Optimal Actuator Design for the Euler-Bernoulli Vibration Model Based on LQR Performance and Shape Calculus

M. Sajjad Edalat zadeh, Dante Kalise, Kirsten A. Morris and Kevin Sturm

Abstract—A method for optimal actuator design in vibration control is presented. The optimal actuator, parametrized as a characteristic function, is found by means of the topological derivative of the LQR cost. An abstract framework is proposed based on the theory for infinite-dimensional optimization of both the actuator shape and the associated control problem. A numerical realization of the optimality condition is developed for the actuator shape using a level-set method for topological derivatives. A numerical example illustrating the design of actuator for Euler-Bernoulli beam model is provided.

Index Terms—optimal actuator location, actuator design, optimal feedback control, active vibration control

I. INTRODUCTION

Actuator location and shape are important design variables for feedback synthesis in active vibration control. Optimal actuator design improves performance of the controller and significantly reduces the cost of control [1], [2], [3]. In the engineering literature, the optimal actuator location problem has gained considerable interest, see [4], [5], [6].

The optimal actuator location theory for linear distributed parameter systems has been extensively discussed in the literature. In [7], it was proven that a linear-quadratic (LQ) optimal actuator location exists if the control operator continuously depends on actuator locations. Similar results have been obtained for H_2 and H_∞ controller design objectives [8], [9]. Other objectives such as maximizing controllability or stability margin have been studied in [10], [11]. A convex optimization method to compute the optimal actuator locations for linear quadratic problems is used in [12]; see [13], [14] for other linear-quadratic algorithms. Controllability-based approaches have been studied in [15], [16], [17].

The theory of optimal actuator shape design has been studied in some recent works. In [18], optimal shape and position of an actuator for the one-dimensional wave equation are discussed. An actuator is placed on a subset $\omega \in [0, \pi]$ with a constant Lebesgue measure $L\pi$ for some $L \in (0, 1)$. The optimal actuator minimizes the norm of a Hilbert Uniqueness Method (HUM)-based control; such control steers the system from a given initial state to zero state in finite time. In [19], optimal actuator shape and position for linear parabolic systems

are discussed, adopting the same approach as in [18], but with initial conditions that have randomized Fourier coefficients. The cost is defined as the average of the norm of HUM-based controls. A quadratic cost function is considered, and shape and topological derivative of this function are derived. Numerical results show significant improvement in the cost and performance of the control. For nonlinear problems, in [20] it is shown that under certain conditions on the nonlinearity and the cost function, an optimal input and actuator design exist for quasi-linear PDEs and optimality equations are obtained. Results apply to nonlinear beam models as well as to the semi-linear wave model in two spatial dimensions [21]. Finally, optimal sensor design is essentially dual to the optimal actuator design problem for linear systems, and has also been studied in [22], [23], [24].

Very few numerical methods have been developed for optimal actuator design in practical problems. In [25], optimal actuator design for linear diffusion equations was discussed using a topological derivative approach. In this paper, we follow a similar approach for optimal actuator design for the Euler-Bernoulli beam model arising in active vibration control. Well-posedness of the abstract problem for a linear evolution equation, and the particular setting for the Euler-Bernoulli model is established. An optimal design is shown to exist over a suitably defined set of admissible shapes for linear quadratic costs. Our contribution is to characterize the actuator as an indicator function over a subdomain, obtaining the optimal actuator shape by resorting to the notion of topological derivative and its associated optimality condition. The main advantage of our approach is that we do not require a prior guess on the number of components of the optimal actuator. We develop a numerical approximation based on the discretization of the control problem linked to the topological derivative. The rest of the paper is structured as follows. In Section II, an abstract formulation of the optimal actuator design problem is described. Section III introduces the control setting associated to the Euler-Bernoulli model for vibration control. Section IV presents a numerical method for the approximation of optimal actuators based on level-set method for topological derivatives. In Section V computational results are discussed.

II. ABSTRACT FORMULATION OF THE OPTIMAL ACTUATOR DESIGN PROBLEM

We consider an abstract evolution equation over $[0, \tau]$

$$\begin{cases} \dot{z}(t) = Az(t) + B(r)u(t), & t \in (0, \tau], \\ z(0) = z_0. \end{cases} \quad (1)$$

Let H and U be Hilbert spaces, $A : D(A) \rightarrow H$ be the state operator, and $z(t)$ the state of the system. The state operator

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A generates a strongly continuous semi-group on H . Let r indicate an actuator design parameter taking values in a topological space K . The input operator $r \mapsto B(r) : K \rightarrow \mathcal{L}(U, H)$ continuously depends on the actuator design parameter r , that is, for all parameters r_0

$$\lim_{r \rightarrow r_0} \|B(r) - B(r_0)\|_{\mathcal{L}(U, H)} = 0. \quad (2)$$

The input u is assumed to belong to a subspace $U_{ad} \subset L^2(0, \tau; U)$ of admissible inputs. The admissible actuator shape set is denoted by K_{ad} and is compact with respect to the topology on K . Given an admissible actuator r , we define the finite-horizon cost

$$J(u, r; z_0) := \frac{1}{2} \int_0^\tau \|z(t)\|_H^2 + \gamma \|u(t)\|_U^2 dt, \quad \gamma > 0. \quad (3)$$

The actuator shape will be designed to minimize $J(u, r; z_0)$ over all admissible inputs, i.e. $J(u, r; z_0) < \infty$, and actuators,

$$\min_{r \in K_{ad}} \min_{u \in U_{ad}} J(u, r; z_0), \quad (4)$$

constrained to eq. (1). Provided that K_{ad} is compact in the topology on K , Theorem 4.1 in [20] guarantees the existence of an optimal control and actuator shape for this problem.

Let $\Omega \subset \mathbb{R}^n$ be a bounded set related to the spatial domain of eq. (1). To characterize the optimal shape in terms of

$$J_1(r; z_0) := \min_{u \in U_{ad}} J(u, r; z_0), \quad (5)$$

our analysis below is restricted to the case in which r corresponds to a subset ω of Ω , which we will later parametrize through an indicator function χ_ω .

III. OPTIMAL ACTUATOR DESIGN FOR THE EULER-BERNOULLI MODEL

We consider a simply supported Euler-Bernoulli beam with Kelvin-Voigt damping coefficient C_d and viscous damping coefficient μ . The state $w(x, t)$ corresponds to the transverse vibrations of the beam with $(x, t) \in [0, 1] \times [0, \tau]$. The actuator shape is characterized by an indicator function $\chi_\omega(x)$, where ω is a Lebesgue measurable subset of $[0, 1]$. The beam vibrations are governed by

$$\begin{cases} \partial_t w + \partial_{xx}(\partial_{xx} w + C_d \partial_{xxt} w) + \mu \partial_t w = \chi_\omega(x) u(t), & t > 0, \\ w(x, 0) = w_0(x), \quad \partial_t w(x, 0) = v_0(x), \\ w(0, t) = w(1, t) = 0, & t \geq 0, \\ \partial_{xx} w(0, t) + C_d \partial_{xxt} w(0, t) = 0, & t \geq 0, \\ \partial_{xx} w(1, t) + C_d \partial_{xxt} w(1, t) = 0, & t \geq 0, \end{cases}$$

where ∂_x denotes the derivative with respect to x ; the derivative with respect to t is indicated similarly. The velocity of the displacement is denoted by $v = \partial_t w$. We define the augmented state $z := (w, v)$. Also, we define the state space $H := H^2(0, 1) \cap H_0^1(0, 1) \times L^2(0, 1)$ with norm

$$\|(w, v)\|_H^2 = \int_0^1 (\partial_{xx} w(x))^2 + w^2(x) + v^2(x) dx, \quad (6)$$

and the closed self-adjoint positive operator

$$\begin{cases} A_0 w := \partial_{xxxx} w, \\ D(A_0) := \{w \in H^4(0, 1) \mid w(0) = w(1) = 0, \\ \partial_{xx} w(0) = \partial_{xx} w(1) = 0\}. \end{cases} \quad (7)$$

The state operator is A in (2) is given by

$$\begin{aligned} A(w, v) &= (v, -A_0(w + C_d v) - \mu v), \\ D(A) &= \{(w, v) \in H \mid v \in H^2(0, 1) \cap H_0^1(0, 1), \\ &\quad w + C_d v \in D(A_0)\}. \end{aligned}$$

Letting $U = \mathbb{R}$ and $U_{ad} = L^2(0, \tau)$, for $r = \chi_\omega$, we define the input operator

$$B(r)u = (0, \chi_\omega u). \quad (8)$$

Note that in the presence of damping (C_d or μ different from zero), the free dynamics are asymptotically stable and $u(t) \equiv 0$ is always an admissible input.

We define $K := L^\infty(0, 1)$. The admissible actuator design set is for some $V > 0$ and $0 < c < 1$,

$$K_{ad} := \{\chi_\omega \in BV(0, 1) : \text{Var}\{\chi_\omega, [0, 1]\} \leq V, |\omega| = c\},$$

where $BV(0, 1)$ denotes the set of functions of bounded variations on $(0, 1)$, and $\text{Var}\{\chi_\omega, [0, 1]\}$ denotes the total variation of χ_ω on $[0, 1]$. Since $BV(0, 1)$ is compactly embedded into $L^1(0, 1)$ and since $\chi_\omega \mapsto \text{Var}(\chi_\omega, [0, 1])$ is lower semi-continuous (see, e.g. [26, p.247, Thm. 6.3.]), it follows that K_{ad} is compact in K with respect to weak-* topology. Also, for every sequence in K_{ad} , there is a subsequence that converges point-wise in $(0, 1)$ to a function in K_{ad} . Thus, if $\chi_\omega \in K_{ad}$ converges to a $\chi_{\omega_0} \in K_{ad}$ in weak-* topology, it converges point-wise. Noting that

$$\|B(\chi_\omega) - B(\chi_{\omega_0})\|_{\mathcal{L}(U, H)} = \left(\int_0^1 (\chi_\omega(x) - \chi_{\omega_0}(x))^2 dx \right)^{\frac{1}{2}}, \quad (9)$$

condition (2) follows from this and the Dominated Convergence Theorem.

For every initial condition $z_0 = (w_0, v_0)$, we define the cost

$$J_1(\chi_\omega; z_0) := \min_u \int_0^\tau \int_0^1 \|(w, v)\|_H^2 + \gamma |u(t)|^2 dt.$$

For a given initial condition z_0 and actuator shape ω , the associated optimal control will be denoted by $\bar{u}_{z_0}^\omega$ and corresponding optimal deflection by $\bar{w}_{z_0}^\omega$. Since the admissible set K_{ad} is compact in K , and the model is linear, hence trivially quasi-linear, a concurrent optimal shape and control for this problem exists [20, Thm. 4.1]. Since the cost function is quadratic, the optimal control is easily written in feedback form. Omitting the dependence of B on the actuator design, for each design, the operator Riccati differential equation

$$\begin{cases} \dot{\Pi}(t) = -A^* \Pi(t) - \Pi(t) A + \frac{1}{\gamma} \Pi(t) B B^* \Pi(t) - I \quad \forall t \in [0, \tau], \\ \Pi(\tau) = 0 \end{cases}$$

has a solution $\Pi(t) \geq 0$. The dependence of Π on the actuator design is also not explicitly indicated. The adjoint variable and optimal control are given, respectively, by

$$p^{\omega, z_0}(t) = 2\Pi(t)\bar{z}(t), \quad \bar{u}^{\omega, z_0}(t) = -\frac{1}{2\gamma} B^* p^{\omega, z_0}(t),$$

where $\bar{z}(t)$ indicates the corresponding system trajectory with this control. In the following, with a slight abuse of notation we write $p = (\bar{p}, \bar{q})$, denoting by $\bar{p}$ the adjoint variable

associated to the vertical displacement w . For B as in (8), it is possible to characterize a topological derivative and necessary conditions for the optimal actuator shape ω . We define the *topological derivative* of $\mathcal{J}_1(\omega) := J_1(\chi_\omega; z_0)$, where we omit the dependence with respect to z_0 .

Definition 3.1 (Topological derivative): Let $b_\varepsilon(\eta_0)$ be the Euclidean open ball centred at η_0 with radius ε , $\bar{b}_\varepsilon(\eta_0)$ be the closure of $b_\varepsilon(\eta_0)$, and $|b_\varepsilon(\eta_0)|$ be the volume of $b_\varepsilon(\eta_0)$. The topological derivative of $\mathcal{J}_1$ at ω in the point $\eta_0 \in \Omega \setminus \partial\omega$ is defined by

$$\mathcal{T} \mathcal{J}_1(\omega)(\eta_0) = \begin{cases} \lim_{\varepsilon \searrow 0} \frac{\mathcal{J}_1(\omega \setminus \bar{b}_\varepsilon(\eta_0)) - \mathcal{J}_1(\omega)}{|b_\varepsilon(\eta_0)|} & \text{if } \eta_0 \in \omega, \\ \lim_{\varepsilon \searrow 0} \frac{\mathcal{J}_1(\omega \cup b_\varepsilon(\eta_0)) - \mathcal{J}_1(\omega)}{|b_\varepsilon(\eta_0)|} & \text{if } \eta_0 \in \Omega \setminus \bar{\omega}. \end{cases}$$

Theorem 1: Let

$$g^\omega(\eta) := \int_0^\tau \bar{u}_{z_0}^\omega(s) \bar{p}_{z_0}^\omega(\eta, s) ds.$$

Then, the topological derivative of $\mathcal{J}_1$ at an open set ω in $\eta_0 \in \Omega \setminus \partial\omega$ is given by

$$\mathcal{T} \mathcal{J}_1(\omega)(\eta_0) := \begin{cases} g^\omega(\eta_0) & \text{if } \eta_0 \in \omega, \\ -g^\omega(\eta_0) & \text{if } \eta_0 \in \Omega \setminus \bar{\omega}, \end{cases} \quad (10)$$

and the necessary optimality condition for the optimal actuator shape is given by

$$\begin{aligned} g^\omega(\eta) &\leq 0 & \text{for all } \eta \in \omega, \\ g^\omega(\eta) &\geq 0 & \text{for all } \eta \in \Omega \setminus \bar{\omega}. \end{aligned} \quad (11)$$

This result follows from [25, Theorem 4.1], where it was derived for a linear-quadratic control problem governed by a linear parabolic equation.

IV. NUMERICAL APPROXIMATION

In this section, we present the numerical approximation of the abstract results of Section III. We develop a finite-dimensional approximation of the dynamics, the associated optimal control problem, and a numerical approximation of the topological derivative.

We consider a sequence of finite-dimensional subspaces $H_N \subset H$, and let P_N indicate the orthogonal projection of $H \rightarrow H_N$, defined by $P_N z = z_N$, so that

$$\langle z_N, \phi \rangle = \langle z, \phi \rangle \quad \forall \phi \in H_N.$$

Let (A_N, B_N) be a sequence of finite-dimensional approximations of (A, B) on H_N , and let S_N indicate the semigroup generated by A_N . The approximation is assumed to satisfy the following standard assumptions for systems governed by PDEs (see, e.g., [27] and [28, Section 4.1]).

Assumption (A1) For each $z_0 \in H$, and $u \in U$,

(i) for each t_1, t_2 , $0 \leq t_1 < t_2$,

$$\lim_{N \rightarrow \infty} \sup_{t_1 \leq t \leq t_2} \|S_N(t) P_N z_0 - S(t) z_0\| \rightarrow 0.$$

(ii) $\lim_{N \rightarrow \infty} \|B_N u - B u\| \rightarrow 0$.

Assumption (A1*) For each $z \in H$,

(i) for each t_1, t_2 , $0 \leq t_1 < t_2$,

$$\lim_{N \rightarrow \infty} \sup_{t \in [t_1, t_2]} \|S_N^*(t) P_N z - S^*(t) z\| \rightarrow 0;$$

(ii) $\lim_{N \rightarrow \infty} \|B_N^* P_N z - B^* z\| \rightarrow 0$.

In our case, let $\{\phi_n\}_{n=1}^N$ be the first N eigenfunctions of the operator A_0 defined in (7). Denote the span of $\{\phi_n\}_{n=1}^N$ by X_N and define $H_N = X_N \times X_N$. The operators (A_N, B_N) are the Galerkin approximation of (A, B) on H_N ; we denote the corresponding approximate state by z_N . This approximation satisfies assumptions (A1)-(A1*); see [28, Eg. 2.26, Eg. 4.8]. We denote by $\mathbf{z}_N$ the coefficients of z_N with respect to the chosen basis; $\mathbf{z}_N$ will be a vector of length $2N$.

The corresponding finite-dimensional quadratic cost function for this approximation is

$$\int_0^\tau \|z_N(\cdot, t)\|_{H_N}^2 + \gamma \|u(t)\|_U^2 dt. \quad (12)$$

The optimal input $\bar{u}$ is obtained by solving the differential Riccati equation for Π

$$\begin{cases} \dot{\Pi}_N(t) = -A_N^\top \Pi_N(t) - \Pi_N(t) A_N + \frac{1}{\gamma} \Pi_N(t) B_N B_N^\top \Pi_N(t) \\ \quad - I_N, \quad \forall t \in [0, \tau), \\ \Pi_N(\tau) = 0. \end{cases}$$

After solving for $\Pi_N(t)$, the optimal cost is given by

$$J_{1,N}(\chi_\omega; z_N(0)) = \mathbf{z}_N^\top(0) \Pi_N(0) \mathbf{z}_N(0),$$

and the optimal input $\bar{u}_N(t)$ is expressed as the feedback

$$\bar{u}_N(t) = -\frac{1}{\gamma} B_N^\top \Pi_N(t) \bar{\mathbf{z}}_N(t), \quad (13)$$

where $\bar{\mathbf{z}}_N$ denotes the coefficients of the optimal state. The adjoint variable associated with the vertical displacement, denoted by $\bar{p}_N$, is given by

$$\bar{p}_N(x, t) = \sum_{n=1}^N \phi_n(x) b_n(t), \quad (14)$$

where $b_n(t)$ are the first N entries of $2\Pi_N(t) \bar{\mathbf{z}}_N(t)$. Assumptions (A1)-(A1*) ensure that $\Pi_N(t)$ converges strongly to $\Pi(t)$, where $\Pi(t)$ solves an infinite-dimensional Riccati equation, for each t , and also that the optimal input $\bar{u}_N(t)$ converges for almost all t and also in $L^2(0, \tau)$ to the optimal control $\bar{u}_{z_0}^\omega$ of the infinite-dimensional system [27, Thm. 5.1].

Having approximated the optimal control problem for a given actuator, we now obtain a numerical realization of the topological derivative associated to the optimal actuator design problem. The discrete quadratic cost function (15) is modified to include a penalty for the actuator size $|\omega|$,

$$\begin{aligned} J_{1,N}(\chi_\omega; \mathbf{z}_N(0)) &:= \min_u \int_0^\tau \|z_N(\cdot, t)\|_{H_N}^2 + \gamma \|u(t)\|_U^2 dt \\ &\quad + \alpha (|\omega| - c)^2. \end{aligned} \quad (15)$$

This penalty introduces a soft constraint on the actuator size, and while it has no effect on the optimal solution of the LQR problem for a given actuator, it introduces an additional term in the topological derivative (10). In fact, given an actuator ω , we define the function

$$g_N^\omega(x) = - \int_0^\tau \bar{u}_N(t) \bar{p}_N(x, t) dt + 2\alpha (|\omega| - c) \quad (16)$$

so that the approximation of the necessary optimality condition (11) for an actuator ω is

$$\begin{aligned} g_N^\omega(x) &\leq 0 & \text{for all } x \in \omega, \\ g_N^\omega(x) &\geq 0 & \text{for all } x \in \Omega \setminus \bar{\omega}. \end{aligned} \quad (17)$$

In order to reach the optimality condition above, we use the level-set method proposed in [29]. Given a general topological derivative, this algorithm generates a gradient-descent type evolution that stops once the optimality condition is achieved. A remarkable advantage of this formulation is that the level-set evolution is versatile enough to allow the generation of multi-component actuators. We characterize the actuator ω by means of continuous level-set function ψ_N , such that

$$\begin{aligned} \psi_N(x) &< 0 & \text{for all } x \in \omega, \\ \psi_N(x) &= 0 & \text{for all } x \in \partial\omega. \\ \psi_N(x) &> 0 & \text{for all } x \in \Omega \setminus \bar{\omega}. \end{aligned} \quad (18)$$

Note that the notation ψ_N is only used to illustrate the dependence of the problem with respect to N . Given an initial guess for ψ_N^0 , the optimal actuator shape is determined by the iteration

$$\psi_N^{i+1} = (1 - \beta_i)\psi_N^i + \beta_i \frac{g_N^{\omega_i}}{\|g_N^{\omega_i}\|}, \quad \omega_i := \{x \in \Omega : \psi_N^i(x) < 0\},$$

where β_i is the step size of the level-set update, which evolves according to the realization of the topological derivative for the current shape. After reaching a stationary point of the topological derivative, characterised by (17); see [29], the level-set evolution stops. Algorithm 1 summarizes all the steps for the calculation of the optimal actuator shape.

Algorithm 1 Level set algorithm for optimal actuator design

Input: ψ_N^0 , $\omega_0 := \{x \in \bar{\Omega} : \psi_N^0(x) < 0\}$, $\beta_0 > 0$, $\mathbf{z}_N(0)$, decrease factor $0 < d < 1$ and tolerance $\varepsilon > 0$.

while $\beta_i \geq \varepsilon$ **do**

$\beta_i \leftarrow \beta_0$

if $J_{1,N}(\omega^+, \mathbf{z}_N(0)) \geq J_{1,N}(\omega_i, \mathbf{z}_N(0))$ and $\beta_i > \varepsilon$ **then**

$\beta_i \leftarrow d\beta_i$

$\psi_N^+ \leftarrow (1 - \beta_i)\psi_N^i + \beta_i \frac{g_N^{\omega_i}}{\|g_N^{\omega_i}\|}$

$\omega^+ \leftarrow \{\psi_N^+ < 0\}$

end if

$(\omega_i, \psi_N^i) \leftarrow (\omega^+, \psi_N^+)$

$i \leftarrow i + 1$

end while

return optimal actuator ω_{opt}

Algorithm 1 includes a line search strategy to find a suitable β_i guaranteeing a decay on the overall cost $J_{1,N}$. The size constraint $|\omega| = c$, represented by the penalty term $\alpha(|\omega| - c)^2$ in (15), is treated through a continuation procedure for α . That is, Algo. 1 is first executed with a small value of α , which is then progressively increased until it becomes large enough so that the optimal actuator matches the size constraint $|\omega| = c$. Every increase of α requires a full execution of Algo. (15) initialized with the previous solution.

V. NUMERICAL EXPERIMENTS

We assess the performance of the proposed method for actuator design. In $\Omega := (0, 1)$, we set $w(x, 0) = \sin(3\pi x)$ and $v(x, 0) = 0$, and the damping parameters $C_d = 10^{-4}$ and $\mu = 10^{-3}$. The system dynamics are discretized by taking the first $N = 40$ eigenmodes of the operator A . The time horizon is set to $\tau = 200$, the control penalty is set to $\gamma = 10^{-3}$ and the volume constraint $c = 0.4$, that is, the actuator is enforced to cover 40% of the domain. The continuation with respect to the volume penalty starts the execution of Algorithm 1 with $\alpha = 0.1$, increasing α by a factor of 10 until reaching $\alpha = 10^4$. For the level-set algorithm, we approximate ψ_N over a uniform grid with 400 nodes, we set $\varepsilon = 10^{-7}$ and $d = 0.9$ ($\beta_0 = 1$). The total CPU time to find the optimal actuator is of 258.54 seconds in a workstation with an Intel(R) Xeon(R) CPU E5-1660, 3.20 GHz, and 64GB RAM.

(a) evolution of the cost $J_{1,N}$ under Algorithm 1.

(b) optimal actuator

Fig. 1: Algorithm performance for the initial condition $w(x, 0) = \sin(3\pi x)$ and $v(x, 0) = 0$ with an actuator volume constraint of 40% of the domain.

The performance of Algorithm 1 is first shown in Fig. 1(a), where we see the decrease of the overall cost $J_{1,N}$ as the shape of the actuator evolves. The initial actuator corresponds to $\omega_0 = [0.1, 0.9]$. The evolution is monotone as ensured by our line search step, and reaches the desired tolerance after 24 iterations, reducing the total cost in three orders of magnitude. The optimal actuator is shown in Figure 1(b), and we observe

a splitting into two components, consistent with the shape of the initial condition under which the actuator was optimized.

(a) controlled displacement

(b) controlled velocity

(c) control signals

Fig. 2: Closed-loop performance of the optimal actuator.

The closed-loop performance with the optimal actuator is shown in Figure 2, and it is compared against the closed-loop performance of a single-component, non-optimized, actuator $w_s = [0.3, 0.7]$ shown in Figure 3.

The convergence of the optimal shape with respect to the discretization of the system dynamics was investigated. If Kelvin-Voigt damping $C_d > 0$, the optimal actuator design problem converges with increasing approximation order. In Figure 4(a), we illustrate this by showing the norm of the Kalman gain $K_N^\omega = -\frac{1}{\gamma} B_N^\top \Pi_N(0)$ for an increasing number

(a) uncontrolled displacement

(b) controlled displacement

(c) controlled velocity

Fig. 3: Closed-loop performance of non-optimized actuator $w_s = [0.3, 0.7]$ of the same volume. With the optimal actuator, better stabilization is achieved with less control action (see Fig. 2 (c)).

of modes in our spectral discretization of the Euler-Bernoulli dynamics. In contrast, when $C_d = 0$, no such convergence is obtained. The fundamental reason is that the infinite-dimensional Riccati operator is no longer a compact operator, see for instance, [7], [22]. This means that no sequence of finite-dimensional approximations to the Riccati operator will

converge in norm and this affects convergence of the optimal actuator design. In this example, the Kalman gain fails to

Fig. 4: Top: Convergence of the Kalman gain and optimal actuator with respect to the discretization depends on the choice of damping parameters. Bottom: In the absence of Kelvin-Voigt damping, the optimal actuator fails to converge, splitting into multiple components as the number of modes increases. In this figure, $N = 40$ is the order of approximation.

converge with respect to the discretization. This leads to a lack of convergence of the actuator, which continues to split into more components as the number of modes grows, see Fig.4.

VI. CONCLUSION AND FUTURE RESEARCH

We have presented an algorithm for computing an LQ-optimal actuator design. We formulate the problem as finding a stationary point of the topological derivative of the cost associated to the LQ cost. A remarkable advantage of such a formulation is that it allows to parametrize the actuator as an indicator function, optimizing its support without any prior assumption on the number of components. We have applied this technique to solve the optimal actuator design problem for a linear Euler-Bernoulli beam model. The improvement in the performance of the system over a reasonable but non-optimal design was demonstrated. Future directions include the optimal actuator shape design for models that include actuator dynamics as well as nonlinearities, and the validation of the proposed methodology in a test-bed, similar to [12].

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