Modeling and Estimation of Ambiguities in Linear Arrays

Athanassios Manikas, *Member, IEEE*, and Christos Proukakis

Abstract—In this paper, the problem of ambiguities inherent in the manifold of any linear array structure is investigated. Ambiguities, which can be classified into *trivial* and *nontrivial*, depending on the ease of their identification, arise when an array cannot distinguish between two different sets of directional sources. Initially, the new concept of an *ambiguous generator set* is introduced; it represents/generates an infinite number of ambiguous sets of directions. Then, by uniformly/nonuniformly partitioning the array manifold curve of a linear array, different ambiguous generator sets can be calculated, and as a direct result, a sufficient condition for the presence of ambiguities is obtained. The theoretical aspects of the investigation are followed by the proposal of an innovative approach that calculates not only all such ambiguities existing in a linear array of arbitrary geometry but the rank of ambiguity in each case as well. The main results presented in the paper are supported by a number of representative examples.

Index Terms—Ambiguities, differential geometry, linear arrays.

NOMENCLATURE

a, A	Scalar.
$\underline{a}, \underline{a}, \underline{A}$	Vector.
$\mathbb{A}, \underline{\underline{A}}$	Matrix.
$\mathbb{I}_d$	$(d \times d)$ identity matrix.
$(\cdot)^T, (\cdot)^H$	Transpose, conjugate transpose.
$ a $	Absolute value of a scalar.
$\ \underline{a}\ $	Euclidean norm of a vector.
$\text{fix}(\bullet)$	Integer part.
$\text{sum}(\bullet)$	Sum of vector elements.
$\exp(\underline{a})$	Element-by-element exponential.
$\text{expm}(\mathbb{A})$	Matrix exponential.
$\text{row}_i\{\mathbb{A}\}$	i th row of a matrix $\mathbb{A}$.
$\underline{a}^k$	Element-by-element k th power.
$r_{i,j}$	Difference between the i th and j th sensor locations.
p	General bearing parameter.
θ	Azimuth bearing.
s	Arc length parameter.
l_m	Manifold length.
l_a	Aperture of a linear array.
$\mathcal{C}^N$	N -dimensional complex space.
$\mathcal{R}^N$	N -dimensional real space.

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The authors are with the Digital Communications Research Section, Department of Electrical and Electronic Engineering, Imperial College of Science, Technology and Medicine, London, U.K.

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$\mathbb{N}^+$ Set of positive integers.
 $\text{tr}(\mathbb{A})$ Trace of matrix $\mathbb{A}$.

I. INTRODUCTION

WHEN finding the directions-of-arrival (DOA's) of narrowband signals using array sensor measurements, it is important to be certain that the problem has a unique solution. If the array has identical responses to two different sets of DOA's, then the ambiguity problem is said to arise. The first attempt to introduce a mathematical framework to deal with the ambiguity problem was by Schmidt [1], who classified the ambiguities according to their "rank" based on the linear dependence between manifold vectors, as rank-1 or rank greater than one. The former (rank-1), which will later be characterized as *trivial*, is comparatively easy to detect and, for this reason, has received much more attention than the really important case, which will be referred to as *nontrivial* ambiguities. Any subsequent research to handle the ambiguity problem was mainly concerned with either the performance of specific array geometries or with the identification of array structures that are free of ambiguities up to a certain rank of ambiguity. Hence, in [2], it was observed that as the aperture of a circular array increases, so does the risk of trivial ambiguities occurring. In [3], the ambiguity problem was examined for the case of linear arrays, and a special type of linear array was identified that does not suffer from trivial ambiguities. In [4], a conjecture is made providing a relatively simple way of identifying whether a set of directions is rank- n ambiguous or not. This would greatly ease the task of identifying ambiguities. However, in [5], the conjecture was shown, through a counterexample, to be, in general, incorrect. In addition, in this paper, a specific class of planar array was presented whose members were shown to be free of both trivial and nontrivial ambiguities. Finally, in [6], a specific class of uniform circular array is shown to be free of rank-2 ambiguities when the sources are coplanar with the array.

In this paper, we attempt to model and calculate ambiguous sets of directions that exist for any linear array geometry. The organization of the paper is as follows. In Section II, the problem of ambiguities is formulated, and a useful classification of ambiguities is given. In Section III, a parametrization of curves, such as the array manifold of a linear array, is presented. In Section IV, the new notion of the ambiguous generator set is introduced. In Section V, a technique is proposed for the identification and estimation of a new class of ambiguous generator set based on uniform partitioning of the array manifold curve. This class exists for any array geometry.

However, if the array is symmetric linear, then a second class of ambiguity is identified based on nonuniform partitioning of the manifold curve. Finally, the paper is concluded in Section VI. Note that in a companion paper [7], the investigation of ambiguities is extended to planar arrays, and an approach is proposed that provides ambiguous sets of DOA's that are not coplanar with the array.

II. PROBLEM FORMULATION AND A CLASSIFICATION OF AMBIGUITIES

It is well known that in the absence of calibration errors, the signal-vector $\underline{x}(t)$ received by an array of N sensors from M ($M < N$) narrowband far-field signal sources with directions $\underline{p} = [p_1, \dots, p_M]^T$, can be modeled as

$$\underline{x}(t) = \mathbf{A}(\underline{p})\underline{m}(t) + \underline{n}(t) \quad (1)$$

where $\underline{m}(t) \in \mathcal{C}^M$ denotes the vector of the complex received baseband signals at the reference point, $\underline{n}(t) \in \mathcal{C}^N$ is the additive noise, and $\mathbf{A}(\underline{p}) \in \mathcal{C}^{N \times M}$ is the matrix with columns the manifold vectors $\underline{a}(p_i) \in \mathcal{C}^N \forall i = 1, \dots, M$, i.e.,

$$\mathbf{A}(\underline{p}) = [\underline{a}(p_1), \underline{a}(p_2), \dots, \underline{a}(p_M)]. \quad (2)$$

The locus of all manifold vectors $\underline{a}(p) \forall p$ is a continuum lying in N -dimensional complex space and is known as the array manifold. Therefore, the array manifold $\{\underline{a}(p), \forall p \in \Omega\}$, where Ω is the parameter space, is essentially a mapping from Ω to the complex N -dimensional space $\mathcal{C}^N$

$$p \mapsto \underline{a}(p). \quad (3)$$

Thus, if the system under consideration is an azimuth-only system, i.e., $p = \theta$, then Ω is the interval $[0, 2\pi]$ on the real line $\mathbb{R}^1$, and the array manifold is a curve in $\mathcal{C}^N$.

Note that the array manifold vector $\underline{a}(p)$ is a function of the locations and characteristics of the array elements and represents the array complex response to a unity power signal impinging on the array from direction p . For a linear array of N omnidirectional sensors, the only parameter of interest is the azimuth θ , i.e., $p = \theta$, and the array manifold vector can be written as

$$\underline{a}(\theta) = \exp(-j\pi \underline{r} \cos \theta) \quad (4)$$

where $\underline{r} \in \mathbb{R}^N$ is the vector of sensor locations in half wavelengths.

The array manifold plays a very important role in the signal subspace algorithms that are applied in direction-finding systems. Signal subspace algorithms search the array manifold to identify the true DOA's as those that satisfy a specified criterion. For instance, MUSIC [1] searches the manifold for those $\underline{a}(p)$ that are closest to the signal subspace in the Euclidean sense.

In this study, we will consider the problem that arises when the mapping $\underline{a}(p)$ is not *one to one*. In this case, the array cannot distinguish between two (or more) different signal environments. For instance, two different sets of signals impinging on the array can provide identical responses at the array output, i.e., the same measurements. Under such

conditions, it is said that the ambiguity problem arises. In such a case, any direction-finding algorithm may be unable to resolve the true directions from the false ones.

At this point, it is important to note that ambiguities arise only as a result of the array geometry, and thus, different array geometries have different sets of ambiguous directions. This implies that an array that is unambiguous for a given set of directions might become ambiguous for the same set if we change its sensor locations even slightly so that a new array geometry is obtained. A typical example of ambiguities associated with a linear array operating in the presence of two sources at 5 and 35° is illustrated in Fig. 1. When, for instance, MUSIC is applied on the array of Fig. 1(a), then only two nulls corresponding to the DOA's of the true sources will arise. If the fourth sensor located at +1 half wavelengths is now moved to the location +1.5 half wavelengths so that the uniform linear array of Fig. 1(c) is obtained, then four nulls appear at directions 5, 35, 109.70, 120.94°, which thus form an ambiguous set of directions.

In general, we can say that every array suffers from ambiguities in some way or another, and sometimes, some ambiguous sets of directions can be easily identified. For instance, it is impossible to distinguish whether a signal is impinging on a linear array from direction-of-arrival θ_1 or from the direction $\theta_2 = 360^\circ - \theta_1$, i.e., the mirror image of θ_1 with respect to 180°. Therefore, if a direction-finding algorithm is applied in the range $[0^\circ, 360^\circ)$, then two directions will be estimated for every true source. Note that in this case

$$\underline{a}(\theta_1) = \underline{a}(\theta_2) \quad (5)$$

and for this reason, the parameter space in the case of linear arrays is confined to $\Omega = [0^\circ, 180^\circ]$. Equation (5) is a special case of

$$\underline{a}(\theta_1) = k\underline{a}(\theta_2) \quad \text{where } \theta_1 \neq \theta_2 \quad (6)$$

for $k = 1$. Equation (6) says that there exists at least one manifold vector $\underline{a}(\theta_1)$, which can be written as a scalar multiple of another manifold vector $\underline{a}(\theta_2)$. Then, the two waves with bearings θ_1, θ_2 are indistinguishable by the array even if $\theta_1, \theta_2 \in \Omega = [0^\circ, 180^\circ)$. This is the simplest type of ambiguity, which is known as a *trivial ambiguity*.

A more complicated type of ambiguity known as a *nontrivial ambiguity* arises when a manifold vector can be written as a linear combination of two or more different manifold vectors. For instance, let $\underline{a}(p_i)$ be a linear combination of $\underline{a}(p_1), \underline{a}(p_2), \dots, \underline{a}(p_{i-1})$ with $i > 2$ so that $\underline{a}(p_i) = k_1\underline{a}(p_1) + k_2\underline{a}(p_2) + \dots + k_{i-1}\underline{a}(p_{i-1})$. In such a case, the array will have identical responses for the sets of bearings $(p_1, p_2, \dots, p_{i-1}), (p_2, p_3, \dots, p_i), (p_1, p_3, p_4, \dots, p_i)$, etc.

For example, consider the nonuniform linear array with sensor locations measured in half wavelengths given by $\underline{r} = [-2, -1, 1, 2]^T$. Let the sources present in the environment be $\underline{\theta} = [60^\circ, 90^\circ, 120^\circ]^T$. For this specific array, $\text{rank}([\underline{a}(0^\circ), \underline{a}(60^\circ), \underline{a}(90^\circ), \underline{a}(120^\circ)]) = 3$, and hence, these four manifold vectors are linearly dependent. As a result, a signal subspace direction-finding algorithm will provide a spurious source-direction at $\theta = 0^\circ$.

Fig. 1. Example of ambiguities in linear arrays. (a) Linear array configuration (in half-wavelengths). (b) True directions: 5 and 35°; directions estimated from MUSIC: 5 and 35°. (c) Linear array configuration (in half-wavelengths). (d) True directions: 5 and 35°; directions estimated from MUSIC: 5, 35, 109.70, and 120.94°.

This type of ambiguity is much more difficult to identify since, unlike a trivial ambiguity, it cannot be detected by a simple search of the manifold.

Note that in this study, the array centroid will be taken as the reference point, implying that $\text{sum}(\underline{r}) = 0$. However, it can be proved that the ambiguities are independent of the choice of the reference point.

III. THE GEOMETRY OF THE CURVES OF THE ARRAY MANIFOLD

Since curves will be extensively used in this paper, it is necessary to provide an appropriate parametrization of curves. This parametrization can be obtained from an area of mathematics called differential geometry; see [8] or [9]. Differential geometry is specifically concerned with the application of calculus to the investigation of the geometric properties of curves imbedded principally in the three-dimensional (3-D) real space $\mathbb{R}^3$. However, notions from differential geometry can be extended (see [10]) to include curves embedded in a complex N -dimensional space such as the manifold curves, which are of interest here.

The most basic feature of a curve, according to differential geometry, is the *arc length* s , which is formally defined as

$$s \stackrel{\text{def}}{=} \int_0^P \left\| \frac{d\mathbf{a}(p)}{dp} \right\| dp \quad (7)$$

where $\mathbf{a}(p)$ must be differentiable at all points, and therefore

$$\frac{ds}{dp} = \left\| \frac{d\mathbf{a}(p)}{dp} \right\|. \quad (8)$$

In the case of the manifold of a linear array of N omnidirectional sensors, the relationship between the arc length s and the azimuth θ , i.e., $p = \theta$ given by (8) is simplified to

$$s(\theta) = \pi \|\underline{r}\| (1 - \cos \theta) \Leftrightarrow \theta(s) = \cos^{-1} \left(1 - \frac{s}{\pi \|\underline{r}\|} \right) \quad (9)$$

where $s = 0$ corresponds to the direction that is parallel to the array line, i.e., to $\theta = 0^\circ$.

Furthermore, since the parameter space is $\Omega = [0, \pi]$, it results that the total length l_m of the array manifold is

$$l_m = \text{manifold length} = s(\pi) = 2\pi \|\underline{r}\|. \quad (10)$$

Note that the manifold curve of a linear array with N omnidirectional sensors was described in great detail in [10] and was found to be a hyperhelix. Such a curve is very attractive since it is uniquely described, except for its position in space, by a set of constant curvatures. The exact values of these curvatures can be computed analytically using the expressions given in [10] and [11], and they form a skew-symmetric matrix known as the Cartan matrix as

$$\mathbb{C} \stackrel{\text{def}}{=} \begin{bmatrix} 0 & -\kappa_1 & 0 & \cdots & 0 & 0 \\ \kappa_1 & 0 & -\kappa_2 & \cdots & 0 & 0 \\ 0 & \kappa_2 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & -\kappa_{d-2} & 0 \\ 0 & 0 & \cdots & \kappa_{d-2} & 0 & -\kappa_{d-1} \\ 0 & 0 & \cdots & 0 & \kappa_{d-1} & 0 \end{bmatrix}$$

where $d = \begin{cases} 2N - m, & \text{if there is no sensor} \\ & \text{at the array centroid} \\ 2N - m - 1, & \text{otherwise} \end{cases}$

with m denoting the number of sensors in symmetrical pairs and κ_i representing the i th curvature. The dimensionality d also represents the number of manifold coordinate vectors $\underline{u}_i(s)$ at any point s , which form the coordinate matrix $\mathbb{U}(s) = [\underline{u}_1(s), \underline{u}_2(s), \dots, \underline{u}_d(s)] \in \mathbb{C}^{N \times d}$.

The manifold coordinate vectors at $s = 0$ are transformed to the coordinate vectors at a point $s \neq 0$ by a continuous differentiable real transformation matrix $\mathbb{F}(s) \in \mathbb{R}^{d \times d}$ (which is known as the “Frame” matrix) as

$$\mathbb{U}(s) = \mathbb{U}(0)\mathbb{F}(s) \quad \text{with} \quad \mathbb{F}(0) = \mathbb{I}_d. \quad (11)$$

Note that the Cartan matrix and the Frame matrix always satisfy the differential equation

$$\mathbb{F}'(s) = \mathbb{F}(s) \cdot \mathbb{C} \quad (12)$$

which implies that the Cartan matrix can be expressed as

$$\mathbb{C} = \mathbb{F}^{-1}(s)\mathbb{F}'(s) = \mathbb{F}^T(s)\mathbb{F}'(s). \quad (13)$$

The array manifold vector at point s can be expressed as a function of the coordinate matrix of the manifold and its curvatures as

$$\underline{a}(s) = \mathbb{U}(s) \cdot \underline{R}_\kappa = \mathbb{U}(0) \cdot \mathbb{F}(s) \cdot \underline{R}_\kappa \quad (14)$$

where the vector $\underline{R}_\kappa$ (which is known as the *radii vector* of the manifold) is given as

$$\underline{R}_\kappa = \begin{cases} [0, -R_2, 0, -R_4, 0, \dots, 0, -R_d]^T & \text{if } d = 2N - m \\ [0, -R_2, 0, -R_4, 0, \dots, 0, -R_{d-1}, 1]^T & \text{if } d = 2N - m - 1 \end{cases} \quad (15)$$

with $R_2 = 1/\kappa_1$, and $R_i = \prod_{j \text{ even}}^{i-2} \kappa_j / \prod_{j \text{ odd}}^{i-1} \kappa_j$, $\forall i \text{ even} > 2$.

In this study, the manifold curves (hyperhelices) will be used to analytically determine and formally define ambiguities in linear arrays.

IV. THE CONCEPT OF AN AMBIGUOUS GENERATOR SET

Before continuing, it is necessary to present the following definitions which will be extensively used.

Definition 1: An ordered set of arc lengths $\underline{s} = [s_1, s_2, \dots, s_m]^T$, where $m \leq N$, is said to be an *ambiguous set of arc lengths* if the matrix $\mathbb{A}(\underline{s}) \in \mathbb{C}^{N \times m}$ with columns the manifold vectors $\underline{a}(s_1), \underline{a}(s_2), \dots, \underline{a}(s_m)$ has rank less than m , i.e., $\text{rank}(\mathbb{A}(\underline{s})) < m$.

Definition 2: If a set of arc lengths $\underline{s} = [s_1, s_2, \dots, s_m]^T$, where $m \leq N$, is ambiguous, then its *rank of ambiguity*¹ is defined as the integer $\rho_a = \text{rank}(\mathbb{A}(\underline{s}))$.

It must be noted that the above definitions need to be extended for $m > N$ since there exist cases where a set of $k \leq N$ manifold vectors intersect the array manifold at $m > N$ points. Therefore, a set $\underline{s} = [s_1, s_2, \dots, s_m]^T$ of arc lengths, where $m > N$, is an ambiguous set of arc lengths if all subsets $\underline{s}_i$ of $\underline{s}$ that contain exactly N elements are themselves ambiguous sets of arc lengths. In this case, the rank of ambiguity of $\underline{s}$ is defined as the integer $\rho_a = \min_{\forall i} (\text{rank}(\mathbb{A}(\underline{s}_i)))$.

Note that the previously introduced definitions can be directly applied to sets of DOA's by simply substituting [using (9)] the arc length s with the azimuth θ .

The theorem that follows is the first result of this paper. It essentially states that if all the elements of an ambiguous set of arc lengths are rotated on the array manifold by the same value, then the resulting set is also an ambiguous set of arc lengths.

Theorem 1: If, for a linear array of N sensors, $\underline{s} = [s_1, s_2, \dots, s_m]^T$, with $N \geq m$, is an ambiguous set of arc lengths with rank of ambiguity ρ_a , then any set $\underline{s}' = [s_1 + \Delta s, s_2 + \Delta s, \dots, s_m + \Delta s]^T$, with $s_1 + \Delta s \geq 0$ and $s_m + \Delta s \leq l_m$, is also an ambiguous set of arc lengths with the same rank of ambiguity ρ_a .

The proof of this theorem can be found in Appendix A, whereas its essential features are illustrated in the following example.

Example 1: Consider the array of Fig. 2(a), which has sensor locations given by $\underline{r} = [-3, -1, -0.5, 2, 2.5]^T$ and manifold length $l_m = 28.4483$. Let four sources be present in the environment with DOA's 53.13, 78.46, 101.54, and 126.87°. If, e.g., MUSIC, is applied, then an extra null will appear at 0°, which implies that the set $\underline{\theta} = [0^\circ, 53.13^\circ, 78.46^\circ, 101.54^\circ, 126.87^\circ]^T$ is an ambiguous set of directions. Using (9), the corresponding ambiguous set of arc lengths is computed to be $\underline{s} = [0, 5.6897, 11.3793, 17.0690, 22.7857]^T$. If this set $\underline{s}$ is rotated by $\Delta s = 2$, then a new set $\underline{s}' = [2, 7.6897, 13.3793, 19.0690, 24.7857]^T$ is obtained, which corresponds to a set of directions $\underline{\theta}' = [30.75^\circ, 62.65^\circ, 86.59^\circ, 109.91^\circ, 137.78^\circ]^T$. If only four sources with directions equal to the first four elements of $\underline{\theta}'$ are present in the environment, then applying MUSIC will result in an extra null at 137.78°, which implies that $\underline{\theta}'$ or, equivalently, $\underline{s}'$, is also an ambiguous set.

¹Note that in [12], the rank of ambiguity of ambiguous set of DOA's is defined as $m - \text{rank}(\mathbb{A}(\underline{s}))$.

Fig. 2. Effects of rotating an ambiguous set. (a) Linear array configuration (in half-wavelengths). (b) True directions: 53.13, 78.46, 101.54, and 126.87°; directions estimated from MUSIC: 0, 53.13, 78.46, 101.54, and 126.87°. (c) True directions: 30.75, 62.65, 86.59, and 109.91°; directions estimated from MUSIC: 30.75, 62.65, 86.59, 109.91, and 137.78°. (d) True directions: 73.13, 98.46, 121.54, and 146.87°; directions estimated from MUSIC: 73.13, 98.46, 121.54, and 146.87°.

Note that the rotation should be carried out in the “arc-length” domain and not in the “directions-of-arrival” domain. For instance, if the directions of the original sources are rotated by $\Delta\theta = 20^\circ$ so that the new set of directions $\theta'' = [73.13^\circ, 98.46^\circ, 121.54^\circ, 146.87^\circ]^T$ is obtained, then as can be seen from Fig. 2(d), this set is not ambiguous. ■

Note that Theorem 1 can easily be generalized to incorporate any rank of ambiguity of the set $\underline{s}$ as well as any number of elements $m > N$. It becomes clear that if one ambiguous set is identified, then by simple rotation, an infinite number of ambiguous sets can be generated, and therefore, two different ambiguous sets may, in fact, be just a rotation of each other. Since all these sets can be generated from a single set, the idea of the *ambiguous generator set* arises and is defined as follows.

Definition 3: An ordered set $\underline{s} = [0, s_1, \dots, s_{m-1}]^T$ of m arc lengths, where $2 \leq m \leq N$, is said to be an *ambiguous generator set of arc lengths* if and only if we have the following.

- All the elements of the set except the first element are nonzero.

- The rank of the $N \times m$ matrix $\mathbf{A}(\underline{s})$, with columns the manifold vectors associated with the elements of the set, is less than m , i.e., $\text{rank}(\mathbf{A}(\underline{s})) = \rho < m$.

- For any subset $\underline{s}_i$ of k elements of $\underline{s}$ with $\rho \leq k < m$, the rank of $\mathbf{A}(\underline{s}_i)$ is equal to ρ .

According to the previous definition, a set of arc lengths $\underline{s} = [s_1, s_2, s_3, s_4]^T$ with $s_1 = 0, s_2, s_3, s_4 \neq 0$ and $\underline{a}(s_3) = c_1 \underline{a}(s_1) + c_2 \underline{a}(s_2)$ is not an ambiguous generator set. This is because $\text{rank}([\underline{a}(s_1), \underline{a}(s_2), \underline{a}(s_3), \underline{a}(s_4)]) = 3$, but $\text{rank}([\underline{a}(s_1), \underline{a}(s_2), \underline{a}(s_3)]) = 2$ (i.e., third condition is not satisfied). On the other hand, the set $\underline{s}_i = [s_1, s_2, s_3]^T$ is an ambiguous generator set since it satisfies all three conditions of Definition 3.

From the above definition, it is obvious that when considering an array, it is an impossible task to try to identify all the ambiguous sets. Therefore, the objective of this paper is to identify *ambiguous generator sets* existing in the manifold of a linear array of arbitrary geometry. In particular, two general classes of ambiguous generator sets will be identified, based on whether the manifold curve is partitioned, according to some specific rules, into “equal” or “unequal” segments,

i.e., uniform or nonuniform partitioning, respectively, of the manifold curve. Although all “uniform” ambiguities can be found by using the framework presented in the following section, there is no answer at this moment as to how all “nonuniform” ambiguities can be estimated. Therefore, in the following section, a new class of “nonuniform” ambiguity that exists in “symmetric linear arrays” will be identified, modeled, and estimated.

V. IDENTIFYING AMBIGUOUS GENERATOR SETS IN MANIFOLD CURVES

A. Based on Uniform Partitions of Hyperhelices

In this section, a technique is proposed for the calculation of *ambiguous generator sets of directions* existing in linear arrays, as well as their associated *rank of ambiguity*. The technique is based on the uniform partitions of the array manifold, which are obtained by dividing the manifold length l_m by the difference between any two array sensor locations. In the case of linear arrays, for which the direction-of-arrival consists of only the azimuth θ , the array manifold is a curve of specified length with well-known properties. If this curve is divided into equal segments, according to the following theorem, then the end points of these segments form an ambiguous set.

Theorem 2: If l_m is the manifold length of a linear array of N sensors with locations $\underline{r} = [r_1, \dots, r_N]^T$ in half wavelengths, then any subset of N elements of the set of arc lengths $s_{i,j}$, where

$$\underline{s}_{i,j} = \left[0, \frac{l_m}{|r_i - r_j|}, 2 \frac{l_m}{|r_i - r_j|}, \dots, c \frac{l_m}{|r_i - r_j|} \right]^T \quad \forall i, j \text{ with } i \neq j \quad (16)$$

is an ambiguous set if i) the last element of $\underline{s}_{i,j}$ is greater than 0 and smaller than l_m , and ii) the number of nonzero entries c is greater than or equal to $N - 1$.

The proof of this theorem can be found in Appendix B.

The requirement that the last element of the set $\underline{s}_{i,j}$ is smaller than the manifold length, combined with the fact that $c \in \mathbb{N}^+$, implies that

$$c \frac{l_m}{|r_i - r_j|} < l_m \Rightarrow c = \begin{cases} \text{fix}(|r_i - r_j|) & \text{if } |r_i - r_j| \notin \mathbb{N}^+ \\ |r_i - r_j| - 1 & \text{if } |r_i - r_j| \in \mathbb{N}^+ \end{cases} \quad (17)$$

where $\text{fix}(\bullet)$ denotes the integer part of a number.

By setting the value of $|r_i - r_j|$ in the above equation to be equal to the biggest possible difference between two array sensor locations, which is obviously the aperture l_a , and by combining the above with the second requirement of the theorem, the following sufficient condition (which has also been rigorously proven in [3]) can be obtained.

A SUFFICIENT condition for the presence of ambiguities in ALL linear arrays is:
 $l_a > N - 1$, where $l_a = \max_{\forall i,j} |r_i - r_j|$

It should be made clear that the above provides a sufficient but by no means necessary condition for the presence of ambiguities. This means that a linear array can possibly suffer from ambiguities even if $l_a \leq N - 1$. Only for the specific case of uniform linear arrays has it been proven that no ambiguities exist if $l_a \leq N - 1$ or, equivalently, if the intersensor spacing is not greater than half a wavelength.

Theorem 2 has an implication that needs to be stressed. It is well known [11] that an increase in the aperture results in better resolution capabilities. However, from the point of view of ambiguities, increasing the aperture may not be a very good idea, at least for signal subspace-type techniques. By increasing the aperture, the number of elements in the set of (16) increases, and once this number becomes greater than $N - 1$, the set becomes ambiguous. Furthermore, an increase of all the intersensor spacings may be considered an even worse idea since an array with sensor locations given by $\underline{r}' = b\underline{r}$, with $b > 1$, might have many differences between sensor locations that result in ambiguous sets of the form of (16). However, it is important to emphasize that there are some cases where, by increasing the aperture and then using some augmentation algorithm [13], [14], ambiguities may be resolved and even, in some situations, may result in an unambiguous DF system.

According to Theorem 2, all subsets of N elements from a set $\underline{s}_{i,j}$ of the form of (16) are ambiguous sets but not necessarily *ambiguous generator sets*. In order for such a subset to be an ambiguous generator set, it must have its first element equal to zero as well as satisfy the third condition of the definition of the ambiguous generator set.

Note that from now on, the vector $\underline{r}$, with elements in the sensor locations (in half wavelengths), will be assumed to be ordered in the sense that $r_i < r_j \Leftrightarrow i < j$. Furthermore, the difference between the i th and j th sensor locations, measured in half wavelengths, will be denoted by $r_{i,j}$, and hence, the aperture of the array will be $r_{1,N}$.

Example 2: Consider an array with sensor locations given by $\underline{r} = [-2.1, -0.9, 0.9, 2.1]^T$. The manifold length l_m of this array is calculated to be 20.3016. Furthermore, the only difference between sensor locations that satisfies the two conditions of Theorem 2 is the aperture $r_{1,4} = |r_1 - r_4|$. In this case, by evaluating (16), the set of arc lengths

$$\underline{s}_{1,4} = [0, 4.8337, 9.6674, 14.5011, 19.3348]^T$$

is obtained, which corresponds to the set of DOA's

$$\underline{\theta}_{1,4} = [0^\circ, 58.41^\circ, 87.27^\circ, 115.38^\circ, 154.80^\circ]^T.$$

According to Theorem 2, any subset of four elements ($N = 4$) from the above set will be an ambiguous set, and the matrix $\mathbb{A}$, which has columns, and the manifold vectors corresponding to such a subset will be singular. This means that if, for instance, MUSIC is employed in a signal environment of three sources with source directions in any three elements of the above set, then five nulls (one for each element of $\underline{s}_{1,4}$) will appear. Furthermore, it can be shown that no matrix $\mathbb{A}$ corresponding to any three elements of $\underline{s}_{1,4}$ is rank deficient. This means that four different ambiguous generator sets can be identified from $\underline{s}_{1,4}$, which are all the subsets of $\underline{s}_{1,4}$ with their first element zero and three nonzero elements of $\underline{s}_{1,4}$.

As expected from Theorem 2, the set $\underline{s}_{1,4}$, which is of the form of (16), is ambiguous since $c \geq N - 1$. However, ambiguous sets might also be defined from Theorem 2 when $c < N - 1$. To see this, consider the set of the form of (16) for the difference $r_{1,3} = r_{2,4} = 3$, which is

$$\underline{s}_{1,3} = \underline{s}_{2,4} = [0, 6.7672, 13.5344]^T \Rightarrow \\ \theta_{1,3} = \theta_{3,4} = [0^\circ, 70.53^\circ, 109.47^\circ]^T.$$

If the matrix $\mathbf{A}$, with columns in the corresponding manifold vectors, is obtained, then $\text{rank}(\mathbf{A}) = 2$, and hence, the above set is an ambiguous set with rank of ambiguity equal to 2. This means that when MUSIC is applied on any two elements of $\underline{s}_{1,3}$, three nulls will appear.

It should also be noted that not all the sets $\underline{s}_{i,j}$ of the form of (16) resulting from a difference $r_{i,j}$ between two sensor locations are ambiguous if $c < N - 1$. To see this, consider the difference $r_{1,2} = r_{3,4} = 1.2$ of the previous array, which results in a set $\underline{s}_{1,2} = [0, 16.9180]^T$, which is unambiguous. ■

The fact that $\underline{s}_{i,j}$ can be ambiguous even if $c < N - 1$ is to be expected from the proof of Theorem 2. In that proof, it was stated that the i th and the j th rows of matrix $\mathbf{A}$, with columns in the manifold vectors corresponding to $\underline{s}_{i,j}$, are equal. If $r_{i,j} = r_{k,l}$, then the i th row and the j th row of $\mathbf{A}$ will be equal, as will the k th and the l th rows. As a result, $\mathbf{A}$ can be rank deficient even if it has less than N rows. In such a case, the ambiguous set will have rank of ambiguity less than $N - 1$.

As a direct result of the previous discussion, it can be said that the uniform linear array with intersensor spacing $r_{1,2}$ greater than 1 suffers from trivial ambiguities. This well-known result is rediscovered by observing that in such a case, the set $\underline{s}_{i,j}$ will contain at least two elements, and the matrix $\mathbf{A}$ will be rank deficient since all its rows will be equal.

Thus far, it has been shown that Theorem 2 can be used in order to identify ambiguous sets inherent in linear arrays of any geometry. When this theorem is focused on specific array geometries, it can produce some more useful results as the following corollaries indicate.

Corollary 1: All the ambiguous sets that exist in a two-element array can be calculated from (16). The proof can be found in Appendix C.

Corollary 2: The set of arc lengths $\underline{g} = [0, \pi\sqrt{2}, 2\pi\sqrt{2}, \dots, c\pi\sqrt{2}]^T$ is an ambiguous set for all three-element symmetric linear arrays, as long as $r_{1,3} > 2$. The proof can be found in Appendix D.

Corollary 3: Let two arrays of N sensors have a common difference r_d between two sensor locations, which results in a set of arc lengths $\underline{g}_d$ of the form of (16). Then, this set is different for the two arrays, but the corresponding ambiguous sets of DOA's are the same. The proof can be found in Appendix E.

Based on Theorem 2 in conjunction with the previous discussion, a new technique, followed by an indicative example, is presented in a step-by-step form. This technique, for a given linear array with sensor locations given by $\underline{r}$, provides a matrix $\underline{\Sigma}$ whose rows are all the ambiguous generator sets arising from uniform partitions of the array manifold. Furthermore,

a column vector $\underline{c}$ is also provided, with its i th element representing the rank of ambiguity associated with the i th ambiguous generator set (i.e., the i th row of $\underline{\Sigma}$).

VI. PROPOSED TECHNIQUE

- 1) Calculate the manifold length $l_m = 2\pi\|\underline{r}\|$, and then take the Hadamard difference between the vector $\underline{r}$ (with elements the sensor locations) and itself, i.e., $\underline{r} \ominus \underline{r}$, which results in a $(N^2 \times 1)$ -dimensional real vector.
- 2) Create a new vector $\hat{\underline{r}}$ by eliminating all the elements of $\underline{r} \ominus \underline{r}$ that are smaller than one. Note that the elements of $\underline{r} \ominus \underline{r}$ that are smaller than one do not satisfy Condition i) of Theorem 2 and, therefore, cannot possibly give rise to ambiguous generator sets. Let d be the dimension of the new vector, i.e., $\hat{\underline{r}} \in \mathbb{R}^{d \times 1}$. Due to the properties of the Hadamard difference, N elements of $\underline{r} \ominus \underline{r}$ will be equal to zero, and half of the remaining elements will be negative. Therefore, $d \leq (N/2)(N - 1)$.
- 3) For each element of $\hat{\underline{r}}$, i.e., for each difference $r_{i,j} > 1$ between two sensor locations, construct the corresponding vector $\underline{s}_{i,j}$ by using (16). Note that there are d different vectors $\underline{s}_{i,j}$ and that each $\underline{s}_{i,j}$ corresponds to a different uniform partition of the array manifold.
- 4) Identify all the vectors $\underline{s}_{i,j}$ that do not satisfy Condition ii) of Theorem 2 (i.e., the number of nonzero elements c in the row is smaller than $N - 1$) and then eliminate those for which

$$\text{rank}(\mathbf{A}(\underline{s}_{i,j})) = c + 1.$$

All the remaining vectors $\underline{s}_{i,j}$ produce ambiguous generator sets.

- 5) Ensure all remaining vectors $\underline{s}_{i,j}$ of Step 4 are of the same length k , where k is the length of the vector with the maximum number of elements, by appending zeros where necessary. It is obvious that the set with the maximum number of elements is the set $\underline{s}_{1,N}$ corresponding to the array aperture. Then, create the matrix $\mathbf{S} \in \mathbb{R}^{d \times k}$ with rows the vectors $\underline{s}_{i,j}^T$.
- 6) For each $\text{row}_i\{\mathbf{S}\}$, identify the ambiguous generator sets based on the following rules:

Rule a) If the nonzero elements of $\text{row}_i\{\mathbf{S}\}$ cannot be found in other rows, then $\binom{c}{N-1}$ ambiguous generator sets can be produced by the elements of $\text{row}_i\{\mathbf{S}\}$. These ambiguous generator sets are all the possible subsets of N elements of $\text{row}_i\{\mathbf{S}\}$ with their first element zero and $N - 1$ nonzero elements of $\text{row}_i\{\mathbf{S}\}$. All ambiguous generator sets constructed in this way have rank of ambiguity $\rho_a = N - 1$. Note that such rows must definitely have $c \geq N - 1$ [see Theorem 2(ii)]; otherwise, they would have been eliminated in Step 4.

Rule b) If the nonzero elements of $\text{row}_i\{\mathbf{S}\}$ can be found in other rows, then ambiguous generator sets with rank of ambiguity less than $N - 1$ might be obtained. This means that all subsets of $\text{row}_i\{\mathbf{S}\}$ with their first element 0 and with length $2, 3, \dots, \min(N, c+1)$ must be considered. These subsets are classified as ambiguous generator sets if the three conditions of the ambiguous

generator set definition are satisfied. Furthermore, for each ambiguous generator set $\underline{g}$, rank of ambiguity $\rho_a = \text{rank}(\mathbf{A}(\underline{g}))$ is estimated. Note that this step clarifies why, in Step 4, it is incorrect to eliminate two identical rows of $\mathbb{S}$, although they will result in the same ambiguous generator sets.

- 7) Create the matrix $\underline{\Sigma}$, whose rows are all the different ambiguous generator sets found in Step 6. If one ambiguous generator set has less than N elements, then complete the corresponding row of $\underline{\Sigma}$ with zeros. In addition, eliminate duplicate rows of $\underline{\Sigma}$. Finally, form the vector $\underline{r}$, with elements of the rank of ambiguity of each ambiguous generator set, as estimated in Step 6.

The above technique was checked using many simulations, and an illustrative example is presented below.

Example 3: The steps of the previously described technique for an array with sensor positions

$$\underline{r} = [-3, -2.2, 1.1, 1.9, 2.2]^T$$

are as follows:

- 1) The manifold length of this array is computed to be $l_m = 30.4589$. The Hadamard difference between $\underline{r}$ and itself is

$$\begin{aligned} \underline{r} \ominus \underline{r} = [0, -0.8, -4.1, -4.9, -5.2, 0.8, 0, -3.3, \\ -4.1, -4.4, 4.1, 3.3, 0, -0.8, -1.1, 4.9, \\ 4.1, 0.8, 0, -0.3, 5.2, 4.4, 1.1, 0.3, 0]^T. \end{aligned}$$

- 2) Eliminating those entries of $\underline{r} \ominus \underline{r}$ that are smaller than unity results in

$$\hat{\underline{r}} = [4.1, 3.3, 4.9, 4.1, 5.2, 4.4, 1.1]^T \Rightarrow d = 7.$$

- 3) The seven row vectors $\underline{s}_{i,j}^T$ are

$$\begin{aligned} \underline{s}_{1,3}^T &= [0, 7.4290, 14.8580, 22.2870, 29.7160] \\ \underline{s}_{2,3}^T &= [0, 9.2300, 18.4600, 27.6900] \\ \underline{s}_{1,4}^T &= [0, 6.2161, 12.4322, 18.6483, 24.8644] \\ \underline{s}_{2,4}^T &= [0, 7.4290, 14.8580, 22.2870, 29.7160] \\ \underline{s}_{1,5}^T &= [0, 5.8575, 11.7150, 17.5725, 23.4300, 29.2875] \\ \underline{s}_{2,5}^T &= [0, 6.9225, 13.8450, 20.7675, 27.6900] \\ \underline{s}_{3,5}^T &= [0, 27.6900]. \end{aligned}$$

- 4) The row vectors $\underline{s}_{i,j}^T$ that have less than $N - 1 = 4$ nonzero elements are examined.

The number of nonzero elements in $\underline{s}_{2,3}^T$ is $c = 3 < N - 1 = 4$.

Furthermore, $\text{rank}(\mathbf{A}(\underline{s}_{2,3})) = 4 = c + 1$, and hence, this row is eliminated.

In addition, $\underline{s}_{3,5}^T$ is eliminated since $c = 1 < 4$ and $\text{rank}(\mathbf{A}(\underline{s}_{3,5})) = 2 = c + 1$. Thus, the remaining sets are $\underline{s}_{1,3}^T$, $\underline{s}_{1,4}^T$, $\underline{s}_{2,4}^T$, $\underline{s}_{1,5}^T$, and $\underline{s}_{2,5}^T$.

- 5) Since $\underline{s}_{1,5}^T$ is the vector with the maximum number of elements (six in this case), it means that $k = 6$, and all the vectors $\underline{s}_{i,j}^T$ are made to have length equal to six.

The matrix $\mathbb{S}$ can now be formed as

$$\mathbb{S} = \begin{bmatrix} 0 & 7.4290 & 14.8580 & 22.2870 & 29.7160 & 0 \\ 0 & 6.2161 & 12.4322 & 18.6483 & 24.8644 & 0 \\ 0 & 7.4290 & 14.8580 & 22.2870 & 29.7160 & 0 \\ 0 & 5.8575 & 11.7150 & 17.5725 & 23.4300 & 29.2875 \\ 0 & 6.9225 & 13.8450 & 20.7675 & 27.6900 & 0 \end{bmatrix}.$$

- 6) In this step, the ambiguous generator sets arising from different uniform partitions of the manifold are calculated. The nonzero elements of $\text{row}_1\{\mathbb{S}\}$ can also be found in $\text{row}_3\{\mathbb{S}\}$, and hence, $\text{row}_1\{\mathbb{S}\}$ is investigated based on Rule 6b. The ambiguous generator set definition is initially checked for the set $\underline{s}_1$, which consists of the five first elements of $\text{row}_1\{\mathbb{S}\}$. This set is certainly ambiguous, and the matrix with columns the corresponding manifold vectors has $\text{rank}(\mathbf{A}(\underline{s}_1)) = 3$. However, any one of the four subsets $\underline{s}'_1$ of $\underline{s}_1$ with four elements of $\text{row}_1\{\mathbb{S}\}$ involving one zero and three nonzero elements is also an ambiguous set. The matrix with columns in the corresponding manifold vectors has $\text{rank}(\mathbf{A}(\underline{s}'_1)) = 3$. Therefore, the set $\underline{s}_1$ satisfies the third condition of the definition of the ambiguous generator set. Similarly, the four subsets $\underline{s}'_1$ are ambiguous generator sets since all their subsets with three elements are unambiguous. Hence, the following five ambiguous generator sets can be defined from $\text{row}_1\{\mathbb{S}\}$, all of which have $\rho_a = 3$:

$$\begin{bmatrix} 0 & 7.4290 & 14.8580 & 22.2870 & 29.7160 \\ 0 & 7.4290 & 14.8580 & 22.2870 & \\ 0 & 7.4290 & 14.8580 & 29.7160 & \\ 0 & 7.4290 & 22.2870 & 29.7160 & \\ 0 & 14.8580 & 22.2870 & 29.7160 & \end{bmatrix}$$

The nonzero elements of $\text{row}_2\{\mathbb{S}\}$ cannot be found in any other row of $\mathbb{S}$, and hence, $\text{row}_2\{\mathbb{S}\}$ is investigated based on Rule 6a. Since the number of nonzero elements in $\text{row}_2\{\mathbb{S}\}$ is $c = 4$, we have $\binom{c}{N-1} = \binom{4}{4} = 1$, which implies that only one ambiguous generator set (which is the same as the set $\underline{s}_{1,4}^T$ of Step 3) can be identified, with $\rho_a = 4$.

Since $\text{row}_3\{\mathbb{S}\}$ is the same as $\text{row}_1\{\mathbb{S}\}$, it follows that the ambiguous generator sets obtained from this row are the ones that were already obtained from $\text{row}_1\{\mathbb{S}\}$.

The nonzero elements of $\text{row}_4\{\mathbb{S}\}$ cannot be found in any other row of $\mathbb{S}$, and hence, $\text{row}_4\{\mathbb{S}\}$ is also investigated based on Rule 6a. This time, $c = 5$, and therefore, $\binom{5}{4} = 5$ ambiguous generator sets can be found by taking all the subsets of N elements of $\text{row}_4\{\mathbb{S}\}$ (with one zero and $N - 1$ nonzero elements).

$$\begin{bmatrix} 0 & 5.8575 & 11.7150 & 17.5725 & 23.4300 \\ 0 & 5.8575 & 11.7150 & 17.5725 & 29.2875 \\ 0 & 5.8575 & 11.7150 & 23.4300 & 29.2875 \\ 0 & 5.8575 & 17.5725 & 23.4300 & 29.2875 \\ 0 & 11.7150 & 17.5725 & 23.4300 & 29.2875 \end{bmatrix}.$$

Each of these sets has $\rho_a = 4$.

Finally, $\text{row}_5\{\mathcal{S}\}$ is also investigated based on Rule 6a. Since $c = 4$, it generates $\binom{4}{4} = 1$ ambiguous generator set. This set is the same as set $s_{2,5}^T$ of Step 3 and has $\rho_a = 4$.

- 7) The matrix $\underline{\underline{\Sigma}}$ with rows in the ambiguous generator sets produced in Step 6 is now constructed, as well as the vector $\underline{\sigma}$ with elements in the rank of ambiguity for the appropriate ambiguous generator set.

$$\underline{\underline{\Sigma}} = \begin{bmatrix} 0 & 7.4290 & 14.8580 & 22.2870 & 29.7160 \\ 0 & 7.4290 & 14.8580 & 22.2870 & 0 \\ 0 & 7.4290 & 14.8580 & 29.7160 & 0 \\ 0 & 7.4290 & 22.2870 & 29.7160 & 0 \\ 0 & 14.8580 & 22.2870 & 29.7160 & 0 \\ 0 & 6.2161 & 12.4322 & 18.6483 & 24.8644 \\ 0 & 5.8575 & 11.7150 & 17.5725 & 23.4300 \\ 0 & 5.8575 & 11.7150 & 17.5725 & 29.2875 \\ 0 & 5.8575 & 11.7150 & 23.4300 & 29.2875 \\ 0 & 5.8575 & 17.5725 & 23.4300 & 29.2875 \\ 0 & 11.7150 & 17.5725 & 23.4300 & 29.2875 \\ 0 & 6.9225 & 13.8450 & 20.7675 & 27.6900 \end{bmatrix}$$

$$\underline{\sigma} = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 4 \\ 4 \\ 4 \\ 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}.$$

In conclusion, 12 ambiguous generator sets can be identified for this array. Five of these have $\rho_a = 3$, whereas the remaining have $\rho_a = 4$. ■

B. Based on Nonuniform Partitions of Hyperhelices

In this section, another class of ambiguous generator set existing only in symmetric² linear arrays will be identified and estimated based not on a *uniform* but on a *nonuniform* partition of the array manifold. This can be achieved using the concept of characteristic points that, as can be shown, partition the manifold curve into $N - 1$ unequal segments and are defined as follows.

Definition 4: A point s on the array manifold, with $0 \leq s \leq 2\pi\|r\|$, is a characteristic point if and only if

$$\text{Re}\{u_1^H(s)\underline{a}(0)\} = 0 \quad (18)$$

where $\underline{u}_1(s)$ denotes the tangent vector at the manifold point s , i.e., $\underline{u}_1(s) = d\underline{a}(s)/ds$.

The above definition indicates an important property of characteristic points, that is, the tangent at any characteristic point is orthogonal (wide sense orthogonality since *only* the real part is zero) to the manifold vector at $s = 0$. The following

² Each sensor has a symmetric counterpart with respect to the array reference point, i.e., $\text{sum}(r^i) = 0$, $\forall i = \text{odd}$.

theorem shows that if the manifold of a symmetric linear array has N characteristic points $s_{c1}, s_{c2}, \dots, s_{cN}$, where $s_{c1} = 0$, then this set is ambiguous. This is the starting point for the identification of a new class of ambiguous generator set having a rank of ambiguity equal to $(N - 1)$.

Theorem 3: If s_{ci} , $i = 1, \dots, N$ (with $s_{c1} = 0$) are the characteristic arc lengths of the manifold of a symmetrical linear array corresponding to the first winding (if N even) or half winding (if N odd) of the associated hyperhelix, then $\det(\mathbb{A}_{cN}) = 0$, where $\mathbb{A}_{cN} = [\underline{a}(s_{c1}), \underline{a}(s_{c2}), \dots, \underline{a}(s_{cN})]$. The proof can be found in Appendix F.

We have seen that the Cartan matrix and the Frame matrix always satisfy the differential equation

$$\mathbb{F}'(s) = \mathbb{F}(s)\mathbb{C} \quad (19)$$

which, with the initial condition $\mathbb{F}(0) = \mathbb{I}_d$, has the solution

$$\mathbb{F}(s) = \text{expm}(s\mathbb{C}). \quad (20)$$

Then, in the case of a symmetric linear array, the characteristic points s_{ci} may be obtained (see Appendix G) as the roots of

$$\text{tr}(\mathbb{C} \text{expm}\{s\mathbb{C}\}) = 0 \quad (21)$$

where $s = \pi\|r\|(1 - \cos \theta)$.

In addition, the number of windings (N odd) or half windings (N even) n_w of this hyperhelical curve can be estimated using the expression

$$n_w = \frac{l_m}{l_w} \quad (22)$$

where l_w indicates the length of one winding (N odd) or of one half winding (N even) and is given as the $(N - 1)$ th positive root (see [9, Th. 1]) of (21).

Thus, a **sufficient** condition for presence of nontrivial ambiguities in the manifold of a **symmetric** linear array is

$$l_m \geq l_w \quad \text{i.e.,} \quad n_w \geq 1$$

Corollary 4: If $0 \leq \theta_1 < \theta_N \leq \pi$ and $\|\cos \theta_1 - \cos \theta_N\| \geq l_w/\pi\|r\| = 2/n_w$, then there exist $N - 2$ bearings $\theta_2, \dots, \theta_{N-1}$ in the region (θ_1, θ_N) such that the manifold vectors corresponding to $\theta_1, \theta_2, \dots, \theta_N$ are linearly dependent.

It is clear from the above discussion that if the array is symmetric, then more rows may be added to the matrix $\underline{\underline{\Sigma}}$, which was presented in the previous section, i.e., more generator sets can be found using a nonuniform partition of the array manifold based on Theorem 3.

For instance, an array with sensor positions $r = [-2.1, -0.9, 0.9, 2.1]^T$ has the following *ambiguous generator sets* matrix, which is estimated based on the uniform

partitioning of its manifold as described in Section VI-A:

$$\underline{\underline{\Sigma}} = \begin{bmatrix} 0 & 6.7672 & 13.5344 & 0 \\ 0 & 4.8337 & 9.6674 & 14.5011 \\ 0 & 4.8337 & 9.6674 & 19.3349 \\ 0 & 4.8337 & 14.5011 & 19.3349 \\ 0 & 9.6674 & 14.5011 & 19.3349 \end{bmatrix} \quad \underline{\sigma} = \begin{bmatrix} 2 \\ 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}. \quad (23)$$

However, because the array is symmetric, (21) can be used to estimate additional ambiguous generator sets (based on nonuniform partitioning). Thus, the Cartan matrix $\mathbb{C}$ of the array is initially formed, i.e.,

$$\mathbb{C} = \begin{bmatrix} 0 & -0.6074 & 0 & 0 \\ 0.6074 & 0 & -0.2056 & 0 \\ 0 & 0.2056 & 0 & -0.2981 \\ 0 & 0 & 0.2981 & 0 \end{bmatrix}$$

and then, the roots of (21), i.e., the set of characteristic points, are estimated.

$$\underline{s}^T = [0, 5.5147, 9.3228, 14.0360, 19.8000].$$

Finally, the following additional ambiguous generator sets, and their associated ranks of ambiguity, should be added as extra rows to the matrix $\underline{\underline{\Sigma}}$ and vector $\underline{\sigma}$ in (23):

$$\text{ambig. gen. sets} = \begin{bmatrix} 0 & 5.5147 & 9.3229 & 14.0360 \\ 0 & 5.5147 & 9.3229 & 19.8000 \\ 0 & 5.5147 & 14.0360 & 19.8000 \\ 0 & 9.3229 & 14.0360 & 19.8000 \end{bmatrix}$$

with

$$\text{rank of ambig.} = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}. \quad (24)$$

In the previous discussion, a new class of ambiguities has been identified, based on the characteristic points. However, there are also other classes of ambiguity based on nonuniform partitions of the manifold curve. For instance, for the array $\underline{r} = [-2.1, -0.9, 0.9, 2.1]^T$, the set of arc lengths $\underline{s} = [0, 4.7549, 9.1754, 14.2903, 19.4052]^T$ is an ambiguous set that, although “nonuniform,” is not the set of characteristic points, i.e., does not belong to the proposed class, and provides the ambiguous generator sets

$$\text{ambig. gen. sets} = \begin{bmatrix} 0 & 4.7549 & 9.1754 & 14.2903 \\ 0 & 4.7549 & 9.1754 & 19.4052 \\ 0 & 4.7549 & 14.2903 & 19.4052 \\ 0 & 9.1754 & 14.2903 & 19.4052 \end{bmatrix}. \quad (25)$$

There is no answer at this moment as to how other nonuniform classes of ambiguous generator sets can be identified.

VII. CONCLUSIONS

In this paper, the notion of the ambiguous generator set, which represents/generates an infinite number of ambiguous sets of directions, is introduced. Then, a completely innovative approach to calculate ambiguous generator sets is constructed. In particular, two general classes of ambiguous generator sets of DOA's are calculated, based on whether the manifold curve

is partitioned, according to some specific rules, uniformly or nonuniformly.

APPENDIX A PROOF OF THEOREM 1

Since $\underline{s} = [s_1, s_2, \dots, s_m]^T$ is an ambiguous set, the matrix $\mathbb{A} \in \mathbb{C}^{N \times m}$ with columns in the corresponding manifold vectors, i.e.,

$$\mathbb{A}(\underline{s}) = [\underline{a}(s_1), \underline{a}(s_2), \dots, \underline{a}(s_m)]$$

is rank deficient. This means that any submatrix $\mathbb{A}_m(\underline{s})$ of $\mathbb{A}(\underline{s})$, which has exactly m rows, is singular. That is

$$\det(\mathbb{A}_m(\underline{s})) = 0.$$

Consider now the set $\underline{s}' = [s_1 + \Delta s, s_2 + \Delta s, \dots, s_m + \Delta s]^T = \underline{s} + \Delta s \cdot \underline{1}_m$, where $\underline{1}_m$ is the m -element column vector with all its elements equal to unity. The manifold vector corresponding to the i th element of $\underline{s}'$ is

$$\underline{a}(s_i + \Delta s) = \underline{a}(s_i) \odot \underline{a}(\Delta s).$$

Therefore, the matrix $\mathbb{A}'$ with columns in the manifold vectors corresponding to the set $\underline{s}'$ is

$$\mathbb{A}' = \mathbb{A}(\underline{s}') = \mathbb{A}(\underline{s} + \Delta s \cdot \underline{1}_m) = \mathbb{A}(\underline{s}) \odot \mathbb{A}(\Delta s \cdot \underline{1}_m).$$

The determinant of any submatrix $\mathbb{A}'_m$ of $\mathbb{A}'$ that has exactly m rows is

$$\begin{aligned} \det(\mathbb{A}'_m) &= \det(\mathbb{A}_m(\underline{s}) \odot \mathbb{A}_m(\Delta s \cdot \underline{1}_m)) \\ &= \det(\mathbb{A}_m(\underline{s})) \cdot \det\left(\underbrace{\text{diag}(\mathbb{A}_m(\Delta s))}_{m \times 1 \text{ vector}}\right) = 0 \end{aligned}$$

and, hence, is singular. This implies that $\mathbb{A}'$ is rank deficient, and therefore, the set $\underline{s}'$ is also an ambiguous set. Furthermore, the rank of ambiguity of $\underline{s}$ and of $\underline{s}'$ is the same. This is because the submatrices of $\mathbb{A}(\underline{s})$ with less than m rows are not singular, which implies that the submatrices of $\mathbb{A}'$ with less than m rows are also not singular. $\square$

APPENDIX B PROOF OF THEOREM 2

Consider a linear array of N sensors with locations $\underline{r} = [r_1, \dots, r_N]^T$.

If the difference between the i th and j th sensor locations, i.e., $r_{i,j} = |r_i - r_j|$, satisfies the following two conditions

- i) $0 < c(l_m/r_{i,j}) < l_m$;
- ii) $c \geq N - 1$;

then we have to prove that any subset of N elements of the set of arc lengths $\underline{s}_{ij}$, where

$$\begin{aligned} \underline{s}_{i,j} &= \left[0, \frac{l_m}{|r_i - r_j|}, 2\frac{l_m}{|r_i - r_j|}, \dots, c\frac{l_m}{|r_i - r_j|} \right]^T \\ &= [s_0, s_1, s_2, \dots, s_c]^T \end{aligned}$$

is an ambiguous set.

Let $\mathbb{A}(\underline{s}_{i,j}) \in \mathcal{C}^{N \times (c+1)}$ be the matrix with columns in the manifold vectors corresponding to the $\underline{s}_{i,j}$, shown in (B.1) at the bottom of the page.

Since $c \geq N - 1$, the number of columns of $\mathbb{A}$ is equal to or greater than N . Hence, by taking a submatrix of $\mathbb{A}$ with exactly N columns, e.g., the N first columns of $\mathbb{A}$, a square submatrix $\mathbb{A}_N \in \mathcal{C}^{N \times N}$ of $\mathbb{A}$ is obtained whose determinant is in (B.2), shown at the bottom of the page.

Consider now the i th and j th rows of $\tilde{\mathbb{A}}_N$ (with $i > j$)

$$\text{row}_i(\tilde{\mathbb{A}}_N) = \left[1, \exp\left(j\pi r_i \frac{2}{r_{i,j}}\right), \dots, \exp\left(j\pi r_i (N-1) \frac{2}{r_{i,j}}\right) \right]$$

$$\text{row}_j(\tilde{\mathbb{A}}_N) = \left[1, \exp\left(j\pi r_j \frac{2}{r_{i,j}}\right), \dots, \exp\left(j\pi r_j (N-1) \frac{2}{r_{i,j}}\right) \right].$$

By using the property that $\exp(-j2\pi k(r_{i,j}/r_{i,j})) = 1$ for any $k \in \mathbb{N}^+$, the i th row of $\tilde{\mathbb{A}}_N$ can be written as in (B.3), shown at the bottom of the next page, i.e., $\text{row}_i(\tilde{\mathbb{A}}_N) = \text{row}_j(\tilde{\mathbb{A}}_N)$, and therefore, $\det(\mathbb{A}_N) = 0$.

It is easy to see that the fact that $\mathbb{A}_N$ was chosen to consist of the first N columns of $\mathbb{A}$ and not any N columns is not restrictive in the least since in every case, the i th and the j th rows of any $N \times N$ submatrix of $\mathbb{A}$ will be equal. Since all submatrices $\mathbb{A}_N$ of $\mathbb{A}$ are singular, it results that $\mathbb{A}$ is rank

$$\begin{aligned} \mathbb{A}(\underline{s}_{i,j}) &= [a(s_0), a(s_1), \dots, a(s_c)] \\ &= \begin{bmatrix} \exp\left(-j\pi r_1 \left(1 - \frac{s_0}{\pi \|\underline{r}\|}\right)\right), & \exp\left(-j\pi r_1 \left(1 - \frac{s_1}{\pi \|\underline{r}\|}\right)\right), & \dots, & \exp\left(-j\pi r_1 \left(1 - \frac{s_c}{\pi \|\underline{r}\|}\right)\right) \\ \vdots & \vdots & \ddots & \vdots \\ \exp\left(-j\pi r_N \left(1 - \frac{s_0}{\pi \|\underline{r}\|}\right)\right), & \exp\left(-j\pi r_N \left(1 - \frac{s_1}{\pi \|\underline{r}\|}\right)\right), & \dots, & \exp\left(-j\pi r_N \left(1 - \frac{s_c}{\pi \|\underline{r}\|}\right)\right) \end{bmatrix} \\ &= \begin{bmatrix} \exp\left(-j\pi r_1 \left(1 - \frac{0}{\pi \|\underline{r}\|}\right)\right), & \exp\left(-j\pi r_1 \left(1 - \frac{l_m}{\pi \|\underline{r}\|}\right)\right), & \dots & \exp\left(-j\pi r_1 \left(1 - \frac{c \frac{l_m}{|r_i - r_j|}}{\pi \|\underline{r}\|}\right)\right) \\ \vdots & \vdots & \ddots & \vdots \\ \exp\left(-j\pi r_N \left(1 - \frac{0}{\pi \|\underline{r}\|}\right)\right), & \exp\left(-j\pi r_N \left(1 - \frac{l_m}{\pi \|\underline{r}\|}\right)\right), & \dots & \exp\left(-j\pi r_N \left(1 - \frac{c \frac{l_m}{|r_i - r_j|}}{\pi \|\underline{r}\|}\right)\right) \end{bmatrix} \end{aligned}$$

and, by using $l_m = 2\pi \|\underline{r}\|$,

$$= \begin{bmatrix} \exp(-j\pi r_1), & \exp\left(-j\pi r_1 \left(1 - \frac{2}{r_{i,j}}\right)\right), & \dots & \exp\left(-j\pi r_1 \left(1 - c \frac{2}{r_{i,j}}\right)\right) \\ \vdots & \vdots & \ddots & \vdots \\ \exp(-j\pi r_N), & \exp\left(-j\pi r_N \left(1 - \frac{2}{r_{i,j}}\right)\right), & \dots & \exp\left(-j\pi r_N \left(1 - c \frac{2}{r_{i,j}}\right)\right) \end{bmatrix} \quad (\text{B.1})$$

$$\begin{aligned} \det(\mathbb{A}_N(\underline{s})) &= \begin{vmatrix} \exp(-j\pi r_1), & \exp\left(-j\pi r_1 \left(1 - \frac{2}{r_{i,j}}\right)\right), & \dots, & \exp\left(-j\pi r_1 \left(1 - (N-1) \frac{2}{r_{i,j}}\right)\right) \\ \vdots & \vdots & \ddots & \vdots \\ \exp(-j\pi r_N), & \exp\left(-j\pi r_N \left(1 - \frac{2}{r_{i,j}}\right)\right), & \dots, & \exp\left(-j\pi r_N \left(1 - (N-1) \frac{2}{r_{i,j}}\right)\right) \end{vmatrix} \\ &= \begin{vmatrix} 1, & \exp\left(j\pi r_1 \frac{2}{r_{i,j}}\right), & \dots, & \exp\left(j\pi r_1 (N-1) \frac{2}{r_{i,j}}\right) \\ \vdots & \vdots & \ddots & \vdots \\ 1, & \exp\left(j\pi r_N \frac{2}{r_{i,j}}\right), & \dots, & \exp\left(j\pi r_N (N-1) \frac{2}{r_{i,j}}\right) \end{vmatrix} \exp(-j\pi(r_1 + \dots + r_N)) \\ &= \underbrace{\begin{vmatrix} 1, & \exp\left(j\pi r_1 \frac{2}{r_{i,j}}\right), & \dots, & \exp\left(j\pi r_1 (N-1) \frac{2}{r_{i,j}}\right) \\ \vdots & \vdots & \ddots & \vdots \\ 1, & \exp\left(j\pi r_N \frac{2}{r_{i,j}}\right), & \dots, & \exp\left(j\pi r_N (N-1) \frac{2}{r_{i,j}}\right) \end{vmatrix}}_{=\tilde{\mathbb{A}}_N} \exp(-j\pi(r_1 + \dots + r_N)) \\ \det(\mathbb{A}_N(\underline{s})) &= \det(\tilde{\mathbb{A}}_N) \exp(-j\pi(r_1 + \dots + r_N)) \quad (\text{B.2}) \end{aligned}$$

deficient, and therefore, any subset of $\underline{s}_{i,j}$ with exactly N elements is an ambiguous set. $\square$

APPENDIX C

PROOF OF COROLLARY 1

Consider the two-element array $\underline{r} = [-r_1, r_1]^T$. Let the set $\underline{s} = [0, s_1]^T$ be an ambiguous set of arc lengths. The matrix $\mathbb{A}$ with columns in the manifold vectors corresponding to $\underline{s}$ is

$$\mathbb{A} = [\underline{a}(0), \underline{a}(s_1)] \\ = \begin{bmatrix} \exp(j\pi r_1) & \exp\left(j\pi r_1 \left(1 - \frac{s_1}{\pi \|\underline{r}\|}\right)\right) \\ \exp(-j\pi r_1) & \exp\left(-j\pi r_1 \left(1 - \frac{s_1}{\pi \|\underline{r}\|}\right)\right) \end{bmatrix}.$$

Since it is assumed that $\underline{s}$ is an ambiguous set, it results that $\mathbb{A}$ is singular and therefore

$$\begin{aligned} \det(\mathbb{A}) &= \exp(j\pi r_1) \cdot \exp\left(-j\pi r_1 \left(1 - \frac{s_1}{\pi \|\underline{r}\|}\right)\right) \\ &\quad - \exp(-j\pi r_1) \exp\left(j\pi r_1 \left(1 - \frac{s_1}{\pi \|\underline{r}\|}\right)\right) = 0 \\ &\Rightarrow \exp\left(j\pi r_1 \left(1 - \left(1 - \frac{s_1}{\pi \|\underline{r}\|}\right)\right)\right) \\ &\quad - \exp\left(-j\pi r_1 \left(1 - \left(1 - \frac{s_1}{\pi \|\underline{r}\|}\right)\right)\right) = 0 \\ &\Rightarrow \exp\left(j\pi r_1 \frac{s_1}{\pi \|\underline{r}\|}\right) - \exp\left(-j\pi r_1 \frac{s_1}{\pi \|\underline{r}\|}\right) = 0 \\ &\Rightarrow \sin\left(\frac{s_1 r_1}{\|\underline{r}\|}\right) = 0 \Rightarrow \frac{s_1 r_1}{\|\underline{r}\|} = k\pi \\ &\quad \text{with } k = 1, 2, 3, \dots \text{ since } s_1 r_1 \neq 0 \\ &\Rightarrow s_1 = k \frac{\pi \|\underline{r}\|}{r_1} = k \frac{2\pi \|\underline{r}\|}{2r_1} = k \frac{l_m}{r_{1,2}} \\ &\quad \text{with } k = 1, 2, 3, \dots \end{aligned}$$

Therefore, the only ambiguous sets of arc lengths that can possibly exist for a two-element array are of the form of

(16). As a result, a necessary and sufficient condition for the presence of ambiguities in a two-element linear array is

$$\frac{l_m}{r_{1,2}} < l_m \Rightarrow r_{1,2} > 1. \quad \square$$

APPENDIX D

PROOF OF COROLLARY 2

For all three-element symmetric linear arrays with sensor positions given by $\underline{r} = [-r_1, 0, r_1]^T$, the manifold length is equal to

$$l_m = 2\pi \|\underline{r}\| = 2\pi \sqrt{(-r_1)^2 + r_1^2} = 2\pi r_1 \sqrt{2}.$$

The set of arc lengths of the form of (16) for the difference between the first and the third sensors, i.e., for $r_{1,3}$, is

$$\begin{aligned} \underline{s} &= \left[0, \frac{l_m}{r_{1,3}}, 2\frac{l_m}{r_{1,3}}, \dots, c\frac{l_m}{r_{1,3}}\right]^T \\ &\Rightarrow \underline{s} = \left[0, \frac{2\pi r_1 \sqrt{2}}{2r_1}, 2\frac{2\pi r_1 \sqrt{2}}{2r_1}, \dots, c\frac{2\pi r_1 \sqrt{2}}{2r_1}\right]^T \\ &\Rightarrow \underline{s} = [0, \pi\sqrt{2}, 2\pi\sqrt{2}, \dots, c\pi\sqrt{2}]^T \end{aligned}$$

and therefore, all three-element symmetric linear arrays suffer from ambiguities of the form of the set $\underline{s}$ given above, as long as $r_{1,3} > N - 1 \Rightarrow r_{1,3} > 2$. $\square$

APPENDIX E

PROOF OF COROLLARY 3

Consider two different arrays, each having N sensors and sensor locations $\underline{r}_1$ and $\underline{r}_2$, respectively. Let there be a common difference r_d between two sensor locations, which, for both arrays, satisfies the conditions of Theorem 2. In this case, the two sets of arc lengths constructed from (16) will be

First array:

$$\underline{s}_1 = \left[0, \frac{2\pi \|\underline{r}_1\|}{r_d}, 2\frac{2\pi \|\underline{r}_1\|}{r_d}, \dots, c\frac{2\pi \|\underline{r}_1\|}{r_d}\right]^T$$

Second array:

$$\underline{s}_2 = \left[0, \frac{2\pi \|\underline{r}_2\|}{r_d}, 2\frac{2\pi \|\underline{r}_2\|}{r_d}, \dots, c\frac{2\pi \|\underline{r}_2\|}{r_d}\right]^T$$

$$\begin{aligned} \text{row}_i(\tilde{\mathbb{A}}_N)^T &= \mathbb{I}_N \cdot \text{row}_i(\tilde{\mathbb{A}}_N)^T \\ &= \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \exp\left(-j2\pi \frac{r_{i,j}}{r_{i,j}}\right) & 0 & \dots & 0 \\ 0 & 0 & \exp\left(-j2\pi 2 \frac{r_{i,j}}{r_{i,j}}\right) & \dots & 0 \\ & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & \exp\left(-j2\pi(N-1) \frac{r_{i,j}}{r_{i,j}}\right) \end{bmatrix} \text{row}_i(\tilde{\mathbb{A}}_N)^T \\ &= \left[1, \exp\left(j\pi(r_i - r_{ij}) \cdot \frac{2}{r_{i,j}}\right), \dots, \exp\left(j\pi(r_i - r_{ij})(N-1) \frac{2}{r_{i,j}}\right)\right]^T \\ &= \left[1, \exp\left(j\pi r_j \cdot \frac{2}{r_{i,j}}\right), \dots, \exp\left(j\pi r_j(N-1) \frac{2}{r_{i,j}}\right)\right]^T \\ &= \text{row}_j(\tilde{\mathbb{A}}_N)^T \end{aligned} \tag{B.3}$$

with $c \geq N - 1$. Since $c \geq N - 1$, the above two sets are ambiguous. Using (9), the corresponding set of DOA's for the first array is

$$\underline{\theta}_1 = \left[0, \cos^{-1} \left(1 - \frac{2\pi \|\underline{r}_1\|/r_d}{\pi \|\underline{r}_1\|} \right), \dots, \cos^{-1} \left(1 - c \frac{2\pi \|\underline{r}_1\|/r_d}{\pi \|\underline{r}_1\|} \right) \right]^T$$

$$\Rightarrow \underline{\theta}_1 = \left[0, \cos^{-1} \left(1 - \frac{2}{r_d} \right), \dots, \cos^{-1} \left(1 - c \frac{2}{r_d} \right) \right]^T$$

whereas, for the second array

$$\underline{\theta}_2 = \left[0, \cos^{-1} \left(1 - \frac{2\pi \|\underline{r}_2\|/r_d}{\pi \|\underline{r}_2\|} \right), \dots, \cos^{-1} \left(1 - c \frac{2\pi \|\underline{r}_2\|/r_d}{\pi \|\underline{r}_2\|} \right) \right]^T$$

$$\Rightarrow \underline{\theta}_2 = \left[0, \cos^{-1} \left(1 - \frac{2}{r_d} \right), \dots, \cos^{-1} \left(1 - c \frac{2}{r_d} \right) \right]^T.$$

The two sets are obviously identical. $\square$

APPENDIX F PROOF OF THEOREM 3

By using (14), the determinant of the matrix $\mathbf{A}_{c_N}$ becomes

$$\begin{aligned} \det(\mathbf{A}_{c_N}) &= \det([\underline{a}(s_{c_1}), \underline{a}(s_{c_2}), \dots, \underline{a}(s_{c_N})]) \\ &= \det(\mathbf{U}(0) \cdot [\mathbf{F}(s_{c_1}) \cdot \underline{R}_\kappa, \mathbf{F}(s_{c_2}) \cdot \underline{R}_\kappa, \dots, \mathbf{F}(s_{c_N}) \cdot \underline{R}_\kappa]) \\ &= \underbrace{\det(\mathbf{U}(0))}_{=\pm 1} \cdot \det([\mathbf{F}(s_{c_1}) \cdot \underline{R}_\kappa, \mathbf{F}(s_{c_2}) \cdot \underline{R}_\kappa, \dots, \mathbf{F}(s_{c_N}) \cdot \underline{R}_\kappa]) \\ &= \pm \det \begin{bmatrix} \underline{R}_\kappa^T \mathbf{F}(s_{c_1}) \\ \underline{R}_\kappa^T \mathbf{F}(s_{c_2}) \\ \vdots \\ \underline{R}_\kappa^T \mathbf{F}(s_{c_N}) \end{bmatrix} \end{aligned}$$

where, in the last equality, the following properties have been used:

- the symmetry of $\mathbf{F}$ with respect to indices i, j satisfying $i - j = 0 \bmod 2$;
- the antisymmetry of $\mathbf{F}$ with respect to indices satisfying $i - j = 1 \bmod 2$;
- $\underline{R}_j = 0$ for j odd.

Furthermore, using the fourth property of Appendix H, it follows that

$$\det(\mathbf{A}_{c_N}) = \pm \det \begin{bmatrix} \underline{a}^H(0) \mathbf{U}(0) \mathbf{F}(s_{c_1}) \\ \underline{a}^H(0) \mathbf{U}(0) \mathbf{F}(s_{c_2}) \\ \vdots \\ \underline{a}^H(0) \mathbf{U}(0) \mathbf{F}(s_{c_N}) \end{bmatrix}$$

which, based on (11), becomes

$$\det(\mathbf{A}_{c_N}) = \pm \det \begin{bmatrix} \underline{a}^H(0) \mathbf{U}(s_{c_1}) \\ \underline{a}^H(0) \mathbf{U}(s_{c_2}) \\ \vdots \\ \underline{a}^H(0) \mathbf{U}(s_{c_N}) \end{bmatrix}.$$

However, the first column of the matrix on the right-hand side of the above equation is the vector

$$[\underline{a}^H(0) [\underline{u}_1(s_{c_1}), \underline{u}_1(s_{c_2}), \dots, \underline{u}_1(s_{c_N})]]^T$$

which, in conjunction with the definition of the characteristic points (18), is the all-zero vector that implies that $\det(\mathbf{A}_{c_N}) = 0$. $\square$

APPENDIX G RELATION BETWEEN (18) AND (21)

For any linear array, we have

$$\begin{aligned} \text{Re}(\underline{a}(0)^H \underline{a}(s)) &\equiv \text{tr}(\text{Re}(\underline{a}(s) \underline{a}(0)^H)) \\ &= \text{tr}(\text{Re}(\mathbf{U}(0) \mathbf{F}(s) \underline{R}_\kappa \underline{R}_\kappa^T \mathbf{F}(0) \mathbf{U}(0)^H)) \\ &= \text{tr} \left(\mathbf{F}(s) \underline{R}_\kappa \underline{R}_\kappa^T \underbrace{\mathbf{F}(0)}_{=\mathbf{I}_d} \underbrace{\text{Re}(\mathbf{U}(0)^H \mathbf{U}(0))}_{=\mathbf{I}_d} \right) \\ &= \text{tr}(\mathbf{F}(s) \underline{R}_\kappa \underline{R}_\kappa^T). \end{aligned} \quad (26)$$

However, at the characteristic points

$$\begin{aligned} \text{Re}(\underline{a}(0)^H \underline{u}_1(s_{c_i})) &= 0 \\ \frac{d}{ds} (\text{Re}(\underline{a}(0)^H \underline{a}(s_{c_i}))) &= 0 \\ \frac{d}{ds} (\text{tr}(\mathbf{F}(s_{c_i}) \underline{R}_\kappa \underline{R}_\kappa^T)) &= 0 \quad [\text{using (26)}]. \end{aligned}$$

The above equation, in the case of symmetric arrays (using the fifth property of Appendix H), becomes

$$\begin{aligned} \frac{d}{ds} (\text{tr}(\mathbf{F}(s_{c_i}))) &= 0 \\ \text{tr}(\mathbf{F}'(s_{c_i})) &= 0 \end{aligned}$$

which implies that the characteristic points are given as the roots of the function

$$\text{tr}(\mathbf{C} \expm(s\mathbf{C})) = 0.$$

APPENDIX H A SUMMARY OF PROPERTIES OF SYMMETRIC LINEAR ARRAY PARAMETERS

- 1) $\text{sum}(\underline{r}_x^n) = 0, \quad \forall n \text{ odd.}$
- 2) $d = N.$
- 3) $\mathbf{U}(s)^H \mathbf{U}(s) = \mathbf{I}_N.$
- 4) $\underline{a}(s)^H \mathbf{U}(s) = \underline{R}_\kappa^T.$
- 5) $\text{tr}(\mathbf{F}(s) \underline{R}_\kappa \underline{R}_\kappa^T) = \text{tr}(\mathbf{F}(s)).$
- 6) $\det(\mathbf{U}(s)) = \pm 1.$

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Christos Proukakis received the Diploma in electrical engineering from the National Technical University of Athens, Athens, Greece, in 1990. He received the M.Sc. degree in communications and signal processing in 1991 and the Ph.D. and D.I.C. degrees in electrical engineering in 1997, all from Imperial College of Science Technology and Medicine, University of London, London, U.K.

His research interests are in the areas of sensor array signal processing and super-resolution spatio-temporal spectral estimation as well as the general

area of digital communications.

Dr. Proukakis was awarded the Frangoulis Memorial studentship in 1992 and was a scholar of the Onassis Foundation from 1993 to 1996. He is a member of the Technical Chamber of Greece.

Athanassios Manikas (M'88) received Ph.D. and D.I.C. degrees from Imperial College, University of London, London, U.K., in 1988 and was appointed a Lecturer at Imperial College the same year.

He is now a Reader in digital communications in the Department of Electrical and Electronic Engineering, Imperial College of Science, Technology, and Medicine, London. He has published extensive set of journal and conference papers relating to his research interests, which are in the general area of communication and signal processing, where he has

developed a wide and deep interest in the topic of super-resolution array processing and the application of array theory to communication systems. He has been involved in array manifold investigations using differential geometry, super-resolution beamformers, direction finding, "near" and "far" field array processing, performance bounds in array systems, array uncertainties and ambiguities, array design, array signal abnormalities and quality factors, higher order signal subspace techniques, H-inf-type robust array processing, robust array receivers, and integrated wireless array communication networks based on CDMA, GSM-like, etc., platforms. Furthermore, he is the Organizer of the Advanced Communications and Signal Processing Laboratory at Imperial College and, over the last eight years, has held a number of important research consultancies for the EU, industry, and government organizations.

Dr. Manikas is a member of IEICE, a corporate member of IEE, and a Chartered Electrical Engineer.