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Thermal Bubble Diagrams Near Zero Energy

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Abstract

The zero four-momentum and equal mass limits are taken for the bubble diagram of scalar fields. It is seen that RTF and ITF are in complete agreement. However contributions from this diagram to both retarded and time-ordered functions do depend on the order of the limits and can be infinite in some cases. This shows explicitly that the relation between the free energy and a derivative expansion of a thermal effective action is generally much more complicated that is the case at zero temperature.

The simplest example in relativistic field theory of a diagram with non-trivial momentum dependence is the bubble diagram of figure (1). It is therefore illuminating

Figure 1: The bubble diagram.

to see exactly what results the standard Feynman rules lead to especially as they turn out to be non-trivial in the zero four-momentum, equal mass limit. This limit is also of special interest because the zero energy limit is closely tied to the infra-red behaviour which plays a vital role in both first and second order phase transitions as, for example, the discussion of the cubic term in the free energy for electroweak theory shows (see [1] and references therein).

However the zero energy limit is problematical. In the context of the calculations done using the Feynman rules of RTF (Real-Time Formalisms) [2, 3, 4, 5, 6, 7] pathological terms of the form $[\delta(K^2 - m^2)]^{N\geq 2}$ appear [6, 8, 9, 10, 11, 12, 13, 14]. One of the motivations of this paper is to show that the obvious problems of RTF at zero energy do not indicate that RTF is flawed but rather that thermal Green functions at zero

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energy are intrinsically more complicated than one would expect from experience with zero temperature field theory. Further, using the 'simple' scalar bubble diagram as an example, it will be clear that both RTF and ITF (Imaginary-Time Formalism) [6, 15] are in complete agreement and show the same difficulties with the zero energy limit. It is hoped that this example will clarify the nature of the calculational problems with this limit. In doing so we will be extending existing calculations of the bubble diagram, [11, 12, 16], to cover all possible variations.

We start by noting the essential properties of two-point Green functions of equilibrium field theory as well as their relation to the results of RTF or ITF calculations. This will also serve to establish the notation used here.

In RTF one has a direct handle on time-ordered Green functions. The connected two-point function in RTF, Π^{ab}, is a two by two matrix where by definition the 11 component is the real-time time-ordered function, Π_t

$$i\Pi^{11}(t,\vec{x}) = i\Pi_t(t,\vec{x}) = \ \mathrm{Tr}\{e^{-\beta H}\mathrm{T}\phi(t,\vec{x})\phi(0,0)\}/\mathrm{Tr}\{e^{-\beta H}\} \tag{1}$$

This is true whatever version of RTF is used (C^*-algebra, Thermo Field Dynamics, path integral methods, etc.).

In ITF one is initially calculating the Euclidean time-ordered function, Π_I. In terms of energy it is only known at imaginary energies which are multiples of $2\pi i/\beta$. Only by making an analytic continuation can one look at the the behaviour of ITF calculations in the neighbourhood of zero energy. Using standard boundary conditions at infinite energies this can be done [17, 18] and the resulting function will be denoted by Π_c. One finds that at real energies ITF is calculating the retarded (R) and advanced (A) Green functions

$$\begin{aligned}\Pi_c(z=E+i\epsilon) &= R(E) \\ \Pi_c(z=E-i\epsilon) &= A(E)\end{aligned} \tag{2}$$

where E will be used to indicate real energies. As functions of time, the retarded and advanced functions are given by

$$iR(t,\vec{x}) = \theta(t)\ \mathrm{Tr}\{e^{-\beta H}[\phi(t,\vec{x})\phi^\dagger(0,0)]\}/\mathrm{Tr}\{e^{-\beta H}\} \tag{3}$$

$$iA(t,\vec{x}) = -\theta(-t)\ \mathrm{Tr}\{e^{-\beta H}[\phi(t,\vec{x})\phi^\dagger(0,0)]\}/\mathrm{Tr}\{e^{-\beta H}\} \tag{4}$$

where only bosonic fields are being considered.

Standard relations between the time-ordered and retarded functions are simple to derive [6] and this then provides a link between the usual results of RTF and ITF calculations by using (1) and (2). In particular relations can be found for truncated diagrams [6, 14]

$$\mathrm{Im}\{[\Pi^{-1}(E)]^{11}\} = \frac{e^{\beta E}+1}{e^{\beta E}-1}\mathrm{Im}\{R(E)\}, \tag{5}$$

$$\mathrm{Re}\{[\Pi^{-1}(E)]^{11}\} = \mathrm{Re}\{R(E)\} \tag{6}$$

$$\mathrm{Im}\{R(E=0)^{-1}\} = 0 \tag{7}$$

where

$$n(E) = \frac{1}{e^{\beta E} - 1} \tag{8}$$

Such relations are derived from the definitions of these full Green functions but they hold even when the contribution from only one single Feynman diagram is considered. This is not surprising as the relations come directly from the definitions of what is meant by equilibrium thermal Green functions (cyclicity of the thermal trace, Kubo-Martin Schwinger condition etc.) which must be respected order by order if an approximation is to be physically realistic.

The usual problem of the infra-red divergence in the Bose-Einstein distribution seems to hit us straight away and we find for the bosonic case

$$\lim_{E\to 0}\{[\Pi^{-1}(E)]^{11} - R(E)^{-1}\} = \lim_{E\to 0}\frac{2}{\beta E}\mathrm{Im}\{R(E)^{-1}\} \tag{9}$$

The divergence in (9) looks very suspicious in view of the RTF divergences we will discuss below. However (9) was derived for arbitrary $\vec{p}, m_1, m_2$ whereas it is the behaviour at equal mass and zero four-momentum that will interest us below. Further, as noted above, $\mathrm{Im}\{R(E=0)\} = 0$ and a more careful analysis [14] shows that

$$\lim_{E\to 0}\{[\Pi^{-1}(E)]^{11} - R(E)^{-1}\} = \int dt \;\; \mathrm{Tr}\{e^{-\beta H}\phi_\mu(t,\vec{x})\phi^\dagger_\nu(0,\vec{0})\}/\mathrm{Tr}\{e^{-\beta H}\}. \tag{10}$$

so that the obvious divergence in (9) has gone and the difference may well be finite. Thus (9) does not seem to tell us much about possible divergences.

The problems which we wish to address here are most evident in the RTF calculations where it is well known that singular terms do appear. These are the $[\delta(K^2 - m^2)]^{N\geq 2}$ terms which appear when N lines in a diagram carry the same four-momentum and are associated with particles of the same mass [6, 8, 9, 10, 11, 12, 13, 14]. This occurs when parts of a diagram are self-energy insertions but in this case it is simple to see that when the usual RTF sum over internal vertex labelings is performed, such contributions cancel [5, 6, 8, 9].

However such singular terms also arise in diagrams where there are external legs carrying zero four-momenta [10, 12]. Such diagrams are physically important, for instance the free energy or effective potential is also the generating functional of all 1PI diagrams in which any external legs carry zero four-momenta, though this is not so straightforward for the thermal case [14].

The simplest example of the $[\delta(K^2 - m^2)]^{N\geq 2}$ singularities comes from the bubble self-energy diagram (1) where two scalar fields run round the loop. The contribution to the time-ordered function, B_t, is easily calculated in RTF and is given by

$$\begin{aligned} -iB_t(E,p;m_1,m_2) &= -iB^{11} \\ &= \frac{(-ig)^2}{2}\int \frac{d^4K}{(2\pi)^4} i\Delta^{11}(K,m_1) i\Delta^{11}(K+P,m_2) \\ &= -iB_{t0} - iB_{t1} - iB_{t2} \end{aligned} \tag{11}$$

where

$$i\Delta^{11}(K,m) = \frac{i}{K^2 - m^2 + i\epsilon} + n(|k_0|)\delta(K^2 - m^2) \tag{12}$$

It is convenient to split it into three terms, B_{tj} being the contribution to the time-ordered functions from the terms in the RTF calculation containing j delta functions and Bose-Einstein factors.

Throughout E will represent real external energies with $P^\mu = (E, \vec{p})^\mu$ the external Minkowskii four-momentum, while $K^\mu = (k^0, \vec{k})^\mu$ denotes Minkowskii loop momentum. The modulus of three-momentum is denoted by $p = |\vec{p}|, k = |\vec{k}|$.

The B_{t2} contains a product of two delta functions. This is only a problem for the RTF calculation if we have set $m_1 = m_2$ and $E, p = 0$ before doing the integrals. Thus avoiding any one of these limits is sufficient to give a well defined integral and this is what will be done. Doing the energy and angular integrals gives

$$B_{t2}(E,p;m_1,m_2) = \frac{-ig^2}{16\pi p}\sum_{\pm}\int dk \frac{k}{\omega_1} n(\omega_1)n(\Omega_2)\theta(\cos\theta^* - 1)|_{\theta=\theta^*} + (1 \leftrightarrow 2) \tag{13}$$

where

$$\cos\theta^* = \frac{1}{2pk}(P^2 + m_1^2 - m_2^2 \pm 2E\omega_1) \tag{14}$$

$$\cos(\theta) = \frac{\vec{p}.\vec{k}}{pk} \tag{15}$$

The dispersion relations, ω_i, Ω_i are defined to be

$$\omega_i = k^2 + m_i^2, \quad \Omega_i = (\vec{p} + \vec{k})^2 + m_i^2, \tag{16}$$

Note that in general the two delta functions can both be non-zero . Thus B_{t2} is often non-zero and well behaved as (13) shows.

If we keep $m_1 \neq m_2$ then as $E, p \to 0$ we find

$$\lim_{E,p\to 0} B_{t2}(E,p;m_1 \neq m_2) = 0 \tag{17}$$

as the argument of the two delta functions can not be satisfied simultaneously. However setting $m = m_1 = m_2$ first gives an interesting result. Taking the limit $E, p \to 0$ the ratio $v = |\vec{v}|$ can be kept fixed, where

$$\vec{v} = \frac{\vec{p}}{E}, \quad \gamma = (1 - v^2)^{-\frac{1}{2}}. \tag{18}$$

We then find

$$\lim_{E,p\to 0}\lim_{m_1\to m_2} B_{t2}(E,p;m_1 \neq m_2) = \frac{-ig^2}{8\pi}\frac{mx_c}{|p|}\theta(v^2-1)\int_1^\infty dx\, n^2(mx_c x). \tag{19}$$

$$\simeq \frac{-ig^2}{8\pi}\theta(v^2-1)\left[\frac{T^2}{mx_c|p|} + \frac{T}{|p|}(-\frac{1}{2} - \zeta_0(1) - \psi(1) + \log(b))\right] \tag{20}$$

where

$$x_c = |v\gamma|. \tag{21}$$

Throughout we shall use $\simeq$ to indicate that a high temperature expansion, $T \gg E, p, m_1, m_2$ has been taken but only terms of $O(T)$ and bigger are given. The integral is generally finite but does not have a simple form so for illustrative purposes the high temperature limit, $T \gg m_1, m_2 \gg E, p$, is given.

In taking the small $E, p, (m_1 - m_2)$ limits, B_{t2}, is generally well defined except at the $E, p = 0$ $m_1 = m_2$ point itself where B_{t2} is infinity or zero. It is also pure imaginary. However the residue of the pole depends on how this point is approached in the E, p plane. So the delta-squared term of the RTF calculation does reflect genuine infinites in the B_{t2} term.

As B_{t2} is imaginary, it is useful to look at the remaining temperature dependent imaginary parts coming from B_{t1} in case it cancels any of the odd behaviour in B_{t2}. While B_{t1} does not contain any explicit $[\delta(K^2 - m^2)]^2$ terms, which provided some of the original motivation for studying B_{t2}, we find

$$\begin{aligned} B_{t1}(E, p; m_1, m_2) &= \frac{g^2}{2} \sum_{\pm} (2\pi)^{-3} \int d^3\vec{k}\, \frac{n(\omega_1)}{2\omega_1} \frac{1}{(E \pm \omega_1)^2 - \Omega_2^2 + i\epsilon} \\ &\quad + \frac{n(\Omega_2)}{2\Omega_2} \frac{1}{(E \pm \Omega_2)^2 - \omega_1^2 + i\epsilon} \end{aligned} \tag{22}$$

from which

$$\mathrm{Im}\{B_{t1}\} = \frac{-ig^2}{16\pi p} \sum_{\pm} \int dk \frac{k}{\omega_1} \frac{1}{2} \left(n(\omega_1) + n(\Omega_2)\right) \theta(\cos\theta^* - 1)|_{\theta=\theta^*} \tag{23}$$

Results are similar to B_{t2}. First

$$\lim_{E,p\to 0} \mathrm{Im}\{B_{t1}(E, p; m_1 \neq m_2)\} = 0. \tag{24}$$

Then we have in the high temperature limit $T \gg m_1, m_2 \gg E, p$

$$\lim_{E,p\to 0} \lim_{m_1 \to m_2} \mathrm{Im}\{B_{t1}\} = \frac{-ig^2}{8\pi} \theta(v^2 - 1) \frac{T}{|p|} \tag{25}$$

In fact it is easy to calculate the exact total temperature dependent imaginary part and this is

$$\begin{aligned} \lim_{E,p\to 0} \lim_{m_1 \to m_2} \mathrm{Im}\{B_{t1+2}\} &= \frac{-ig^2}{8\pi} \theta(v^2 - 1) \frac{T}{|p|} n(\beta m x_c) & (26) \\ &\simeq \frac{-ig^2}{8\pi} \theta(v^2 - 1) \left[\frac{T^2}{m x_c |p|} - \frac{T}{2|p|} \right] \\ x_c &= |v\gamma| & (27) \end{aligned}$$

Both B_{t1} and B_{t2} contribute to this divergent result despite the fact that B_{t1} had only a single delta function in its integrand. However, away from $m_1 = m_2$ and $E, p \to 0$ the result is finite. For instance taking the $v = \infty$ limit in (26) (i.e. p small but not zero) shows that its divergence only occurs at the zero four-momentum point and is not associated with just the zero energy limit as (9) might suggest. So the delta-squared term in B_{t2} is therefore flagging a deeper problem, the existence of genuine divergences present in the total result for the imaginary part of the time-ordered function, $\mathrm{Im}\{B_t\}$, at the zero four-momentum equal mass point.

Having established this situation with the imaginary part of the time-ordered result as obtained in an RTF calculation, it is interesting to see what is happening with other results obtained from the same diagram at the same point. So we will look at the real part of the time-ordered function obtained in RTF, at the ITF results from which we will extract the advanced and retarded Green functions, and check the raltion between them. This will enable us to see if RTF calculations or time-ordered functions are unreliable in this limit and whether ITF calculations or retarded Green functions have similar problems.

For the specific case of the bubble diagram in ITF one initially calculates $B_I(2\pi i n/\beta, p; m_1, m_2)$ which is only defined at discrete values of the energy, $2\pi i n/\beta$, $n \in \mathcal{Z}$. The only real value of energy which is directly accessible is $E = 0$. In particular, it should be noted that in the ITF calculation setting $m_1 = m_2, E, p = 0$ before integrating does gives a unique and well defined result. There is no divergence in the integrand as there was with RTF and this result will be noted below, (33). It in the other ITF calculations discussed here, the $m_1 = m_2, E, p = 0$ limit is taken after the loop energy integration.

The ITF function, B_I, can be continued to one defined at general complex energies which is denoted by B_c. It is found to be

$$B_c(z \in \mathcal{C}, p; m_1, m_2) = (T = 0) \ + \\ \frac{g^2}{2} \sum_{\pm} (2\pi)^{-3} \int d^3\vec{k} \left[\frac{n(\omega_1)}{2\omega_1} \frac{1}{(z \pm \omega_1)^2 - \Omega_2^2} \ + \frac{n(\Omega_2)}{2\Omega_2} \frac{1}{(z \pm \Omega_2)^2 - \omega_1^2} \right] \tag{28}$$

where z is the complex external energy parameter. The retarded, $R(E, p; m_1, m_2)$, and advanced, $A(E, p; m_1, m_2)$, functions correspond to continuations of B_c to either side of the real energy axis, $z \to E \pm i\epsilon$ [17, 18]. By inspection the $i\epsilon$ terms are different from the RTF B^{11} calculation in which the energy integral has been performed. This allows the imaginary part of the ITF calculation to differ from the RTF result. We find for general p, m_1, m_2 and zero energy that

$$\begin{aligned} B_c(z = 0 + i\epsilon, p; m_1, m_2) &= R(E = 0, p; m_1, m_2) = A(E = 0, p; m_1, m_2) \\ &= B_I(z = 0, p; m_1, m_2) \\ &= \mathrm{Re}\{B^{11}(E = 0, p; m_1, m_2)\}. \end{aligned} \tag{29}$$

The last result is obtained by inspection. These results are consistent with the identities (6), (7) and (9). Note that not surprisingly the results obtained using B_I at the zero energy point are identical with the case where an analytic continuation is made to general complex energies and then $E \to 0$ is taken before any other limit. However, these results are questionable if the functions are badly behaved at any point. As we have already seen problems when $E, p = 0, m_1 = m_2$, it remains to be shown that the above relations hold at that point.

The next stage is to look at the $E, p = 0, m_1 = m_2$ limit of the ITF calculation of B_c. The limit can be taken in various ways and it is easily found that

$$\begin{aligned}
\lim_{m_1 \to m_2} \lim_{E,p \to 0} B_c(E, p; m_1, m_2) &= (T = 0) + \frac{g^2}{4\pi^2} \frac{d}{dm^2} \int_0^\infty dk \, \frac{k^2}{\omega} n(\omega) \\
&= (T = 0) - \frac{g^2}{8\pi^2} \int_m^\infty d\omega \, \frac{1}{k} n(\omega) \qquad (30) \\
&\simeq -\frac{g^2}{16\pi} \frac{T}{m} \qquad (31) \\
&= \lim_{p \to 0} \lim_{m_1 \to m_2} \lim_{E \to 0} B_c(E, p; m_1, m_2) \qquad (32) \\
&= \lim_{p \to 0} \lim_{E \to 0} \lim_{m_1 \to m_2} B_c(E, p; m_1, m_2) \\
&= \lim_{p, m_1 \to m_2} B_I(z = 0, p; m_1, m_2) \\
&= B_I(z = 0, p = 0; m_1 = m_2) \qquad (33)
\end{aligned}$$

Note that these B_c results coincide with those from the direct, no analytic continuation ITF calculation, i.e. B_I (33), however the equal mass and zero three-momentum limits are taken. In particular, it agrees with the result obtained when setting $m_1 = m_2$ and $E = p = 0$ before doing the energy sum which we have denoted as (33) with no limit symbols. The only one that causes problems is when $m_1 = m_2$ is taken first and then we take $E \to 0$ with or after $\vec{p} \to 0$. If we are to get a single answer we must take E to zero before the energy integral is done or before both of the other limits have been taken. If this is not done then more complicated results are obtained and I find [11, 12]

$$\begin{aligned}
&\lim_{p,E \to 0, p/E = v} \lim_{m_1 \to m_2} B_c(E, p; m_1, m_2) \\
&= (T = 0) - \frac{g^2}{8\pi^2} \int_1^\infty d\omega \, n(\beta\omega) \frac{k}{\omega^2 + v^2\gamma^2 m^2} \qquad (34) \\
&\simeq -\frac{g^2}{16\pi} \frac{1}{1+\gamma} \frac{T}{m} \qquad (35)
\end{aligned}$$

In the case $v = \infty$ ($E \to 0$ before $\vec{p} \to 0$) (33) is recovered.

This general result can be compared against calculations where either E or p are kept non-zero and not small but in which, apart from this, the usual limit is taken. These calculations give

$$\begin{aligned}
B_c(E, p = 0, m_1 = m_2) &= (T = 0) - \frac{g^2}{8\pi^2} \int_1^\infty d\omega \, n(\beta\omega) \frac{k}{\omega^2 - E^2/4} \qquad (36) \\
B_r(E = 0, p, m_1 = m_2) &= (T = 0) - \frac{g^2}{16\pi^2} \frac{m}{p} \int_1^\infty d\omega \, n(\beta\omega) \log\left[\left(\frac{p+2k}{p-2k}\right)^2\right] \qquad (37)
\end{aligned}$$

In the zero four-momentum limit the above results do indeed reproduce (34) for $v = 0$ and $v = \infty$ respectively.

We can also look at the imaginary part of B_c in the case where $m_1 = m_2$ is taken first (34). This expression for B_c is real for $|v| < 1$ but for $|v| > 1$ the integrand has a pole and an imaginary part is generated. The small $i\epsilon$ present in the energy, and hence in v, tells us which way round the pole to go and we find

$$\lim_{p,E\to 0, p/E=v} \lim_{m_1\to m_2} \mathrm{Im}\{B_c(z=E+i\eta|\epsilon|,p,m_1,m_2)\} = \frac{ig^2\eta}{16\pi}\theta(v^2-1)n(\beta m x_c)\frac{E}{p} \quad (38)$$

Note that this ITF calculation of the bubble diagram's contribution to the retarded function in the zero four-momentum and equal mass limit is not divergent unlike the the time ordered function. However it is not generally zero as (7) suggests. While it is linear in energy near zero energy, as the imaginary part of the retarded function should be [14], there is a competing $1/p$ factor as $p \to 0$ which keeps the result finite. It is only when the $v \to \infty$ limit of (38) is taken do we get zero for the imaginary part of B_r and so agreement with the spectral representation result (7). This is not surprising as (7) was derived for fixed p and $E=0$ which corresponds to the $v=\infty$ limit.

One can then use the spectral function results (5) together with these ITF calculations of B_c to tell us how the imaginary part of the time-ordered function behaves. It is completely consistent with the direct RTF calculation of the time ordered function (26). In particular the time-ordered bubble diagram is found to be infinite for $v > 1$. However it is not the factor of $1/E$ for small energies in (9) that is responsible for the divergence, this is canceled by the retarded result being linear in E. Rather it is the $1/p$ behaviour at small p that is leading to the singularities in the time-ordered function.

By inspection of (22) and (28), which differ only in the $i\epsilon$ factors, the real parts of the time-ordered function calculated using RTF, B^{11}, and the retarded function calculated in ITF from B_c, are seen to be equal even in the tricky limit. Thus it is clear that RTF and ITF are in complete agreement with each other. If one choose to, one could calculate the retarded function in RTF or the time ordered function using ITF for the same diagram, and the same results would be obtained.

On the other hand if we look at a different approaches to the zero four-momentum point, v finite, then we see that the retarded function has a non-zero, finite value but the old infra red divergence in the Bose-Einstein function ensures that the imaginary part of the time-ordered function is blowing up in the region $1 < |v| < \infty$. It is also zero for time-like limits $v < 1$.

There is one approach to the $E, p, (m_1 - m_2) \to 0$ limit that has not been considered so far and that is taking $p \to 0$ first and then taking the remaining limits in either order or simultaneously. The result is found to be

$$\begin{aligned}
&\lim_{E,(m_1\to m_2)\to 0} \lim_{p\to 0} B_c(E,p;m_1,m_2) \\
&= (T=0) - \frac{g^2}{8\pi^2}\int_0^\infty dk \left\{ \frac{k^2}{\omega^3}n(\omega) + \right. \\
&\qquad \left. q^2\beta m^2\, n(\omega)[1+n(\omega)]\frac{k^2}{\omega^2[q^2m^2-\omega^2]} \right\} \\
&= (T=0) - \frac{g^2}{8\pi^2}\int_m^\infty d\omega\, n(\omega) \left\{ \frac{k}{\omega^2} + \right. \\
&\qquad \left. q^2m^2\frac{m^4q^2+2\omega^4-3\omega^2m^2}{k\omega^2[m^2q^2-\omega^2]^2} \right\}
\end{aligned} \quad (39)$$

$$\simeq -\frac{g^2}{16\pi}\frac{T}{m}\left[1-\frac{1}{1+(1-q^2/4)^{1/2}}\right] \qquad (40)$$

where

$$q=\frac{2(m_1-m_2)}{E}. \qquad (41)$$

The $q=0$ limit means that $E\to 0$ only after the other two and gives the same answer as the $v=0$ limit of (34). This shows that the relative order of the $m_1\to m_2$ and $p\to 0$ is irrelevant, it only matters how they are taken relative to $E\to 0$ as has be found elsewhere. The $q=\infty$ limit is the same as (30) as it should be.

Lastly we can take the high temperature limit first which allows us to keep E,p,m_1,m_2 arbitrary relative to one another and yet get a closed expression for the integral. The leading term is $O(T)$ and is given by making the replacement $n(\omega)\to T/\omega$ (e.g. see appendix of [19]). Thus by using [20]

$$\begin{aligned}&\int_0^\infty dy\frac{y}{y^2+b^2}\log\left[\frac{y^2+2ay\cos(t)+a^2}{y^2-2ay\cos(t)+a^2}\right]\\ &= \frac{1}{2}\pi^2-\pi t+\pi\arctan\left[\frac{(a^2-b^2)\cos(t)}{(a^2+b^2)\sin(t)+2ab}\right] \qquad (42)\\ &\quad a,b>0,\quad 0<t<\pi\end{aligned}$$

we find that

$$\begin{aligned}B_c(E,p,m_1,m_2) &\simeq -\frac{g^2}{32\pi}\frac{T}{p}\arctan\left[\frac{2B(c+1)}{(c+1)^2-B^2}\right] +(1\leftrightarrow 2) \qquad (43)\\ B &= \frac{1}{2}\frac{p}{m_1}\left(1+\frac{m_1^2-m_2^2}{P^2}\right)\\ c^2 &= \frac{E^2}{P^2}-\frac{E^2}{4m_1^2}\left(1+\frac{m_1^2-m_2^2}{P^2}\right)^2\end{aligned}$$

Comparing this result against the high temperature results already obtained independent of (42) (only (40) was derived directly from (42)) we see that there is complete agreement.

It is therefore clear that both time-ordered and retarded functions are not analytic near zero-four momenta when the masses are equal. In particular whenever the zero energy limit is last, after or taken with the $m_1=m_2,p$ limits, various answers can be obtained. If the masses are kept different [11, 16] or three-momentum non-zero till last then a unique answer is obtained. The results are summarised in figure (2). Starting at

Figure 2: The ω integrand of B_c for the different possible orders of the limits.

the centre the energy integral or sum has been performed. The limits are taken in the order indicated as one moves out along a radius. The result obtained for B_c is shown at the end of a radius. In figure (2) it is given as the integrand for a remaining ω integral where a factor of $-g^2/8\pi^2$ has also been taken out c.f. (30),(34),(39). The high temperature equivalents are given in figure (3) c.f. (31),(35),(40). Either RTF or ITF

Figure 3: The high temperature results of B_c for the different possible orders of the limits.

can be used to calculate these results.

The answer obtained when the $E \to 0$ limit is not taken last is of particular note because it is also the answer obtained if no analytic continuation is done in ITF, where $E, p = 0$ and $m_1 = m_2$ are all set at the very start before any integration is done. In this case the order of the limits is irrelevant because the integrand is analytic. It is not clear why these limits coincide but it means one can find this result by keeping $m_1 \neq m_2$ or $p \neq 0$ till the end. The results obtained in these cases are the same as the one used in free energy calculations, (33).

The real part of the retarded function is always finite as $E, p \to 0$. However the imaginary part is not always zero its appearance reflecting the lack of analyticity despite the fact that it is linear in the energy, (38) and [14]. Thus there are peculiarities hiding in this limit which are not obvious from the ITF formalism usually used to calculate the retarded functions.

The time-ordered function, B^{11} of RTF, is found to have the same real part as the retarded and so it is also finite. However the imaginary part shows the effect of the problematic δ^2 term of RTF which results in the same $|\vec{p}|^{-1}$ behaviour as is found in the retarded function as $E, p \to 0$ if $E < p$ and provided $m_1 = m_2$. Thus the δ^2 reflects a genuine divergence in the time-ordered function. It is, however, zero and well behaved when $m_1 \neq m_2$ or $p \neq 0$.

The problems with the bubble diagram and its various limits may seem to be mere technicalities but they do have real do have real implications for calculations of physical quantities. One example of interest is the use of zero four-momentum diagrams to calculate approximations to the finite temperature effective action [14]. A detailed discussion given in [21]. This might be done in order to obtain some dynamical information such as is used in studying the electroweak phase transition (see [1] and references therein). Usually a derivative expansion on the effective action is being performed. At zero temperature the lowest order term is just the effective potential and such an expansion is well defined [2]. The effective potential is thus generated by the zero-momentum 1PI diagrams. The various results for the zero four-momentum limit of diagrams at finite temperature therefore calls into question the simple derivative expansion of thermal effective actions but only when there are equal masses or suitable self-interactions in the theory. This means that the coefficients of the expansion depend on the order in which the time and space derivative expansions are taken. Thus, the results given here suggest that the lowest order term is only necessarily the standard free energy if the time derivative expansion was done first i.e. a static limit taken, $E \to 0$ first. This does not seem to be very relevant to dynamics where one might expect a homogeneous, $p \to 0$ limit to be more relevant but the example of the bubble diagram suggests that the lowest order term is quite different in this case. The calculations performed here and in [11, 16] stress the role of equal masses in this problem of zero four-momentum limits.

Finally note that RTF and ITF give completely consistent results for the bubble diagram (c.f. comments in [22, 23, 24]). Either can be used to calculate any of the functions discussed here. While the RTF has obvious singularities in the zero four-momentum equal mass limit, the $[\delta(K^2 - m^2)]^{N\geq 2}$, the same singularities are hiding in ITF calculations of time-ordered functions. Related problems appear in calculations of the retarded function in the unexpected non-zero, if finite, imaginary parts.

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