ANTENNA ARRAY MANIFOLD: A SIMPLIFIED REPRESENTATION

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ABSTRACT

The array manifold vector is one of the most important parameters of an array system. For a general non-linear array of sensors, (i.e. a two or three dimensional array geometry), the locus of the manifold vectors is a hypersurface embedded in a multidimensional complex space, with parameters of interest the azimuth and elevation angles. In this paper, a simplified representation of this complex hypersurface on a real two-dimensional plane is presented. This simplification/mapping, using concepts from differential geometry, maintains the main characteristics of the original array manifold surface and has the potential of providing new grounds for design and analysis of array systems. In this study, both planar and three-dimensional arrays of omnidirectional sensors will be considered.

1. INTRODUCTION

Array systems for signal detection, direction finding, superresolution beamforming, etc., are strongly dependent on the properties and parameters of the array manifold [1][2][3][4][5][6]. The manifold of an array of N sensors as a function of the azimuth and elevation angles is a surface embedded in an N -dimensional complex space. The performance of an array system is a function of the properties of this complex hypersurface which in turn are functions of the array geometry itself. However, it is very difficult to handle this complex hypersurface and therefore, in this paper, a theoretical framework is presented which simplifies the representation by mapping the hypersurface onto a real plane. The proposed mapping preserves the main properties/characteristics of the original array manifold and has the potential of providing new grounds for design and analysis of array systems. For the development of the mapping, concepts from differential geometry will be used and therefore, initially, the main differential geometry parameters of the array manifold will be defined. We believe that this mapping, as well as the proposed parameters, will provide useful and much needed insight into various array related problems such as detection and resolution, ambiguities and ultimately into array design.

For planar and three-dimensional arrays of N sensors, the array manifold is the locus of the manifold vector $\underline{a}(\theta, \phi)$ defined as follows:

$$\underline{a}(\theta, \phi) = \underline{\gamma}(\theta, \phi) \odot \exp(-j\underline{\underline{r}}^T \underline{k}) \quad \forall (\theta, \phi) \quad (1)$$

where (θ, ϕ) are the (azimuth, elevation) angles, $\underline{\underline{r}} \in R^{3 \times N}$ is the sensor position matrix in half wavelengths,

$\underline{k} = \pi[\cos\phi\cos\theta, \cos\phi\sin\theta, \sin\phi]^T \in R^{3 \times 1}$ denotes the wavenumber vector, $\odot$ represents the Hadamard product and $\underline{\gamma}(\theta, \phi) \in C^{N \times 1}$ is the directional gain of each sensor. For a point (θ, ϕ) on the manifold surface the most important parameters are:

- The manifold metric, $\mathbb{G}$, which is a function of the derivatives of the manifold vector of Eq.(1).
- The Gaussian curvature, K , which is a function of the the manifold metric $\mathbb{G}$.

If the point (θ, ϕ) on the manifold surface is also taken as a point on a curve on the manifold surface, then the following two parameters, which are also functions of the metric $\mathbb{G}$, are of interest:

- The arc length, s , of the curve up to that point.
- The geodesic curvature, κ_g , at that point.

Manifold Metric

The manifold metric is a 2×2 real matrix with elements known in Differential Geometry as the first fundamental coefficients (or metric coefficients) and is defined as follows:

$$\mathbb{G} = \begin{bmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi} \end{bmatrix} = \begin{bmatrix} \|\underline{a}_\theta\|^2 & \text{Re}\{\underline{a}_\theta^H \underline{a}_\phi\} \\ \text{Re}\{\underline{a}_\phi^H \underline{a}_\theta\} & \|\underline{a}_\phi\|^2 \end{bmatrix} \quad (2)$$

where

$$\underline{a}_\theta = \frac{d\underline{a}(\theta, \phi)}{d\theta} \in C^{N \times 1} \quad \text{and} \quad \underline{a}_\phi = \frac{d\underline{a}(\theta, \phi)}{d\phi} \in C^{N \times 1} \quad (3)$$

Gaussian Curvature

The shape of the surface at a (θ, ϕ) point is described by the Gaussian curvature, K , which, by using [7], we have proven that it can expressed as a function of the metric $\mathbb{G}$ and its derivatives as follows:

$$K(\theta, \phi) = -\frac{1}{\sqrt{\det(\mathbb{G})}} \left(\frac{d\sqrt{\det(\mathbb{G})}}{d\theta} \frac{\Gamma_{\theta\phi}^\phi}{d\theta} - \frac{d\sqrt{\det(\mathbb{G})}}{d\phi} \frac{\Gamma_{\theta\theta}^\phi}{d\theta} \right) \quad (4)$$

where $\Gamma_{\theta\phi}^\phi, \Gamma_{\theta\theta}^\phi$ are two elements of 2×2 real matrix $\mathbb{G}^{-1} \frac{d\mathbb{G}}{dp}$ with $p = \theta$ or ϕ .

Note that, in Differential Geometry terms, the elements of the matrix $\mathbb{G}^{-1} \frac{d\mathbb{G}}{dp}$ are known as the eight Christoffel symbols of the second kind, i.e.

$$\underline{\Gamma}_{2p} \triangleq \mathbb{G}^{-1} \frac{d\mathbb{G}}{dp} = \begin{bmatrix} \Gamma_{\theta p}^\theta & \Gamma_{\phi p}^\theta \\ \Gamma_{\theta p}^\phi & \Gamma_{\phi p}^\phi \end{bmatrix} \quad \text{where } p = \theta \text{ or } \phi \quad (5)$$

with $\Gamma_{\theta\phi}^p = \Gamma_{\phi\theta}^p$.

Arc Length

The arc length, s , can be interpreted in physical terms as the total distance covered by a person travelling along a certain path. On the array manifold surface, it is the total distance travelled along a curve on the surface from a 'initial' point to the current point under consideration (θ, ϕ) . Its expression as a function of the manifold metric is:

$$s = \int_t \sqrt{\frac{dp^T}{dt} \mathbf{G} \frac{dp}{dt}} dt \quad (6)$$

where $\underline{p}(t) = [\theta(t) \ \phi(t)]^T$ and, hence, $\frac{dp}{dt} = \left[\frac{d\theta}{dt} \ \frac{d\phi}{dt} \right]^T$.

Geodesic Curvature

The final parameter of interest for the mapping is the *geodesic curvature* of a curve on the manifold surface. In order to understand the significance of this parameter the geodesicity concept should be introduced. It is well known that the curve with the minimum length between two points in a Euclidian space is a straight line. This concept is extended to a curve connecting two points on a surface and the curve of minimum length belonging to the surface is called a geodesic curve (equivalent to a straight line in a plane). By considering a curve on a surface connecting two points, the closeness of this curve to a geodesic curve can be assessed by means of the geodesic curvature, κ_g , with $\kappa_g = 0$ corresponding to a geodesic curve. This curvature is given by the following expression (for proof see author's internal report [8]):

$$\kappa_g(s) = \sqrt{\det(\mathbf{G})} \left[-\frac{d\phi}{ds} \right]^T \left\{ \left(\Gamma_{2\theta} \frac{d\theta}{ds} + \Gamma_{2\phi} \frac{d\phi}{ds} \right) \frac{dp}{ds} + \frac{d^2 p}{ds^2} \right\} \quad (7)$$

where s is the arc length of the curve under consideration. Note that for curves, on the surface, of constant elevation i.e. θ -curves, Equation 7 is simplified using the properties $\frac{d\theta}{ds} = 0$ and $\frac{d\phi}{ds} = \frac{1}{g_{\phi\phi}}$, to the following expression:

$$\kappa_{g\theta} = \sqrt{\frac{\det(\mathbf{G})}{g_{\theta\theta}^3}} \Gamma_{\theta\phi} \quad (8)$$

A similar expression can be found also for curves on the surface of constant azimuth (ϕ -curves), i.e.

$$\kappa_{g\phi} = -\sqrt{\frac{\det(\mathbf{G})}{g_{\phi\phi}^3}} \Gamma_{\phi\theta} \quad (9)$$

2. MANIFOLD SURFACE REPRESENTATION

The main idea is to propose a mapping to represent a complex surface (embedded in a multidimensional space) by a simpler surface. The simplest surface, from a conceptual point of view, is the real plane. In order to be useful, this mapping should preserve the main characteristics of the manifold surface, for example the geodesic properties of the parameter curves (θ -curves or ϕ -curves) i.e. the mapping should be *geodesic*. From *Beltrami's* theorem [7], only surfaces of constant Gaussian curvature can be geodesically mapped onto the plane (which has a constant Gaussian curvature of zero) and so this becomes a prerequisite for the existence of a manifold surface mapping.

Such a representation would simplify the analysis of array manifold surfaces and consequently of array systems. Based on the above discussion, a mapping is proposed below in the form of a theorem.

THEOREM 1 - Consider an arbitrary point, (θ, ϕ) , on the manifold surface of an array of N sensors. The image of the point (θ, ϕ) on the real parameter plane is given by :

$$\underline{D}(\theta, \phi) = \begin{bmatrix} x_0(\phi) + \int_0^\theta \kappa_{g\theta}^{-1}(\vartheta, \phi) \cos(\vartheta) d\vartheta \\ y_0(\phi) + \int_0^\theta \kappa_{g\theta}^{-1}(\vartheta, \phi) \sin(\vartheta) d\vartheta \end{bmatrix} \quad (10)$$

where $\kappa_{g\theta}(\theta, \phi)$ is the geodesic curvature of the θ -lines on the manifold and $x_0(\phi)$ and $y_0(\phi) \in \mathbb{R}$ are integration constants such that :

$$x_0 = -\frac{1}{2} \int_0^\pi \kappa_{g\theta}^{-1}(\vartheta, \phi) \cos(\vartheta) d\vartheta \\ \text{and } y_0 = -\frac{1}{2} \int_0^\pi \kappa_{g\theta}^{-1}(\vartheta, \phi) \sin(\vartheta) d\vartheta$$

The locus of the images $\underline{D}(\theta, \phi) \forall (\theta, \phi)$ is known as the *development* of the manifold and has the following properties :

i) It exists if and only if the Gaussian curvature of the manifold surface is constant.

ii) The arc length $s_d(\theta, \phi_0)$ of θ -curves on the development is given by:

$$s_{d\theta}(\theta, \phi_0) = \int_\theta \kappa_{g\theta}^{-1}(\vartheta, \phi_0) d\vartheta \quad (11)$$

iii) The curvature of the θ -curves on the development is equal to the geodesic curvature $\kappa_{g\theta}(\theta, \phi_0)$ of the θ -curves on the manifold surface.

The proof of the three points mentioned above is given in Appendix 1. Note that the integration constants have the effect of shifting the points on the plane to a common centre that is $\underline{D}(\theta, \phi) + \underline{D}(\theta + \pi, \phi) = \underline{0}$. Furthermore, from now on, an array manifold surface will be called *developable* if the development $\underline{D}(\theta, \phi)$ exists, i.e. if the first property of Theorem 1 is satisfied.

3. APPLICATION TO ARRAYS OF OMNIDIRECTIONAL SENSORS

In order to demonstrate the feasibility and potential benefit of the above development concept, arrays of omnidirectional sensors have been considered (i.e. $\underline{\gamma}(\theta, \phi) = [1, 1, \dots, 1]^T$) and two classes of arrays with developable manifold surfaces have been identified. The first class is that of planar arrays and the second is a family of three-dimensional array geometries known as 3D-grid arrays defined below.

DEFINITION - 3D-grid arrays. A three-dimensional array geometry is said to be a 3D-grid array if and only if the following expression is satisfied:

$$\underline{\mathbf{r}} \underline{\mathbf{r}}^T = \rho^2 \mathbb{I}_3 \quad \text{where } \rho \in \mathbb{R} \quad (12)$$

An example of a 3D-grid array is the 8-element cube array with all sides equal to one half-wavelengths. Note that for planar arrays:

$$\underline{\mathbf{r}} \underline{\mathbf{r}}^T = \begin{bmatrix} c_1 & c_2 & 0 \\ c_2 & c_3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{where } \mathbf{L} = \begin{bmatrix} c_1 & c_2 \\ c_2 & c_3 \end{bmatrix} \in \mathbb{R}^{2 \times 2}$$

Table 1: Analytical Expressions of Differential Geometry Parameters for Developable Arrays of Omnidirectional Sensors		
	3D-grid Arrays	Planar Arrays
$\mathbf{G}$	$\rho^2 \pi^2 \begin{bmatrix} \cos^2 \phi & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\theta\phi} & g_{\phi\phi} \end{bmatrix}$
$\det(\mathbf{G})$	$\rho^4 \pi^4 \cos^2 \phi$	$\frac{\pi^4}{4} \det(\mathbf{L}) \sin^2 2\phi$
$\underline{\Gamma}_{2\theta}$	$\begin{bmatrix} 0 & -\tan \phi \\ \cos \phi \sin \phi & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & -\tan \phi \\ \frac{1}{\tan \phi} & 0 \end{bmatrix}$
$\underline{\Gamma}_{2\phi}$	$\begin{bmatrix} -\tan \phi & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} -\tan \phi & 0 \\ 0 & \frac{1}{\tan \phi} \end{bmatrix}$
K	$\frac{1}{\rho^2 \pi^2}$	0
$\kappa_{g\theta}$	$\frac{\tan \phi}{\rho \pi}$	$\sqrt{\frac{\det(\mathbf{G})}{g_{\theta\theta}}} \frac{1}{\tan \phi}$
$\kappa_{g\phi}$	0	0
$\underline{D}(\theta, \phi_0)$	$\frac{\rho \pi}{\tan \phi_0} \begin{bmatrix} \sin \theta \\ -\cos \theta \end{bmatrix}$	$\frac{\pi \cos \phi_0}{\sqrt{\det(\mathbf{L})}} \begin{bmatrix} \int_0^\theta f(\vartheta) \cos \vartheta d\vartheta - \frac{1}{2} \int_0^\pi f(\vartheta) \cos \vartheta d\vartheta \\ \int_0^\theta f(\vartheta) \sin \vartheta d\vartheta - \frac{1}{2} \int_0^\pi f(\vartheta) \sin \vartheta d\vartheta \end{bmatrix}$ with $f(\theta) = \sqrt{(c_3 \cos^2 \theta - 2c_2 \cos \theta \sin \theta + c_1 \sin^2 \theta)^3}$

It is important to point out that the Gaussian curvature of a general three-dimensional array is not always constant and hence the manifold hypersurface may not always be developable. However 3D-grid arrays and planar arrays are always developable.

Table 1 summarises the results of the evaluation of the differential geometry parameters introduced in this paper for the two classes of arrays of omnidirectional sensors. Table 1 shows that the geodesic curvature of the ϕ -curves, $\kappa_{g\phi}$, is zero and hence the representation of these curves should be straight lines on the real plane. For the 3D-grid array class, the geodesic curvature of the θ -curves, $\kappa_{g\theta}$, is independent of the parameter θ and hence the representation of these curves on the plane will be circles. This is confirmed by the expression for $\underline{D}(\theta, \phi_0)$. Figures 1 and 2 provide the development of a representative example of a 3D-grid array and that of a planar array. Note that in a direction finding system using the arrays with developments presented in Figures 1 and 2, the signal eigenvectors of the data covariance matrix can also be mapped on the development as well as ambiguous sets of directions [5], uncertainty spheres [1], etc... and this can be the starting point for designing new array processing algorithms working in a two-dimensional real plane rather than an N -dimensional complex space.

4. CONCLUSION

In this paper, the main differential geometry parameters of the manifold surface of a non-linear array of sensors have been identified and analytical expressions have been provided. The concept of the development has been proposed, under certain conditions. This is an alternative representation of the manifold surface on a real plane, maintaining its main differential geometry characteristics. The proposed concepts have been applied to three-dimensional and planar arrays of omnidirectional sensors and a real two-dimensional representation of the array

manifold surface has been plotted. This representation can be used along with other array systems' theories to provide further insight to the antenna array design.

5. REFERENCES

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6. APPENDICES

Appendix 1: Proof of Theorem 1

i) Beltrami's theorem states that if a surface is mapped to another surface and the mapping is such that geodesic curves in one surface are mapped onto geodesic curves in the other surface (geodesic mapping) then if one surface has a constant Gaussian curvature the other must also have a constant Gaussian curvature in order to have a distortionless mapping. Since the destination surface of our mapping, i.e. the plane, has a constant curvature (equal to zero) the condition for the existence of a distortionless mapping is that the array manifold surface should have a constant curvature.

ii) Considering the development of the manifold θ -curves for a constant ϕ_0 , $\underline{D}_\theta(\theta) = \underline{D}(\theta, \phi_0) \forall \theta$, the arc length of $\underline{D}_\theta(\theta)$ is given by:

$$s_{d\theta}(\theta) = \int_\theta \left\| \frac{d\underline{D}_\theta(\vartheta)}{d\vartheta} \right\| d\vartheta$$

Using the Leibnitz rule of integral differentiation on Eq.(10):

$$\frac{d\underline{D}_\theta(\theta)}{d\theta} = \kappa_{g\theta}^{-1}(\theta, \phi_0) \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$

and hence $\left\| \frac{d\underline{D}_\theta(\theta)}{d\theta} \right\| = \frac{1}{\kappa_{g\theta}(\theta, \phi_0)}$ proving Eq.(11).

iii) The curvature of a curve is the magnitude of the derivative of the normalised tangent vector with respect to its arc length hence, considering the development of the manifold θ -curves for a constant ϕ_0 , let $\underline{u}(\theta)$ be its normalised tangent vector:

$$\underline{u}(\theta) = \frac{d\underline{D}_\theta(\theta)}{d\theta} / \left\| \frac{d\underline{D}_\theta(\theta)}{d\theta} \right\| = \pm \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$

Differentiating Eq.(11) leads to $ds_{d\theta} = \frac{d\theta}{\kappa_{g\theta}(\theta, \phi_0)}$ and hence $\frac{d\theta}{ds_{d\theta}} = \kappa_{g\theta}(\theta, \phi_0)$.

The magnitude of the derivative of the normalised tangent vector with respect to its arc length hence becomes:

$$\left\| \frac{d\underline{u}(\theta)}{ds_{d\theta}} \right\| = \left\| \frac{d\underline{u}(\theta)}{d\theta} \frac{d\theta}{ds_{d\theta}} \right\| = \left\| \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix} \kappa_{g\theta}(\theta, \phi_0) \right\| = \kappa_{g\theta}(\theta, \phi_0)$$

The curvature of the development is thus equal to the geodesic curvature of the manifold curve. $\square$

Figure 1: θ -curves and ϕ -curves development of the manifold of a 3D-grid array with sensors located on the eight vertices of a cube of side one half-wavelength and with array centroid the center of the cube.

Figure 2: θ -curves and ϕ -curves development of the manifold of a planar array of 5-sensor located at the

$$\text{positions } \underline{r} = \begin{bmatrix} -0.4 & -0.3 & 0.2 & 0.1 & 0.4 \\ 0.5 & -0.2 & -0.3 & 0.2 & -0.2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

in half wavelength.