\mathbf{16})
\end{aligned} \tag{D.15}$$

Again, these projections cannot depend on the undetermined pieces of the connection (i.e. the $(\mathbf{20},\mathbf{4}) + (\mathbf{4},\mathbf{20})$ parts).

6 dimensions

For the $H_6 \simeq Sp(8)$ case we have the representations

$$E \sim \mathbf{27} \qquad S \sim \mathbf{8} \qquad J \sim \mathbf{48} \tag{D.16}$$

and the components of an $Sp(8)$ compatible connection lie in the space

$$E^* \times \mathrm{ad}(Sp(8)) \sim \mathbf{27} \times \mathbf{36} = \mathbf{27} + \mathbf{36} + \mathbf{315} + \mathbf{594} \tag{D.17}$$

For $\chi \in S$

$$D \cdot \chi \in \mathbf{27} \times \mathbf{8} = \mathbf{8} + \mathbf{48} + \mathbf{160} \tag{D.18}$$

We are interested in the projections onto S and J, so again we look at the tensor products

$$\begin{aligned} S \otimes S^* &= \mathbf{8} \times \mathbf{8} = \mathbf{1} + \mathbf{27} + \mathbf{36} \\ J \otimes S^* &= \mathbf{48} \times \mathbf{8} = \mathbf{27} + \mathbf{42} + \mathbf{315} \end{aligned} \tag{D.19}$$

Clearly, all of the representations appearing here which are common with the connection are those in the torsion of the connection which has the decomposition

$$E^* \oplus K \sim \mathbf{27} + \mathbf{36} + \mathbf{315} \tag{D.20}$$

and so the projection only depends on the torsion components of the compatible connection.

Similarly for $\varphi \in J$

$$D \cdot \varphi \in \mathbf{27} \times \mathbf{48} = \mathbf{8} + \mathbf{48} + \mathbf{160} + \mathbf{288} + \mathbf{792} \tag{D.21}$$

Looking to project onto the S and J parts we consider

$$\begin{aligned} S \otimes J^* &= \mathbf{8} \times \mathbf{48} = \mathbf{27} + \mathbf{42} + \mathbf{315} \\ J \otimes J^* &= \mathbf{48} \times \mathbf{48} = \mathbf{1} + \mathbf{27} + \mathbf{36} + \mathbf{315} + \mathbf{308} + \mathbf{792} + \mathbf{825} \end{aligned} \tag{D.22}$$

Again, these projections cannot depend on the undetermined part of the connection (i.e. the **594** part).

7 dimensions

For the $H_7 = SU(8)$ case we have the representations

$$E \sim \mathbf{28} + \bar{\mathbf{28}} \qquad S \sim \mathbf{8} + \bar{\mathbf{8}} \qquad J \sim \mathbf{56} + \bar{\mathbf{56}} \tag{D.23}$$

and the components of an $SU(8)$ compatible connection lie in the space

$$\begin{aligned} E^* \times \mathrm{ad}(SU(8)) &\sim (\mathbf{28} + \bar{\mathbf{28}}) \times \mathbf{63} \\ &= \mathbf{28} + \bar{\mathbf{28}} + \mathbf{36} + \bar{\mathbf{36}} + \mathbf{420} + \bar{\mathbf{420}} + \mathbf{1280} + \bar{\mathbf{1280}} \end{aligned} \tag{D.24}$$

Looking at the projections, we may focus on the $\mathbf{8} \subset S$ and $\mathbf{56} \subset J$, as the other parts follow as the complex conjugate representations (the tangent space and connection are real representations). So for $\chi \in \mathbf{8}$ we have

$$D \cdot \chi \in (\mathbf{28} + \bar{\mathbf{28}}) \times \mathbf{8} = \bar{\mathbf{8}} + \mathbf{56} + \mathbf{168} + \bar{\mathbf{216}} \tag{D.25}$$

Our interest is in the $\bar{\mathbf{8}}$ and $\mathbf{56}$ projections, so we look at

$$\begin{aligned} \bar{\mathbf{8}} \times \bar{\mathbf{8}} &= \bar{\mathbf{28}} + \bar{\mathbf{36}} \\ \mathbf{56} \times \bar{\mathbf{8}} &= \mathbf{28} + \mathbf{420} \end{aligned} \tag{D.26}$$

and as all of these representations are contained in the torsion

$$E^* \oplus K \sim \mathbf{28} + \bar{\mathbf{28}} + \mathbf{36} + \bar{\mathbf{36}} + \mathbf{420} + \bar{\mathbf{420}} \tag{D.27}$$

the projection only depends on the torsion components of the compatible connection. Similarly for $\varphi \in \mathbf{56}$

$$D \cdot \varphi \in (\mathbf{28} + \bar{\mathbf{28}}) \times \mathbf{56} = \mathbf{8} + \bar{\mathbf{56}} + \mathbf{216} + \mathbf{504} + \mathbf{1008} + \mathbf{1344} \tag{D.28}$$

Looking to project onto the $\mathbf{8}$ and $\bar{\mathbf{56}}$ parts we consider

$$\begin{aligned} \mathbf{8} \times \bar{\mathbf{56}} &= \bar{\mathbf{28}} + \bar{\mathbf{420}} \\ \bar{\mathbf{56}} \times \bar{\mathbf{56}} &= \mathbf{28} + \mathbf{420} + \mathbf{1176} + \mathbf{1512} \end{aligned} \tag{D.29}$$

Again, the representations which are common between these and the connection are all contained in the torsion, so these projections are uniquely determined by the torsion of the connection.

8 dimensions ?

For an $E_{8(8)} \times \mathbb{R}^+$ generalised geometry, the natural local symmetry group is $H_8 = SO(16)$ [56]. Inspired by our $GL(d, \mathbb{R})$ decompositions and the embedding tensor formalism, we would guess the $E_{8(8)} \times \mathbb{R}^+$ representations

$$E \sim \mathbf{248}_{+1} \qquad N \sim \mathbf{3875}_{+2} \qquad K \sim \mathbf{1}_{-1} + \mathbf{3875}_{-1} \tag{D.30}$$

Some decompositions and some more guesses lead us to the $H_8 = SO(16)$ representations

$$E \sim \mathbf{120} + \mathbf{128}^+ \qquad S \sim \mathbf{16} \qquad J \sim \mathbf{128}^- \tag{D.31}$$

and the components of an $SO(16)$ compatible connection then lie in the space

$$\begin{aligned} E^* \times \mathrm{ad}(SO(16)) &\sim (\mathbf{120} + \mathbf{128}^+) \times \mathbf{120} \\ &= \mathbf{1} + \mathbf{120} + \mathbf{135} + \mathbf{1820} + \mathbf{5304} + \mathbf{7020} + \mathbf{128}^+ + \mathbf{1920}^- + \mathbf{13312}^+ \end{aligned} \tag{D.32}$$

We would then have for $\chi \in S$

$$D \cdot \chi \in (\mathbf{120} + \mathbf{128}^+) \times \mathbf{16} = \mathbf{16} + \mathbf{128}^- + \mathbf{560} + \mathbf{1344} + \mathbf{1920}^+ \tag{D.33}$$

Our interest is in the **16** and $\mathbf{128}^-$ projections, so we look at

$$\begin{aligned} \mathbf{16} \times \mathbf{16} &= \mathbf{1} + \mathbf{120} + \mathbf{135} \\ \mathbf{128}^- \times \mathbf{16} &= \mathbf{128}^+ + \mathbf{1920}^- \end{aligned} \tag{D.34}$$

The torsion decomposes under $SO(16)$ as

$$E^* \oplus K \sim (\mathbf{120} + \mathbf{128}^+) + (\mathbf{1} + \mathbf{135} + \mathbf{1820} + \mathbf{1920}^-) \tag{D.35}$$

while the undetermined parts of the connection have the decomposition

$$U \sim \mathbf{5304} + \mathbf{7020} + \mathbf{13312}^+ \tag{D.36}$$

As for the $d \leq 7$ cases, we see that the projection only depends on the torsion components of the compatible connection. Similarly for $\varphi \in \mathbf{128}^-$

$$\begin{aligned} D \cdot \varphi \in (\mathbf{120} + \mathbf{128}^+) \times \mathbf{128}^- &= \mathbf{16} + \mathbf{560} + \mathbf{4368} + \mathbf{11440} \\ &\quad + \mathbf{128}^- + \mathbf{1920}^+ + \mathbf{13312}^- \end{aligned} \tag{D.37}$$

Looking to project onto the $\mathbf{16}$ and $\mathbf{128}^-$ parts we consider

$$\begin{aligned} \mathbf{16} \times \mathbf{128}^- &= \mathbf{128}^+ + \mathbf{1920}^- \\ \mathbf{128}^- \times \mathbf{128}^- &= \mathbf{1} + \mathbf{120} + \mathbf{1820} + \mathbf{8008} + \mathbf{6435}^- \end{aligned} \tag{D.38}$$

Again, the representations which are common between these and the connection are all contained in the torsion, so these projections are uniquely determined by the torsion of the connection.

E. Spinor Decompositions

In this appendix the details of the spinor decompositions used in chapter 7 are presented. For the purposes of looking at supersymmetric backgrounds, the spinors on the external Minkowski space are taken to be Grassman valued, while the internal spinors are commuting.

$Spin(3,1) \times Spin(6) \subset Spin(9,1)$

We can use a complex decomposition of the (real) Cliff$(9,1;\mathbb{R})$ gamma matrices as

$$\Gamma^\mu = \gamma^\mu \otimes 1, \qquad \Gamma^m = i\gamma^{(4)} \otimes \gamma^m, \tag{E.1}$$

Choosing the real representation of the Cliff$(4,\mathbb{R})$ and the imaginary anti-symmetric representation of the Cliff$(6;\mathbb{R})$ gamma matrices γ^m, we have the chiral spinor decompositions

$$\begin{aligned} \varepsilon^+ &= \eta^+ \otimes \chi^+ + \eta^- \otimes \chi^-, \\ \varepsilon^- &= \eta^+ \otimes \chi^- + \eta^- \otimes \chi^+, \end{aligned} \tag{E.2}$$

where $\gamma^{(6)}\chi^+ = i\chi^+$, $\gamma^{(4)}\eta^+ = -i\eta^+$, $\chi^- = (\chi^+)^*$ and $\eta^- = (\eta^+)^*$.

$Spin(3,1) \times Spin(7) \subset Spin(10,1)$

We take essentially the same complex decomposition of the Cliff$(10,1;\mathbb{R})$ gamma matrices

$$\Gamma^\mu = \gamma^\mu \otimes 1, \qquad \Gamma^m = i\gamma^{(4)} \otimes \gamma^m, \tag{E.3}$$

Choosing the real representation of the Cliff$(4,\mathbb{R})$ and the imaginary anti-symmetric representation of the Cliff$(7;\mathbb{R})$ gamma matrices γ^m, we have the spinor decompositions

$$\varepsilon^+ = \eta^+ \otimes \chi + \eta^- \otimes \chi^* \tag{E.4}$$

where $\gamma^{(4)}\eta^+ = -i\eta^+$ and $\eta^- = (\eta^+)^*$.

Bibliography

[1] A. Coimbra, C. Strickland-Constable and D. Waldram, "Supergravity as Generalised Geometry I: Type II Theories," JHEP **1111** (2011) 091 [arXiv:1107.1733 [hep-th]].

[2] A. Coimbra, C. Strickland-Constable and D. Waldram, "$E_{d(d)} \times \mathbb{R}^+$ Generalised Geometry, Connections and M theory," arXiv:1112.3989 [hep-th].

[3] A. Coimbra, C. Strickland-Constable and D. Waldram, "Supergravity as Generalised Geometry II: $E_{d(d)} \times \mathbb{R}^+$ and M theory," to appear.

[4] G. Veneziano, "Construction of a crossing - symmetric, Regge behaved amplitude for linearly rising trajectories," Nuovo Cim. A **57**, 190 (1968).

[5] M. B. Green, J. H. Schwarz and E. Witten, "Superstring Theory. Vol. 1: Introduction," Cambridge, Uk: Univ. Pr. (1987) 469 P. (Cambridge Monographs On Mathematical Physics)

[6] M. B. Green, J. H. Schwarz and E. Witten, "Superstring Theory. Vol. 2: Loop Amplitudes, Anomalies And Phenomenology," Cambridge, Uk: Univ. Pr. (1987) 596 P. (Cambridge Monographs On Mathematical Physics)

[7] J. Polchinski, "String theory. Vol. 1: An introduction to the bosonic string," Cambridge, UK: Univ. Pr. (1998) 402 p

[8] J. Polchinski, "String theory. Vol. 2: Superstring theory and beyond," Cambridge, UK: Univ. Pr. (1998) 531 p

[9] B. Zwiebach, "A first course in string theory," Cambridge, UK: Univ. Pr. (2009) 673 p

[10] K. Becker, M. Becker and J. H. Schwarz, "String theory and M-theory: A modern introduction," Cambridge, UK: Cambridge Univ. Pr. (2007) 739 p

[11] C. G. Callan, Jr., E. J. Martinec, M. J. Perry and D. Friedan, "Strings in Background Fields," Nucl. Phys. B **262**, 593 (1985).

[12] M. B. Green and J. H. Schwarz, "Anomaly Cancellation in Supersymmetric D=10 Gauge Theory and Superstring Theory," Phys. Lett. B **149**, 117 (1984).

[13] J. Dai, R. G. Leigh and J. Polchinski, "New Connections Between String Theories," Mod. Phys. Lett. A **4**, 2073 (1989).

[14] J. Polchinski, "Dirichlet Branes and Ramond-Ramond charges," Phys. Rev. Lett. **75**, 4724 (1995) [hep-th/9510017].

[15] C. G. Callan, Jr., J. A. Harvey and A. Strominger, "World sheet approach to heterotic instantons and solitons," Nucl. Phys. B **359**, 611 (1991).

[16] P. K. Townsend, "The eleven-dimensional supermembrane revisited," Phys. Lett. B **350**, 184 (1995) [hep-th/9501068].

[17] E. Witten, "String theory dynamics in various dimensions," Nucl. Phys. B **443**, 85 (1995) [hep-th/9503124].

[18] E. Cremmer, B. Julia and J. Scherk, "Supergravity Theory in Eleven-Dimensions," Phys. Lett. B **76**, 409 (1978).

[19] M. J. Duff and K. S. Stelle, "Multimembrane solutions of D = 11 supergravity," Phys. Lett. B **253**, 113 (1991).

[20] R. Gueven, "Black p-brane solutions of D = 11 supergravity theory," Phys. Lett. B **276**, 49 (1992).

[21] M. J. Duff, P. S. Howe, T. Inami and K. S. Stelle, "Superstrings in D=10 from Supermembranes in D=11," Phys. Lett. B **191**, 70 (1987).

[22] P. K. Townsend, "D-branes from M-branes," Phys. Lett. B **373**, 68 (1996) [hep-th/9512062].

[23] P. Horava and E. Witten, "Heterotic and type I string dynamics from eleven-dimensions," Nucl. Phys. B **460**, 506 (1996) [hep-th/9510209].
P. Horava and E. Witten, "Eleven-dimensional supergravity on a manifold with boundary," Nucl. Phys. B **475**, 94 (1996) [hep-th/9603142].

[24] K. Kikkawa and M. Yamasaki, "Casimir Effects in Superstring Theories," Phys. Lett. B **149**, 357 (1984).

[25] E. Alvarez and M. A. R. Osorio, "Duality Is an Exact Symmetry of String Perturbation Theory," Phys. Rev. D **40**, 1150 (1989).

[26] K. S. Narain, "New Heterotic String Theories in Uncompactified Dimensions < 10," Phys. Lett. B **169**, 41 (1986).

[27] K. S. Narain, M. H. Sarmadi and E. Witten, "A Note on Toroidal Compactification of Heterotic String Theory," Nucl. Phys. B **279**, 369 (1987).

[28] T. H. Buscher, "A Symmetry of the String Background Field Equations," Phys. Lett. B **194**, 59 (1987).

[29] A. Font, L. E. Ibanez, D. Lust and F. Quevedo, "Strong - weak coupling duality and nonperturbative effects in string theory," Phys. Lett. B **249**, 35 (1990).

[30] A. Sen, "Electric magnetic duality in string theory," Nucl. Phys. B **404**, 109 (1993) [hep-th/9207053].
A. Sen, "Magnetic monopoles, Bogomolny bound and SL(2,Z) invariance in string theory," Mod. Phys. Lett. A **8**, 2023 (1993) [hep-th/9303057].

[31] J. H. Schwarz and A. Sen, "Duality symmetric actions," Nucl. Phys. B **411**, 35 (1994) [hep-th/9304154].
"Duality symmetries of 4-D heterotic strings," Phys. Lett. B **312**, 105 (1993) [hep-th/9305185].

[32] C. M. Hull and P. K. Townsend, "Unity of superstring dualities," Nucl. Phys. B **438**, 109 (1995) [hep-th/9410167].

[33] E. Cremmer, B. Julia, "The $SO(8)$ Supergravity," Nucl. Phys. **B159**, 141 (1979).

[34] J. Polchinski and E. Witten, "Evidence for heterotic - type I string duality," Nucl. Phys. B **460**, 525 (1996) [hep-th/9510169].

[35] P. H. Ginsparg, "Comment on Toroidal Compactification of Heterotic Superstrings," Phys. Rev. D **35**, 648 (1987).

[36] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," AIP Conf. Proc. **484**, 51 (1999) [Int. J. Theor. Phys. **38**, 1113 (1999)].

[37] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, "Large N field theories, string theory and gravity," Phys. Rept. **323**, 183 (2000) [hep-th/9905111].

[38] J. Khoury and H. L. Verlinde, "On open - closed string duality," Adv. Theor. Math. Phys. **3**, 1893 (1999) [hep-th/0001056].

[39] P. Candelas, G. T. Horowitz, A. Strominger and E. Witten, "Vacuum Configurations for Superstrings," Nucl. Phys. B **258**, 46 (1985).

[40] B. R. Greene and M. R. Plesser, "Duality In Calabi-Yau Moduli Space," Nucl. Phys. B **338**, 15 (1990).

[41] P. Candelas, M. Lynker and R. Schimmrigk, "Calabi-Yau Manifolds in Weighted P(4)," Nucl. Phys. B **341**, 383 (1990).

[42] A. Strominger, S. -T. Yau and E. Zaslow, "Mirror symmetry is T duality," Nucl. Phys. B **479**, 243 (1996) [hep-th/9606040].

[43] K. Dasgupta, G. Rajesh and S. Sethi, "M theory, orientifolds and G - flux," JHEP **9908**, 023 (1999) [hep-th/9908088].

[44] T. R. Taylor and C. Vafa, "R R flux on Calabi-Yau and partial supersymmetry breaking," Phys. Lett. B **474**, 130 (2000) [hep-th/9912152].

[45] S. B. Giddings, S. Kachru and J. Polchinski, "Hierarchies from fluxes in string compactifications," Phys. Rev. D **66**, 106006 (2002) [hep-th/0105097].

[46] J. Polchinski and M. J. Strassler, "The String dual of a confining four-dimensional gauge theory," hep-th/0003136.
I. R. Klebanov and M. J. Strassler, "Supergravity and a confining gauge theory: Duality cascades and chi SB resolution of naked singularities," JHEP **0008**, 052 (2000) [hep-th/0007191].
J. M. Maldacena and C. Nunez, "Towards the large N limit of pure N=1 superYang-Mills," Phys. Rev. Lett. **86**, 588 (2001) [hep-th/0008001].

[47] H. Samtleben, "Lectures on Gauged Supergravity and Flux Compactifications," Class. Quant. Grav. **25**, 214002 (2008) [arXiv:0808.4076 [hep-th]].

[48] S. Kachru, M. B. Schulz, P. K. Tripathy and S. P. Trivedi, "New supersymmetric string compactifications," JHEP **0303**, 061 (2003) [hep-th/0211182].

[49] K. S. Narain, M. H. Sarmadi and C. Vafa, "Asymmetric Orbifolds," Nucl. Phys. B **288**, 551 (1987).

[50] S. Hellerman, J. McGreevy and B. Williams, "Geometric constructions of nongeometric string theories," JHEP **0401**, 024 (2004) [hep-th/0208174].

[51] A. Dabholkar and C. Hull, "Duality twists, orbifolds, and fluxes," JHEP **0309**, 054 (2003) [hep-th/0210209].

[52] A. Flournoy, B. Wecht and B. Williams, "Constructing nongeometric vacua in string theory," Nucl. Phys. B **706**, 127 (2005) [hep-th/0404217].

[53] C. M. Hull, "A geometry for non-geometric string backgrounds," JHEP **0510**, 065 (2005) [arXiv:hep-th/0406102].

[54] A. Dabholkar and C. Hull, "Generalised T-duality and non-geometric backgrounds," JHEP **0605**, 009 (2006) [hep-th/0512005].

[55] B. de Wit and H. Nicolai, "$D = 11$ Supergravity With Local $SU(8)$ Invariance," Nucl. Phys. B **274**, 363 (1986).

[56] H. Nicolai, "$D = 11$ Supergravity with Local $SO(16)$ Invariance," Phys. Lett. B **187**, 316 (1987).

[57] M. J. Duff, "Duality Rotations In String Theory," Nucl. Phys. B **335**, 610 (1990).

[58] A. A. Tseytlin, "Duality Symmetric Formulation Of String World Sheet Dynamics," Phys. Lett. B **242**, 163 (1990).
A. A. Tseytlin, "Duality symmetric closed string theory and interacting chiral scalars," Nucl. Phys. B **350**, 395 (1991).

[59] M. J. Duff and J. X. Lu, "Duality Rotations In Membrane Theory," Nucl. Phys. B **347**, 394 (1990).

[60] A. Lukas and B. A. Ovrut, "U duality symmetries from the membrane world volume," Nucl. Phys. B **502**, 191 (1997) [hep-th/9704178].

[61] W. Siegel, "Two vierbein formalism for string inspired axionic gravity," Phys. Rev. **D47** (1993) 5453-5459. [hep-th/9302036].
W. Siegel, "Superspace duality in low-energy superstrings," Phys. Rev. **D48** (1993) 2826-2837. [hep-th/9305073].

[62] K. Koepsell, H. Nicolai and H. Samtleben, "An exceptional geometry for $d = 11$ supergravity?," Class. Quant. Grav. **17**, 3689 (2000) [arXiv:hep-th/0006034].

[63] B. de Wit and H. Nicolai, "Hidden symmetries, central charges and all that," Class. Quant. Grav. **18**, 3095 (2001) [hep-th/0011239].

[64] C. Hillmann, "Generalized $E_{7(7)}$ coset dynamics and $D = 11$ supergravity," JHEP **0903**, 135 (2009). [arXiv:0901.1581 [hep-th]],
C. Hillmann, "$E_{7(7)}$ and $d = 11$ supergravity," [arXiv:0902.1509 [hep-th]].

[65] B. Julia, "Infinite Lie Algebras In Physics," In *Baltimore 1981, Proceedings, Unified Field Theories and Beyond*, 23-41

B. Julia, "Kac-moody Symmetry Of Gravitation And Supergravity Theories," LPTENS-82-22.

[66] H. Nicolai, N. P. Warner, "The Structure Of N=16 Supergravity In Two-dimensions," Commun. Math. Phys. **125**, 369 (1989).

[67] S. 'y. Mizoguchi, "E(10) symmetry in one-dimensional supergravity," Nucl. Phys. B **528**, 238 (1998) [hep-th/9703160].

[68] P. C. West, "E_{11} and M theory," Class. Quant. Grav. **18**, 4443 (2001) [hep-th/0104081].

[69] P. C. West, "Hidden superconformal symmetry in M theory," JHEP **0008**, 007 (2000) [hep-th/0005270].

[70] P. C. West, "E(11), SL(32) and central charges," Phys. Lett. B **575**, 333 (2003) [hep-th/0307098].

[71] C. M. Hull, "Gravitational duality, branes and charges," Nucl. Phys. B **509**, 216 (1998) [hep-th/9705162].

[72] E. A. Bergshoeff, M. de Roo and O. Hohm, "Can dual gravity be reconciled with E11?," Phys. Lett. B **675**, 371 (2009) [arXiv:0903.4384 [hep-th]].

[73] P. C. West, "The IIA, IIB and eleven-dimensional theories and their common E(11) origin," Nucl. Phys. B **693**, 76 (2004) [hep-th/0402140].
F. Riccioni and P. C. West, "The E(11) origin of all maximal supergravities," JHEP **0707**, 063 (2007) [arXiv:0705.0752 [hep-th]].
A. Kleinschmidt, I. Schnakenburg and P. C. West, "Very extended Kac-Moody algebras and their interpretation at low levels," Class. Quant. Grav. **21**, 2493 (2004) [hep-th/0309198].
F. Riccioni and P. C. West, "E(11)-extended spacetime and gauged supergravities," JHEP **0802**, 039 (2008) [arXiv:0712.1795 [hep-th]].

[74] D. Steele and P. West, "E11 and Supersymmetry," JHEP **1102**, 101 (2011) [arXiv:1011.5820 [hep-th]].

[75] T. Damour, M. Henneaux and H. Nicolai, "E_{10} and a "small tension expansion" of M theory," Phys. Rev. Lett. **89**, 221601 (2002) [hep-th/0207267].

[76] T. Damour, A. Kleinschmidt and H. Nicolai, "Hidden symmetries and the fermionic sector of eleven-dimensional supergravity," Phys. Lett. B **634**, 319 (2006) [hep-th/0512163].

[77] S. de Buyl, M. Henneaux and L. Paulot, "Extended E(8) invariance of 11-dimensional supergravity," JHEP **0602**, 056 (2006) [hep-th/0512292].

[78] M. J. Duff and J. T. Liu, "Hidden space-time symmetries and generalized holonomy in M theory," Nucl. Phys. B **674**, 217 (2003) [hep-th/0303140].

[79] C. Hull, "Holonomy and symmetry in M theory," hep-th/0305039.

[80] A. Keurentjes, "The Topology of U duality (sub)groups," Class. Quant. Grav. **21**, 1695 (2004) [hep-th/0309106].

[81] N. Hitchin, "Generalized Calabi-Yau manifolds," Quart. J. Math. Oxford Ser. **54**, 281 (2003) [arXiv:math.dg/0209099].

[82] M. Gualtieri, "Generalized Complex Geometry," Oxford University DPhil thesis (2004) [arXiv:math.DG/0401221] and [arXiv:math.DG/0703298].

[83] D. Roytenberg, "Courant algebroids, derived brackets and even symplectic supermanifolds", Ph.D. Thesis, U.C. Berkeley, arXiv:math/9910078 .

[84] Z. Chen, M. Stienon and P. Xu, "On Regular Courant Algebroids", arXiv:0909.0319.

[85] A. Alekseev and P. Xu, "Derived brackets and Courant algebroids", (2001). unpublished manuscript.

[86] M. Gualtieri, "Branes on Poisson varieties", in *The Many Facets of Geometry, A Tribute to Nigel Hitchin*, edited by Oscar Garcia-Prada, Jean Pierre Bourguignon and Simon Salamon, OUP, 2010, arXiv:0710.2719 [math.DG].

[87] I. T. Ellwood, "NS-NS fluxes in Hitchin's generalized geometry," JHEP **0712**, 084 (2007) [arXiv:hep-th/0612100].

[88] M. Grana, R. Minasian, M. Petrini and A. Tomasiello, "Supersymmetric backgrounds from generalized Calabi-Yau manifolds," JHEP **0408** (2004) 046 [arXiv:hep-th/0406137]. M. Grana, R. Minasian, M. Petrini and A. Tomasiello, "Generalized structures of $N = 1$ vacua," JHEP **0511** (2005) 020 [arXiv:hep-th/0505212].

[89] C. Jeschek and F. Witt, "Generalised G_2-structures and type IIB superstrings," JHEP **0503** (2005) 053 [arXiv:hep-th/0412280].

[90] P. Berglund and P. Mayr, "Non-perturbative superpotentials in F-theory and string duality," arXiv:hep-th/0504058.

[91] A. Tomasiello, "Generalized structures of ten-dimensional supersymmetric solutions," JHEP **1203**, 073 (2012) [arXiv:1109.2603 [hep-th]]. M. Petrini and A. Zaffaroni, "A Note on Supersymmetric Type II Solutions of Lifshitz Type," arXiv:1202.5542 [hep-th].

[92] M. Grana, J. Louis and D. Waldram, "Hitchin functionals in N=2 supergravity," JHEP **0601**, 008 (2006) [hep-th/0505264].

[93] N. Hitchin, "The geometry of three-forms in six and seven dimensions," J. Diff. Geom. **55** (2000), no.3 547 [arXiv: math.DG/0010054].

[94] N. Hitchin, "Stable forms and special metrics," in "Global Differential Geometry: The Mathematical Legacy of Alfred Gray", M.Fernandez and J.A.Wolf (eds.), Contemporary Mathematics **288**, American Mathematical Society, Providence (2001) [arXiv:math.DG/0107101].

[95] O. Ben-Bassat, "Mirror symmetry and generalized complex manifolds," J. Geom. Phys. **56** (2006) 533 [arXiv:math.ag/0405303].

[96] C. Jeschek, "Generalized Calabi-Yau structures and mirror symmetry," hep-th/0406046.

[97] M. Grana, J. Louis and D. Waldram, "SU(3) x SU(3) compactification and mirror duals of magnetic fluxes," JHEP **0704**, 101 (2007) [hep-th/0612237].

[98] M. Grana, R. Minasian, M. Petrini and D. Waldram, "T-duality, Generalized Geometry and Non-Geometric Backgrounds," JHEP **0904**, 075 (2009) [arXiv:0807.4527 [hep-th]].

[99] U. Lindstrom, "Generalized N = (2,2) supersymmetric nonlinear sigma models," Phys. Lett. B **587**, 216 (2004) [arXiv:hep-th/0401100],
U. Lindstrom, R. Minasian, A. Tomasiello and M. Zabzine, "Generalized complex manifolds and supersymmetry," Commun. Math. Phys. **257**, 235 (2005) [arXiv:hep-th/0405085],
A. Kapustin, "Topological strings on noncommutative manifolds," Int. J. Geom. Meth. Mod. Phys. **1**, 49 (2004) [arXiv:hep-th/0310057],
A. Kapustin and Y. Li, "Topological sigma-models with H-flux and twisted generalized complex manifolds," arXiv:hep-th/0407249.
V. Pestun and E. Witten, "The Hitchin functionals and the topological B-model at one loop," arXiv:hep-th/0503083.

[100] C. M. Hull, "Generalised Geometry for M-Theory," JHEP **0707**, 079 (2007) [arXiv:hep-th/0701203].

[101] P. P. Pacheco, D. Waldram, "M-theory, exceptional generalised geometry and superpotentials," JHEP **0809**, 123 (2008). [arXiv:0804.1362 [hep-th]].

[102] M. Grana, J. Louis, A. Sim, D. Waldram, "$E_{7(7)}$ formulation of $N = 2$ backgrounds," JHEP **0907**, 104 (2009). [arXiv:0904.2333 [hep-th]].

[103] M. Grana, F. Orsi, "N=1 vacua in Exceptional Generalized Geometry," [arXiv:1105.4855 [hep-th]].

[104] D. Baraglia, "Leibniz algebroids, twistings and exceptional generalized geometry," [arXiv:1101.0856 [math.DG]].

[105] C. M. Hull, "Global aspects of T-duality, gauged sigma models and T-folds," JHEP **0710**, 057 (2007) [hep-th/0604178].

[106] C. M. Hull and R. A. Reid-Edwards, "Gauge symmetry, T-duality and doubled geometry," JHEP **0808**, 043 (2008) [arXiv:0711.4818 [hep-th]].

[107] C. M. Hull and R. A. Reid-Edwards, "Non-geometric backgrounds, doubled geometry and generalised T-duality," JHEP **0909**, 014 (2009) [arXiv:0902.4032 [hep-th]].

[108] C. M. Hull, "Doubled Geometry and T-Folds," JHEP **0707**, 080 (2007) [hep-th/0605149].

[109] C. Hull, B. Zwiebach, "Double Field Theory," JHEP **0909** (2009) 099. [arXiv:0904.4664 [hep-th]].

[110] C. Hull, B. Zwiebach, "The Gauge algebra of double field theory and Courant brackets," JHEP **0909** (2009) 090. [arXiv:0908.1792 [hep-th]].

[111] O. Hohm, C. Hull, B. Zwiebach, "Background independent action for double field theory," JHEP **1007** (2010) 016. [arXiv:1003.5027 [hep-th]].

[112] O. Hohm, C. Hull, B. Zwiebach, "Generalized metric formulation of double field theory," JHEP **1008** (2010) 008. [arXiv:1006.4823 [hep-th]].

[113] P. West, "E_{11}, generalised space-time and IIA string theory," Phys. Lett. **B696** (2011) 403-409. [arXiv:1009.2624 [hep-th]].

[114] D. S. Berman and M. J. Perry, "Generalized Geometry and M theory," JHEP **1106**, 074 (2011) [arXiv:1008.1763 [hep-th]].
D. S. Berman, H. Godazgar and M. J. Perry, "$SO(5,5)$ duality in M-theory and generalized geometry," Phys. Lett. B **700**, 65 (2011) [arXiv:1103.5733 [hep-th]].
D. S. Berman, H. Godazgar, M. J. Perry and P. West, "Duality Invariant Actions and Generalised Geometry," arXiv:1111.0459 [hep-th].

[115] D. S. Berman, H. Godazgar, M. Godazgar and M. J. Perry, "The Local symmetries of M-theory and their formulation in generalised geometry," arXiv:1110.3930 [hep-th].

[116] O. Hohm, S. K. Kwak, "Frame-like Geometry of Double Field Theory," J. Phys. A **A44** (2011) 085404. [arXiv:1011.4101 [hep-th]].

[117] I. Jeon, K. Lee, J. -H. Park, "Differential geometry with a projection: Application to double field theory," JHEP **1104** (2011) 014. [arXiv:1011.1324 [hep-th]],
I. Jeon, K. Lee, J. -H. Park, "Stringy differential geometry, beyond Riemann," [arXiv:1105.6294 [hep-th]].

[118] O. Hohm, S. K. Kwak and B. Zwiebach, "Double Field Theory of Type II Strings," JHEP **1109**, 013 (2011) [arXiv:1107.0008 [hep-th]].

[119] A. Rocen, P. West, "E_{11}, generalised space-time and IIA string theory: the R-R sector," [arXiv:1012.2744 [hep-th]].

[120] O. Hohm and B. Zwiebach, "On the Riemann Tensor in Double Field Theory," JHEP **1205**, 126 (2012) [arXiv:1112.5296 [hep-th]].

[121] D. Andriot, "Heterotic string from a higher dimensional perspective," Nucl. Phys. B **855**, 222 (2012) [arXiv:1102.1434 [hep-th]].

[122] O. Hohm and S. K. Kwak, "Double Field Theory Formulation of Heterotic Strings," JHEP **1106**, 096 (2011) [arXiv:1103.2136 [hep-th]].

[123] G. Dall'Agata, N. Prezas, H. Samtleben and M. Trigiante, "Gauged Supergravities from Twisted Doubled Tori and Non-Geometric String Backgrounds," Nucl. Phys. B **799**, 80 (2008) [arXiv:0712.1026 [hep-th]].

[124] G. Aldazabal, W. Baron, D. Marques and C. Nunez, "The effective action of Double Field Theory," JHEP **1111**, 052 (2011) [Erratum-ibid. **1111**, 109 (2011)] [arXiv:1109.0290 [hep-th]].

[125] D. Geissbuhler, "Double Field Theory and N=4 Gauged Supergravity," JHEP **1111**, 116 (2011) [arXiv:1109.4280 [hep-th]].

[126] B. de Wit, H. Samtleben, M. Trigiante, "On Lagrangians and gaugings of maximal supergravities," Nucl. Phys. **B655**, 93-126 (2003). [hep-th/0212239];

B. de Wit, H. Samtleben, M. Trigiante, "The Maximal D=4 supergravities," JHEP **0706**, 049 (2007). [arXiv:0705.2101 [hep-th]].

[127] A. Le Diffon, H. Samtleben, M. Trigiante, "N=8 Supergravity with Local Scaling Symmetry," JHEP **1104**, 079 (2011). [arXiv:1103.2785 [hep-th]].

[128] O. Hohm and S. K. Kwak, "Massive Type II in Double Field Theory," JHEP **1111**, 086 (2011) [arXiv:1108.4937 [hep-th]].

[129] M. Grana and D. Marques, "Gauged Double Field Theory," JHEP **1204**, 020 (2012) [arXiv:1201.2924 [hep-th]].

[130] I. Jeon, K. Lee and J. -H. Park, "Incorporation of fermions into double field theory," JHEP **1111**, 025 (2011) [arXiv:1109.2035 [hep-th]].

[131] O. Hohm and S. K. Kwak, "N=1 Supersymmetric Double Field Theory," JHEP **1203**, 080 (2012) [arXiv:1111.7293 [hep-th]].

[132] D. Andriot, M. Larfors, D. Lust, P. Patalong, "A ten-dimensional action for non-geometric fluxes," [arXiv:1106.4015 [hep-th]].

[133] D. Andriot, O. Hohm, M. Larfors, D. Lust and P. Patalong, "Non-Geometric Fluxes in Supergravity and Double Field Theory," arXiv:1204.1979 [hep-th].

[134] D. C. Thompson, "Duality Invariance: From M-theory to Double Field Theory," JHEP **1108**, 125 (2011) [arXiv:1106.4036 [hep-th]].
N. B. Copland, "Connecting T-duality invariant theories," [arXiv:1106.1888 [hep-th]].
N. B. Copland, "A Double Sigma Model for Double Field Theory," JHEP **1204**, 044 (2012) [arXiv:1111.1828 [hep-th]].
I. Jeon, K. Lee and J. -H. Park, "Supersymmetric Double Field Theory: Stringy Reformulation of Supergravity," Phys. Rev. D **85**, 081501 (2012) [arXiv:1112.0069 [hep-th]].
B. Heidenreich, L. McAllister and G. Torroba, "Dynamic SU(2) Structure from Seven-branes," JHEP **1105**, 110 (2011) [arXiv:1011.3510 [hep-th]].

B. Heidenreich, "SL(2,R) covariant conditions for N=1 flux vacua," JHEP **1110**, 057 (2011) [arXiv:1107.3163 [hep-th]].
D. S. Berman, E. T. Musaev and M. J. Perry, "Boundary Terms in Generalized Geometry and doubled field theory," Phys. Lett. B **706**, 228 (2011) [arXiv:1110.3097 [hep-th]].
E. Malek, "U-duality in three and four dimensions," arXiv:1205.6403 [hep-th].

[135] C. Jeschek, F. Witt, "Generalised geometries, constrained critical points and Ramond-Ramond fields," [math/0510131 [math.DG]].

[136] N. J. Hitchin, "Lectures on special Lagrangian submanifolds," arXiv:math/9907034.

[137] I. Ya. Dorfman, "Dirac structures of integrable evolution equations" Physics Lett. **A 125**, 240 (1987).

[138] T. J. Courant, "Dirac manifolds", Trans. Amer. Math. Soc. **319**, 631 (1990).

[139] Y. Kosmann-Schwarzbach, "Derived brackets," Lett. Math. Phys. **69**, 61 (2004) [arXiv:math/0312524].

[140] M. Gualtieri, "Generalized geometry and the Hodge decomposition", Lecture at the String Theory and Geometry workshop, August 2004, Oberwolfach, arXiv:math.dg/0409093.

[141] E. Bergshoeff, R. Kallosh, T. Ortin, D. Roest, A. Van Proeyen, "New formulations of D = 10 supersymmetry and D8 - O8 domain walls," Class. Quant. Grav. **18**, 3359-3382 (2001) [hep-th/0103233].

[142] S. F. Hassan, "SO(d,d) transformations of Ramond-Ramond fields and space-time spinors," Nucl. Phys. **B583** (2000) 431-453. [hep-th/9912236],
S. F. Hassan, "T duality, space-time spinors and RR fields in curved backgrounds," Nucl. Phys. **B568** (2000) 145-161. [hep-th/9907152].

[143] D. Brace, B. Morariu, B. Zumino, "T duality and Ramond-Ramond backgrounds in the matrix model," Nucl. Phys. **B549**, 181-193 (1999). [hep-th/9811213].

[144] M. Fukuma, T. Oota, H. Tanaka, "Comments on T dualities of Ramond-Ramond potentials on tori," Prog. Theor. Phys. **103** (2000) 425-446. [hep-th/9907132].

[145] E. Cremmer, H. Lu, C. N. Pope, K. S. Stelle, "Spectrum generating symmetries for BPS solitons," Nucl. Phys. **B520**, 132-156 (1998). [hep-th/9707207].

[146] J. P. Gauntlett, S. Pakis, "The Geometry of D = 11 killing spinors," JHEP **0304**, 039 (2003). [hep-th/0212008].

[147] E. Cremmer, "$N = 8$ Supergravity", in *Unification of fundamental particle interactions*, p. 137, eds. S. Ferrara, J. Ellis, P. van Nieuwenhuizen, Plenum (1980);
B. Julia, "Group disintegrations", in *Superspace & Supergravity*, p. 331, eds. S.W. Hawking and M. Rocek, Cambridge University Press (1981).

[148] G. Aldazabal, E. Andres, P. G. Camara and M. Grana, "U-dual fluxes and Generalized Geometry," JHEP **1011**, 083 (2010) [arXiv:1007.5509 [hep-th]].

[149] M. Grana, "Flux compactifications in string theory: A Comprehensive review," Phys. Rept. **423**, 91 (2006) [hep-th/0509003].

[150] M. Berger, "Sur les groupes d'holonomie homogène des variétés à connexion affines et des variétés riemanniennes," Bull. Soc. Math. France **83** (1955), 279-330.

[151] D. Joyce, "Compact Riemannian 7-manifolds with holonomy G2. Part II," J. Diff. Geom. **43** (1996) 329.

[152] H. Baum, K. Lärz and T. Leistner "On the full holonomy group of special Lorentzian manifolds," arXiv:1204.5657v2 [math-dg].

[153] M. Gabella, J. P. Gauntlett, E. Palti, J. Sparks and D. Waldram, "AdS(5) Solutions of Type IIB Supergravity and Generalized Complex Geometry," Commun. Math. Phys. **299**, 365 (2010) [arXiv:0906.4109 [hep-th]].

[154] A. Strominger, "Superstrings with Torsion," Nucl. Phys. B **274**, 253 (1986).

[155] C. M. Hull, "Compactifications Of The Heterotic Superstring," Phys. Lett. B **178**, 357 (1986).

[156] E. Calabi, "The space of Kähler metrics," in *Proceedings of the International Congress of Mathematicians, Amsterdam, 1954, vol. 2*, pp. 206–207. North-Holland, Amsterdam, 1956.
E. Calabi, "On Kähler manifolds with vanishing canonical class," in *Algebraic geometry and topology, a symposium in honour of S. Lefschetz*, pp. 78–89. Princeton University Press, Princeton, 1957.
S.-T. Yau, "On Calabi's conjecture and some new results in algebraic geometry," *Proceedings of the National Academy of Sciences of the U.S.A.* **74** (1977) 1798–1799.
S.-T. Yau, "On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equations. I.," *Communications on pure and applied mathematics* **31** (1978) 339–411.

[157] V. Braun, Y. -H. He, B. A. Ovrut and T. Pantev, "A Heterotic standard model," Phys. Lett. B **618**, 252 (2005) [hep-th/0501070].

[158] V. Braun, Y. -H. He, B. A. Ovrut and T. Pantev, "The Exact MSSM spectrum from string theory," JHEP **0605**, 043 (2006) [hep-th/0512177].

[159] J. M. Maldacena and C. Nunez, "Supergravity description of field theories on curved manifolds and a no go theorem," Int. J. Mod. Phys. A **16**, 822 (2001) [hep-th/0007018].

[160] P. Candelas and D. J. Raine, "Spontaneous Compactification and Supersymmetry in d=11 Supergravity," Nucl. Phys. B **248**, 415 (1984).
P. Candelas, "Compactification And Supersymmetry Of Chiral N=2 D = 10 Supergravity," Nucl. Phys. B **256**, 385 (1985).
B. de Wit, D. J. Smit and N. D. Hari Dass, "Residual Supersymmetry of Compactified D=10 Supergravity," Nucl. Phys. B **283**, 165 (1987).

[161] A. Hanany and B. Kol, "On orientifolds, discrete torsion, branes and M theory," JHEP **0006**, 013 (2000) [hep-th/0003025].

[162] R. L. Bryant, "Metrics with exceptional holonomy," Ann. of Math. **126** (1987), 525-576.
R. L. Bryant and S. M. Salamon, "On the construction of some complete metrics with exceptional holonomy," Duke Math. J. **58** (1989), 829-850.

[163] D. Joyce, "Compact Manifolds With Special Holonomy," (Oxford University Press, 2000).
D. Joyce, "Compact Riemannian 7-manifolds with holonomy G2. Part I," J. Diff. Geom. **43** (1996) 291.

[164] J. P. Gauntlett, N. Kim, D. Martelli and D. Waldram, "Five-branes wrapped on SLAG three cycles and related geometry," JHEP **0111**, 018 (2001) [hep-th/0110034].

[165] M. Henneaux, A. Kleinschmidt and H. Nicolai, "Real forms of extended Kac-Moody symmetries and higher spin gauge theories," Gen. Rel. Grav. **44**, 1787 (2012) [arXiv:1110.4460 [hep-th]].

[166] B. A. Ovrut and D. Waldram, "Membranes and three form supergravity," Nucl. Phys. B **506**, 236 (1997) [hep-th/9704045].

[167] J. P. Gauntlett, D. Martelli, S. Pakis and D. Waldram, "G structures and wrapped NS5-branes," Commun. Math. Phys. **247**, 421 (2004) [hep-th/0205050].
J. P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, "Supersymmetric AdS(5) solutions of M theory," Class. Quant. Grav. **21**, 4335 (2004) [hep-th/0402153].

[168] V. Mathai and J. M. Rosenberg, "T duality for torus bundles with H fluxes via noncommutative topology," Commun. Math. Phys. **253**, 705 (2004) [hep-th/0401168].
P. Bouwknegt, K. Hannabuss and V. Mathai, "Nonassociative tori and applications to T-duality," Commun. Math. Phys. **264**, 41 (2006) [hep-th/0412092].

[169] P. Grange and S. Schafer-Nameki, "T-duality with H-flux: Non-commutativity, T-folds and G x G structure," Nucl. Phys. B **770**, 123 (2007) [hep-th/0609084].