Wake transition in the flow around two circular cylinders in staggered arrangements

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The wake transition of the flow around two circular cylinders placed in staggered arrangements with fixed streamwise separation of $5D$ and cross stream separation varying from $0D$ to $3D$ has been studied. The wake transition is compared to that of a single isolated cylinder. Linear stability analysis utilising Floquet theory and direct numerical simulations, using the spectral/hp element method were carried out. It is found that besides modes A and B, mode C can also appear in the wake transition, depending on the relative positioning of the cylinders. The structure of mode C is analysed and the non-linear character of the bifurcation for this mode is investigated.

1. Introduction

The flow around circular cylinders has been extensively studied due to its practical importance in engineering and scientific relevance in fluid mechanics. On the engineering side, there are a number of applications in mechanical, civil and naval engineering that employ circular-cylindrical structures, such as heat exchangers, chimneys and off-

Figure 1: Schematic drawing of two circular cylinders in a staggered arrangement.

shore platforms. In scientific terms, the flow around circular cylinders presents various important physical phenomena, such as separation, vortex shedding and transition.

In the next two subsections, some of the relevant research that has been carried out on two different aspects of this flow is reviewed. These aspects are the wake transition in bluff body flows and the flow around circular cylinders in staggered arrangements, illustrated in figure 1.

1.1. Wake transition

In the last two decades, much effort has focused on the study of the three-dimensional transition that takes place in the von Kármán wake that appears in the flow downstream of a single cylinder. This line of research in its contemporary form was instigated by the seminal work of Williamson (1988), in which two distinct stages were identified in the wake transition, spanning $160 \leq Re \leq 300$. The limits of these stages were identified by discontinuities in the curve obtained when the Strouhal number (St) was plotted against the Reynolds number (Re) and these discontinuities correspond to the manifestation of different three-dimensional structures in the wake. The first to appear is called mode A, and it has a spanwise wavelength of approximately 4 diameters. The second is called mode B, and its spanwise wavelength is close to 1 diameter. A number of papers on additional experiments and direct numerical simulations (DNS) for this Re range have

been published since then with the aim of reproducing these findings, see for example Wu *et al.* (1994), Zhang *et al.* (1995) and Thompson *et al.* (1996). A great leap forward occurred with the work of Barkley & Henderson (1996), who performed high accuracy Floquet stability analyses of two-dimensional time periodic base flows, precisely identifying the critical Reynolds numbers and also characterising each of the modes that take part in the wake transition. Williamson (1996) presented an extensive study on the wake transition, setting the most significant results presented so far against some new experimental data in order to show the different symmetries of the modes and to put forward physical mechanisms to explain the origin of the instabilities. The non-linear characterisation of the modes was first assessed by Henderson & Barkley (1996), who found that mode A became unstable through a subcritical bifurcation. This work was extended by Henderson (1997), where it was concluded that mode B bifurcated through a supercritical route. Various other computational papers have expanded the research on the non-linear features of the first stages of the wake transition, see for instance Persillon & Braza (1998) and Posdziech & Grundmann (2001).

Techniques similar to that presented in Barkley & Henderson (1996) were used to perform Floquet stability analysis of the flow around other bluff bodies. Robichaux *et al.* (1999) analysed the stability of the wake of the flow around a square cylinder, and found that, besides modes A and B, the geometrical configuration they used presented a third mode of intermediary wavelength, which was called mode S. The authors suggested that this was a period doubling bifurcation, i.e. an instability whose Floquet multiplier is real and negative. Later, however, Blackburn & Lopez (2003) showed that this was in fact a quasi-periodic mode (the Floquet multiplier is complex), and that the mistake was due to the method used in Robichaux *et al.* (1999), which only allowed the identification of the real part of the Floquet multiplier. This quasi-periodic mode was also observed in the

wake of a circular cylinder in Blackburn & Lopez (2003) and Barkley & Henderson (1996), but it was stable in the Re range investigated in the latter. Marques *et al.* (2004), using bifurcation theory, showed that only three modes, modes A, B and the quasi-periodic, could possibly be observed in a two-dimensional wake with Z_2 symmetry.

However, for bluff body wakes that do not present this symmetry, a period-doubling mode has also been found. As far as we are aware, the first evidence of this period-doubling mode, which is commonly called mode C, appears in the computations and experiments in Zhang *et al.* (1995) when a tiny trip wire was placed close to a circular cylinder. More recently Sheard *et al.* (2003) identified this same mode in the flow around bluff rings by means of Floquet stability calculations. This was extended in Sheard *et al.* (2004), in which a non-linear characterisation of mode C was made by means of DNS. Subsequently, additional computational results were compared to experimental data in Sheard *et al.* (2005*b*) and, more recently, Sheard *et al.* (2005*a*) has suggested a physical mechanism to try to explain mode C behaviour.

1.2. *The flow around circular cylinders in staggered arrangements*

An important aspect of the flow around bluff bodies such as circular cylinders is that, if two or more bodies are placed in close proximity in the fluid stream, the flow field and fluid loads change dramatically when compared to the flow around an isolated body. This phenomenon is known as flow interference. Comparatively few papers on the flow around groups of cylinders have been published, despite the fact that in all the applications cited at the beginning of this section it is common to have circular-cylindrical structures grouped together. The usual approach is to consider only a pair of cylinders of equal diameter, since in such configurations the number of parameters is kept to a reasonable quantity for a proper systematic analysis and the basics of the interference mechanisms can be more easily understood.

An example of this can be found in Zdravkovich (1977), in which an extensive review of flow interference between two circular cylinders in tandem, side-by-side and staggered arrangements was conducted. For all these configurations, discussions about pressure distributions, velocity profiles, flow regimes and patterns, drag and lift forces and Reynolds number effects were presented. In relation to staggered arrangements, Zdravkovich (1977) investigated the physical origin of the non-zero lift forces that are observed for these configurations. Later, Sumner *et al.* (2000) put forward a taxonomy for the flow regimes observed in their experiments with staggered arrangements. Based on flow visualisation and force measurements, nine different patterns were identified according to the relative position of the cylinders. This work was extended by Akbari & Price (2005), who reproduced the flow regimes using computational simulations and scrutinised the physical features observed in each.

As far as we are aware, no computational work addressing the three-dimensional aspects of the flow around staggered arrangements of circular cylinders has yet been published. This paper focuses on the changes that occur in the transition in the wake when the flow around staggered arrangements is compared to the flow around a single cylinder. The instabilities that arise at the beginning of the transition are characterised by means of linear stability analysis of two-dimensional periodic base flows and also fully three-dimensional simulations.

The paper is organised as follows. In section 2, the numerical methods used to calculate the base flows and their stability are described. Details of the parameters used in the simulations are given in section 3 and in section 4 the results are presented and analysed. Finally, in section 5, conclusions are drawn regarding the effect of the relative position of the cylinders on the ways the wake transition occurs.

2. Overview

The methodology adopted in this paper can be divided into three distinct parts. The first refers to simulations of two-dimensional time-dependent flows by means of numerical solution of the incompressible Navier-Stokes equations. The second comprises stability calculations of two-dimensional time-periodic flows in respect to three-dimensional perturbations, and this is achieved using Floquet theory. Finally, the third part consists of three-dimensional simulations of selected cases, in order to analyse the non-linear behaviour of some of the bifurcations found. In the next three subsections, an overview of the mathematical formulation and computational implementation of each of these techniques is provided.

2.1. Two-dimensional base flow calculation

We consider a two-dimensional incompressible viscous flow, which is governed by the Navier-Stokes equations. These equations can be written in non-dimensional form using the cylinder diameter as the reference length and the free-stream flow speed U_∞ as the reference speed:

$$\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla) \mathbf{u} - \nabla p + \frac{1}{Re} \nabla^2 \mathbf{u}, \quad (2.1)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (2.2)$$

where $\mathbf{u} \equiv (u, v, w)$ is the velocity field, t is the time, p is the static pressure and Re is the Reynolds number. The pressure is assumed to be scaled by the constant density ρ .

These equations were discretized following the Spectral/hp method as described in Karniadakis & Sherwin (2005). To generate the base flows, equations (2.1) and (2.2) were solved in two dimensions, using a stiffly stable splitting scheme (Karniadakis *et al.* (1991)).

2.2. Floquet stability analysis

For this part, a similar approach as that utilised in Barkley & Henderson (1996) was used. We considered periodic two-dimensional base flows $\mathbf{U}(x, y, t)$, with period T , and checked their stability with respect to an infinitesimal three-dimensional perturbation $\mathbf{u}'(x, y, z, t)$. The equations that govern the perturbation evolution to leading order are the linearized Navier-Stokes equations:

$$\frac{\partial \mathbf{u}'}{\partial t} = -(\mathbf{U} \cdot \nabla) \mathbf{u}' - (\mathbf{u}' \cdot \nabla) \mathbf{U} - \nabla p' + \frac{1}{Re} \nabla^2 \mathbf{u}', \quad (2.3)$$

$$\nabla \cdot \mathbf{u}' = 0, \quad (2.4)$$

where $p'(x, y, z, t)$ is the pressure perturbation. The velocity boundary conditions that were imposed on this system are $u' = 0$ on boundaries where Dirichlet conditions are specified for the base flow, and $\partial u' / \partial n = 0$ on boundaries where Neumann conditions are specified for the base flow.

The right hand side of (2.3) subjected to the incompressibility constraint of equation (2.4) can be represented by an operator $\mathbf{L}$, so we can write the evolution equation in the compact form:

$$\frac{\partial \mathbf{u}'}{\partial t} = \mathbf{L}(\mathbf{u}'). \quad (2.5)$$

The operator $\mathbf{L}(\mathbf{u}')$ is T -periodic because it depends on the base flow $\mathbf{U}(x, y, t)$, which is T -periodic. Therefore, the stability of (2.5) can be investigated using Floquet analysis. The solutions of (2.5) can be decomposed into a sum of solutions of the form $\tilde{\mathbf{u}}(x, y, z, t)e^{\sigma t}$, where $\tilde{\mathbf{u}}(x, y, z, t)$, which are called Floquet modes, are T -periodic solutions. The complex exponents σ are the Floquet exponents, and the sign of their real parts determines the stability of the system. However, in Floquet-type problems it is more usual to consider the Floquet multiplier $\mu \equiv e^{\sigma T}$ instead of the Floquet exponent. If the Floquet multiplier is located inside the unit circle ($|\mu| < 1$), then the solution will

decay exponentially with time, and if the multiplier is located outside the unit circle ($|\mu| > 1$), the solution will grow exponentially with time, rendering the system unstable.

If we consider a system that is homogeneous in the spanwise direction z , a further simplification can be made. A perturbation of the velocity field can be expressed in terms of the Fourier integral:

$$\mathbf{u}'(x, y, z, t) = \int_{-\infty}^{\infty} \hat{\mathbf{u}}(x, y, \beta, t) e^{i\beta z} d\beta \quad (2.6)$$

and similarly for p' . Since the equation (2.5) is linear, modes with different β do not couple, and perturbations of the form

$$\mathbf{u}'(x, y, z, t) = (\hat{u} \cos \beta z, \hat{v} \cos \beta z, \hat{w} \sin \beta z), \quad (2.7)$$

$$p'(x, y, z, t) = \hat{p} \cos \beta z \quad (2.8)$$

remain of this form under this operator. Thus, the Floquet modes $\tilde{\mathbf{u}}(x, y, z, t)$ will necessarily be of this same form. As the velocity components $(\hat{u}, \hat{v}, \hat{w})$ and pressure $\hat{p}$ depend only on x , y and t , the three-dimensional stability problem can be reduced to a series of two-dimensional stability problems, each with a different value of β . Consequently, the stability of such two-dimensional incompressible periodic flows in respect to three-dimensional perturbations can be analysed by computing the Floquet multipliers and corresponding modes as a function of Re and β .

To calculate the Floquet modes, an operator representing the evolution of the system in one period was constructed:

$$\mathbf{u}'_{n+1} = \mathbf{A}(\mathbf{U}) \mathbf{u}'_n, \quad (2.9)$$

where $\mathbf{u}'_n = \mathbf{u}'(x, y, z, t_0 + nT)$ was the perturbation vector after n periods. The eigenvalues of $\mathbf{A}$ are the Floquet multipliers of $\mathbf{L}$ and the eigenfunctions of $\mathbf{A}$ are the Floquet modes at some instant in time t_0 , where t_0 depends on the initial phase of the base flow $\mathbf{U}$ used to construct $\mathbf{A}$.

The action of operator $\mathbf{A}$ on the perturbation $\mathbf{u}'$ was computed by integrating the linearized Navier-Stokes equations, using the same schemes that were used to compute the base flow. Two changes were necessary in the integration algorithm. The first was that a linear operator $-(\mathbf{U} \cdot \nabla)\mathbf{u}' - (\mathbf{u}' \cdot \nabla)\mathbf{U}$ had to be used to calculate the advection terms. In this operator, the values of $\mathbf{U}$ were computed by means of a Fourier interpolation of the time slices resulting from the base flow calculation. The second change was to replace the operator ∇ by $(\partial/\partial x, \partial/\partial y, -i\beta)$, and compute the three velocity components $(\hat{u}, \hat{v}, \hat{w})$ and $\hat{p}$ on the two-dimensional domains.

The Arnoldi method was used to compute the Floquet multipliers of largest magnitude. We briefly describe this method here; for a more complete discussion the reader should refer to Saad (1992). Essentially, this is an orthogonal projection method of a matrix $\mathbf{A}$ on to a k -dimensional Krylov subspace $\mathcal{K}_k \equiv \text{span}\{\mathbf{u}, \mathbf{A}\mathbf{u}, \mathbf{A}^2\mathbf{u}, \dots, \mathbf{A}^{k-1}\mathbf{u}\}$. Given an orthonormal basis $\mathbf{Q}_k = [\mathbf{v}_0 | \mathbf{v}_1 | \dots | \mathbf{v}_{k-1}]$ of the Krylov subspace $\mathcal{K}_k$, it is possible to decompose the matrix $\mathbf{A}$ in the following way:

$$\mathbf{A}\mathbf{Q}_k = \mathbf{Q}_k\mathbf{H}_k + h_{k,k-1}\mathbf{v}_k\mathbf{e}_{k-1}^H. \quad (2.10)$$

$\mathbf{H}_k$ is a Hessenberg matrix, whose component on row i and column j is denoted as $h_{i,j}$, $\mathbf{v}_k$ is a unitary vector orthogonal to the basis $\mathbf{Q}_k$ and $\mathbf{e}_{k-1}$ is a versor pointing in the direction of the $k-1$ component. Multiplying both sides of eq. (2.10) on the left by $\mathbf{Q}_k^H$ and using the fact that $\mathbf{Q}_k$ is orthonormal results in

$$\mathbf{Q}_k^H\mathbf{A}\mathbf{Q}_k = \mathbf{H}_k. \quad (2.11)$$

The eigenvalues $\lambda_i^{(k)}$ of the Hessenberg matrix $\mathbf{H}_k$ are approximations of the eigenvalues of the matrix $\mathbf{A}$, and the approximate eigenvector of $\mathbf{A}$ associated with $\lambda_i^{(k)}$, also called the Ritz eigenvector, can be calculated using the expression

$$\mathbf{w}_i^{(k)} = \mathbf{Q}_k\mathbf{y}_i^k, \quad (2.12)$$

where $\mathbf{y}_i^k$ is the eigenvector of $\mathbf{H}_k$ associated with the eigenvalue $\lambda_i^{(k)}$. Some of the Ritz eigenvalues are good approximations of the eigenvalues of $\mathbf{A}$ and the quality of the approximation normally improves as k increases. An efficient way of estimating the residual norm ϵ_i of the approximations is to use the expression

$$\epsilon_i = h_{k,k-1} \left| \hat{\mathbf{e}}_{k-1}^H \mathbf{y}_i^{(k-1)} \right|. \quad (2.13)$$

An in-house implementation of a restarted Arnoldi algorithm was used to compute the Floquet multipliers of largest magnitude. First of all, given a first perturbation vector $\mathbf{u}'_0$ and using the expression (2.9), a Krylov subspace $\mathbf{T}_{k+1}$ of dimension $k+1$ was generated:

$$\mathbf{T}_{k+1} = \{\mathbf{u}'_0, \mathbf{u}'_1, \mathbf{u}'_2, \dots, \mathbf{u}'_k\}.$$

Next, a QR factorisation for the matrix $\mathbf{T}_{k+1}$ was calculated, using the modified Gram-Schmidt procedure, as presented in Golub & Van Loan (1996).

With the orthonormal basis $\mathbf{Q}_{k+1}$ for the Krylov subspace and the upper triangular matrix $\mathbf{R}_{k+1}$ ($\mathbf{T}_{k+1} = \mathbf{Q}_{k+1}\mathbf{R}_{k+1}$), it is possible to derive a simple expression to calculate the Hessenberg matrix $\mathbf{H}_k$ correspondent to a k -dimensional Krylov subspace $\mathcal{K}_k = \text{span}\{\mathbf{u}'_0, \mathbf{u}'_1, \dots, \mathbf{u}'_{k-1}\}$. First, we multiply both sides of eq. (2.11) on the right by $\mathbf{R}_k$, and we get

$$\mathbf{Q}_k^H \mathbf{A} \mathbf{T}_k = \mathbf{H}_k \mathbf{R}_k. \quad (2.14)$$

Since $\mathbf{T}_k$ is generated by the Krylov sequence $\{\mathbf{u}'_0, \mathbf{A}\mathbf{u}'_0, \mathbf{A}^2\mathbf{u}'_0, \dots, \mathbf{A}^{k-1}\mathbf{u}'_0\}$, we can perform the decomposition

$$\mathbf{A} \mathbf{T}_k = \mathbf{Q}_{k+1} \bar{\mathbf{R}}_k^{(k+1)}, \quad (2.15)$$

where $\bar{\mathbf{R}}_k^{(k+1)}$ is a $(k+1) \times k$ matrix composed of the k last columns of $\mathbf{R}_{k+1}$. Substituting (2.15) into (2.14), we get

$$\mathbf{Q}_k^H \mathbf{Q}_{k+1} \bar{\mathbf{R}}_k^{(k+1)} = \mathbf{H}_k \mathbf{R}_k. \quad (2.16)$$

Because the vectors that form the $\mathbf{Q}$ matrices are orthogonal, the left hand side of (2.16) can be written

$$\mathbf{Q}_k^H \mathbf{Q}_{k+1} \bar{\mathbf{R}}_k^{(k+1)} = \begin{bmatrix} \mathbf{I}_k & \mathbf{0} \end{bmatrix} \bar{\mathbf{R}}_k^{(k+1)} = \check{\mathbf{R}}_k^{(k+1)},$$

so $\check{\mathbf{R}}_k^{(k+1)}$ is a $k \times k$ matrix made up of the last k columns and first k lines of matrix $\mathbf{R}_{k+1}$.

Using the fact that both $\check{\mathbf{R}}_k^{(k+1)}$ and $\mathbf{R}_k$ are submatrices of $\mathbf{R}_{k+1}$, the final expression, $\check{\mathbf{R}}_k^{(k+1)} = \mathbf{H}_k \mathbf{R}_k$, can be written in index form ($r_{i,j}$ and $h_{i,j}$ are elements on row i and column j of the matrices $\mathbf{R}_{k+1}$ and $\mathbf{H}_k$ respectively):

$$r_{i,j+1} = \sum_{l=0}^j h_{i,l} r_{l,j},$$

which can be rearranged in order to give the values of the matrix $\mathbf{H}_k$

$$h_{i,j} = \frac{1}{r_{j,j}} \left(r_{i,j+1} - \sum_{l=0}^{j-1} h_{i,l} r_{l,j} \right). \quad (2.17)$$

Therefore, to resume the description of the implementation, after doing a QR factorisation for the matrix $\mathbf{T}_{k+1}$, the matrix $\mathbf{H}_k$ was calculated using (2.17). The eigenvalues and eigenvectors of $\mathbf{H}_k$ were calculated using the LAPACK routines (Anderson *et al.* 1999), and the residual was estimated using (2.13).

In practice, a maximum value for the dimension of the Krylov subspace was specified, and the actual k started with 1. In every iteration the value of k was incremented, the Krylov subspace was updated, the calculations were carried out and the residual was calculated. If the Krylov subspace dimension reached the specified maximum size before convergence, a new vector in the Krylov sequence was generated and the oldest vector was discarded, thus keeping the Krylov subspace size constant and equal to the maximum size specified. This is equivalent to restarting the method with the second oldest vector in the sequence. Once the required eigenvalues had converged, the Ritz approximations of the correspondent eigenvectors of $\mathbf{A}$ were computed using eq. (2.12). If the converged Floquet multiplier was a complex pair, the Ritz eigenvectors were calculated using the

same approach as the LAPACK routines (Anderson *et al.* 1999), using one vector for the real part and one for the imaginary part, thus making it possible to fully reconstruct the Floquet mode.

This implementation has the advantage of providing an explicit restart without the disadvantage of resetting the size of the Krylov subspace to 1 again, as in the method presented by Saad (1992). When compared to Implicit Restarted Arnoldi methods (Lehoucq & Sorensen 2000), the method presented here is easier to implement. The calculation of the action of the operator $\mathbf{A}(\mathbf{U})$ over the perturbation, which was actually an integration in time, was much more expensive than the QR decomposition of the matrix $\mathbf{T}_{k+1}$, therefore the fact that the decomposition must be made in every iteration was not a significant drawback when considering the total computational time.

2.3. *Three-dimensional simulations and non-linear characterisation of the transition*

Linear stability calculations can predict the critical Reynolds numbers and unstable mode topology and wavelength. However, in order to comprehend the evolution of the instability beyond its onset, non-linear effects have to be taken into account, and full three-dimensional simulations must be carried out.

The three-dimensional computational calculations utilised the same framework outlined in section 2.1. However, the spanwise direction was treated using a Fourier expansion in the discretization. This approach was introduced by Karniadakis (1990) and has been often employed since then in cases where the geometry has one homogeneous direction. The meshes utilised in the two-dimensional simulations can be re-used, since the three-dimensional domains for this type of calculation consist of two-dimensional discretizations repeated in the spanwise direction. In addition, the way the solution algorithm is developed makes the parallelization of the code straightforward and efficient.

The three-dimensional simulation results enabled a non-linear analysis of the transi-

tions to be conducted. The Landau equation has often been successfully used in hydrodynamics as a low dimensional model to describe the non-linear behaviour of transitions close to their respective critical points (see for example Sheard *et al.* 2004; Henderson 1997; Provansal *et al.* 1987; Dušek *et al.* 1994; Noack & Eckelmann 1994). The idea behind this equation is that for a small perturbation amplitude A the linear term determines the behaviour of dA/dt . If the system is unstable the modulus of A grows and the higher order terms become important at a certain point. Here, the Landau equation is written up to third order, using the same notation as in Sheard *et al.* (2004):

$$\frac{dA}{dt} = (\sigma + i\omega) A - l(1 + ic)|A|^2 A + \dots, \quad (2.18)$$

where $A(t)$ is the complex amplitude of the perturbation mode being considered.

The term $(\sigma + i\omega)$ in (2.18) is the eigenvalue obtained from the linear stability analysis, and in the present case is the Floquet exponent. To first order approximation, the perturbation grows at a rate given by σ , so the flow is stable if σ is negative and unstable otherwise. In addition, still to first order approximation, the perturbation oscillates with angular frequency ω , which is non-zero for a Hopf bifurcation. The parameter l is the real part of the third order coefficient and the classification of the transition depends directly on its sign. If $l > 0$, the transition is supercritical, thus non-hysteretic, and the behaviour of the transition until saturation should be adequately described by the Landau equation truncated at 3rd order. The reason for this is that the real part of the 3rd order term is negative and counter-balances the growth provoked by the 1st order term, eventually making the instability reach a saturated state, so no higher order terms are necessary. On the other hand, if $l < 0$, the proper description of the transition must be made using terms of at least 5th order, as both 1st and 3rd order terms promote the growth of A and the numerical and experimental observations indicate that this growth must saturate at some point. In this case, the transition is said to be subcritical and often shows a hys-

teretic behaviour. Lastly, c is called the Landau constant and it modifies the oscillation frequency at saturation.

If the complex amplitude A is written in polar form, $A(t) = \rho(t)e^{i\Phi(t)}$, and some simple algebra is performed, the real and imaginary parts of (2.18) become

$$\frac{d \log(\rho)}{dt} = \sigma - l\rho^2, \quad (2.19)$$

$$\frac{d\Phi}{dt} = \omega - lc\rho^2. \quad (2.20)$$

Given (2.19), a graph of $d \log(|A|)/dt$ against $|A|^2$ can be plotted and used to determine the values of σ and l . The value where the curve intercepts the y -axis is σ and the slope of the curve close to the y -axis gives $-l$. If the cubic Landau model is enough to describe the transition, then this plot should be linear; if not, then this is an indication that at least 5th order terms are necessary to describe the transition.

Last, the amplitude A must be defined in terms of flow variables. In the present work, this amplitude is defined by

$$|A(t)| = \left[\int_{\Omega} |\hat{\mathbf{u}}_1(x, y, t)|^2 \right]^{1/2},$$

where Ω is the two-dimensional cross-section of the domain being considered and $\hat{\mathbf{u}}_1(x, y, t)$ is the coefficient at time t of the Fourier expansion term whose wavenumber is the same as the instability in question. A similar expression was used in Henderson & Barkley (1996) and Henderson (1997); the only difference is a multiplicative constant.

3. Numerical simulations

Figure 1 shows a schematic drawing of the geometry investigated in this research. The two cylinders have the same diameter D . For all cases, the streamwise distance between the centres of the cylinders, L_x , is $5.0D$. The transversal distance, L_y , was varied from $0.0D$ to $3.0D$, with intervals of $0.5D$. Calculations were also carried out for the single

cylinder case, which served as a benchmark for comparison with the two-cylinder cases. The Reynolds numbers simulated were 200, 250, 300 and 350, covering the wake transition range for the flow around a single cylinder.

Quadrilateral meshes were generated for the computations; an example is shown in figure 2. The size of the domain and the discretization approach were chosen based on a previous work (Carmo 2005), in which an extensive convergence study was made considering the flow around a single cylinder and using the same numerical method. The upstream boundary (boundary at the left-hand side in figure 2) is located $36D$ from the upstream cylinder centre, the side boundaries (top and bottom boundaries in figure 2) are located $45D$ from the closest cylinder centre and the downstream boundary is located $95D$ from the downstream cylinder. The domains extended far downstream in order to consider the vortex interaction that takes place in the far wake, as reported in section 4.1. The number of elements for the mesh with a single cylinder was 466, and for the meshes with two cylinders the number of elements varied from 568 to 721, depending on L_y . The meshes were the same for all Re , varying only according to the geometrical configuration.

8th-degree polynomial expansions were used as basis functions for the spatial discretization, and the integration in time was second-order accurate. The value of the non-dimensional time-step varied from 0.004 to 0.006, depending only on Re . Uniform stream velocity boundary conditions ($u = 1$, $v = 0$) were imposed at the upstream boundary and upper and lower surfaces of the computational domain. At the cylinder walls, a no-slip condition ($u = 0$, $v = 0$) was imposed, and at the downstream border Neumann boundary conditions, $\partial u / \partial \mathbf{n} = 0$, $\partial v / \partial \mathbf{n} = 0$ were applied.

The periodic base flows were calculated using direct numerical simulation, employing the method described in section 2.1. The first step in the process was to let the simula-

Figure 2: Example of a mesh used in the calculations.

tions reach an asymptotic, time periodic state. To quantify the periodicity of the system, the peak values of the lift coefficient time history of the upstream cylinder during ten shedding cycles were taken. If the ratio between the standard deviation and the mean of these values was less than 0.5%, then the simulation was considered to have reached a periodic state. Subsequently, 32 equally-spaced snapshots of the velocity solution were recorded per cycle. These snapshots were used to generate the base flow for the stability calculations. In the single cylinder cases, only one vortex shedding cycle was considered for the Floquet calculations. However, for all cases with two cylinders, two vortex shedding cycles were used for the Floquet calculations, because the vortex interaction that took place in the far wake had period $2T$, where T is the vortex shedding period (see section 4.1).

For all stability calculations, a Krylov subspace of dimension of 10 was used, and the perturbation initialised with the field $(u', v') = (f, f)$, where f is the positive number

that causes the field to have a unitary norm. The calculations were carried out until the residual of the biggest eigenvalue reached a value less than 10^{-5} .

The numerical method used here has been previously utilized both for direct numerical simulations (see the examples and references in Karniadakis & Sherwin 2005) and Floquet calculations (Barkley & Henderson 1996; Robichaux *et al.* 1999; Sheard *et al.* 2003), in which convergence for these applications has been verified. However, in the present work, it was necessary to determine whether the size and discretization of the domains were adequate for the cases investigated in the present work.

For this reason, additional simulations were carried out considering the cases ($L_y = 0.0D$, $Re = 200$), ($L_y = 0.0D$, $Re = 350$), ($L_y = 3.0D$, $Re = 200$), ($L_y = 3.0D$, $Re = 350$). Two refinements were considered separately: the first was to increase the polynomial degree from 8 to 9, and the second to extend the original mesh by $5D$ on all four external boundaries. The base flow was calculated and the Floquet stability analysis was carried out for $\beta = 2.0D$, using the base flow generated with the largest mesh. The parameters monitored in the base flow results were the Strouhal number (St), the mean drag coefficient and the RMS of the lift coefficient, all of which were evaluated for both cylinders, and from the Floquet calculations the value of the Floquet multiplier was also tested for convergence. In all cases, all parameters varied less than 1% when the refined meshes were compared to the reference meshes.

For the three-dimensional simulations, the meshes and parameters of the corresponding two-dimensional cases were re-used, and Fourier expansions employed in the spanwise direction z . Periodic boundary conditions were set on the boundary planes located at the extreme values of z .

In the remainder of this paper, all the distances will be given in diameters, unless stated otherwise.

Figure 3: Instantaneous vorticity contours, $Re = 200$. Contours vary from $\omega_z D/U_\infty \leq -2.2$ (light contours) to $\omega_z D/U_\infty \geq 2.2$ (dark contours).

4. Results

4.1. Base flow calculations

The vortex shedding regime in the base flow changed significantly in the range of L_y investigated. The changes in the regime can be observed in figure 3, which shows instantaneous vorticity contours for different L_y . The Strouhal number (St) was calculated by performing a Fast Fourier Transform of the lift coefficient time history of both cylinders. For every case in the present work, the value of the Strouhal number for the upstream cylinder was the same as the value obtained for the downstream one.

Analysing figure 3a, it can be seen that, for $L_y = 0.0$, the vortices shed by the upstream cylinder hit the downstream one. We believe that the pressure and velocity fluctuations

imposed near the second body synchronised the vortex shedding. For $0.5 \leq L_y \leq 1.0$, the vortices shed from the upstream cylinder did not directly impinge upon the downstream cylinder, but interacted with the vortex being shed from the lower surface of the downstream cylinder, stretching and pulling it from the near wake of the second cylinder, as shown in figure 3b. In all cases with $L_y \leq 1.0$, the vortex interactions that took place in the near wake of the downstream cylinder were notably different from those in the near wake of the upstream cylinder, as we can see in figures 3a and 3b. In the upstream near wake, the interaction between one vortex and the shear layer of opposite sign vorticity on the other side of the cylinder, which, according to the model proposed in Gerrard (1966), eventually leads to the shedding of a vortex formed in the previous half cycle, was always very clear. On the other hand, the shedding in the near wake of the downstream cylinder seemed to be forced by the influence of the upstream wake. The wake of the downstream cylinder was much wider and no strong interaction between vortices and shear layers of opposite sign were observed in the near wake of the downstream cylinder.

We can see in figure 3d that, for configurations with $L_y \geq 2.0$, the interaction between vortices of the near wakes of the downstream and upstream cylinders became weaker and the vortices originating from the downstream cylinder were not deformed nor significantly pulled towards the upstream wake. In addition, the shedding mechanisms in both near wakes are very similar, corresponding qualitatively to what happens in the near wake of the flow around a single cylinder. Nevertheless, a synchronisation of the wakes due to the proximity of the vortices can still be observed.

The arrangement with $L_y = 1.5$ could not be classified into either of the two regimes presented above. The flow around this configuration is in the transition range between these two regimes, being very unstable in terms of the patterns observed in the wake downstream and corresponding to a peak in the absolute value of the mean lift coeffi-

cient of the downstream cylinder. In figure 3c, we see that the vortex formed from the bottom of the downstream cylinder (the positive, dark vortex) is split, and part of it merges with the wake of the upstream cylinder, while the rest follows the direction of the downstream cylinder wake. However, it does not occur regularly every cycle; on some occasions the complete vortex is pulled towards the upstream cylinder wake. For $Re \geq 250$, the behaviour of the vortex in question in each cycle appears random.

For every configuration with two cylinders, irrespective of the values of L_y and Re , different kinds of vortex interactions were observed in the far wake. This interaction took place in a limited region, and gave rise to a different vortex configuration in the wake downstream, which will here be referred to as a secondary wake. The secondary wake had a period of $2T$, where T was the period of vortex shedding, because its formation always involved the merging of vortices from two successive cycles. Figure 4 shows instantaneous vorticity contours focusing on the vortex interaction region that arises in the far wake, for three different configurations. It should be noted that the type of interaction is very different in each of the illustrated cases, although all of them involve vortex merging. For $L_y \leq 1.5$, this vortex interaction region appears between $10D$ and $20D$ downstream of the front cylinder, while for configurations of higher L_y , when two distinct wakes are observed, this region appears further downstream, at distances as far as $30D$ downstream of the front cylinder. Similar patterns in the far wake were observed in smoke visualisations performed by Zdravkovich (1972) for different tandem and staggered arrangements.

4.2. Floquet stability calculations

4.2.1. Single cylinder

In order to validate the code and to have a benchmark for the other cases, the first stage of the stability calculations consisted of the Floquet analysis of the flow around a single cylinder. For this particular configuration, we used 32 equispaced time slices taken

(a) $L_y = 0.0$ (b) $L_y = 1.5$ (c) $L_y = 3.0$

Figure 4: Instantaneous vorticity contours, $Re = 250$. Contours vary from $\omega_z D/U_\infty \leq -2.2$ (light contours) to $\omega_z D/U_\infty \geq 2.2$ (dark contours) for (a) and (b) and from $\omega_z D/U_\infty \leq -1.1$ (light contours) to $\omega_z D/U_\infty \geq 1.1$ (dark contours) for (c).

from a single vortex shedding period T to use in a temporal Fourier representation of the base flow. The values of the Floquet multiplier, μ , for these calculations are shown in figure 5, and they compare well with those presented in Barkley & Henderson (1996).

Figure 5: Floquet multiplier μ as a function of the spanwise wavenumber β for various Re . Flow around a single cylinder.

Two different unstable modes are present in the considered Re range. Following the nomenclature suggested by Williamson (1988), they are called modes A and B. Mode A, which corresponds to the peaks of lowest β in each of the curves shown in figure 5, is, in terms of Re , the first to appear, and is already unstable for $Re = 200$. For this mode, the wavelength of maximum growth rate is around $4.0D$. Mode B corresponds to the peak of higher β , having a wavelength of maximum growth rate of about $0.8D$, and, among the cases calculated, it is only unstable for $Re = 300$ and $Re = 350$, although a peak can already be seen in the stable region for $Re = 250$.

An important difference between modes A and B is their spatio-temporal symmetries (Barkley & Henderson 1996; Blackburn *et al.* 2005). The three-dimensional structure of the normalised Floquet mode associated with a spanwise wave number β can be reconstructed by considering the perturbation in the vector form:

$$\mathbf{u}'(x, y, z, t) = [\tilde{u}(x, y, t) \cos \beta z] \hat{\mathbf{i}} + [\tilde{v}(x, y, t) \cos \beta z] \hat{\mathbf{j}} + [\tilde{w}(x, y, t) \sin \beta z] \hat{\mathbf{k}} \quad (4.1)$$

The perturbation vorticity field can be calculated using the modified nabla operator

$\nabla_\beta = (\partial/\partial x, \partial/\partial y, -i\beta)$, leading to:

$$\omega(x, y, z, t) = [\tilde{\omega}_x(x, y, t) \sin \beta z] \hat{\mathbf{i}} + [\tilde{\omega}_y(x, y, t) \sin \beta z] \hat{\mathbf{j}} + [\tilde{\omega}_z(x, y, t) \cos \beta z] \hat{\mathbf{k}} \quad (4.2)$$

The T -periodic base flow obeys a reflectional symmetry about the wake centreline ($y = 0$), when time is advanced by $T/2$. Following Robichaux *et al.* (1999), this symmetry is called RT symmetry (R for reflectional and T for translation in time). Mode A has the following RT symmetry:

$$\text{Mode A: } \begin{cases} \tilde{u}(x, y, z, t) = \tilde{u}(x, -y, z, t + T/2) \\ \tilde{v}(x, y, z, t) = -\tilde{v}(x, -y, z, t + T/2) \\ \tilde{w}(x, y, z, t) = \tilde{w}(x, -y, z, t + T/2) \end{cases} \quad (4.3)$$

which is the same as the symmetry of the two-dimensional base-flow. In contrast, mode B has the opposite symmetry of mode A:

$$\text{Mode B: } \begin{cases} \tilde{u}(x, y, z, t) = -\tilde{u}(x, -y, z, t + T/2) \\ \tilde{v}(x, y, z, t) = \tilde{v}(x, -y, z, t + T/2) \\ \tilde{w}(x, y, z, t) = -\tilde{w}(x, -y, z, t + T/2) \end{cases} \quad (4.4)$$

Most of the published results regarding these modes focus on the streamwise vorticity component. Using eq. (4.3), the following expression for the x -component vorticity ω_x on mode A is derived:

$$\tilde{\omega}_x(x, y, z, t) = -\tilde{\omega}_x(x, -y, z, t + T/2) \quad (4.5)$$

For mode B, using eq. (4.4), the streamwise vorticity is found to obey the symmetry:

$$\tilde{\omega}_x(x, y, z, t) = \tilde{\omega}_x(x, -y, z, t + T/2) \quad (4.6)$$

In figure 6 streamwise vorticity contours evaluated in time on a vertical line $2D$ downstream of the cylinder are presented. The symmetry of the form and absolute strength of the vortices can be clearly observed. For mode A, the vorticity has different sign either

Figure 6: x -vorticity of the unstable eigenvector on the line $x = 2.0$. Light regions correspond to negative and dark to positive streamwise vorticity of the eigenvector. Time is non-dimensionalized using the shedding period.

side of the wake centreline, for mode B, the vorticity has the same sign throughout the cycle.

4.2.2. Staggered arrangements

Having reviewed the single cylinder case, we now consider the stability of the periodic flow around pairs of cylinders in staggered arrangements. For these cases, two shedding periods were considered, taking into account the $2T$ -periodicity of the far wake. Thus, 64 time slices of the two-dimensional simulations were used to generate each periodic base flow. Due to its transitional character, the case with $L_y = 1.5$ did not present a reasonably periodic wake for $Re \geq 250$. Therefore no Floquet stability results are presented for this case, since the level of uncertainty of the calculations would be too high for a proper analysis.

For small L_y , more specifically for $L_y = 0.0$ and 0.5 , the shape of the curves μ versus β were very similar to the single cylinder case. Figure 7 shows the values of μ for the configurations with $L_y = 0.0$ and $L_y = 0.5$. The two peaks, one for mode A and one for mode B, are clearly evident.

In order to examine the structure of the modes in greater detail, figure 8 shows stream-

Figure 7: Floquet multiplier μ as a function of the spanwise wavenumber β for various Re .

wise vorticity contours of the unstable eigenvectors. Figure 8a shows structures characteristics of mode A. Of note is that the instability first develops in the near wake of the upstream cylinder, but then grows stronger when it reaches the downstream cylinder. Also of note is the higher energy in the region of vortex interaction that initiates the secondary wake. It seems that the vortex merging and straining that occur in this region helps to reinforce mode A structures. The results are different for mode B. In figure 8b, we see that the instability first develops in the near wake of the upstream cylinder, as in the mode A case. However, the streamwise vortical structures do not seem to gain significant strength when they impinge upon the downstream cylinder. Also, no mode B structures were observed in the vortex interaction region that initiates the secondary wake.

For $L_y = 1.0$, the transition deviates from that observed for the flow around a single cylinder. A peak of intermediate wave number dominates, as can be seen in figure 9a. This corresponds to the asymmetric period-doubling mode C, which will be described and analysed in more detail in section 4.3. The peak relating to mode B is only visible for $Re = 350$, nevertheless it was verified that this mode is already unstable for $Re = 300$,

Figure 8: x -vorticity of eigenvectors, flow around two cylinders with $L_y = 0.0$, $Re = 300$. Dark contours correspond to positive x -vorticity and light contours to negative x -vorticity.

but that the corresponding Floquet multiplier is smaller than that relating to mode C. Figure 9b shows the neutral stability curves for this configuration. The critical Reynolds number for which mode A became unstable was $Re_A = 178$, hence lower than the value $Re_A = 188.5$ for a single cylinder (Barkley & Henderson 1996), and the corresponding wavelength was $L_{zA} = 3.86$, which is also smaller than the one found for a single cylinder, $L_{zA} = 3.96$ (Barkley & Henderson 1996). Mode C became unstable for $Re_C = 191$ and the instability had a wavelength $L_{zC} = 1.55$.

For configurations in which two distinct wakes could be observed ($2.0 \leq L_y \leq 3.0$), mode C was also detected. The graphs on the left hand side of figure 10 exhibit three distinct peaks, and the peak relating to mode C appears in a region of intermediate wave numbers, between the peaks associated with modes A and B.

The right hand side of figure 10 shows the neutral stability curves for the same configurations. For the three cases, mode A bifurcates at a slightly higher Re and roughly the same L_z that it does in the single cylinder case; Re_A varied from 190 ($L_y = 2.0$) to 194 ($L_y = 3.0$) and L_{zA} varied from 3.94 ($L_y = 2.0$) to 3.98 ($L_y = 3.0$). The same trend is observed for the mode B bifurcation, with Re_B varying from 265 ($L_y = 2.0$) to 263 ($L_y = 3.0$) and $L_{zB} = 0.83$ for the three cases; the values for the single cylinder case

Figure 9: Flow around a pair of cylinders with $L_y = 1.0$. (a) Floquet multiplier μ as a function of the spanwise wave number β for various Re . (b) Neutral stability curves (in the region on the right hand side of the curves the two-dimensional periodic flow is unstable with respect to three-dimensional perturbations).

reported in Barkley & Henderson (1996) were $Re_B = 259$ and $L_{zB} = 0.822$. Nevertheless, the neutral stability curve for mode C changed considerably when these cases are compared. Also in the same figure it can be noticed that mode becomes unstable later at a progressively increasing Re as L_y increases, and the region of mode C instability becomes thinner. This behaviour is not surprising, given that the larger the distance between the cylinders, the closer the flow is to the single cylinder case, for which no mode C instability is observed. Another indication of the increasing similarities with the isolated cylinder case for higher L_y is the fact that for $L_y = 3.0$ a second unstable eigenvalue emerged in the calculations for $Re = 350$, corresponding to a mode B instability originating in the near wake of the downstream cylinder.

The next sub-section is specially dedicated to the analysis of the mode C three-dimensional instability.

(a) $L_y = 2.0$ (b) $L_y = 2.5$ (c) $L_y = 3.0$

Figure 10: Floquet multiplier μ as a function of the spanwise wave number β for various Re (left hand side) and neutral stability curves (right hand side). Flow around staggered arrangements.

4.3. Mode C

In this section, the intermediate wave number mode that appears in the wake transition for $L_y \geq 1.0$ is examined. This mode is in fact a mode C instability (Zhang *et al.* 1995; Sheard *et al.* 2003), as will be shown later, so this designation will be used from now on. Figure 11 shows instantaneous x -vorticity contours of an eigenvector of mode C. A significant difference between this mode and modes A and B is that mode C originates in the near wake of the downstream cylinder, while modes A and B first appear in the near wake of the upstream cylinder and are also observed in the near wake of the downstream cylinder. This region where mode C develops has a dominant frequency equal to the shedding frequency, as opposed to the far wake region, which has a dominant frequency of half the shedding frequency. The Floquet multiplier for mode C was real and positive; however, it should be remembered that two shedding periods were considered in the calculations, in other words, twice a Poincaré map (Marques *et al.* 2004). In order to verify if the mode in question was a synchronous or a period-doubling mode, some additional calculations were carried out, considering only one shedding period, which corresponds to one Poincaré map. The results gave real negative Floquet multipliers, therefore the mode is a period-doubling instability, just like the mode C instability reported by Sheard *et al.* (2003).

Comparing figures 11a and 11b, it is possible to see that the streamwise vorticity contours of mode C have striking differences in shape depending on the shedding mode. For $L_y = 1.0$, the positive vortices shed from the downstream cylinder in the base flow are pulled away from the body by the vortices that are shed from the upstream cylinder. Consequently, the negative vortices form an isolated vortex line, where mode C develops. This is an extreme case of wake asymmetry. For $L_y \geq 2.0$ (figure 11b), there is a complete wake being formed after of the downstream cylinder, so the level of asymmetry is much

(a) $L_y = 1.0$, $Re = 200$ (b) $L_y = 2.0$, $Re = 300$

Figure 11: Instantaneous streamwise vorticity contours of the eigenvector (greyscale) superposed to spanwise vorticity contours of the base flow (lines), $\beta = 1.571$. Light regions correspond to negative and dark to positive streamwise vorticity of the eigenvector. The solid and dashed lines represent positive and negative z -vorticity contours of the base flow respectively.

weaker, although doubtlessly present. In such cases mode C streamwise vorticity can be observed in both sides of the wake.

Figure 12 shows the time history of the x -vorticity contours of the eigenvector of mode C on a vertical line $2D$ aft of the downstream cylinder ($7D$ aft the upstream cylinder). The first important aspect to notice is that this mode is not symmetric in relation to the wake centreline (in figure 12, the wake centreline is $y = 2$). Therefore, the RT symmetry is broken, as there is no reflectional symmetry in this mode. A second aspect of note

Figure 12: Normalised x -vorticity of the unstable eigenvector on the line $x = 7.0$, configuration $L_y = 2.0$, $Re = 300$ and $\beta = 1.571$. Light regions correspond to negative and dark to positive streamwise vorticity of the eigenvector. Time is non-dimensionalized using the shedding period.

is that this mode has a period of $2T$, where T is the shedding period, in contrast with modes A and B, which are T -periodic.

Figure 13 shows plots of vorticity iso-surfaces of the base flow combined with the unstable Floquet mode for modes A, B and C. The three-dimensional structures of modes A and B are very similar to those observed in the flow around a single cylinder (Henderson 1997). For mode A, the streamwise vorticity is stronger in the primary vortex cores, whereas for mode B it is stronger in the braid shear layers. This similarity in the structures is not surprising given the already mentioned similarities in the symmetry and wavelength. Apart from the different wavelength and its period-doubling nature, mode C presents a structure similar to mode B, in the sense that it is stronger in the braid shear layers. The asymmetrical nature of mode C is also clear in the figure, as the stronger streamwise vorticity is observed in the side of the wake closest to the vortex street formed from the upstream cylinder.

Figure 14 shows two snapshots of spanwise vorticity iso-surfaces of the unstable mode C summed to the base flow, taken at time instants separated by one flow period apart.

The period-doubling character of mode C makes the waviness of the vortex cores on the same location at consecutive periods to have a phase difference of 180° , as can be observed looking to the far left vortices in figure 14. Another evidence of the period-doubling character displayed in figure 14 is the alternation of the sign of the spanwise vorticity, from one period to the next, in the shear layers close to the cylinder .

As previously mentioned, the period-doubling mode C was also identified by Sheard *et al.* (2003), in stability calculations of the flow around bluff rings with aspect ratio (ratio between the ring diameter and the cross section diameter) greater than 4.0. In this work they found a maximum growth rate for mode C for spanwise wavelengths between $1.6D$ and $1.7D$, which agrees well with the results of the present investigation. They also reported that the smaller the aspect ratio was, the earlier (in terms of Re) mode C became unstable. The same behaviour was observed in our staggered cylinder arrangement, if we replace the aspect ratio by L_y , as discussed in section 4.2.2. Furthermore, striking similarities between the three-dimensional structure of mode C are found when comparing figure 13 to the results presented in Sheard *et al.* (2004). The mode C wavelength found in the present results is also reasonably close to the value of $1.8D$ reported in Zhang *et al.* (1995). In this study, mode C appeared in their experiments when a thin control wire was placed near the cylinder, and they reproduced this result in computational simulations by enforcing zero velocity in the point of the mesh that corresponded to the coordinate of the centre of the control wire. We note that asymmetry is again involved in the base flow.

4.4. Non-linear analysis

In order to investigate the non-linear evolution of mode C, fully three-dimensional simulations were also performed. The configurations $L_y = 1.0$, $L_y = 2.0$, $L_y = 2.5$ and $L_y = 3.0$ were tested, and Reynolds numbers slightly higher than critical Re for mode C

(a) Mode A, $L_z = 4.000$ (b) Mode B, $L_z = 0.838$ (c) Mode C, $L_z = 1.571$

Figure 13: Three-dimensional structures of the unstable modes, configuration $L_y = 2.0$, $Re = 300$. The represented flow fields are linear combinations of the base-flow and the unstable Floquet mode and the flow is from right to left. Translucent green and yellow surfaces are iso-surfaces of negative and positive ω_z , respectively. Solid blue and red surfaces are iso-surfaces of negative and positive ω_x respectively.

(a) Snapshot at time t (b) Snapshot at time $t + T$

Figure 14: Iso-surfaces of spanwise vorticity, unstable mode summed to base flow, configuration $L_y = 2.0$, $Re = 300$, $\beta = 1.571$. Light grey iso-surfaces are negative ω_z and dark grey are positive ω_z . Flow is from right to left.

bifurcation (Re_c) for each configuration were selected. For each case, the periodic span of the domain was the wavelength of the mode C instability at its onset and the initial condition was a time slice of the two-dimensional periodic solution summed to the unstable Floquet mode multiplied by a factor small enough as to make the perturbation initial growth to follow the linear prediction. This method of choosing the periodic length and the initial condition was also adopted in Henderson & Barkley (1996) and Henderson (1997), and has the advantage of enabling the observation of the evolution of the mode in a rather clear way. In all calculations reported in this subsection, 8 Fourier modes were used in the discretization in the spanwise direction.

Figure 15: Results for three-dimensional simulation of case $L_y = 1.0$, $Re = 195$, $L_z = 1.55$. (a) displays the graph of the derivative of the amplitude logarithm against the square of the amplitude. (b) shows the growth and saturation of the amplitude in the wavelength corresponding to the mode C instability (symbols) and growth predicted by linear calculations (continuous line). The x axis is non-dimensional time.

4.4.1. $L_y = 1.0$

In figure 15a, it can be seen that the curve of the derivative of the amplitude logarithm against the square of the amplitude has a positive slope close to the y -axis, hence the bifurcation for mode C was subcritical for the configuration with $L_y = 1.0$. It is worth noticing how small is the range of $|A|^2$ for which the curve has a positive slope when compared to other cases of subcritical bifurcation (see, for example, figure 17a of this paper and the cases of subcritical bifurcation in Sheard *et al.* (2004)). After this range, the curve follows an approximately straight line of negative slope. Further evidence of the subcritical character can be found in figure 15b, which shows that, before saturation, the actual growth of the mode energy is clearly higher than the prediction given by the linear calculations. This configuration did not present hysteresis close to the critical point, despite the fact that the bifurcation was subcritical.

4.4.2. $L_y = 2.0$ and $L_y = 2.5$

The mode C bifurcation in these two configurations presented a supercritical character, as can be seen in the graphs plotted in figures 16a and 16c, which have negative slopes for their entire range. The curves in these figures are linear until fairly close to saturation, therefore, the non-linear evolution of the mode within this linear range can be described by a Landau equation with terms up to third order. The coefficients of the Landau equation were calculated for both cases, and in figures 16b and 16d it can be seen that the results of the simulations agree very well with the Landau predictions until close to saturation. Since the difference between the two curves is very small, we believe that the introduction of the 5th order term in the equation should be enough to provide a proper description of the instability evolution. The 3rd order Landau equations predict a higher value of the saturated amplitude than the results given by the simulations, both for $L_y = 2.0$ and $L_y = 2.5$, therefore we conclude that the coefficient of the 5th order term should be negative.

The insets on figures 16b and 16d show that the energy approaches zero gradually when the Re is decreased towards the critical value. Diagrams with this shape are typical of supercritical bifurcations.

4.4.3. $L_y = 3.0$

For this particular configuration, the mode C bifurcation showed a subcritical character, as could be inferred from the curve in figure 17a, which exhibits a positive slope close to the y -axis and whose non-rectilinear shape indicates that terms of order higher than three must be considered in the Landau equation to adequately describe the non-linear evolution of the instability. Another subcritical attribute of the bifurcation can also be discerned analysing figure 17b, where the energy relative to mode C wavelength, obtained from the three-dimensional simulations, grows at a higher rate than that predicted by

the linear theory. The inset in this same figure shows the hysteresis present in the vicinity of the critical point.

In order to check if the period-doubling mode C leads to a period-doubling cascade in the route to turbulence, some additional three-dimensional simulations were performed using a periodic spanwise wavelength of $12.0D$. Runs with $Re = 350$ were carried out for all configurations. Extra simulations were also performed for $(L_y = 1.0, Re = 200)$, $(L_y = 2.0, Re = 200)$ and $(L_y = 3.0, Re = 250)$, since for these intermediate cases only modes A and C were unstable and we were interested to know how these modes interacted in the absence of mode B.

Velocity time histories at diverse points in the wake were monitored, and for all configurations, no sign of a period-doubling cascade was observed. The spectra of the velocity in x and y directions was broad banded, with a dominant peak at the vortex shedding frequency. In the cases with $Re = 200$ and $Re = 250$, a secondary peak at half of the shedding frequency was observed in the spectrum of the x and y velocity at points located in the near wake of the downstream cylinder, but this peak was not observed further downstream and it was absent in all points monitored in the cases with $Re = 350$. Therefore, we concluded that modes A and B were dominant when the full non-linear interactions are considered, and this was verified by looking to the evolution of the vorticity iso-surfaces in two shedding cycles, as no mode C structures could be observed. By and large, it seems that the transition to turbulence occurs through spatio-temporal chaos, as in the isolated cylinder case (Henderson 1997). The same behaviour was observed in the flow around bluff rings (Sheard *et al.* 2005*b*).

(a) $L_y = 2.0$, $Re = 185$, $L_z = 1.65$ (b) $L_y = 2.0$, $Re = 185$, $L_z = 1.65$ (c) $L_y = 2.5$, $Re = 205$, $L_z = 1.45$ (d) $L_y = 2.5$, $Re = 205$, $L_z = 1.45$

Figure 16: Results for three-dimensional simulation of cases $L_y = 2.0$, $Re = 185$, $L_z = 1.65$ ((a) and (b)) and $L_y = 2.5$, $Re = 205$, $L_z = 1.45$ ((c) and (d)). (a) and (c) show the derivative of the amplitude logarithm against the square of the amplitude. (b) and (d) show the growth and saturation of the amplitude in the wavelength corresponding to the mode C instability (symbols), growth predicted by linear calculations (continuous line) and growth predicted by the Landau equation (dashed line). The x axis is non-dimensional time. The insets of these figures show diagrams of the supercritical bifurcations.

Figure 17: Results for three-dimensional simulation of case $L_y = 3.0$, $Re = 215$, $L_z = 1.45$. (a) shows the derivative of the amplitude logarithm against the square of the amplitude. (b) shows the growth and saturation of the amplitude in the wavelength corresponding to the mode C instability (symbols) and growth predicted by linear calculations (continuous line). The x axis is non-dimensional time. The inset of (b) show a diagram of the subcritical bifurcation.

5. Conclusion

Results have been presented from direct numerical simulations and Floquet stability analysis of the flow around two circular cylinders in staggered arrangements. Using plots of the Floquet multiplier versus spanwise wave number and neutral stability curves, we have identified the different modes that take part in the wake transition in such flows. With the help of vorticity contours, we have characterised the symmetries and periodicity of these modes. In addition, the non-linear behaviour of mode C has been investigated utilising three-dimensional simulations.

The basic modes in the wake transition in the flow around a single circular cylinder, modes A and B, were also found in the staggered arrangements considered in this inves-

tigation. This enforces the thesis that these modes are fundamental in the transition to turbulence in vortex street wakes, even in the case of multiple cylinders.

Another important finding of this study was the presence of mode C in the wake transition for some asymmetrical arrangements. The linear stability results for this mode agree well with previous investigations (Sheard *et al.* 2003; Zhang *et al.* 1995) in many aspects, such as type of perturbation symmetry, wavelength and periodicity. Interestingly, while modes A and B first appear in the near wake of the upstream cylinder, mode C initiates in the near wake of the downstream cylinder.

Also the non-linear analysis produced some interesting results. Mode C bifurcation was subcritical for $L_y = 1.0$, supercritical for $L_y = 2.0$ and $L_y = 2.5$ and again subcritical for $L_y = 3.0$. However, since the vortex shedding regime for $L_y = 1.0$ is different from that for $L_y \geq 2.0$, the configuration $L_y = 1.0$ must be considered separately if we want to isolate the three-dimensional effects.

For configurations $L_y = 2.0$ and $L_y = 2.5$ mode C presented a supercritical character, but it bifurcated before mode A for $L_y = 2.0$ and after mode A for $L_y = 2.5$. For $L_y = 3.0$, it bifurcated after mode A, but through a subcritical route. Bearing this in mind, we deduce that the character of the bifurcation of mode C is not directly related to the order the modes bifurcate.

For $L_y \geq 1.0$, all three modes appear in the Re range investigated, and the fact that mode C appears for $L_y = 1.0$ and for $L_y \geq 2.0$ indicates that the vortex shedding regime and the presence of mode C are not directly related, although both depend on the geometrical configuration of the bodies. In addition, the fact that mode C appeared for two substantially different wakes ($L_y = 1.0$ and $L_y \geq 2.0$) corroborates the hypothesis that mode C is a fundamental mode in the transition of time-periodic wakes with a certain degree of Z_2 asymmetry (Sheard *et al.* 2003, 2004). This mode seems to replace

the quasi-periodic mode that emerges in Z_2 symmetric wakes (Blackburn & Lopez 2003; Marques *et al.* 2004), spanning the same wave number range.

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