Reliability-based Fracture Analysis for Plate Bending Problems with the Dual Boundary Element Method

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Abstract. A novel methodology is presented for the efficient reliability analysis of plate structures containing cracks with the Dual Boundary Element Method (DBEM). The derivatives of the DBEM plate formulations for the Crack Surface Displacement Extrapolation (CSDE) method have been derived for the first time and are used as part of an Implicit Differentiation Method (IDM) for the efficient calculation of Stress Intensity Factor (SIF) sensitivities. A numerical example is investigated in which results from the presented CSDE methodology are compared to those from the J-integral. The SIF sensitivities were used with the First-Order Reliability Method (FORM) to determine the reliability of a plate structure containing a crack. Results indicate that the proposed CSDE methodology is capable of providing estimates for reliability that are very similar to those provided by the J-integral. Given that the proposed CSDE methodology can be easier to implement, it could prove to be an effective alternative to the J-integral for the efficient reliability analysis of plate structures containing cracks.

INTRODUCTION

The Dual Boundary Element Method (DBEM) was developed as an effective means of modelling cracks in structures. In contrast to previous methods, mixed-mode crack problems could be solved with the DBEM in a single-region formulation; significantly reducing model complexity and the number of equations needed. Due to the effectiveness of the DBEM in fracture analysis, it has been extensively used in the research community mainly for problems involving 2D problems. Examples include topology optimisation [1], reliability analysis of structures subjected to fatigue [2], shape optimisation of hydrofoils [3], and the modelling of creep in concrete [4].

Despite its popularity for 2D problems, the DBEM has received less attention for plate problems due to the increased level of complexity; plates have more degrees of freedom (5 vs. 2), they can be subjected to the combined loading of tensile, bending, and pressure forces, and they have finite thickness along which stress can vary. Plate problems are arguably just as important to consider as 2D problems. Many engineering structures can be thought of as plates; aircraft wings and fuselages are composed of many plates joined together. The DBEM for plate problems was first developed by Dirgantara and Aliabadi [5]. This approach has since been applied to a wide range of problems by the research community, such as dynamic fracture analysis [6], the modelling of Lamb waves for Structural Health Monitoring (SHM) [7], fracture analysis with large deflection [8], and fatigue crack growth in assembled structures [9]. A comprehensive overview of the DBEM for 2D and plate problems, can be found in a book by Aliabadi [10]. Due to the widespread relevance of plates for engineering applications, they are the focus of this paper.

The calculation of the sensitivities of Stress Intensity Factors (SIFs) with respect to design variables is an important part of determining the reliability of a structure containing flaws. Among the most common methods for calculating Stress Intensity Factors (SIFs) with the DBEM is the crack Surface Displacement Extrapolation (CSDE) method and the J-integral. Since the DBEM can very accurately model displacements and stresses in the internal domain of a structure, it is very effective when used with the J-integral. However, due to its complexity it can be difficult to implement. The CSDE on the other hand is significantly easier to implement but can require the use of special quarter-point elements at crack tips to ensure sufficient accuracy. The sensitivities of SIFs with the DBEM can be calculated numerically using the relatively crude Finite Difference Method (FDM), or analytically using the more efficient Implicit Differentiation Method (IDM) which involves deriving the derivatives of the DBEM fundamental solutions. To the knowledge of the authors, there is only one example of SIF sensitivities for plates being calculated with an IDM: Morse et al. [11] derived the derivatives of the J-integral for plates to calculate SIF sensitivities for a plate with a centre crack subjected to complex loading conditions. Results were found to compare well with the FDM and the IDM was found to be significantly more efficient than the FDM. Although previous work has involved the derivatives of the J-integral being used to calculate SIF sensitivities for plates, the derivatives of the CSDE method have yet to be used for this purpose. Due to the fact that the CSDE method is significantly easier to implement it could

prove to be an effective alternative to the J-integral for the calculation SIF sensitivities for plate structures.

In summary, this paper presents a novel methodology for the calculation of SIF sensitivities for plates with the CSDE method. The derivatives of the DBEM plate formulations for the CSDE method are derived for the first time and are used as part of an IDM for the calculation of SIF sensitivities. The methodology behind the CSDE method will first be introduced, followed by the newly derived IDM formulations. A numerical example will then be presented featuring reliability analyses on a square plate subjected to complex loading conditions that required the use of the SIF sensitivities. The accuracy of the CSDE method will then be compared to J-integral.

METHODOLOGY

DBEM-based IDM for Plate Bending Problems

For conciseness, the DBEM integral equations and their derivatives for plate bending are omitted. The DBEM integral equations for plate bending can be found in [10], while their derivatives with respect to some geometric parameter Z_m can be found in [11].

The IDM involves taking the derivative of the system of equations from DBEM with respect to some geometric parameter Z_m (e.g. plate length). The derivative of the system of equations is:

$$\mathbf{A}\mathbf{X}_{,m} = [\mathbf{F}_{,m} - \mathbf{A}_{,m}\mathbf{X}] \quad (1)$$

where $\mathbf{A}$ and $\mathbf{X}$ can be obtained from DBEM. Because the right-hand side of equation (1) is known, LU decomposition can be used to obtain the unknown derivatives of boundary displacements and tractions $\mathbf{X}_{,m}$.

The calculation time of the matrix $\mathbf{A}_{,m}$ is significantly less than might be expected since when both the collocation node and the field point lie on a fixed boundary that is unaffected by a change in the geometric parameter Z_m , the corresponding elements in $\mathbf{A}_{,m}$ are zero. This has the effect of significantly improving the efficiency of the proposed methodology.

Stress Intensity Factor Evaluation

The CSDE method is used in this work for the calculation of SIFS and their sensitivities. It involves the linear extrapolation of the SIFs near the crack tip. A DBEM mesh of a crack tip with discontinuous quadratic elements can be seen in Figure 1. Some of the nodes near the crack tip, e.g. nodes A, B, C, and D, are labelled.

FIGURE 1: Nodes near a crack tip. The global coordinate system is shown.

The SIFs can be calculated at a certain distance r from the crack tip shown in Figure 1 using the following equations:

$$K_I^m(r) = \frac{Eh\sqrt{2\pi}}{8\sqrt{r}} |\Delta u_2(r)| \quad K_{II}^m(r) = \frac{Eh\sqrt{2\pi}}{8\sqrt{r}} |\Delta u_1(r)| \quad (2)$$

$$K_I^b(r) = \frac{Eh^3\sqrt{\pi/2}}{48\sqrt{r}}|\Delta w_2(r)| \quad K_{II}^b(r) = \frac{Eh^3\sqrt{\pi/2}}{48\sqrt{r}}|\Delta w_1(r)| \quad K_{III}^b(r) = \frac{5Eh\sqrt{\pi/2}}{24(1+\nu)\sqrt{r}}|\Delta w_3(r)| \quad (3)$$

where the superscripts m and b denote membrane and bending SIFs respectively. w_1 and w_2 denote rotations in the directions x_1 and x_2 respectively, and w_3 denotes displacement in the direction x_3 (out-of-plane). u_1 and u_2 are the displacements in the directions x_1 and x_2 respectively. Δ indicates the difference across the crack tip, e.g. $|\Delta u_1(r=L/2)| = |u_1^C - u_1^D|$, where u_1^C and u_1^D are the values of u_1 evaluated at nodes C and D respectively in Figure 1.

By conducting linear extrapolation using equations (2) and (3), the SIFs at the crack tip can be obtained:

$$K_\alpha^m = R_{CD}K_\alpha^{m,CD} - R_{AB}K_\alpha^{m,AB} \quad K_i^b = R_{CD}K_i^{b,CD} - R_{AB}K_i^{b,AB} \quad (4)$$

where the subscript α can take values of I or II , and the subscript i can take values of I , II , or III . R_{AB} and R_{CD} are defined as: $R_{AB} = r_{CD}/(r_{AB} - r_{CD})$ and $R_{CD} = r_{AB}/(r_{AB} - r_{CD})$, where r_{AB} and r_{CD} are the distances of the nodes on the lines AB and CD respectively to the crack tip. The maximum values of the SIFs through the thickness of the plate are:

$$K_I^{max} = \frac{K_I^m}{h} + \frac{6}{h^2}K_I^b \quad K_{II}^{max} = \frac{K_{II}^m}{h} + \frac{6}{h^2}K_{II}^b \quad K_{III}^{max} = \frac{3}{2h}K_{III}^b \quad (5)$$

Stress Intensity Factor Sensitivity Evaluation

By taking the derivatives of equations (2) and (3) with respect to some geometric parameter Z_m , the derivatives of the SIFs can be calculated at a certain distance r from the crack tip:

$$K_{I,m}^m(r) = \frac{Eh\sqrt{2\pi}}{8\sqrt{r}} \left[\frac{\Delta u_2(r)}{|\Delta u_2(r)|} \Delta u_{2,m}(r) - \frac{r_{,m}}{2r} |\Delta u_2(r)| \right] \quad (6)$$

$$K_{II,m}^m(r) = \frac{Eh\sqrt{2\pi}}{8\sqrt{r}} \left[\frac{\Delta u_1(r)}{|\Delta u_1(r)|} \Delta u_{1,m}(r) - \frac{r_{,m}}{2r} |\Delta u_1(r)| \right] \quad (7)$$

$$K_{I,m}^b(r) = \frac{Eh^3\sqrt{\pi/2}}{48\sqrt{r}} \left[\frac{\Delta w_2(r)}{|\Delta w_2(r)|} \Delta w_{2,m}(r) - \frac{r_{,m}}{2r} |\Delta w_2(r)| \right] \quad (8)$$

$$K_{II,m}^b(r) = \frac{Eh^3\sqrt{\pi/2}}{48\sqrt{r}} \left[\frac{\Delta w_1(r)}{|\Delta w_1(r)|} \Delta w_{1,m}(r) - \frac{r_{,m}}{2r} |\Delta w_1(r)| \right] \quad (9)$$

$$K_{III,m}^b(r) = \frac{5Eh\sqrt{\pi/2}}{24(1+\nu)\sqrt{r}} \left[\frac{\Delta w_3(r)}{|\Delta w_3(r)|} \Delta w_{3,m}(r) - \frac{r_{,m}}{2r} |\Delta w_3(r)| \right] \quad (10)$$

By conducting linear extrapolation using equations (6-10), the sensitivities of the SIFs at the crack tip can be obtained:

$$K_{\alpha,m}^m = (R_{CD,m}K_\alpha^{m,CD} + R_{CD}K_{\alpha,m}^{m,CD}) - (R_{AB,m}K_\alpha^{m,AB} + R_{AB}K_{\alpha,m}^{m,AB}) \quad (11)$$

$$K_{i,m}^b = (R_{CD,m}K_i^{b,CD} + R_{CD}K_{i,m}^{b,CD}) - (R_{AB,m}K_i^{b,AB} + R_{AB}K_{i,m}^{b,AB}) \quad (12)$$

where the subscript α can take values of I or II , and the subscript i can take values of I , II , or III . $R_{AB,m}$ and $R_{CD,m}$ are defined as: $R_{AB,m} = [r_{CD,m} - R_{AB}(r_{AB,m} - r_{CD,m})]/(r_{AB} - r_{CD})$ and $R_{CD,m} = [r_{AB,m} - R_{CD}(r_{AB,m} - r_{CD,m})]/(r_{AB} - r_{CD})$, where $r_{AB,m}$ and $r_{CD,m}$ are the derivatives of the distances of the nodes on the lines AB and CD respectively to the crack tip. By taking the derivatives of equation (5), the sensitivities of the maximum values of the SIFs through the thickness of the plate are:

$$K_{I,m}^{max} = \frac{K_{I,m}^m}{h} + \frac{6}{h^2}K_{I,m}^b \quad K_{II,m}^{max} = \frac{K_{II,m}^m}{h} + \frac{6}{h^2}K_{II,m}^b \quad K_{III,m}^{max} = \frac{3}{2h}K_{III,m}^b \quad (13)$$

NUMERICAL EXAMPLE

To test the accuracy of the proposed CSDE methodology, a numerical example featuring a square steel plate with a crack was investigated and results were compared to the J-integral method. The plate can be seen in Figure 2, and a zoomed-in view of the crack at the top-left hole can be seen in Figure 3. The plate has straight edges and curved corners. Through-thickness holes are present at the corners. The plate is subjected to uniform pressure over its entire upper surface and combined biaxial tension and bending on the straight edges. The plate is simply supported such that the straight outer edges cannot displace in the x_3 direction. Reliability analyses were conducted on this plate using the 16 random variables seen in Table I. The First-Order Reliability Method (FORM) was used to calculate reliability indices of the plate for a range of Coefficients of Variation (CoV) of the crack length a between 0.05 and 0.5. A good introduction to the FORM and other reliability methods is [12].

A DBEM mesh consisting of 132 quadratic elements was used. The four straight outer edges were composed of 12 elements each. The four outer curved corners were composed of 3 elements each. The top-left hole was composed of 30 elements, and the other three holes were composed of 4 elements each. The crack originating from the upper-left hole was composed of 15 elements on each surface. This DBEM mesh was found to provide sufficient converge within 1% when calculating the stress intensity factors at the crack tip with both the CSDE method and the J-integral.

FIGURE 2: Square steel plate with a crack subjected to uniform pressure and biaxial tension and bending. The crack can be seen originating from the top-left hole.

FIGURE 3: A zoomed-in view of the crack originating from the top-left hole.

TABLE I: The 16 random variables and their probability distributions.

Z_i	X_i	Parameter	Description	Distribution	Mean	CoV
Z_1	X_1	L	Plate length	Normal	1 m	0.05
Z_2	X_2	W	Plate width	Lognormal	1 m	0.05
Z_3	X_3	r_1	Corner radius	Lognormal	0.1 m	0.05
Z_4	X_4	r_2	Hole radius	Normal	0.025 m	0.05
Z_5	X_5	θ_1	Angle to crack	Lognormal	45°	0.05
Z_6	X_6	θ_2	Angle of crack	Normal	30°	0.05
Z_7	X_7	a	Crack length	Lognormal	0.05 m	Variable
Z_8	X_8	h	Plate thickness	Normal	0.1 m	0.05
Z_9	X_9	t_1	Membrane traction in direction x_1	Lognormal	8 MN/m	0.10
Z_{10}	X_{10}	t_2	Membrane traction in direction x_2	Normal	9 MN/m	0.10
Z_{11}	X_{11}	p_1	Bending moment in direction x_1	Lognormal	1 MN	0.10
Z_{12}	X_{12}	p_2	Bending moment in direction x_2	Lognormal	1.5 MN	0.10
Z_{13}	X_{13}	q_3	Uniform pressure	Normal	1 MN/m ²	0.10
Z_{14}	X_{14}	E	Young's modulus	Lognormal	206.8 GPa	0.20
Z_{15}	X_{15}	ν	Poisson's ratio	Lognormal	0.29	0.20
Z_{16}	-	G_o	Strain energy release rate required for crack growth onset	Lognormal	200 J/m ²	0.20

The failure condition of this plate is with regards to the onset of crack growth. The Limit State Function (LSF) g in this case is:

$$g(\mathbf{Z}) = G_o - G_{eff}(\mathbf{X}) \quad (14)$$

where G_o is the strain energy release rate required for the onset of crack growth, and G_{eff} is the effective strain energy release rate at the crack tip. $\mathbf{Z}$ is composed of the random variables that influence g : $\mathbf{Z} = (L, W, r_1, r_2, \theta_1, \theta_2, a, h, t_1, t_2, p_1, p_2, q_3, E, \nu, G_o)$, while the vector $\mathbf{X}$ contains the random variables that influence G_{eff} : $\mathbf{X} = (L, W, r_1, r_2, \theta_1, \theta_2, a, h, t_1, t_2, p_1, p_2, q_3, E, \nu, G_o)$. Ideally, G_o should be found experimentally. However, for demonstration purposes, G_o is chosen arbitrarily.

Dirgantara and Aliabadi [13] proposed an equation for G_{eff} :

$$G_{eff}(\mathbf{X}) = G_I^m(\mathbf{X}) + T_I G_I^b(\mathbf{X}) + T_{II} \left(G_{II}^m(\mathbf{X}) + G_{II}^b(\mathbf{X}) + G_{III}^b(\mathbf{X}) \right), \quad 0 \leq T_I \leq 1, \quad 0 \leq T_{II} \leq 1 \quad (15)$$

where T_I and T_{II} are coefficients ideally found from experiments. However, in this work they were both arbitrarily chosen to be 0.5. The individual components e.g. G_I^m , G_I^b , etc. are calculated using the individual stress intensity factors e.g. K_I^m , K_I^b , etc. Their expressions can be found in [13]. The sensitivity of G_{eff} with respect to some geometric variable Z_m is:

$$G_{eff,m}(\mathbf{X}) = G_{I,m}^m(\mathbf{X}) + T_I G_{I,m}^b(\mathbf{X}) + T_{II} \left(G_{II,m}^m(\mathbf{X}) + G_{II,m}^b(\mathbf{X}) + G_{III,m}^b(\mathbf{X}) \right), \quad 0 \leq T_I \leq 1, \quad 0 \leq T_{II} \leq 1 \quad (16)$$

By using equations (15) and (16) with the LSF seen in equation (14), the FORM can be used to obtain estimates for the reliability of the structure with regards to the onset of crack growth. The results of this can be seen in Figure 4 for a range of CoV values for a (CoV_a) from 0.05 to 0.5. The left axis presents the reliability index β , and the right axis presents the probability of failure P_F . The relationship between these two parameters is:

$$P_R = 1 - P_F = 1 - \Phi(-\beta) = \Phi(\beta) \quad (17)$$

where Φ is the CDF of the standard normal distribution.

FIGURE 4: Reliability indices and probabilities of failure calculated using the FORM with the CSDE method and the J-integral.

It can be seen in Figure 4 that, as expected, the reliability index decreases and P_F increases as CoV_a increases. Furthermore, the CSDE and the J-integral compare very well for each value of CoV_a . The average difference in β between the two methods was less than 1% at 0.23%.

CONCLUSION

In conclusion, a novel methodology was presented for the calculation of Stress Intensity Factor (SIF) sensitivities with the Crack Surface Displacement Extrapolation (CSDE) method. A numerical example featuring a square steel plate with a crack subjected to complex loading conditions was investigated. Reliability analyses were conducted on this plate with the First-Order Reliability Method (FORM) using the SIF sensitivities from the CSDE method and the J-integral. Results suggest that the CSDE and the J-integral compare well and the average difference between the two methods when calculating reliability indices was only 0.23%. These results indicate that the proposed CSDE methodology possesses a similar level of accuracy to the J-integral for the calculation of SIF sensitivities. Given that the proposed CSDE methodology is easier to implement, it could prove to be an effective alternative to the J-integral for the calculation of the SIF sensitivities for plate bending problems.

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