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# A bound for the orders of centralizers of irreducible subgroups of algebraic groups

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**Abstract.** We prove that if $G$ is a connected semisimple algebraic group of rank $r$, and $H$ is a subgroup of $G$ that is contained in no proper parabolic subgroup, then we have $|C_G(H)| < c^r|Z(G)|$, where $c$ is an absolute constant ($c = 16$ if all simple factors of $G$ are classical, and $c \leq 197$ in general).

## 1 Introduction

Let $G$ be a connected semisimple algebraic group of rank $r$ over an algebraically closed field $K$. A subgroup $H$ of $G$ is *$G$-irreducible* if it is contained in no proper parabolic subgroup of $G$. Such a subgroup $H$ has finite centralizer in $G$ by [4, Lemma 2.1]. In this note, we give a bound for the order of the centralizer $C_G(H)$. In the case where $H$ is connected, this and much more was done in [5].

**Theorem 1.** *Let $G$ be a connected semisimple algebraic group of rank $r$ over an algebraically closed field $K$, and let $H$ be a $G$-irreducible subgroup. Then there is a constant $c \leq 197$ such that*

$$|C_G(H)| < c^r|Z(G)|.$$

For the case where all the simple factors of $G$ are classical, the proof shows that the constant $c$ can be improved to 16 (see Lemmas 2.3 and 2.4). Example (1) below shows that $c$ must be at least 4. It is possible that the theorem holds with $c = 4$, but we have not attempted to achieve this degree of precision.

Theorem 1 has been used in a number-theoretic application in [1].

**Examples.** (1) Let $G = \mathrm{SO}_n(K)$ with $\mathrm{char}(K) \neq 2$, and let $v_1, \dots, v_n$ be an orthonormal basis of the underlying orthogonal space. Then $G$ has an elementary abelian subgroup $H \cong 2^{n-1}$ consisting of elements that send each $v_i \mapsto \pm v_i$. It is easy to see that $H$ is $G$-irreducible, and $|C_G(H)| = |H| = 2^{n-1}$. When $n$ is odd, this is equal to $4^r$, where $r$ is the rank of $G$.

(2) Let $G = \mathrm{Sp}_{2r}(K)$ with $\mathrm{char}(K) \neq 2$. Then $G$ has an irreducible subgroup $H = \mathrm{Sp}_2(K)^r$, and $|C_G(H)| = |Z(H)| = 2^r$.

(3) Let $s$ be a prime and $H = s^{1+2a}$ an extraspecial group. There is an irreducible embedding of $H$ in $\mathrm{SL}_n(K)$, where $n = s^a$ and $\mathrm{char}(K) \neq s$. Hence we have $\bar{H} = s^{2a} < G = \mathrm{PGL}_n(K)$ and $|C_G(\bar{H})| = |\bar{H}| = s^{2a} = n^2$. Our proof of Theorem 1 shows that $n^2$ is actually the correct bound for $G$ simple of type $A_{n-1}$ (see Lemma 2.3).

(4) Here are some examples for $G$ simple of exceptional type (see for example [2, Theorem 3]):

$$G = E_8 : H = 2^5 \quad \text{with } C_G(H) = 2^{5+10},$$
$$G = E_6 : H = 3^3 \quad \text{with } C_G(H) = 3^{3+3},$$
$$G = G_2 : H = 2^3 \quad \text{with } C_G(H) = H.$$

So, for example, there is an irreducible subgroup $H = (2^5)^l < G = E_8^l$ such that $|C_G(H)| = 2^{15l} = c^{\mathrm{rk}(G)}$, where $c = 2^{15/8}$.

## 2 Proof of the theorem

We prove Theorem 1 in a series of lemmas.

**Lemma 2.1.** *Suppose the conclusion of Theorem 1 holds in the case where $G$ is simple of adjoint type. Then the conclusion holds in general.*

*Proof.* Let $G = G_1 \cdots G_k$, a commuting product of simple algebraic groups $G_i$. Let $\bar{G} = G/Z(G) = \bar{G}_1 \times \cdots \times \bar{G}_k$, the direct product of adjoint groups $\bar{G}_i$, and let $\pi\colon G \mapsto \bar{G}$ be the natural map. Let $H < G$ be $G$-irreducible and $\bar{H} = \pi(H)$. Then $\pi$ maps $C_G(H) \mapsto C_{\bar{G}}(\bar{H})$, so $|C_G(H)| \leq |C_{\bar{G}}(\bar{H})||Z(G)|$. Moreover, we have $C_{\bar{G}}(\bar{H}) = \prod C_{\bar{G}_i}(\bar{H}_i)$, where $\bar{H}_i$ is the projection of $\bar{H}$ in $\bar{G}_i$. By hypothesis, $|C_{\bar{G}_i}(\bar{H}_i)| < c^{r_i}$, where $r_i = \mathrm{rank}(G_i)$, and so

$$|C_{\bar{G}}(\bar{H})| \leq \prod_1^k c^{r_i} = c^r,$$

and the lemma follows. □

In view of the previous lemma, we assume from now on that $G$ is simple of adjoint type. Let $H < G$ be $G$-irreducible, and let $F = C_G(H)$.

**Lemma 2.2.** *The group $F$ consists of semisimple elements.*

*Proof.* Suppose false, and let $f \in F$ be an element with non-identity unipotent part $u$. Then $H \leq C_G(u)$, which is contained in a parabolic subgroup, contradicting the irreducibility of $H$. □

**Lemma 2.3.** *Suppose* $G = \mathrm{PGL}_n(K)$. *Then* $|C_G(H)| \leq n^2$.

*Proof.* Let $\hat{G} = \mathrm{SL}_n(K) = \mathrm{SL}(V)$, and let $\hat{H}$, $\hat{F}$ and $\hat{C}$ be the preimages in $\hat{G}$ of $H$, $F$ and $C_G(F)$, respectively. Also let $Z = Z(\hat{G})$. Note that $C_{\hat{G}}(\hat{H}) = Z$ since $\hat{H}$ acts irreducibly on $V$.

Observe first that $[\hat{H}, \hat{F}, \hat{F}] = [\hat{F}, \hat{H}, \hat{F}] = 1$, and therefore $[\hat{F}, \hat{F}, \hat{H}] = 1$. Hence $\hat{F}' \leq C_{\hat{G}}(\hat{H}) = Z$, and so $F$ is abelian. Let $1 \neq f \in F$, and let $\hat{f}$ be a preimage of $f$. Then $C_{\hat{G}}(\hat{f}) = \prod \mathrm{GL}_{m_i}(K) \cap \hat{G}$, where $\sum m_i = n$. As $C_G(f)$ is irreducible, its preimage in $\hat{G}$ must permute the factors transitively, and it follows that $C_G(f) = (\mathrm{GL}_m(K)^r.r \cap \hat{G})/Z$ for some $r$ dividing $n$, and $f$ has order dividing $r$. Hence $F$ is abelian of exponent dividing $n$.

For $c \in \hat{C}$, there is a map $\chi_c \in \mathrm{Hom}(\hat{F}, Z)$ given by $\chi_c(f) = [c, f]$ for all $f \in \hat{F}$. The map $\pi\colon \hat{C} \mapsto \mathrm{Hom}(\hat{F}, Z)$ sending $c \mapsto \chi_c$ is a homomorphism.

Assume now that $\hat{F}$ is abelian. For $\chi \in \mathrm{Hom}(\hat{F}, Z)$, define

$$V_\chi = \{v \in V : vf = \chi(f)v \text{ for all } f \in \hat{F}\}.$$

Then $V \downarrow \hat{F} = \bigoplus_1^t V_{\chi_i}$, where $V_{\chi_i} \neq 0$ for all $i$. For $c \in \hat{C}$, we have $V_{\chi_i} c = V_{\chi_i \chi_c}$, and as $\hat{C}$ is irreducible, this action of $\hat{C}$ permutes the set $\{V_{\chi_i} : 1 \leq i \leq t\}$ transitively. Replacing each $f \in \hat{F}$ by a scalar multiple, we may take $\chi_1$ to be trivial (i.e. $\chi_1(f) = 1$ for all $f \in \hat{F}$). It follows that $|F|$ is at most the order of the transitive group $\pi(\hat{C})$. Being transitive and abelian, this group has order $t$, and hence $|F| \leq t \leq n$, giving the conclusion in this case.

Now assume that $\hat{F}$ is non-abelian. Let $V \downarrow \hat{F} = \bigoplus_1^t W_i$, where $W_i$ are the homogeneous components. As above, $\{W_1, \ldots, W_t\}$ is permuted transitively by $\pi(\hat{C})$. The action of $\hat{F}$ on the homogeneous component $W_1$ has order at most $(\dim W_1)^2|Z|$ by [3, Theorem 2.31], and hence

$$|F| \leq t(\dim W_1)^2 |\pi(\hat{C})| = t^2 (\dim W_1)^2 = n^2.$$

This completes the proof. □

**Lemma 2.4.** *Suppose* $G = \mathrm{PGSp}_n(K)$ *or* $\mathrm{PGO}_n(K)$. *Then* $|C_G(H)| \leq 4^{2r}$, *where* $r = \mathrm{rank}(G)$.

*Proof.* Let $\hat{G} = \mathrm{Sp}_n(K)$ or $\mathrm{SO}_n(K)$, $Z = Z(\hat{G})$, and $V = K^n$. Let $\hat{F}$ be the preimage in $\hat{G}$ of $F = C_G(H)$.

If $F$ contains an element $f$ of odd prime order, then $C_G(f)$ is connected and has a central torus, hence cannot be irreducible. It follows that $F$ is a 2-group.

Next we show that $\hat{F}$ has exponent dividing 4. Suppose then that $\hat{F}$ contains an element $f$ of order 8, with image $\bar{f} \in F$. Let $\omega \in K$ be a primitive 8-th root of unity, and let $E_{\omega^j}$ be the $\omega^j$-eigenspace of $f$ on $V$ for $0 \leq j \leq 7$. We can assume that $E_\omega \neq 0$, and hence also $E_{\omega^{-1}} \neq 0$.

Let $\bar{g} \in C_G(\bar{f})$, with preimage $g \in \hat{G}$. Then $f^g = \pm f$. If $f^g = f$, then $g$ stabilizes every eigenspace $E_{\omega^j}$; for $j \neq 0, 4$, these are all totally singular. And if $f^g = -f$, then $g$ swaps $E_\omega$ and $E_{-\omega}$, hence stabilizes $E_\omega + E_{-\omega}$, which is also totally singular. We conclude that $C_G(\bar{f})$ stabilizes a totally singular subspace of $V$, hence is a reducible subgroup of $G$, a contradiction.

Hence $\hat{F}$ has exponent dividing 4, as claimed. Since $\hat{F}$ is contained in the normalizer of a maximal torus by [7, II, 5.16], we have

$$|\hat{F}| \leq 4^r |W(G)|_2, \tag{2.1}$$

where $W(G)$ is the Weyl group and $r$ is the rank of $G$. Since $|W(G)| = 2^{r-\delta} r!$ with $\delta \in \{0, 1\}$, it follows that $|W(G)|_2 \leq 4^r$. Hence $|\hat{F}| \leq 4^{2r}$, as required. □

**Lemma 2.5.** *Suppose $G$ is of exceptional type, of rank $r$. Then $|C_G(H)| \leq c^r$, where $c$ is as in the table below.*

| $G$ | $G_2$ | $F_4$ | $E_6$ | $E_7$ | $E_8$ |
|---|---|---|---|---|---|
| $c$ | 8.5 | 30.7 | 10 | 17.3 | 197 |

*Proof.* First assume that $G \neq E_8$. We claim that the non-identity elements of prime-power order in $F$ can only have the following possible orders:

| $G$ | $G_2$ | $F_4$ | $E_6$ | $E_7$ |
|---|---|---|---|---|
| Poss. $o(f)$ | 2, 3 | 2, 3, 4 | 2, 3 | 2, 3, 4 |

Table 1

To see this, let $1 \neq f \in F$ have prime-power order. If $C_G(f)$ is connected, then as it is irreducible, it is semisimple. Then the order $o(f)$ is equal to a coefficient of the expression for the highest root in terms of simple roots (see for example [6, (4.5)]), which gives the conclusion in these cases. This deals with $G = G_2$, or $F_4$, as these are simply connected; hence all their semisimple element centralizers are connected by [7, II, 3.9]. Now consider $G = E_7$ with $C = C_G(f)$ disconnected. Then $|C/C^0| = 2$ by [7, II, 4.4]. In the simply connected cover $\hat{G}$, a preimage $\hat{f}$ of $f$ satisfies $\hat{f}^g = z\hat{f}$ for some $g \in \hat{G}$, where $Z(\hat{G}) = \langle z \rangle$ of order 2, and hence $f$ is a 2-element. Let $C^0 = DT$, where $D$ is a semisimple group and $T$ a central torus, and write $f = dt$ with $d \in Z(D)$, $t \in T$. Let $g \in C \setminus C^0$. Then $C_T(g)^0 = 1$ as $C$ is irreducible in $G$, and hence $g$ acts on $T$ as an involution. It follows that $t^g = t^{-1}$, and hence $t^2 = d^g d^{-1} \in Z(D)$. If $t^2$ has order greater than 2, then it must have order 4 and be contained in a factor $A_3$, $A_3A_3$ or $A_7$ of $D$. In the first case, $d^g d^{-1}$ has order 1 or 2 (being a product of two elements of the same order in $Z(A_3)$), a contradiction. In the second case, $t \in C_G(A_3A_3) = A_1$, so $t^2 \in Z(A_1)$ again has order at most 2. Finally, if $D = A_7$, then $T = 1$ and $f = d \in Z(D)$, which has order 4. This establishes the claim for $G = E_7$, and the argument for $G = E_6$ is similar.

From the previous paragraph, we can list the possible centralizers of the elements of $F$ of orders specified in the following table; the table also gives the traces of their actions on the adjoint module $L(G)$:

| $G$ | $G_2$ | $F_4$ | $E_6$ | $E_7$ |
|---|---|---|---|---|
| $o(f)$ | $2, 3$ | $3$ | $2$ | $3$ |
| $C_G(f)$ | $A_1A_1, A_2$ | $A_2A_2$ | $A_1A_5$ | $A_2A_5$ |
| $\mathrm{tr}_{L(G)}(f)$ | $-2, 5$ | $-2$ | $-2$ | $-2$ |

From Table 1, we see that $F$ is a $\{2, 3\}$-group, hence is solvable, and so there exist Sylow 2- and 3-subgroups $P_2$, $P_3$ of $F$ such that $F = P_2P_3$. We can bound the orders of $P_2$ and $P_3$ as in the previous proof. Since $P_2$ has exponent dividing 4 and is contained in the normalizer of a maximal torus, we have

$$|P_2| \leq 4^r |W(G)|_2, \tag{2.2}$$

where $W(G)$ is the Weyl group of $G$. Similarly,

$$|P_3| \leq 3^r |W(G)|_3.$$

We can use the trace values given above to reduce these bounds for some cases. For example, consider $P_3 < E_7$. If $|P_3| = 3^a$, then since the trace of every non-identity element of $P_3$ is $-2$, we have

$$\dim C_{L(E_7)}(P_3) = \frac{1}{3^a}\big(133 - 2(3^a - 1)\big).$$

The right-hand side can only be a non-negative integer if $a \leq 3$, and hence we have $|P_3| \leq 3^3$. Similar calculations give the following bounds for other cases:

| $G$ | $G_2$ | $F_4$ | $E_6$ | $E_7$ |
|---|---|---|---|---|
| Bound | $\|P_2\| \leq 2^3, \|P_3\| \leq 3^2$ | $\|P_3\| \leq 3^3$ | $\|P_2\| \leq 2^4$ | $\|P_3\| \leq 3^3$ |

The bounds in the conclusion now follow from these together with (2.2): for example, consider $G = E_7$. Here we have

$$|F| = |P_2||P_3| \leq 4^7 \cdot |W(E_7)|_2 \cdot 3^3,$$

and this is less than $c^7$ for $c = 17.3$.

Now consider $G = E_8$. For $1 \neq f \in F$, $C_G(f)$ is irreducible and connected, so as above, $f$ has order equal to a coefficient of the highest root, hence to $2, 3, 4, 5$ or 6. Moreover, any element of order 5 in $f$ has centralizer $A_4A_4$ and trace $-2$ on $L(G)$.

Observe that $F$ is a $\{2, 3, 5\}$-group. Suppose first that $F$ is solvable so that $F = P_2P_3P_5$, where each $P_i$ is a Sylow $i$-subgroup. If $|P_5| = 5^a$, then

$$\dim C_{L(E_8)}(P_5) = \frac{1}{5^a}\big(248 - 4(5^a - 1)\big),$$

which forces $a \leq 3$. Hence, as above,

$$|F| = |P_2||P_3||P_5| \leq 4^8 \cdot |W(E_8)|_2 \cdot 3^8 \cdot |W(E_8)|_3 \cdot 5^3 < 147^8,$$

giving the conclusion.

Now suppose that $F$ is non-solvable. Any non-abelian composition factor of $F$ is a simple $\{2, 3, 5\}$-group, and inspection of the simple groups shows that the only possibilities are $A_5$, $A_6$ and $U_4(2)$. However, $U_4(2)$ is excluded, as it has an element of order 12 which is not in the list of possible orders of elements of $F$. Let $R$ be the solvable radical of $F$ (i.e. the largest solvable normal subgroup). Then $F/R$ has socle $F_1 \times \cdots \times F_t$, a direct product of non-abelian simple groups $F_i$, each isomorphic to $A_5$ or $A_6$. If $t \geq 2$, then $F/R$ has an element of order 15, which is not possible. Therefore, $t = 1$ and $F/R$ has socle $A_5$ or $A_6$. Then $F$ has a solvable subgroup $J$ of index 5 or 10. As above, we have $|J| < 147^8$, and hence

$$|F| \leq 147^8 \cdot 10 < 197^8.$$

This completes the proof of the lemma. □

The proof of Theorem 1 is now complete.

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