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A model-free approach to continuous-time finance

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Abstract

We present a pathwise approach to continuous-time finance based on causal functional calculus. Our framework does not rely on any probabilistic concept. We introduce a definition of continuous-time self-financing portfolios, which does not rely on any integration concept and show that the value of a self-financing portfolio belongs to a class of nonanticipative functionals, which are pathwise analogs of martingales. We show that if the set of market scenarios is *generic* in the sense of being stable under certain operations, such self-financing strategies do not give rise to arbitrage. We then consider the problem of hedging a path-dependent payoff across a generic set of scenarios. Applying the transition principle of Rufus Isaacs in differential games, we obtain a pathwise dynamic programming principle for the superhedging cost. We show that the superhedging cost is characterized as the solution of a path-dependent equation. For the Asian option, we obtain an explicit solution.

1 INTRODUCTION

Continuous-time finance theory (Merton, 1992) was developed using probabilistic concepts such as the Ito integral, the martingale representation theorem, and the Markov property to characterize and compute the value of contingent claims given a probabilistic model describing the

evolution of asset prices. The *uncertainty* on the choice of this probabilistic model has attracted much attention in the last two decades, with many attempts to develop a "robust" approach to continuous-time finance (Avellaneda et al., 1995; Bartl et al., 2019; Biagini et al., 2017; Burzoni et al., 2021), (Bick & Willinger, 1994; Lochowski et al., 2018; Nutz, 2015; Nutz & Soner, 2016; Lyons, 1995) or to seek foundations not directly based on probabilistic modeling (Vovk, 2015; Lochowski et al., 2018). In these approaches, probabilistic concepts are not completely absent: one considers either a family of probabilistic models -representing *model ambiguity*- or an 'outer measure' as in the game-theoretic formulation of Vovk (2015), Lochowski et al. (2018). In particular, the gain of a strategy is defined as an integral whose definition relies at some level on Ito integration.

A possible approach for incorporating Knightian uncertainty in continuous-time finance, which avoids recourse to probabilistic assumptions is one inspired by the pathwise Ito calculus introduced by Föllmer (1981), which was used by Bick and Willinger (1994) and Lyons (1995) to develop model-free approaches to option pricing, assuming continuous price paths with finite quadratic variation. In an insightful expository (Föllmer and Schied, 2013, §5), Föllmer and Schied sketched this nonprobabilistic, pathwise framework (see also Bick and Willinger (1994)). They argue that, if price paths are continuous, then they need to have finite and nonzero quadratic variation, otherwise this gives rise to arbitrage opportunities (free lunches). In particular, one may exihibit a self-financing strategy whose value at T along a price path x is given by

$$(x(T) - x(0))^2 - [x](T) = \int_0^T 2(x(t) - x(0))dx(t), \tag{1}$$

where Equation (1) would become non-negative for all continuous paths with zero quadratic variation (i.e., $[x] = 0$) and strictly positive for all such paths meeting the condition $x(T) \neq x(0)$. In such a setting, it is then natural to use Föllmer 's pathwise Ito calculus (Föllmer, 1981) or its extension to path-dependent functionals, the causal functional calculus (Chiu & Cont, 2022).

However, if we are *uncertain* that price paths would evolve continuously, then paths of zero quadratic variations would no longer give rise to such arbitrage opportunities. In this case, for càdlàg paths, which admit at least one discontinuity, we have

$$[x](T) \geq (\Delta x(t))^2 > 0, \tag{2}$$

for some $t \leq T$ and it is now possible for Equation (1) to go negative. As we shall argue below, when the continuity assumption is removed, the quadratic variation assumption is not necessary to avoid arbitrage and we do not need to make a priori assumptions on the (p-)variation of price paths.

We present in this work a pathwise approach to continuous-time finance, based on causal functional calculus (Chiu & Cont, 2022). Our framework does not rely on any probabilistic concept. We first argue that the set of price trajectories should be stable under certain operations, which leads us to the concept of *generic* set of price paths. All functionals of price paths are defined on such generic sets, which constitute a canonical domain for causal functional calculus.

We then introduce a local definition of continuous-time self-financing portfolios which does not rely on any integration concept and show that the value of a self-financing portfolio belongs to a class of nonanticipative functionals, which are pathwise analogs of martingales. We show that if the set of market scenarios is generic, such self-financing strategies do not give rise to arbitrage. This absence of arbitrage holds on all generic domains that include, but are not limited to, paths of pth-order variation for any $p \geq 2$. In contrast to related results established using the measure-theoretic "game" approach of Vovk (2015) (see also Lochowski et al. (2018)), we are able to work

with the classical notion of arbitrage, rather than passing to an asymptotic relaxation that may not necessarily be implementable by a self-financing trading strategy.

For nonlinear payoffs, we show that a perfect hedge does not exist in general. We adopt a primal approach to superhedging on bounded generic subset. In particular, we solve the model-free superhedging problem over the above set of scenarios using a minimax approach, in the spirit of Isaacs's tenet of transition (Isaacs, 1951), and provide a verification theorem for the optimal cost-to-go functional. As an example, we study the case of Asian options and obtain explicit solution.

Related superhedging problems have been studied using probabilistic approaches or so-called robust approaches based on quasi-sure analysis using a family of probability measures (Bartl et al., 2019; Lochowski et al., 2018; Nutz, 2015; Nutz & Soner, 2016). In contrast to these approaches, our approach is purely pathwise and does not appeal to any probabilistic assumptions. Finally, our hedging strategy comes as a by-product, whereas in the quasi-sure approach, it is not straightforward to compute the optimal strategy (Nutz, 2015).

2 | NOTATIONS

Denote D to be the Skorokhod space of $\mathbb{R}^m$-valued positive càdlàg functions

$$t \longmapsto x(t) := (x_1(t), \dots, x_m(t))' \tag{3}$$

on $\mathbb{R}_+ := [0, \infty)$ and for $p \in 2\mathbb{N}$, we denote $D(\mathbb{R}_+, \mathbb{R}^m \otimes^p)$ the Skorokgod space of $\mathbb{R}^m \otimes^p$-valued càdlàg functions on $\mathbb{R}_+ := [0, \infty)$. Denote C, $\mathbb{S}$, BV, respectively, the subsets of continuous functions, step functions, locally bounded variation functions in D. $x(0-) := x_0 > 0$ and $\Delta x(t) := x(t) - x(t-)$. The path $x \in D$ stopped at $(t, x(t))$ (respectively $(t, x(t-))$)

$$s \longmapsto x(s \wedge t) \tag{4}$$

shall be denoted by $x_t \in D$ (respectively $x_{t-} := x_t - \Delta x(t) \mathbb{I}_{[t,\infty)} \in D$). We write $(D, \mathfrak{d}_{J_1})$ when D is equipped with a complete metric $\mathfrak{d}_{J_1}$, which induces the Skorokhod (a.k.a. J_1) topology.

Let $\pi := (\pi_n)_{n \geq 1}$ be a fixed sequence of partitions $\pi_n = (t_0^n, \dots, t_{k_n}^n)$ of $[0, \infty)$ into intervals $0 = t_0^n < \dots < t_{k_n}^n < \infty$; $t_{k_n}^n \uparrow \infty$ with vanishing mesh $|\pi_n| \downarrow 0$ on compacts. By convention, $\max(\emptyset \cap \pi_n) := 0$, $\min(\emptyset \cap \pi_n) := t_{k_n}^n$. Since π is fixed, we will avoid superscripting π.

For any $p \in 2\mathbb{N}$, we say that $x \in D$ has finite pth-order variation $[x]_p$ if

$$\sum_{\pi_n \ni t_i \leq t} (x(t_{i+1}) - x(t_i))^{\otimes p} \tag{5}$$

converges to $[x]_p$ in the Skorokhod J_1 topology in $D(\mathbb{R}_+, \mathbb{R}^m \otimes^p)$. In light of Chiu and Cont (2018), we remark that in the special case $p = 2$, this definition is equivalent to that of Föllmer (1981). We refer to Cont and Perkowski (2019) for a discussion of pth-order variation for continuous paths. We denote V_p the set of càdlàg paths of finite pth-order variations,

$$t'_n := \max\{t_i < t | t_i \in \pi_n\}, \tag{6}$$

and the following piecewise constant approximations of x by

$$x^n := \sum_{t_i \in \pi_n} x(t_{i+1}) \mathbb{1}_{[t_i, t_{i+1})}. \tag{7}$$

We let $\Omega \subset D$ be *generic* (Definition 3.1) and define our *domain* as

$$\Lambda := \{(t, x_t) | t \in \mathbb{R}_+, x \in \Omega\}. \tag{8}$$

3 | CAUSAL FUNCTIONAL CALCULUS

Causal functional calculus (Chiu & Cont, 2022) is a calculus for nonanticipative functionals defined on sets of càdlàg paths satisfying certain stability properties. In this section, we summarize some key definitions and results; we refer to Chiu and Cont (2022) for a detailed exposition.

Definition 3.1 (Generic sets of paths). A nonempty subset $\Omega \subset D$ is called *generic* if Ω satisfies the following closure properties under operations: (we recall Equation (7) for the definition of x^n)

 (i) For every $x \in \Omega, T > 0, \exists N(T) \in \mathbb{N}; x^n \in \Omega, \quad \forall n \geq N(T)$.
(ii) For every $x \in \Omega, t \geq 0, \exists$ convex neighborhood $\Delta x(t) \in \mathcal{U}$ of 0;

$$x_{t-} + e \mathbb{1}_{[t,\infty)} \in \Omega, \quad \forall e \in \mathcal{U}. \tag{9}$$

Example 3.2. Examples of generic subsets include $\mathbb{S}, BV, D$, and V_p for $p \in 2\mathbb{N}$. Generic subsets are closed under finite intersections. All subsets of C are not generic.

Definition 3.3 (Strictly causal functionals). Let $F : \Lambda \to \mathbb{R}$ and denote $F_-(t, x_t) = F(t, x_{t-})$. F is called *strictly causal* if $F = F_-$.

We associate with the sequence of partitions π a topology on the space Λ of càdlàg paths called the π-topology, introduced in Chiu and Cont (2022).

Definition 3.4 (Continuous functionals). We denote by $C(\Lambda)$ the set of maps $F : \Lambda \to \mathbb{R}$ which satisfy

1.
   (a) $\lim_{s \uparrow t; s \leq t} F(s, x_{s-}) = F(t, x_{t-})$,
   (b) $\lim_{s \uparrow t; s < t} F(s, x_s) = F(t, x_{t-})$,
   (c) $t_n \longrightarrow t; t_n \leq t'_n \implies F(t_n, x^n_{t_n-}) \longrightarrow F(t, x_{t-})$,
   (d) $t_n \longrightarrow t; t_n < t'_n \implies F(t_n, x^n_{t_n}) \longrightarrow F(t, x_{t-})$,
2.
   (a) $\lim_{s \downarrow t; s \geq t} F(s, x_s) = F(t, x_t)$,
   (b) $\lim_{s \downarrow t; s > t} F(s, x_{s-}) = F(t, x_t)$,
   (c) $t_n \longrightarrow t; t_n \geq t'_n \implies F(t_n, x^n_{t_n}) \longrightarrow F(t, x_t)$,
   (d) $t_n \longrightarrow t; t_n > t'_n \implies F(t_n, x^n_{t_n-}) \longrightarrow F(t, x_t)$,

for all $(t, x_t) \in \Lambda$. A functional is called *left (respectively right) continuous* if it satisfies 1.(a)–(d) (respectively 2.(a)–(d)).

By analogy with the concept of "regulated functions" we define:

Definition 3.5 (Regulated functionals). A functional $F : \Lambda \to \mathbb{R}$ is *regulated* if there exists $\widetilde{F} \in C(\Lambda)$ such that $\widetilde{F}_- = F_-$. $\widetilde{F}$ is then unique by Proposition 3.4.2(b).

Remark 3.6. Since $C(\Lambda)$ is an algebra, we remark the set of regulated functionals forms an algebra.

We now introduce functional derivatives, following Cont and Fournié (2010), Dupire (2019):

Definition 3.7 (Horizontal differentiability). $F : \Lambda \longmapsto \mathbb{R}$ is called *differentiable in time* if

$$\mathcal{D}F(t, x_t) := \lim_{h \downarrow 0} \frac{F(t + h, x_t) - F(t, x_t)}{h} \tag{10}$$

exists for all $(t, x_t) \in \Lambda$.

Definition 3.8 (Vertical differentiability). $F : \Lambda \longmapsto \mathbb{R}$ is called *vertically differentiable* if for every $(t, x_t) \in \Lambda$, the map

$$e \longmapsto F\left(t, x_t + e \mathbb{I}_{[t,\infty)}\right) \tag{11}$$

is differentiable at 0. We define $\nabla_x F(t, x_t) := (\nabla_{x_1} F(t, x_t), \dots, \nabla_{x_m} F(t, x_t))'$;

$$\nabla_{x_i} F(t, x_t) := \lim_{\epsilon \to 0} \frac{F\left(t, x_t + \epsilon \mathbf{e}_i \mathbb{I}_{[t,\infty)}\right) - F(t, x_t)}{\epsilon}. \tag{12}$$

Definition 3.9 (Differentiability). A functional is called *differentiable* if it is horizontally and vertically differentiable.

Remark 3.10. All definitions above extend to multidimensional functions on Λ whose components satisfy the respective conditions.

Lemma 3.11. *A function on Λ is strictly causal if and only if it is differentiable in space with vanishing derivative.*

Proof. We refer to Chiu and Cont (2022, §4). $\qquad \square$

Definition 3.12 (Classes $\mathcal{S}$ and $\mathcal{M}$). A continuous and differentiable functional F is of *class $\mathcal{S}$* if $\mathcal{D}F$ is right continuous and locally bounded, $\nabla_x F$ is left continuous and strictly causal. If in addition, $\mathcal{D}F$ vanishes, then F is of *class $\mathcal{M}$*.

Denote $\mathcal{M}(\Lambda)$ the set of all functionals of class $\mathcal{M}$ and $\mathcal{M}_0(\Lambda)$ the subset of $\mathcal{M}(\Lambda)$ with vanishing initial values.

Definition 3.13 (Pathwise integral). Let $\phi : \Lambda \longmapsto \mathbb{R}^m$; ϕ_- be left continuous. For every $x \in \Omega$, define

$$\mathbf{I}\left(t, x_t^n\right) := \sum_{\pi_n \ni t_i \leq t} \phi\left(t_i, x_{t_i-}^n\right) \cdot (x(t_{i+1}) - x(t_i)). \tag{13}$$

If $\mathbf{I}(t, x_t) := \lim_n \mathbf{I}(t, x_t^n)$ exists and $\mathbf{I}$ is continuous, then ϕ is called *integrable* and $\mathbf{I} := \int_0^{\cdot} \phi \, dx$ is called the *pathwise integral*.

We remark that, if $\Omega \subset QV$, then integrands of the type $\nabla f \circ x, f \in C^2(\mathbb{R}^d)$ (Föllmer, 1981) and their path-dependent analogs $\nabla F \circ x, F \in \mathbb{C}^{1,2}(\mathbb{R}^d)$ (Chiu & Cont, 2022) are integrable. The following result (Chiu and Cont, 2022, §5) characterizes class $\mathcal{M}(\Lambda)$ as the class of functionals admitting a representation as pathwise integral.

Theorem 3.14 (Representation theorem). *A functional $F : \Lambda \to \mathbb{R}$ is a pathwise integral if and only if $F \in \mathcal{M}_0(\Lambda)$:*

$$F \in \mathcal{M}_0(\Lambda) \iff \exists \phi : \Lambda \to \mathbb{R}^m, \phi_- \text{ left-continuous}; \tag{14}$$

$$F(t, x_t) = \int_0^t \phi_- dx, \quad \forall (t, x_t) \in \Lambda. \tag{15}$$

4 | MARKET SCENARIOS, SELF-FINANCING STRATEGIES, AND ARBITRAGE

We consider a frictionless market with $d > 0$ tradable assets, and one numeraire whose price is identically 1. We denote x to be the price paths of tradable assets and $x \in \Omega$, where Ω is generic (Definition 3.1).

A trading strategy is a pair (ϕ, ψ) of regulated functionals $\phi : \Lambda \mapsto \mathbb{R}^d$ and $\psi : \Lambda \mapsto \mathbb{R}$. The value V of the portfolio is given by

$$V(t, x_t) := \widetilde{\phi}(t, x_t) \cdot x(t) + \widetilde{\psi}(t, x_t). \tag{16}$$

The number of shares in assets and the quantity in numeraire held immediately before the portfolio revision at time t will be denoted by ϕ_- and ψ_-.

A key concept in continuous-time finance is the concept of *self-financing* strategy (Bick & Willinger, 1994, §2). This concept is usually defined in a probabilistic setting, by equating the changes in the portfolio value V with a *gain process* defined as a stochastic integral $\int \phi \, dx$. An arbitrage strategy is then defined as a riskless self-financing strategy, which may lead to nonzero profit in certain scenarios. The notion of arbitrage thus hinges upon the definition of self-financing strategy. We aim to address the following fundamental questions:

- What is meant by a *self-financing strategy*?
- What is an *arbitrage* strategy?

We propose a new approach to the notion of self-financing strategy based on *local* properties, without involving any use of (pathwise or stochastic) integration notions.

Definition 4.1 (Self-financing strategy). A trading strategy or portfolio (ϕ, ψ) is called *self-financing* if for every $(t, x) \in \Lambda$,

(i) $\Delta\widetilde{\phi}(t, x_t) \cdot x(t) + \Delta\widetilde{\psi}(t, x_t) = 0$,
(ii) $(\widetilde{\phi}(t + h, x_t) - \widetilde{\phi}(t, x_t)) \cdot x(t) + \widetilde{\psi}(t + h, x_t) - \widetilde{\psi}(t, x_t) = 0$ for all $h > 0$.

Both conditions correspond to the property that the proceeds from any change in the asset positions is reflected in the change in the cash position. However, the important point is that we only require this in two situations:

(i) an instantaneous change in the asset positions, and
(ii) a change in the asset/cash position while asset prices remain constant.

As we will show, through piecewise constant approximation these two situations cover the case of all continuous-time strategies under minimal regularity properties.

Remark 4.2. If (ϕ, ψ) is self-financing, then the value of the portfolio may also be expressed as

$$V(t, x_t) = \phi(t, x_{t-}) \cdot x(t) + \psi(t, x_{t-}). \tag{17}$$

We remark here that interchanging (16) and (17) for the definition of a portfolio value would not have any effect for self-financing portfolios.

Theorem 4.3 (Gain of a self-financing strategy as a pathwise integral). *Let V be the portfolio value associated with the trading strategy (ϕ, ψ). Then (ϕ, ψ) is self-financing if and only if $V \in \mathcal{M}(\Lambda)$, $\nabla_x V = \phi_-$. In that case*

$$V(t, x_t) = V(0, x_0) + \int_0^t \phi(s, x_{s-}) dx. \tag{18}$$

Proof. If (ϕ, ψ) is self-financing, we may first use Equation (17) to deduce that $\nabla_x V = \phi_-$, which is left continuous and strictly causal. From Equation (16) and the fact that $C(\Lambda)$ is an algebra (i.e., Proposition 3.4), we see that V is continuous. We then apply Equation (16) and Definition 4.1(ii) to deduce that $\mathcal{D}V$ is vanishing. Hence, $V \in \mathcal{M}(\Lambda)$ and Equation (18) follows from Theorem 3.14.

On the other hand, if $V \in \mathcal{M}(\Lambda)$, then V is continuous. By the continuity of V, Equation (17) and Proposition 3.4.2(b), we first obtain Equation (16), hence Definition 4.1(i). Since $\mathcal{D}V$ vanishes, by Chiu and Cont (2022, Lem.5.1), we obtain

$$V(t + h, x_t) - V(t, x_t) = \int_t^{t+h} \mathcal{D}V(s, x_t) ds = 0. \tag{19}$$

Resorting once again to Equation (16), we also obtain Definition 4.1.(ii), hence (ϕ, ψ) is self-financing. $\square$

Proposition 4.4. *Let $V \in \mathcal{M}(\Lambda)$, then the following properties are equivalent:*

(i) *V is the value of a self-financing trading strategy (ϕ, ψ).*
(ii) *$\nabla_x V$ is regulated.*

Proof. (i) implies (ii) follows from Definition 4.1, Definition 3.5, and Theorem 4.3. Assume (ii) holds, let ϕ be the continuous version of $\nabla_x V$ and put

$$\psi(t, x_t) := V(t, x_t) - \phi(t, x_t) \cdot x(t), \tag{20}$$

then ψ is continuous (i.e., $C(\Lambda)$ is an algebra) and V is the portfolio value associated with the trading strategy (ϕ, ψ). Taking Δ from Equation (20), we obtain

$$\Delta V - \nabla_x V \Delta x = x \cdot \Delta \phi + \Delta \psi. \tag{21}$$

By Theorem 3.14, we deduce the LHS of Equation (21) vanishes and obtain Equation (17), hence $\nabla_x V = \phi_-$, the proof is complete by Theorem 4.3. □

Remark 4.5. In view of Theorem 4.3, Proposition 4.4, and Equation (20), we may call a functional V *self-financing* if $V \in \mathcal{M}$ with regulated $\nabla_x V$.

Definition 4.6 (Arbitrage). A self-financing strategy (ϕ, ψ) with value V is called an *arbitrage* on $[0, T]$ if

$$\forall x \in \Omega, \qquad V(T, x_T) - V(0, x_0) \geq 0 \tag{22}$$

and there exists $x \in \Omega$ such that $V(T, x_T) - V(0, x_0) > 0$.

Lemma 4.7. *Let $M \in \mathcal{M}_0(\Lambda)$. If there exists $T > 0$;*

$$M(T, x_T) \geq 0 \tag{23}$$

for all $x \in \Omega$, then for every $x \in \Omega$, the map

$$t \longmapsto M(t, x_t) \tag{24}$$

is non-negative for $t \leq T$.

Proof. If $M \in \mathcal{M}_0(\Lambda)$, then $\mathcal{D}M$ vanishes, by Chiu and Cont (2022, Lem. 5.1), we obtain

$$M(t, x_t) = M(t, x_t) + \int_t^T \mathcal{D}M(s, x_t)ds = M(T, x_T) \geq 0 \tag{25}$$

for all $t \leq T$, where the last inequality is due to $x_t \in \Omega$. □

Theorem 4.8 (Fair game property). *Let $M \in \mathcal{M}_0(\Lambda)$. If there exists $T > 0$ such that*

$$M(T, x_T) \geq 0 \tag{26}$$

for all $x \in \Omega$, then $M(T, x_T) \equiv 0$.

Proof. Let $T > 0$; $M(T, x_T) \geq 0 \ \forall \ x \in \Omega$. By Lemma 4.7, we first obtain

$$M(t, x_t) \geq 0 \tag{27}$$

for all $t \leq T$, $x \in \Omega$. Suppose there exists $\omega \in \Omega$;

$$M(T, \omega_T) > 0. \tag{28}$$

By the continuity of M and Theorem 3.14, it follows

$$M(T, \omega_T^n) = \sum_{\pi_n \ni t_i \leq T} \nabla_x M\left(t_i, \omega_{t_i-}^n\right) (\omega(t_{i+1}) - \omega(t_i)) > 0 \tag{29}$$

for all n sufficiently large. Define

$$t_n^* := \min\left\{ t_i \in \pi_n \,|\, M\left(t_i, \omega_{t_i}^n\right) > 0 \right\}, \tag{30}$$

then $t_n^* \leq T$. By Equations (27) and (29), the left continuity of M and the fact that $\omega^n \in \Omega$, we obtain

$$M\left(t_n^*, \omega_{t_n^*}^n\right) > M\left(t_n^*, \omega_{t_n^*-}^n\right) = 0, \tag{31}$$

hence

$$M\left(t_n^*, \omega_{t_n^*}^n\right) = \nabla_x M\left(t_n^*, \omega_{t_n^*-}^n\right) \Delta\omega\left(t_n^*\right) > 0. \tag{32}$$

Definition 3.1(ii) implies that there exists $\epsilon > 0$;

$$\omega^* := \omega_{t_n^*-}^n - \epsilon \Delta\omega\left(t_n^*\right) \mathbb{I}_{[t_n^*, \infty)} \in \Omega, \tag{33}$$

hence

$$M\left(t_n^*, \omega_{t_n^*}^*\right) = \nabla_x M\left(t_n^*, \omega_{t_n^*-}^n\right)\left(-\epsilon \Delta\omega\left(t_n^*\right)\right) < 0, \tag{34}$$

which is a contradiction to Equation (27). □

Using these results we can now show that if the set of market scenarios is a generic set of paths, arbitrage in the sense of Definition 3.1 does not exist:

Corollary 4.9. *Arbitrage does not exist in a generic market.*

Proof. It is a direct consequence of Definition 4.6 and Theorem 4.3 and 4.8. □

Remark 4.10. As previously discussed, the set $\mathbb{S}$ of piecewise-constant paths, the space $D([0, \infty), \mathbb{R}_+^m)$ of positive càdlàg paths or the space V_p of càdlàg paths with finite pth-order variation for $p \in 2\mathbb{N}$ are examples of generic sets of paths, to which the above result applies. However, unlike the results of Lochowski et al. (2018), Schied and Voloshchenko (2016), and Vovk (2015), the proof of the above result does not involve any assumption on the variation index of the path.

5 | WHEN DOES A PAYOFF ADMIT A PERFECT HEDGE?

In this section, we define path-dependent payoff as functionals and prove that a payoff can be perfectly hedged in a generic market if and only if it is linear. We give an explicit example of such a payoff: the Asian option with zero strike.

For $u, v \in \mathbb{R}^l$, we write $u > v$ if $u_i > v_i$ for all i. We call v positive if $v > 0$. Let Ω be a generic set of paths. In order for the operation

$$x_{t-} + \mathfrak{e}(t, x_{t-})\mathbb{1}_{[t,\infty)} \in \Omega \tag{35}$$

to be closed, $\mathfrak{e}(t, x_{t-})$ may not take arbitrary values, this motivates the following:

Definition 5.1 (Admissible perturbation). A regulated function $\mathfrak{e} : \Lambda \to \mathbb{R}^d$ is called an *admissible* perturbation if for every $x \in \Omega, t \geq 0$,

$$x_{t-} + \mathfrak{e}(t, x_{t-})\mathbb{1}_{[t,\infty)} \in \Omega \tag{36}$$

We denote $\mathcal{E}$ to be the set of all admissible perturbations on Λ.

Example 5.2. $\mathfrak{e} := 0$ is admissible. If Ω is either $\mathbb{S}$, V_p, or D, then every $\mathbb{R}^d$-valued regulated function $\mathfrak{e}$ satisfying

$$\mathfrak{e}(t, x_{t-}) > -x(t-), \tag{37}$$

for all $x \in \Omega, t \geq 0$ is admissible.

Definition 5.3 (Nondegenerate). A subset Ω is called *nondegenerate*, if there exists $\mathfrak{e}^1, \dots, \mathfrak{e}^d \in \mathcal{E}$ where

$$\mathfrak{e}^i_j(t, x_{t-}) \begin{cases} \neq 0, & \text{if } i = j. \\ = 0, & \text{otherwise;} \end{cases} \tag{38}$$

for every $x \in \Omega, t \geq 0$.

Remark 5.4. If Ω is either $\mathbb{S}$, V_p, or D, then Ω is nondegenerate. In the sequel, we shall assume that Ω is nondegenerate.

Definition 5.5 (Payoff). A *payoff* with maturity $T > 0$ is a functional $H : \Omega \to \mathbb{R}$ such that

(i) $H(x) = H(x_T)$ for all $x \in \Omega$.
(ii) For every $x \in \Omega, t \geq 0$ and the map

$$e \longmapsto H(x_{t-} + e\mathbb{1}_{[t,\infty)}) \tag{39}$$

is continuous on every convex neighborhood $\mathcal{U} \subset \mathbb{R}^d$ of 0 satisfying Equation (9).
(iii) The functional $(t, x_t) \longmapsto H(x_t)$ is continuous on Λ and for every $\mathfrak{e} \in \mathcal{E}$, the functional

$$(t, x_t) \in \Lambda \longmapsto H(x_{t-} + \mathfrak{e}(t, x_{t-})\mathbb{1}_{[t,\infty)}), \tag{40}$$

is regulated.

Example 5.6. Let $d = 1, T > 0, K \geq 0$ and let V be the value of a self-financing portfolio. Then

(a) $H(x) := (\frac{1}{T} \int_0^T x(t)dt - K)^+$,
(b) $H(x) := (\sup_{s \leq T} x(s) - x(T))^+$,
(c) $H(x) := (V(T, x_T) - K)^+$,

satisfy Definition 5.5.

Proof. We first compute $H(x_{t-} + e\mathbb{1}_{[t,\infty)})$ and obtain

$$(a) \quad \left(\frac{1}{T} \left(\int_0^{t \wedge T} xds + (T-t)(x(t-) + e)\mathbb{1}_{[0,T]} \right) - K \right)^+,$$

$$(b) \quad \left(\sup_{s < t} x_T(s) - x_T(t-) - e\mathbb{1}_{[0,T]} \right)^+,$$

$$(c) \quad \left(V(t, x_{(t \wedge T)-}) + \nabla_x V(t, x_{t-})e\mathbb{1}_{[0,T]} - K \right)^+, \tag{41}$$

which are all continuous in e and we obtain Definition 5.5(ii). If we replace e with $\Delta x(t)$ and observe in (b) that

$$\left(\sup_{s < t} x_T(s) - x_T(t) \right)^+ = \left(\sup_{s \leq t} x_T(s) - x_T(t) \right)^+, \tag{42}$$

we see that $(t, x_t) \mapsto H(x_t)$ is continuous. If we replace e with $e \in \mathcal{E}$, by the admissibility of e and Remark 3.6, we obtain Definition 5.5(iii). □

Definition 5.7 (Vertically affine functionals). A payoff $H : \Omega \to \mathbb{R}$ is called *vertically affine* if for every $x \in \Omega, t \geq 0$ and convex neighborhood $\mathcal{U} \subset \mathbb{R}^d$ of 0 satisfying Equation (9), the map

$$e \longmapsto H(x_{t-} + e\mathbb{1}_{[t,\infty)}) \tag{43}$$

is affine on $\mathcal{U}$.

Remark 5.8. If $K = 0$, the payoffs in Example 5.6(i) and (iii) are vertically affine.

Definition 5.9 (Perfect hedge). A payoff $H : \Omega \to \mathbb{R}$ with maturity $T > 0$ is said to admit a *perfect hedge* on Ω if there exists a self-financing portfolio with value V such that

$$\forall x \in \Omega, \qquad V(T, x_T) = H(x_T). \tag{44}$$

Theorem 5.10. *Every vertically affine payoff admits a perfect hedge on Ω.*

Proof. If H is vertically affine, then $e \longmapsto H(x_{t-} + e\mathbb{1}_{[t,\infty)})$ is an affine map. Since Ω is generic, it follows there exists a convex neighborhood $\Delta x(t) \in \mathcal{U} \subset \mathbb{R}^d$ of 0 satisfying Equation (9) and we obtain a constant c and a $\phi \in \mathbb{R}^d$;

$$H(x_{t-} + e\mathbb{1}_{[t,\infty)}) = c(t, x_{t-}) + \phi(t, x_{t-}) \cdot e, \tag{45}$$

on $\mathcal{U}$, hence $c(t, x_{t-}) = H(x_{t-})$ and

$$H(x_t) - H(x_{t-}) = \phi(t, x_{t-}) \cdot \Delta x(t). \tag{46}$$

Since it holds for every $x \in \Omega$ and $t \geq 0$, it follows from Definition 5.5(iii) that

$$V(t, x_t) := H(x_t), \tag{47}$$

is continuous on Λ, $\mathcal{D}V$ vanishes and by Equation (46) and Lemma 3.11, $\nabla_x V(t, x_t) = \phi(t, x_{t-})$, which is strictly causal and V is of class $\mathcal{M}$. It remains to show that ϕ is regulated. Since Ω is nondegenerate, there exists everywhere nonvanishing $\mathfrak{e}^i \in \mathcal{E}$, $i = 1, \ldots, d$;

$$H(x_{t-} + \mathfrak{e}^i(t, x_{t-})\mathbb{I}_{[t,\infty)}) - H(x_{t-}) = \phi(t, x_{t-}) \cdot \mathfrak{e}^i(t, x_{t-}). \tag{48}$$

Since $(\mathfrak{e}^i_i) \neq 0$, it follows from Definition 5.5(iii), Remark 3.6, and Equation (48) that ϕ is regulated. By Proposition 4.4 and Remark 4.5, V is self-financing and hence the claim follows. □

Corollary 5.11. *A payoff admits a perfect hedge on a generic set of paths Ω if and only if it is vertically affine.*

Proof. The if part follows from Theorem 5.10. If H admits a perfect hedge then there exists $V \in \mathcal{M}(\Lambda)$; $H(x_T) = V(T, x_T)$ on Ω. It follows that

$$H(t, x_{t-} + e\mathbb{I}_{[t,\infty)}) = V(t, x_{t-}) + \nabla_x V(t, x_{t-})e. \tag{49}$$

□

Example 5.12 (Asian option with $K = 0$). The Asian option with strike $K = 0$, that is, for example, 5.6(i) is vertically affine and the perfect hedge is computed as

$$\nabla_x V(t, x_t) = \frac{T - t}{T},$$

$$V(t, x_t) = \frac{1}{T}\left(\int_0^{t \wedge T} x(s)ds + (T - t)x(t) \right), \tag{50}$$

$$V(0, x_0) = x(0).$$

We remark here that the perfect hedge is model independent.

6 | HEDGING STRATEGY FOR NON-LINEAR PAYOFFS: ASIAN OPTION

In the previous section, we have established that a perfect hedge may not exist for nonlinear payoffs, thereby justifying the search for an alternative approach. A well-studied paradigm for valuation in the absence of perfect replicating strategies is super-hedging across a set of market scenarios (Avellaneda et al., 1995; Lyons, 1995).

Here we deploy this idea in a model-free manner on a *bounded* generic set of paths. Let Ω be generic. We define, for constants $0 \leq a < b$,

$$\Omega_a^b := \{x \in \Omega | a < x(t) < b\}. \tag{51}$$

Observe that Ω_a^b is again generic, hence is itself free of arbitrage in the sense of Definition 4.6. Also, the superhedging price obtained on this set is a proper arbitrage-free price from the standpoint of Ω. We denote

$$\Omega_a^b(x_t) := \left\{ z \in \Omega_a^b | z_t = x_t \right\}, \qquad \mathcal{L} := \{\nabla_x V | V \text{ is self-financing}\}. \tag{52}$$

Definition 6.1 (Superhedging price and strategy). Let H be a payoff defined on Ω with maturity $T > 0$ and V be self-financing (Remark 4.5) that dominates H on Ω_a^b, that is,

$$V(T, x_T) \geq H(x_T), \tag{53}$$

for all $x \in \Omega_a^b$. If for every other self-financing W that dominates H on Ω_a^b, we have

$$W(0, x_0) \geq V(0, x_0), \tag{54}$$

then $V(0, x_0)$ is called the superhedging price of H and $\nabla_x V$ is a superhedging strategy for the payoff H.

A superhedging strategy, if it exists, may not be unique. We first develop the notion of optimal strategy (which, if it exists, will be unique), in the spirit of Isaacs's tenet of transition in differential games (Isaacs, 1951, p3). Our approach here is to construct a (cost-to-go) functional $U \in S$ (Definition 3.12) such that for all $0 \leq s \leq t \leq T$ and $x \in \Omega_a^b$, the followings hold:

$$U(s, x_s) = \min_{\phi \in \mathcal{L}} \sup_{z \in \Omega_a^b(x_s)} \left\{ U(t, z_t) - \int_s^t \phi dz \right\}, \tag{55}$$

$$U(T, x_T) = H(x_T).$$

Lemma 6.2. *Let $U \in S$ be a functional that satisfies Equation (55). Then the map*

$$h \longmapsto U(s + h, x_s) \tag{56}$$

is decreasing on $[0, \infty)$.

Proof. We have

$$U(s, x_s) \geq \min_{\phi \in \mathcal{L}} \left\{ U(t, z_t) - \int_s^t \phi dz \right\} \tag{57}$$

for all $z \in \Omega_a^b(x_s)$, this holds, in particular for all z stopped at s. It follows

$$U(s, x_s) \geq \min_{\phi \in \mathcal{L}} U(t, z_s) = U(t, x_s). \tag{58}$$

$\square$

Remark 6.3. (Hedging strategy)

Thus if U satisfies Equation (55), then $V(T, x_T) := U_0 + \int_0^T \nabla_x U dx$ solves Equation (53), meeting condition (54) and the solution is unique up to Ω_a^b due to

$$U_1(t, x_t) = \min_{\phi \in \mathcal{L}} \sup_{z \in \Omega_a^b(x_t)} \left\{ H(T, z_T) - \int_t^T \phi dz \right\} = U_2(t, x_t). \tag{59}$$

In particular, $U(t_0, x_{t_0})$ is the superhedging price to hedge starting at time $0 \leq t_0 < T$. The relationship with the value of the hedging portfolio V (see also Remark 4.5) is

$$V(t, x_t) = U(t_0, x_{t_0}) + \int_{t_0}^t \nabla_x U dx = U(t, x_t) - \int_{t_0}^t \mathcal{D} U ds, \tag{60}$$

hence at maturity time T, the final portfolio value is

$$V(T, x_T) := H(T, x_T) - \int_{t_0}^T \mathcal{D} U ds \geq H(T, x_T), \tag{61}$$

where the last inequality is due to Lemma 6.2 and the final PnL is $\int_{t_0}^T -\mathcal{D} U ds$.

We now use the following Minimax Theorem to prove a verification theorem.

Theorem 6.4 (Minimax). *If $M \in \mathcal{M}$; then*

$$\min_{\phi \in \mathcal{L}} \max_{z \in \Omega_a^b(x_s)} \left\{ \int_s^t (\nabla_x M - \phi) dz \right\} = 0 = \max_{z \in \Omega_a^b(x_s)} \min_{\phi \in \mathcal{L}} \left\{ \int_s^t (\nabla_x M - \phi) dz \right\} \tag{62}$$

Proof. We first have

$$c := \inf_{\phi \in \mathcal{L}} \sup_{z \in \Omega_a^b(x_s)} \left\{ \int_s^t (\nabla_x M - \phi) dz \right\}$$

$$\leq \max_{z \in \Omega_a^b(x_s)} \left\{ \int_s^t (\nabla_x M - \nabla_x M) dz \right\} = 0. \tag{63}$$

If $c < 0$, then there exists an $\epsilon > 0$ such that

$$\int_s^t (\phi_\epsilon - \nabla_x M) dz \geq -(c + \epsilon) > 0, \tag{64}$$

which gives an arbitrage. It follows from Theorem 4.8 that $c = 0$ and hence the infimum and supremum are attained, respectively, by $\phi := \nabla_x M$ and any z. The case of maximin follows similar lines of proof. □

We obtain, as a corollary, yet another property functionals of class $\mathcal{M}$, reminiscent of the martingale property.

Corollary 6.5. *Define for $H : \Lambda \longmapsto \mathbb{R}$*

$$\mathbb{E}(H(t, x_t)|x_s) := \begin{cases} \min\limits_{\phi \in \mathcal{L}} \sup\limits_{z \in \Omega_a^b(x_s)} \left\{ H(t, z_t) - \int_s^t \phi dz \right\}, & \text{if RHS exists} \\ \infty, & \text{otherwise.} \end{cases} \tag{65}$$

Then for every $M \in \mathcal{M}(\Lambda)$, we have

$$\mathbb{E}(M(t, x_t)|x_s) = M(s, x_s). \tag{66}$$

Theorem 6.6 (Verification theorem). *Let $U \in S(\Lambda), \nabla_x U \in \mathcal{L}$; U satisfies*

$$\sup_{z \in \Omega_a^b(x_t)} \int_t^T \mathcal{D}U(s, z_s)ds = 0, \tag{67}$$

$$U(T, x_T) = H(x_T),$$

for all $t \le T$ and $x \in \Omega_a^b$. Then $\phi := \nabla_x U$ is a superhedging strategy for H on Ω_a^b and achieves the optimum in Equation (55).

Proof. We first obtain

$$c := \inf_{\phi \in \mathcal{L}} \sup_{z \in \Omega_a^b(x_s)} \left\{ \int_s^t \mathcal{D}U(r, z_r)dr + \int_s^t (\nabla_x U - \phi)dz \right\}$$

$$\le \min_{\phi \in \mathcal{L}} \max_{z \in \Omega_a^b(x_s)} \left\{ \int_s^t (\nabla_x U - \phi)dz \right\} = 0, \tag{68}$$

due to Lemma 6.2 and Theorem 6.4. It remains to show that $c \ge 0$.

$$c \ge \sup_{z \in \Omega_a^b(x_s)} \inf_{\phi \in \mathcal{L}} \left\{ \int_s^t \mathcal{D}U(r, z_r)dr + \int_s^t (\nabla_x U - \phi)dz \right\}$$

$$\ge \sup_{z \in \Omega_a^b(x_s)} \left\{ \int_s^t \mathcal{D}U(r, z_r)dr \right\} + \max_{z \in \Omega_a^b(x_s)} \min_{\phi \in \mathcal{L}} \left\{ \int_s^t (\nabla_x U - \phi)dz \right\} = 0, \tag{69}$$

by Equation (67) and Theorem 6.4 (Minimax). The infimum is attained by $\phi := \nabla_x U$. ☐

Example 6.7 (Asian option). Let Ω be either BV, V_p; $p \in 2\mathbb{N}$ or D. The optimal cost-to-go functional is

$$U(t, x_t) = H^+(t, x_t)p(x(t)) + H^-(t, x_t)(1 - p(x(t))) \tag{70}$$

where

$$H^+(t, x_t) = \left(\frac{1}{T} \left(\int_0^t x(s)ds + b(T - t) \right) - K \right)^+,$$

$$H^-(t, x_t) = \left(\frac{1}{T}\left(\int_0^t x(s)ds + a(T-t)\right) - K\right)^+,$$

$$p(x(t)) = \frac{x(t) - a}{b - a}, \tag{71}$$

and the optimal strategy is

$$\nabla_x U(t, x_t) = \frac{H^+(t, x_t) - H^-(t, x_t)}{b - a}. \tag{72}$$

Proof. We first see that U is of class S with $U(T, x_T) = H(x_T)$. For $z \in \Omega_a^b(x_t)$, we have

$$\mathcal{D}U(s, z_s) = \mathcal{D}H^+(s, z_s)p(z) + \mathcal{D}H^-(s, z_s)(1 - p(z)), \tag{73}$$

where

$$\mathcal{D}H^+(s, z_s) = \frac{z(s) - b}{T}\mathbb{1}_{\{H^+ > 0\}},$$

$$\mathcal{D}H^-(s, z_s) = \frac{z(s) - a}{T}\mathbb{1}_{\{H^- > 0\}}. \tag{74}$$

Since $H^+ = 0$ implies $H^- = 0$ and that $H^- > 0$ implies $H^+ > 0$, it follows

$$\mathcal{D}U(s, z_s) = \frac{(z(s) - b)(z(s) - a)}{T(b - a)}\mathbb{1}_{\{H^+ > 0\}}\mathbb{1}_{\{H^- = 0\}} \leq 0. \tag{75}$$

For sufficiently small $\epsilon > 0$, we construct a path $z^\epsilon \in \Omega_a^b(x_t)$:

$$z^\epsilon(s) := \begin{cases} x(t), & s \in [t, t + \epsilon) \\ b - \epsilon, & [t + \epsilon, \infty), \end{cases} \tag{76}$$

and observe that

$$0 \geq \int_t^T \mathcal{D}U(s, z_s^\epsilon)ds \geq -\epsilon\left(1 - \frac{\epsilon}{b - a}\right), \tag{77}$$

hence $\sup\limits_{z \in \Omega_a^b(x_t)} \int_t^T \mathcal{D}U(s, z_s)ds = 0$ and we obtained Equation (67) in Theorem 6.6. $\square$

Remark 6.8. Note that if $K = 0$, we obtain the perfect hedge in Example 5.12 (50) as a special case. If we set $a = 0$ and let $b \uparrow \infty$, then Equation (70) converges to the superhedging price on Ω of the Asian option

$$U(t, x_t) = \left(\frac{1}{T}\int_0^t x(s)ds - K\right)^+ + x(t)\frac{T - t}{T}. \tag{78}$$

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