Statistical Inference of the Equivalent Initial Flaw Size for Assembled Plate Structures with the Dual Boundary Element Method

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Abstract

The statistical inference of the Equivalent Initial Flaw Size Distribution (EIFSD) is developed for the first time using the Dual Boundary Element Method (DBEM) for assembled shear deformable plate structures. As part of this inference procedure, Bayesian updating is employed to enable the continuous refinement of the EIFSD via data obtained by many simulated routine inspections of a stiffened panel from a fleet of aircraft. Fatigue crack growth is modelled using an incremental crack growth procedure that only requires modelling of the boundary of the 2.5D structure with line elements and requires no remeshing during crack growth simulations. Stochastic Kriging is employed to account for the stochastic nature of fatigue crack growth and to offset the high computational cost associated with modelling complex built-up structures. To demonstrate the efficiency of the proposed inference methodology, a numerical example featuring a stiffened panel subjected to complex loading in the form of combined tension and bending is presented. Once the EIFSD has been inferred, it can be used to optimise the intervals between routine aircraft inspections via the use of reliability analysis techniques as part of a combined reliability-EIFS approach. It is demonstrated that the proposed methodology offers the capability to reduce the costs associated with inspections.

Keywords: Equivalent initial flaw size distribution; Fatigue crack growth; Plate bending; Assembled structure; Dual boundary element method

1. Introduction

Current damage tolerance design specifications for aircraft state that the engineer should assume that initial flaws in the aircraft's structures are of a size that is the maximum that could remain undetected after the use of a Non-Destructive Inspection (NDI) technique [1]. This can lead to conservative estimates for the fatigue life of the structure, resulting in the structure being over-engineered. It can also result in inspections being carried out more frequently than required. This is particularly important for aircraft since inspections can be expensive, requiring 1000s of man-hours to complete. Ideally, the frequency of inspections should be minimised while maintaining a very high level of safety. This can be achieved by obtaining more accurate estimates for fatigue life.

In terms of accuracy, an ideal means of estimating fatigue life would be to model the growth of very small flaws from their actual initial sizes to a maximum permissible crack size (e.g. the critical crack size). However, these initial flaws typically have sizes of the order of several micrometers ($\sim 10^{-6}$ m), making them difficult and

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Nomenclature

Acronym	Description	Acronym	Description
<u>General</u>		<u>Reliability</u>	
BU	Bayesian Updating	LSF	Limit State Function
COV	Coefficient of Variation	MCS	Monte Carlo Simulation
DBEM	Dual Boundary Element Method	FORM	First-Order Reliability Method
EIFS	Equivalent Initial Flaw Size	SORM	Second-Order Reliability Method
EIFSD	EIFS Distribution	<u>Error</u>	
EPS	Equivalent Pre-Crack Size	MAE	Mean Absolute Error
MLE	Maximum Likelihood Estimation	MAPE	Mean Absolute Percentage Error
MSD	Multiple-Site Damage	RMSE	Root Mean Square Error
NDI	Non-Destructive Inspection	RRSE	Root Relative Squared Error
<u>Symbol</u>	<u>Description</u>	<u>Symbol</u>	<u>Description</u>
<u>EIFSD</u>		$\hat{\theta}$	Possible EIFSD
N	Number of cycles	$\hat{\mu}$	Possible EIFSD location parameter
K_{eff}	Effective stress intensity factor	$\hat{\sigma}$	Possible EIFSD shape parameter
C_0	Paris law coefficient for $R = 0$	$\hat{M}$	Possible EIFSD location parameter set
C	Paris law coefficient	$\hat{\Sigma}$	Possible EIFSD shape parameter set
m	Paris law exponent	P^{int}	Number of integration points
t	Applied traction	i	Index for $\hat{M}$ ($i = 1, 2, \dots, P^{int}$)
t_{max}	Maximum applied traction	j	Index for $\hat{\Sigma}$ ($j = 1, 2, \dots, P^{int}$)
p	Applied bending moments	l	Index for inspections ($l = 1, 2, \dots, 500$)
p_{max}	Maximum applied bending moments	<u>Material</u>	
R	Applied load ratio	E	Young's modulus
a_0	Initial crack size	ν	Poisson's ratio
a	Final crack size	γ	Coefficient for the Walker equation
θ	"Actual" EIFSD	<u>Reliability</u>	
μ	"Actual" EIFSD location parameter	P_F	Probability of failure
σ	"Actual" EIFSD shape parameter	P_R	Reliability

expensive to accurately measure since typical NDI techniques can reliably detect flaws of several millimetres at best. Further difficulties arise due to the complexities associated with determining which of these many initial flaws will grow to form a crack. The growth of such small cracks would need to be modelled using a model suitable for short-crack growth. However, due to the significant influence of the material's microstructure on short-crack growth, this is difficult.

An approach that avoids the above difficulties is the Equivalent Initial Flaw Size (EIFS) concept. An EIFS is not an actual flaw size but can be thought of as a crack growth model calibration parameter. By analysing crack sizes of a particular type (e.g. crack starting from fastener holes) throughout the life of a structure or other structures under similar loading conditions, the EIFS can be estimated using a crack growth model suitable for long-crack growth such as the Paris law. Once obtained, the EIFS can be used to estimate the fatigue life of the structure by using the same long-crack growth model to estimate its growth to some maximum permissible crack size. The EIFS found for this structure can also be used as the EIFS for other similar structures under similar loading conditions [1].

Some of the most common methods used to determine EIFS include the back-extrapolation method [2–11], Equivalent Pre-Crack Size (EPS) [8, 12], and the Kitagawa-Takahashi diagram [6, 13–15]. A common feature of these methods is that the EIFS is treated in a deterministic manner. However, since fatigue crack growth is essentially a stochastic process, a probabilistic description of the EIFS should be obtained that takes into account different sources of uncertainty. These sources of uncertainty could include uncertainty in the Paris law constants C and m , as well as uncertainty in the applied stress σ .

A stochastic approach has been taken by a number of authors for the inference of the EIFSD [16–22]. The stochastic approach treats the EIFS as a model calibration parameter and statistical methods such as Maximum Likelihood Estimation (MLE) [17, 20, 21] or Bayesian updating [16, 18–20, 22] are used to infer its distribution based on crack size observations at various times. The methods presented in [20–22] involve uncertainty from multiple sources being represented by a single noise term. A more robust approach would involve allowing individual sources of uncertainty to be individually quantified [16, 17, 19]. This is the approach taken in this current work with Bayesian updating.

While EIFS can be used to obtain more accurate estimates for fatigue life, it cannot be used on its own to determine inspection intervals. Reliability analysis, on the other hand, can be used to determine inspection intervals by estimating the probability of a crack exceeding some maximum allowable size (the probability of failure). However, obtaining a suitable amount of data to conduct these reliability analyses from routine inspections of an aircraft is very challenging. To overcome this problem, a combined reliability-EIFS approach can be employed to determine optimal inspection intervals for aircraft structures. A significant benefit of the EIFS approach is that it can be inferred using data obtained from a fleet of aircraft, not just a single aircraft. This benefit, coupled with the continual refinement of the EIFS with Bayesian updating using data from new inspections, enables the EIFS to be inferred in a very efficient manner, and allows aircraft with less flight-hours to benefit from the data obtained from aircraft with more flight-hours. In addition, an advantage of Bayesian updating is that it allows the engineer to include subjective engineering judgement in the updating procedure in the form of a prior distribution, potentially allowing the EIFSD to be accurately inferred with fewer inspections. Therefore, this combined reliability-EIFS approach to optimising inspection intervals has the capability to reduce costs without compromising on safety, while also requiring less data from inspections.

The Boundary Element Method (BEM) is a numerical computational method for structural analysis that has found increasing use in the engineering industry over the past few decades. One of the most attractive uses of the BEM is for crack growth simulation. The Dual Boundary Element Method (DBEM) was developed specifically to improve the modelling of cracks. The only remeshing requirement in the DBEM for crack growth simulation is that new elements are added to the previous crack tip. Therefore, no remeshing of the original model is required. The DBEM is also able to provide a continuous modelling of the internal domain of the structure, providing high resolution stresses and displacements. This is especially useful for the calculation of stress intensity factors using methods such as the J-integral. Due to the benefits of the DBEM and Bayesian updating, they have often been used together. Cordeiro et al. [23], employed Bayesian updating to quantify estimates, based on experimental data, for the parameters that govern the cohesive laws during the nonlinear fracture of concrete. Results were successfully validated against DBEM models. Morse et al. [16] employed Bayesian updating and the DBEM for the statistical inference of the EIFSD for a lap-joint structure. It was found that the EIFSD could be accurately determined with data from only 50 inspections. A comprehensive overview and history of the application of the BEM and the DBEM to structural analysis can be found in a

book by Aliabadi [24]

Past examples of EIFS calculation have predominately involved applications to relatively simple structures such as fasteners holes in lap joints [2, 10, 16, 25], fatigue test specimens [6, 8, 12, 14, 15], and in 3D cylindrical structures [19, 20]. EIFS methods have not yet been developed for built-up structures subjected to complex loading conditions, such as stiffened panels. This is particularly important for aircraft since vital aircraft structures such as the wings and the fuselage are subjected to complex loading conditions in the form of combined tension and bending loads. Built-up structures offer several challenges - (1) there is the possibility of having multiple crack sites (also known as Multi-Site Damage or MSD), (2) the modelling of such structures can be computationally expensive, and (3) the cracks can grow in a complex manner due to the interaction of the complex geometry and loading conditions. The aim of this work is to develop EIFS techniques to accurately predict the fatigue life of built-up structures.

In this work, a novel reliability-EIFS approach is developed for the first time for the efficient optimisation of inspection intervals of a built-up aircraft structure. As part of this approach, a new methodology for the statistical inference of the EIFS Distribution (EIFSD) for a built-up structure with Multi-Site Damage (MSD) is developed. Bayesian updating (or Bayesian inference) is used as part of this inference procedure to allow for the continual refinement of the EIFSD using new information, such as data obtained from routine inspections of a structural component from a fleet of aircraft. To reduce the computational cost associated with modelling built-up structures with complex loading conditions, and to reflect the stochastic nature of fatigue crack growth, stochastic (or regression) Kriging is used as part of the Bayesian inference procedure. Outputs from a DBEM model of a stiffened panel are used to train the stochastic Kriging models. The DBEM enables the automatic calculation of crack growth direction, and requires no remeshing during the crack growth procedure. The use of the DBEM therefore allows the growth of multiple cracks in complex structures to be modelled in a very efficient manner. New data is provided to the Bayesian updating procedure in the form of simulated routine inspections on a built-up structure from a fleet of the same aircraft. Once the EIFSD has been inferred it can be used to optimise the intervals between inspections via the use of reliability analysis techniques. Offering the capability to reduce the costs associated with inspections without compromising on safety, while also requiring less data from inspections. An example of this is shown at the end of this paper.

2. Methodology

2.1. Statistical Inference of the EIFSD for Built-Up Structures

In this work, Bayesian updating is used to infer the EIFSD of an assembled plate structure. Bayesian updating uses new information, such as new data obtained from inspections, to update prior assumptions regarding the probability of an event. Baye's rule is:

$$P(H|D) = \frac{P(D|H)P(H)}{P(D)} \quad (1)$$

where $P(H|D)$ is the *posterior*, the probability that our hypothesis H is true given some new data D . $P(D|H)$ is the *likelihood* and is the probability of observing the new data D given that our hypothesis H is true. $P(H)$ is the *prior* and is the probability of our hypothesis H before obtaining the new data D . Finally, $P(D)$ is the *marginal likelihood* and is the total probability of observing the new data D . It can also be thought of as a normalising factor that ensures that the integral of $P(H|D)$ over the domain under investigation is equal to 1.

Lognormal or Weibull distributions are often used to model distributions for variables such as crack length and fatigue life [26]. Therefore, the EIFSD in this work is assumed to have a lognormal distribution. Let θ be a random variable that represents the EIFS. Therefore, $\theta \sim \ln\mathcal{N}(\mu, \sigma)$, where μ is the location parameter and σ is the shape parameter. Bayesian updating will be used to infer the statistical parameters μ and σ of the EIFSD using new data obtained from inspections in the presence of multiple sources of uncertainty. In total, 500 inspections were modelled using the DBEM, each inspection is assumed to identify one crack. It is assumed that all of the cracks are detected as soon as they reach length of 20 mm. This length represents a typical crack length capable of being detected by Non-Destructive Inspection (NDI). For each inspection, the number of load cycles N_i^{ins} ($i = 1, 2, \dots, 500$) that the structure has been subjected to is also recorded.

Let $\hat{\mathbf{M}}$ and $\hat{\Sigma}$ be random variables that represent a range of possible values for the real location and shape parameters of the EIFSD. $\hat{\mathbf{M}}$ can take a range of evenly-spaced values $\hat{\mu}_1, \hat{\mu}_2, \dots, \hat{\mu}_{P^{int}}$, while $\hat{\Sigma}$ can take a range of evenly-spaced values $\hat{\sigma}_1, \hat{\sigma}_2, \dots, \hat{\sigma}_{P^{int}}$, where P^{int} is the number of integration points. In this work $P^{int} = 60$. The likelihood that a pair of possible values $(\hat{\mu}_i, \hat{\sigma}_j)$ ($i, j = 1, 2, \dots, P^{int}$) are the real location and shape parameters μ and σ of the EIFSD given the data from some inspection k is:

$$L(\hat{\mu}_i, \hat{\sigma}_j | N_k^{ins}, \mathbf{Y}) = f_{N|\hat{\mu}_i, \hat{\sigma}_j, \mathbf{Y}}(N_k^{ins} | \hat{\mu}_i, \hat{\sigma}_j, \mathbf{Y}) = \frac{1}{N_k^{ins} \sqrt{2\pi\beta_{ij}^2}} \exp\left(-\frac{[\log(N_k^{ins}) - \alpha_{ij}]^2}{2\beta_{ij}^2}\right) \quad (2)$$

where $\mathbf{Y}$ is a vector that contains the uncertainty of parameters that influence crack growth ($C, m, t_{max}, p_{max}, R$). $f_{N|\hat{\mu}_i, \hat{\sigma}_j, \mathbf{Y}}$ is a distribution for the number of cycles required to grow a crack from an EIFSD with statistical parameters $\hat{\mu}_i$ and $\hat{\sigma}_j$ to 20mm given $\mathbf{Y}$. It is assumed to follow a lognormal distribution with location and shape parameters α_{ij} and β_{ij} respectively. The two parameters α_{ij} and β_{ij} were estimated with Monte Carlo sampling using the EIFSD parameters $\hat{\mu}$ and $\hat{\sigma}$ and the uncertainty $\mathbf{Y}$. The combination of $\hat{\mu}_i$ and $\hat{\sigma}_j$ describes a possible distribution for the EIFS. This possible distribution is defined as $\hat{\theta} \sim \ln\mathcal{N}(\hat{\mu}_i, \hat{\sigma}_j)$ and can be seen in Figure 1. The procedure outlined above can also be seen in Figure 1.

Figure 1: Procedure for obtaining the likelihood $L(\hat{\mu}_i, \hat{\sigma}_j | N_k^{ins}, \mathbf{Y})$.

With only one value for N^{ins} as seen in equation (2), it is possible to estimate the likelihood of $\hat{\mu}_i$. However, more values for N^{ins} are required to obtain an accurate estimate for the likelihood of $\hat{\sigma}_j$. Therefore, the data from more than one inspection is necessary. Given the data from l inspections, the likelihood that a pair of guesses $\hat{\mu}_i$ and $\hat{\sigma}_j$ are the real location and shape parameters μ and σ of the EIFSD is:

$$L(\hat{\mu}_i, \hat{\sigma}_j | \mathbf{N}_l^{ins}, \mathbf{Y}) = \prod_{k=1}^l L(\hat{\mu}_i, \hat{\sigma}_j | N_k^{ins}, \mathbf{Y}) \quad (3)$$

where $\mathbf{N}_l^{ins}$ is a vector of size l that contains the values N_k^{ins} ($k = 1, \dots, l$). The individual likelihood functions for $\hat{\mu}_i$ and $\hat{\sigma}_j$ are:

$$L(\hat{\mu}_i | \mathbf{N}_l^{ins}, \mathbf{Y}) = \int_{\hat{\sigma}_1}^{\hat{\sigma}^{Pint}} L(\hat{\mu}_i, \hat{\sigma}_j | \mathbf{N}_l^{ins}, \mathbf{Y}) f_{\hat{\Sigma} | \mathbf{N}_{l-1}^{ins}, \mathbf{Y}}(\hat{\sigma} | \mathbf{N}_{l-1}^{ins}, \mathbf{Y}) d\hat{\sigma} \quad (4)$$

$$L(\hat{\sigma}_j | \mathbf{N}_l^{ins}, \mathbf{Y}) = \int_{\hat{\mu}_1}^{\hat{\mu}^{Pint}} L(\hat{\mu}, \hat{\sigma}_j | \mathbf{N}_l^{ins}, \mathbf{Y}) f_{\hat{M} | \mathbf{N}_{l-1}^{ins}, \mathbf{Y}}(\hat{\mu} | \mathbf{N}_{l-1}^{ins}, \mathbf{Y}) d\hat{\mu} \quad (5)$$

where $f_{\hat{M} | \mathbf{N}_{l-1}^{ins}, \mathbf{Y}}$ and $f_{\hat{\Sigma} | \mathbf{N}_{l-1}^{ins}, \mathbf{Y}}$ are the prior estimates for the probability distributions of $\hat{M}$ and $\hat{\Sigma}$. There are no prior distributions available for the very first iteration of Bayesian updating, and so they will need to be chosen based on previous experience. However, if this experience is not available then uniform distributions can be assigned as the initial prior distributions. Posterior estimates for the probability distributions of $\hat{M}$ and $\hat{\Sigma}$ are:

$$f_{\hat{M} | \mathbf{N}_l^{ins}, \mathbf{Y}}(\hat{\mu}_i | \mathbf{N}_l^{ins}, \mathbf{Y}) = \frac{L(\hat{\mu}_i | \mathbf{N}_l^{ins}, \mathbf{Y}) f_{\hat{M} | \mathbf{N}_{l-1}^{ins}, \mathbf{Y}}(\hat{\mu}_i | \mathbf{N}_{l-1}^{ins}, \mathbf{Y})}{\int_{\hat{\mu}_1}^{\hat{\mu}^{Pint}} L(\hat{\mu} | \mathbf{N}_l^{ins}, \mathbf{Y}) f_{\hat{M} | \mathbf{N}_{l-1}^{ins}, \mathbf{Y}}(\hat{\mu} | \mathbf{N}_{l-1}^{ins}, \mathbf{Y}) d\hat{\mu}} \quad (6)$$

$$f_{\hat{\Sigma} | \mathbf{N}_l^{ins}, \mathbf{Y}}(\hat{\sigma}_j | \mathbf{N}_l^{ins}, \mathbf{Y}) = \frac{L(\hat{\sigma}_j | \mathbf{N}_l^{ins}, \mathbf{Y}) f_{\hat{\Sigma} | \mathbf{N}_{l-1}^{ins}, \mathbf{Y}}(\hat{\sigma}_j | \mathbf{N}_{l-1}^{ins}, \mathbf{Y})}{\int_{\hat{\sigma}_1}^{\hat{\sigma}^{Pint}} L(\hat{\sigma} | \mathbf{N}_l^{ins}, \mathbf{Y}) f_{\hat{\Sigma} | \mathbf{N}_{l-1}^{ins}, \mathbf{Y}}(\hat{\sigma} | \mathbf{N}_{l-1}^{ins}, \mathbf{Y}) d\hat{\sigma}} \quad (7)$$

where $f_{\hat{M} | \mathbf{N}_l^{ins}, \mathbf{Y}}(\hat{\mu}_i | \mathbf{N}_l^{ins}, \mathbf{Y})$ is the posterior estimate for the distribution of $\hat{M}$ given the data $\mathbf{N}_l^{ins}$ and the uncertainty $\mathbf{Y}$ evaluated at $\hat{\mu}_i$. Similar is true for $f_{\hat{\Sigma} | \mathbf{N}_l^{ins}, \mathbf{Y}}(\hat{\sigma}_j | \mathbf{N}_l^{ins}, \mathbf{Y})$. The integrations seen in equations (4)-(7) were performed using the trapezoidal rule with P^{int} evenly spaced integration points of both $\hat{\mu}$ and $\hat{\sigma}$. This provided a total of $(P^{int})^2$ pairs of $\hat{\mu}$ and $\hat{\sigma}$. Once the data from a new inspection is obtained, the process repeats through equations (2)-(7) for every combination of $\hat{\mu}_i$ and $\hat{\sigma}_j$ and the posterior of the previous iteration becomes the prior of the new iteration and the posterior of the new iteration is calculated. The repetition of this process continues until the total number of inspections is reached (which is 200 in this work).

The posterior distributions obtained from equations (6) and (7) can be used to obtain an estimate for the EIFSD. It is assumed in this work that the means of the posterior distributions $f_{\hat{M} | \mathbf{N}_l^{ins}, \mathbf{Y}}$ and $f_{\hat{\Sigma} | \mathbf{N}_l^{ins}, \mathbf{Y}}$ are estimates for the real location and shape parameters of the EIFSD:

$$\mu_l^{BU} = E(\hat{M}) = \int_{\hat{\mu}_1}^{\hat{\mu}^{Pint}} \hat{\mu} f_{\hat{M} | \mathbf{N}_l^{ins}, \mathbf{Y}}(\hat{\mu} | \mathbf{N}_l^{ins}, \mathbf{Y}) d\hat{\mu} \quad (8)$$

$$\sigma_l^{BU} = E(\hat{\Sigma}) = \int_{\hat{\sigma}_1}^{\hat{\sigma}^{Pint}} \hat{\sigma} f_{\hat{\Sigma} | \mathbf{N}_l^{ins}, \mathbf{Y}}(\hat{\sigma} | \mathbf{N}_l^{ins}, \mathbf{Y}) d\hat{\sigma} \quad (9)$$

where μ_l^{BU} and σ_l^{BU} are estimates of μ and σ from Bayesian updating after obtaining data from l inspections. Ideally, the estimated parameters, μ_l^{BU} and σ_l^{BU} , should more closely match those of the actual EIFSD, μ and σ , as l increases.

2.2. Reliability Analysis

In the field of reliability analysis, the boundary between succeeding or failing to meet a certain set of criteria can be represented mathematically by a limit state function (LSF) $g(\mathbf{Z})$. For example, if we are looking at the probability of a structure failing due to load, the LSF will be:

$$g(\mathbf{Z}) = R - S(\mathbf{X}) \quad (10)$$

where R is the resistance of the structure to some load S . $\mathbf{Z}$ is a vector of length q containing values for each of the q design variables that influence g , while $\mathbf{X}$ ($\mathbf{X} \subseteq \mathbf{Z}$) is a vector of length $q - 1$ that contains values for

each of the $q - 1$ design variables that influence S . The vector $\mathbf{Z}$ is the same as the vector $\mathbf{X}$ except it includes the design variable R . The probability that the set of criteria has failed to be met is termed the *probability of failure* P_F , while the probability that the set of criteria has been successfully met is termed *reliability* P_R . In the example outlined above, these probabilities would correspond to the probabilities of the structure failing or being safe under the load S respectively. Reliability can be determined by evaluating the following integral:

$$P_R = 1 - P_F = P\{g(\mathbf{Z}) > 0\} = \int_{g(\mathbf{Z}) > 0} f_{\mathbf{Z}}(\mathbf{Z}) d\mathbf{Z} \quad (11)$$

where $f_{\mathbf{Z}}(\mathbf{Z})$ is the joint PDF of $\mathbf{Z}$. P_R and P_F are obtained by integrating over the failure region ($g(\mathbf{Z}) < 0$) and the safe region ($g(\mathbf{Z}) > 0$) respectively. All of the design variables are assumed to be mutually independent. The integration seen in equation (11) can be difficult if there are many variables in $\mathbf{Z}$ or if the boundary $g(\mathbf{Z}) = 0$ is non-linear.

There are several methods that can be used to evaluate the integral in equation (11). The most widely known are Monte Carlo simulation (MCS), the First-Order Reliability Method (FORM), and the Second-Order Reliability Method (SORM) [27]. MCS is the simplest of the three methods and involves randomly sampling from the distributions of the variables in $\mathbf{Z}$ and using these samples to evaluate $g(\mathbf{Z})$. The proportion of samples that result in $g(\mathbf{Z}) > 0$ provides an estimate for the probability of failure P_F . FORM and SORM attempt to simplify the integration by approximating the boundary $g(\mathbf{Z}) = 0$ as first-order and second-order Taylor series expansions respectively. As a result, FORM and SORM are significantly faster than MCS, but less accurate.

Reliability analysis, in combination with the EIFSD inference procedure described in section 2.1, can be used to determine optimal inspection intervals. An example of this is presented at the end of this paper.

2.3. The Dual Boundary Element Method for Built-Up Structures

In this work, fatigue crack growth is modelled using an automatic incremental crack growth procedure with the Dual Boundary Element Method (DBEM) for assembled plate structures. A good source of information on the DBEM with assembled plate structures is [24]. The crack growth procedure is capable of automatically growing cracks in an incremental manner from a specified initial crack size to a specified final crack size. The individual crack growth increments are modelled as piecewise-flat discontinuous elements. The angle of growth of each increment is determined automatically using the maximum principal stress criterion. The mode I and II membrane stress intensity factors and the mode I, II, and III bending stress intensity factors needed for the maximum principal stress criterion are calculated at the new crack tip with the J-integral method after each growth increment. This procedure requires no re-meshing of the original model and so offers an efficient means to model fatigue crack growth.

The Dual Boundary Element Method (DBEM) for plate structures subjected to both bending and tension was first developed by Dirgantara and Aliabadi [28]. In this approach, Reissner plate theory is used to model bending and shear, while 2D plane stress theory is used to model the behaviour of the membrane of the plate. The DBEM equations as presented in [16] were used to model the behaviour of the membrane in this work. The DBEM equations for Reissner plate theory can be written in compact form as:

$$\begin{aligned} \mathbf{C}\mathbf{w} + \sum_{m=1}^{N_e} \left(\int_{\Gamma_m} \mathbf{T}^b \boldsymbol{\psi}_e^T d\Gamma_m \right) \mathbf{w}^m + \sum_{m=1}^{N_c} \left(\int_{\Gamma_m} \mathbf{T}^b \boldsymbol{\psi}_c^T d\Gamma_m \right) \Delta \mathbf{w}^m \\ = \sum_{m=1}^{N_e} \left(\int_{\Gamma_m} \mathbf{U}^b \boldsymbol{\psi}_e^T d\Gamma_m \right) \mathbf{p}^m + q_3 \sum_{m=1}^{N_e} \left(\int_{\Gamma_m} \mathbf{B}^b d\Gamma_m \right) \end{aligned} \quad (12)$$

for the displacement boundary integral equation, and:

$$\begin{aligned} & \frac{1}{2} \mathbf{p} + \sum_{m=1}^{N_e} \mathbf{n} \left(\int_{\Gamma_m} \mathbf{S}^b \psi_e^T d\Gamma_m \right) \mathbf{w}^m + \sum_{m=1}^{N_c} \mathbf{n} \left(\int_{\Gamma_m} \mathbf{S}^b \psi_c^T d\Gamma_m \right) \Delta \mathbf{w}^m \\ &= \sum_{m=1}^{N_e} \mathbf{n} \left(\int_{\Gamma_m} \mathbf{D}^b \psi_e^T d\Gamma_m \right) \mathbf{p}^m + q_3 \sum_{m=1}^{N_e} \mathbf{n} \left(\int_{\Gamma_m} \mathbf{B}^* d\Gamma_m \right) + \mathbf{n} \frac{q_3 \nu}{(1-\nu)\lambda^2} \end{aligned} \quad (13)$$

for the traction boundary integral equation.

In the above equations, N_e and N_c denote the number of elements on the external boundary and on the crack surfaces respectively. $\mathbf{w}^m$ is a (3×1) vector that contains rotations w_1 and w_2 , and vertical displacement w_3 for element m . $\Delta \mathbf{w}^m$ denotes the displacement discontinuity across the crack surfaces for an element m that lies on a crack surface. $\mathbf{p}^m$ is also a (3×1) vector that contains bending tractions p_1 and p_2 , and shear traction p_3 for element m . $\mathbf{T}^b$, $\mathbf{U}^b$, $\mathbf{B}^b$, $\mathbf{S}^b$, $\mathbf{D}^b$, and $\mathbf{B}^*$ are fundamental solutions evaluated along Γ_m (the surface of element m). $\mathbf{n}$ is a (2×1) vector containing n_1 and n_2 which are the components of the outward normal to the crack surface. q_3 is a uniform load applied over the surface of the plate. ν and λ denote the Poisson's ratio and the shear factor respectively ($\lambda = \sqrt{10}/h$, where h is the thickness of the plate). ψ_e^T contains the continuous quadratic shape functions for the external boundary, while ψ_c^T contains the discontinuous quadratic shape functions for the crack surfaces. More detailed forms of the equations above can be seen in [28].

Using equations (12) and (13), as well as the membrane equations in [16], a system of equations $\mathbf{H}\mathbf{u} = \mathbf{G}\mathbf{t}$ can be formed. Where, the vector $\mathbf{u}$ contains displacements and rotations, and the vector $\mathbf{t}$ contains membrane, bending, and shear tractions. This system of equations can then be rewritten as $\mathbf{A}\mathbf{X} = \mathbf{F}$, where the vector $\mathbf{X}$ contains all of the unknown displacements, rotations, and tractions. The matrix $\mathbf{A}$ contains the coefficients from $\mathbf{H}$ and $\mathbf{G}$. The vector $\mathbf{F}$ contains the known displacements, rotations, and tractions multiplied by the corresponding coefficients from $\mathbf{H}$ and $\mathbf{G}$. Details of this procedure can be seen in [16].

In the case of a single plate, the procedure described above is relatively simple. However, for an assembled plate structure composed of several connected plates the procedure becomes more complex since the joint/junction lines between the plates needs to be considered. Figure 2 shows an example of two plates joined together. In this case the junction line lies along the x_2 -axis. Each plate has a local set of coordinates (x_1^i, x_2^i, x_3^i) ($i = 1, 2, \dots, N$) with N being the number of assembled plates along the junction line under consideration.

Figure 2: Assembled plate structure and local coordinates. The edges of each plate are numbered 1-4. A crack is present in plate 2.

The displacements/tractions in one plate will influence the displacements/tractions in the other plate. A series of equations can be derived for displacement/traction compatibility/equilibrium along a junction line

between two plates. The compatibility/equilibrium equations for displacements are:

$$u_1^1 \cos(\theta^1) - w_3^1 \sin(\theta^1) = u_1^2 \cos(\theta^2) - w_3^2 \sin(\theta^2) = \dots = u_1^N \cos(\theta^N) - w_3^N \sin(\theta^N) \quad (14)$$

$$u_1^1 \sin(\theta^1) + w_3^1 \cos(\theta^1) = u_1^2 \sin(\theta^2) + w_3^2 \cos(\theta^2) = \dots = u_1^N \sin(\theta^N) + w_3^N \cos(\theta^N) \quad (15)$$

$$u_2^1 = u_2^2 = \dots = u_2^N \quad (16)$$

$$w_1^1 = w_1^2 = \dots = w_1^N \quad (17)$$

$$w_2^1 = w_2^2 = \dots = w_2^N = 0 \quad (18)$$

The compatibility equations for tractions are:

$$\sum_{i=1}^N \left(t_1^i \cos(\theta^i) - p_3^i \sin(\theta^i) \right) = 0 \quad (19)$$

$$\sum_{i=1}^N \left(t_1^i \sin(\theta^i) + p_3^i \cos(\theta^i) \right) = 0 \quad (20)$$

$$\sum_{i=1}^N t_2^i = 0 \quad (21)$$

$$\sum_{i=1}^N p_\alpha^i = 0 \quad (22)$$

where θ^i is the angle between the x_1^i axis and the global x_1 axis. For the assembled plates seen in Figure 2 it is assumed that the coordinate system for plate 1 coincides with the global coordinate system. Therefore, x_1^1 is equal to x_1 in this Figure. The above compatibility/equilibrium equations also assume that there is no offset between the membranes of the two plates [24].

The **A** matrix for the assembled structure seen in Figure 2 will be composed of the individual **A** matrices of its constituent plates. An example of this can be seen in Figure 3 for an assembled structure consisting of two plates joined together. The rows of this matrix indicate equations corresponding to collocation points, while the columns of this matrix indicate equations corresponding to field points. Each plate is composed of the same number of elements. In this Figure it can be seen that there is an $(N_B \times N_B)$ submatrix representing the **A** matrix of plate 1, and another $(N_B \times N_B)$ submatrix representing the **A** matrix of plate 2. N_B is the number of equations corresponding to the boundary of each plate. Since each plate is composed of the same number of elements, they each have the same number of equations corresponding to their boundary. The individual **A** matrices for plates 1 and 2 can be split into four parts. Each part includes equations that correspond to field points on a particular edge of the plates. These parts are labelled 1-4 in Figure 3, and correspond to edges 1-4 for each of the plates as seen in Figure 2. Each of the individual **A** matrices also include a submatrix that contains equations related to each plate's contribution to the junction line. These submatrices are labelled as "J". N_J is the number of equations corresponding to the junction line between the two plates. Below the individual **A** matrices there are four small square submatrices labelled "E" that are composed of the displacement and traction compatibility/equilibrium equations (14)-(22). This is because edge 2 of plate 1 is physically connected to edge 4 of plate 2. More information on multi-region formulation for plates can be seen in [29].

Figure 3: The $\mathbf{A}$ matrix for the assembled structure composed of two plates seen in Figure 2.

To model the growth of a crack, more elements are added to the mesh and so more rows and columns will need to be added to the $\mathbf{A}$ matrix of the assembled structure. For a single region problem this is quite simple and is described in [16]. For a multi-region problem, as is the case in this work, it is more complicated. New rows and columns will need to be added to the $\mathbf{A}$ matrix of the assembled structure seen in Figure 3. Assume that there is a crack present in plate 2. The additional rows and columns resulting from the growth of this crack after the first and second growth increments are represented by the shaded regions labelled "C1a-c" and "C2a-c" respectively. N_{C1} and N_{C2} are the number of equations added as a result of the first and second growth increments respectively. This process can be repeated for following growth increments.

Once the $\mathbf{A}$ matrix for the assembled structure has been generated, it can be used in the system of equations $\mathbf{A}\mathbf{X} = \mathbf{F}$ to calculate the unknowns in vector $\mathbf{X}$. The general system of equations for an assembled structure consisting of n plates could be written as:

$$\begin{bmatrix}
\mathbf{A}^1 & \dots & 0 & 0 & 0 \\
& \ddots & & & \\
\vdots & \mathbf{A}^j & \vdots & \vdots & \vdots \\
& & \ddots & & \\
0 & \dots & \mathbf{A}^n & \mathbf{A}^{n,C1a} & \mathbf{A}^{n,C2a} \\
\mathbf{E}^{1j} & \dots & \mathbf{E}^{1j} & \dots & 0 \\
0 & \dots & \mathbf{E}^{jn} & \dots & \mathbf{E}^{jn} \\
\hline
0 & \dots & \mathbf{A}^{n,C1c} & \mathbf{A}^{n,C1b} & \\
\hline
0 & \dots & \mathbf{A}^{n,C2c} & & \mathbf{A}^{n,C2b}
\end{bmatrix} \times \begin{bmatrix} \mathbf{x}^1 \\ \vdots \\ \mathbf{x}^j \\ \vdots \\ \mathbf{x}^n \\ 0 \\ 0 \\ \mathbf{x}^{n,C1} \\ \mathbf{x}^{n,C2} \end{bmatrix} = \begin{bmatrix} \mathbf{F}^1 \\ \vdots \\ \mathbf{F}^j \\ \vdots \\ \mathbf{F}^n \\ 0 \\ 0 \\ \mathbf{F}^{n,C1} \\ \mathbf{F}^{n,C2} \end{bmatrix} \quad (23)$$

In equation (23) the j 'th plate is joined to the 1st and n 'th plates. $\mathbf{A}^j$ is the individual $\mathbf{A}$ matrix for the j 'th plate and $\mathbf{E}^{jn}$ is a submatrix consisting of the equilibrium/compatibility equations for the junction line between the j 'th and n 'th plates. A crack is present in the n 'th plate and has been grown by two increments. $\mathbf{A}^{n,C1a}$, $\mathbf{A}^{n,C1b}$, and $\mathbf{A}^{n,C1c}$ are submatrices containing new equations added as a result of the first crack growth increment. Similar is true for the second crack growth increment. $\mathbf{x}^{n,C1}$ and $\mathbf{x}^{n,C2}$ correspond to the unknown boundary conditions for new elements added as a result of the first and second crack increments respectively.

It was determined in this work that the condition number of the $\mathbf{A}$ matrix seen in equation (23) was around 7×10^8 . A good rule of thumb is that a digit of precision is lost due to conditioning for every power of 10 in the condition number. Double precision was used in this work which provided 15 significant digits of precision. Hence, the remaining level of precision was about 7 significant digits. For the purpose of this work this level of precision was more than adequate.

Since both $\mathbf{A}$ and $\mathbf{F}$ are known, LU decomposition can be used to evaluate the vector $\mathbf{X}$. However, in the case of an assembled plate structure composed of many plates, $\mathbf{A}$ is a unsymmetric sparse matrix and so the use of standard LU decomposition becomes impractical. In this work, the *multifrontal* method for unsymmetric sparse matrices [30] is used to conduct the LU decomposition. The multifrontal method works by carrying out partial factorisations of the many small dense submatrices (also called *fronts*) that make up the large sparse matrix. The multifrontal method is able to carry out the partial factorisations of many fronts at once using parallel computing, significantly reducing the computational cost of the LU decomposition. Also, during the crack growth procedure the original $\mathbf{A}$ matrix does not need to be LU decomposed again, only the new rows and columns added during the crack growth procedure need to be LU decomposed. This significantly improves the efficiency of the crack growth procedure with the DBEM.

2.4. Crack Growth Procedure

The number of cycles N required to grow a crack from an initial size a_i to a final size a_f is:

$$N = \int_{a_i}^{a_f} \frac{1}{da/dN} da \quad (24)$$

where da/dN is the crack growth rate. da/dN will be a function of the effective stress intensity factor range ($\Delta K_{eff} = K_{eff}^{max} - K_{eff}^{min}$). An equation for K_{eff} was proposed by Dirgantara and Aliabadi [31]. A slightly modified version which separates the mode-I components from the mode-II and mode-III components is used in this work:

$$K_{eff}^2 = (K_I^m)^2 + T_I \frac{\pi}{3} (K_I^b)^2 + T_{II} \left((K_{II}^m)^2 + \frac{\pi}{3} (K_{II}^b)^2 + \frac{\pi(1+\nu)}{5} (K_{III}^b)^2 \right) \quad (25)$$

where the superscripts m and b denote membrane and bending stress intensity factor components respectively. The membrane crack deformation modes K_I^m and K_{II}^m , and the bending crack deformation modes K_I^b , K_{II}^b , and K_{III}^b are defined in Figure 4.

Figure 4: Membrane and bending crack deformation modes [32].

The coefficients T_I and T_{II} ($0 \leq T_I \leq 1$, $0 \leq T_{II} \leq 1$) in equation (25) quantify the relative influence of the mode-I and mode-II stress intensity factor components on K_{eff} . T_I and T_{II} are dependent on the combination of loads being investigated. Therefore, an ideal way to determine their values would be through empirically-obtained data. However, in the absence of empirical data the following relations are used in this work:

$$T_I = \sqrt{\frac{\left| \frac{K_I^b}{h} \right|}{\left| \frac{K_I^b}{h} \right| + |K_I^m|}} \quad (26)$$

$$T_{II} = \sqrt{\frac{\left| \frac{K_{II}^b}{h} \right|}{\left| \frac{K_{II}^b}{h} \right| + |K_{II}^m|}} \quad (27)$$

where h is the thickness of the plate.

The Paris law constants C and m are dependent on the material used. C is also dependent on the stress ratio $R = \sigma_{min}/\sigma_{max}$. To account for this dependency, the Walker equation is used in this work:

$$\frac{da}{dN} = C(\Delta K_{eff})^m \quad (28)$$

where C is:

$$C = \frac{C_0}{(1 - R)^{m(1-\gamma)}} \quad (29)$$

where C_0 is the value of C for $R = 0$, and γ is a material constant that indicates how strongly the stress ratio R affects the crack growth rate C in the material.

As discussed previously, an automatic incremental crack growth procedure is used in this work with the DBEM to model fatigue crack growth in assembled structures. This procedure requires no re-meshing and so

is an efficient means of modelling crack growth. It can be used to evaluate the total number of cycles N seen in equation (24). In this procedure, cracks are automatically grown in an incremental manner from a specified initial crack size a_i to a specified final crack size a_f . K_{eff} at the crack tips is calculated via the J-integral. The individual crack growth increments are modelled as piecewise-flat discontinuous elements. The propagation angle of each increment is calculated using the maximum principal stress criterion. The number of cycles ΔN required to grow a crack by one increment Δa is:

$$\Delta N = \int_{a_0}^{a_1} \frac{1}{da/dN} da \quad (30)$$

where a_0 and a_1 are the lengths of the crack before and after the current growth increment respectively ($\Delta a = a_1 - a_0$). The value of Δa can be specified by the user. Smaller values of Δa ensure that the modelled crack follows the path of a real crack more closely, but this is more computationally expensive since more increments need to be used.

The integration in equation (30) required the calculation of da/dN at crack lengths of a_0 and a_1 . This required the effective stress intensity factor range ΔK_{eff} to be calculated at the crack tips corresponding to these crack lengths using the J-integral. Once ΔK_{eff} was calculated for these two crack lengths, the values of ΔK_{eff} for crack lengths between these two lengths can be calculated via interpolation. These ΔK_{eff} values can be converted into da/dN values using the Paris law (equation 28) with the Walker equation (equation 29). A graph of da/dN vs. a values is therefore obtained for crack lengths between a_0 and a_1 and the integration of this graph can be conducted using numerical integration. Simpson's rule with eight subdivisions was used in this work, as this number was found to provide accurate convergence for ΔN .

2.5. Stochastic Kriging

Stochastic or regression Kriging is useful for noisy deterministic problems when repeat-samples are not available. Due to the incremental nature of modelling crack growth there will be some noise inherent in the simulation. Therefore, stochastic Kriging is used in this work. The object-orientated Design and Analysis of Computer Experiments (ooDACE) Matlab toolbox [33] is used in this work for creating and running the stochastic Kriging models.

Kriging can be thought of as a Gaussian process $Y(\mathbf{x})$:

$$Y(\mathbf{x}) = f(\mathbf{x}) + Z(\mathbf{x}) \quad (31)$$

where $\mathbf{x}$ is a vector of length d that contains sample values for d variables, $f(\mathbf{x})$ is a regression function, Z is a Gaussian process with zero mean, variance σ^2 , and correlation matrix Ψ . The regression function $f(\mathbf{x})$ can be thought of as the mean of the Gaussian process Y . There are different versions of Kriging defined based on the form of $f(\mathbf{x})$. If $f(\mathbf{x}) = 0$ then this version of Kriging is termed *simple* Kriging. *Ordinary* Kriging is when $f(\mathbf{x}) = \alpha_0$, where α_0 is an unknown constant to be determined. *Universal* Kriging is a third version when the regression function is a polynomial:

$$f(\mathbf{x}) = \sum_{i=1}^p \alpha_i b_i(\mathbf{x}) \quad (32)$$

where α_i ($i = 1, \dots, p$) are the coefficients, and $b_i(\mathbf{x})$ are the basis functions of the polynomial. Choosing the correct form for the regression function can be difficult. Therefore, simple Kriging or ordinary Kriging is often used.

To create a Kriging model a set of design points is required. A $(n \times d)$ matrix $\mathbf{X}$ can be defined that contains these design points such that $\mathbf{X} = [\mathbf{x}^1, \dots, \mathbf{x}^n]^T$, where n is the number of design points. A $(n \times 1)$ vector containing the outputs corresponding to these n design points can be defined as $\mathbf{y} = [y^1, \dots, y^n]^T$. The regression function $f(\mathbf{x})$ evaluated at each of the sample points in $\mathbf{X}$ can be included in a $(n \times p)$ matrix $\mathbf{F}$:

$$\mathbf{F} = \begin{bmatrix} b_1(\mathbf{x}^1) & \dots & b_p(\mathbf{x}^1) \\ \vdots & \ddots & \vdots \\ b_1(\mathbf{x}^n) & \dots & b_p(\mathbf{x}^n) \end{bmatrix} \quad (33)$$

The $(n \times n)$ correlation matrix Ψ is defined as:

$$\Psi = \begin{bmatrix} \psi(\mathbf{x}^1, \mathbf{x}^1) & \dots & \psi(\mathbf{x}^1, \mathbf{x}^n) \\ \vdots & \ddots & \vdots \\ \psi(\mathbf{x}^n, \mathbf{x}^1) & \dots & \psi(\mathbf{x}^n, \mathbf{x}^n) \end{bmatrix} \quad (34)$$

where ψ is the correlation function. Matérn correlation functions have been shown to perform well for engineering problems. Therefore, in this work the correlation function for a pair of points $(\mathbf{x}, \mathbf{x}')$ is the Matérn correlation function with $\nu = 3/2$:

$$\psi(\mathbf{x}, \mathbf{x}') = \left(1 + \sqrt{3}l\right) \exp\left(-\sqrt{3}l\right) \quad (35)$$

where $l = \sqrt{\sum_{i=1}^d \theta_i (x_i - x'_i)^2}$. θ_i ($i = 1, \dots, d$) are hyperparameters and can be found using Maximum Likelihood Estimation (MLE). The value of the hyperparameters describe how fast the correlation drops to zero for each of the d variables.

The Kriging predictor can be derived as:

$$\hat{y}(\mathbf{x}) = \mathbf{M}(\mathbf{x})\boldsymbol{\alpha}^* + \mathbf{r}(\mathbf{x})\Psi^{-1}(\mathbf{y} - \mathbf{F}\boldsymbol{\alpha}^*) \quad (36)$$

where $\mathbf{M}(\mathbf{x})$ is a $(1 \times p)$ vector containing the basis functions of the regression function evaluated at some trial point $\mathbf{x}$: $\mathbf{M} = [b_1(\mathbf{x}), \dots, b_p(\mathbf{x})]^T$. $\mathbf{r}(\mathbf{x})$ is a $(1 \times n)$ vector of correlations between the trial point $\mathbf{x}$ and the n samples in $\mathbf{X}$: $\mathbf{r}(\mathbf{x}) = [\psi(\mathbf{x}, \mathbf{x}^1), \dots, \psi(\mathbf{x}, \mathbf{x}^n)]$. $\boldsymbol{\alpha}^*$ is a $(p \times 1)$ vector of coefficients found using MLE.

In stochastic Kriging the sampling uncertainty inherent in stochastic simulations is modelled as a separate Gaussian process $\xi(\mathbf{x})$ with zero mean and covariance matrix Σ : $\xi \sim \mathcal{GP}(0, \Sigma)$. The Kriging predictor for stochastic Kriging is then:

$$\hat{y}(\mathbf{x}) = \mathbf{M}(\mathbf{x})\boldsymbol{\alpha}^* + \mathbf{r}(\mathbf{x})\left(\Psi + \frac{1}{\sigma^2}\Sigma\right)^{-1}(\bar{\mathbf{y}} - \mathbf{F}\boldsymbol{\alpha}^*) \quad (37)$$

where the $(n \times n)$ matrix $\frac{1}{\sigma^2}\Sigma$ resembles signal-to-noise ratios, and $\bar{\mathbf{y}}$ is a $(n \times 1)$ vector containing the average outputs for many repeated simulation runs at the design points in $\mathbf{X}$. If there is no noise (the elements of Σ are zero) then equation (37) will be equal to equation (36). The matrix Σ can be created using repeated simulations at the design points in $\mathbf{X}$:

$$\Sigma = \begin{bmatrix} \text{var}(\mathbf{y}^1) & & 0 \\ & \ddots & \\ 0 & & \text{var}(\mathbf{y}^n) \end{bmatrix} \quad (38)$$

where $\text{var}(\mathbf{y}^i)$ is the variance of the output of repeated simulations at design point $\mathbf{x}^i$. For noisy deterministic simulations for which no repeat simulations are available the noise can be assumed to be uniformly distributed.

Therefore, the matrix Σ consists of the scalar value 10^λ on its diagonal such that $\Sigma = 10^\lambda I_n$, where I_n is the $(n \times n)$ identity matrix. A higher value of λ indicates there is more noise in the simulations. The value of λ can be estimated using MLE.

3. Numerical Example - Built-up Structure

To demonstrate the efficiency of the proposed EIFSD inference methodology, a built-up structure in the form of a stiffened panel was investigated. The plate and stiffeners are made out of aluminium 6061-T6 with Young's modulus $E = 68.9$ GPa, Poisson's ratio $\nu = 0.33$, and material constant $\gamma = 0.68$ in the Walker equation (see equation 29). Material properties for aluminium 6061-T6 can be found in [34]. The plate is simply supported along its edges such that the out-of-plane displacement w is zero along all of its edges. The plate is subjected to tractions t and bending moments p along all of its edges as can be seen in Figure 5. The stiffened panel has two aluminium I-stiffeners bonded to its surface. Figure 6 shows a diagram of the stiffeners. As can be seen in Figure 5, the stiffened panel contains a square central hole with rounded corners labelled 1-4. It is assumed that cracks will most likely form at the corners of this hole. Therefore, the inference of the EIFS will be conducted for cracks growing out of these corners. The values of the various dimensions seen in Figures 5-7 are presented in Table 1.

Figure 5: Assembled plate structure used in the numerical example. The four corners of the central hole are labelled 1-4.

Figure 6: Dimensions of the stiffeners seen in Figure 5.

Figure 7: Dimenions of the central hole seen in Figure 5. The four corners are labelled 1-4.

Table 1: The dimensions of the stiffened panel used in the numerical example

Dimension	Value
L_1	0.18 m
L_2	0.64 m
L_3	0.04 m
L_4	0.04 m
W	0.50 m
r	0.08 m
h_1	2.5 mm
h_2	5.0 mm
h_3	7.5 mm

The stiffened panel was modelled using DBEM as 13 individual plates bonded together. A total of 200 quadratic elements were used to model the stiffened panel when no cracks are present. The curved corners seen in Figure 7 are composed of six elements each, and the straight edges connecting them are composed of two elements each. Therefore, in the case where there are no cracks present, the central hole is composed of 32 elements. The BEM mesh used when there are no cracks present can be seen in Figure 8.

Figure 8: The BEM mesh used when there are no cracks present. The element end-nodes are denoted by large dots, and the element centre-nodes are denoted by small dots.

In the case where there is a crack present, the number of elements on the curved corner at which the crack is present is doubled from 6 to 12. The number of elements composing the initial crack is calculated based on

the length of the initial crack. During the crack growth procedure, two elements (one element for each crack surface) are automatically added to the crack tip of the previous increment. An example of the DBEM mesh for an initial crack at corner 1 (see Figure 7) is presented in Figure 9. The initial crack has a length of 5 mm and is composed of ten elements (five for each crack surface) of length 1 mm each.

Figure 9: The DBEM mesh used to model an initial crack at corner 1 (see Figure 7). Sub-figure (a) shows a zoomed-out view, and sub-figure (b) shows zoomed-in view.

3.1. Simulation of Inspection Data

The inference of the EIFSD was carried out under the presence of uncertainty in various crack growth and loading variables, including the Paris law constants C_0 and m , the maximum applied loading parameters t_{max} and p_{max} , and the applied load ratio R . It is assumed that t and p are correlated such that $t = 10p$. Therefore, $R = t_{min}/t_{max} = p_{min}/p_{max}$. Each of the five variables listed above is assumed to follow a lognormal distribution. In this work, three arbitrary levels of uncertainty were considered in these five variables: low uncertainty, medium uncertainty, and high uncertainty. The EIFSD will be inferred under each level of uncertainty. It is expected that the posterior estimate for the EIFSD from Bayesian updating will converge at a faster rate when there is low uncertainty, and at a slower rate when there is high uncertainty. Therefore, more inspection data will be required for converge using high-uncertainty data than low-uncertainty data. The distributions of the five variables can be seen in Table 2 for each of the three levels of uncertainty.

Table 2: Random variables and their distributions.

Variable	Distribution	Mean	Coefficient of Variation (COV)		
			Low	Medium	High
C_0	Lognormal	$1.027 \times 10^{-9} \frac{\text{m/cycle}}{(\text{MPa}\sqrt{\text{m}})^m}$	0.05	0.10	0.15
m	Lognormal	2.389	0.010	0.015	0.020
t_{max}	Lognormal	10 MPa	0.01	0.03	0.05
p_{max}	Lognormal	1 MPa	0.01	0.03	0.05
R	Lognormal	0.1	0.01	0.03	0.05

The data used for Bayesian updating was obtained from simulated inspections of a particular aircraft component from a fleet of aircraft - in this case the assembled plate structure seen in Figure 5. A total of 500

inspections were simulated in this work. If the inspection data is obtained from a fleet of aircraft, as is possible with the EIFS method, this number of inspections is reasonable. It is assumed that a crack is detected in an inspection when its size reaches 20 mm. This size is a typical detectable size for common NDI techniques. Therefore, if there are multiple cracks present the assembled structure during an inspection, only one crack will be detected (the one that has reached 20 mm first). In each inspection only the number of cycles at which the inspection was carried out was recorded, as it is assumed that the size of the detected crack is equal to 20 mm. Using this inspection data, the EIFSD can be inferred. A number of factors were randomised for each of the 500 simulated inspections. These were:

- (1) The location of the cracks present.
- (2) The number of cracks present.
- (3) The initial sizes of the cracks.
- (4) The values assigned to C_0 , m , t_{max} , p_{max} , and R .

Regarding factor 1: As stated previously, it is expected that cracks will most likely form at the four corners of the central hole seen in Figures 5 and 7.

Regarding factor 2: It is assumed that only one crack can form at each corner, e.g. there could be two cracks, one at corner 1 and the other at corner 4. The maximum number of cracks that could be present in an inspection is four. However, due to the complexity of modelling the growth of multiple simultaneous cracks and the large number of possible crack combinations, the number of simultaneous cracks has been limited to two. For the purpose of demonstration, this limitation is acceptable.

Regarding factor 3: The number and location of the cracks for each inspection were randomly chosen, the initial sizes of these cracks were also randomised. Since only computational analyses are used in this work, there is no "actual" EIFSD and therefore it can be arbitrarily chosen. Inspection data was created by randomly sampling from this "actual" EIFSD and then this "actual" EIFSD was "forgotten" about and the Bayesian updating methodology described in section 2.1 was then used to infer it using this inspection data. This "actual" EIFSD was given a lognormal distribution with a mean of 5 mm and a standard deviation of 0.5 mm.

Regarding factor 4: Each inspection was given different values for C_0 , m , t_{max} , p_{max} , and R . These values were randomly sampled from the distributions seen in Table 2. The variation of these values between inspections is due to uncertainties in material properties and loading conditions.

Out of a total of 1500 inspections (500 for each level of uncertainty), a total of 984 inspections only contained one crack, while 516 contained two cracks. The average number of cracks per inspection was 1.34 and the standard deviation was 0.48. Figure 10 shows the initial crack sizes of the fastest-growing crack from each inspection vs. the number of cycles required for that crack to grow to 20 mm for each of the 500 inspections at the three levels of uncertainty. It can be seen that the variation in the number of cycles increases as the uncertainty is increased, which makes sense.

Figure 10: Initial crack sizes of the fastest-growing crack from each inspection vs. the number of cycles required for that crack to grow to 20 mm. The data from all 500 inspections is shown for each of the three levels of uncertainty.

3.2. Stochastic Kriging Model Creation and Evaluation

In this work, Monte Carlo sampling is used to determine the parameters α_{ij} and β_{ij} of the distribution $f_{N|\hat{\mu}_i, \hat{\sigma}_j, \mathbf{Y}}$ which is required to evaluate the likelihood $L(\hat{\mu}_i, \hat{\sigma}_j | N_k^{ins}, \mathbf{Y})$ as part of the Bayesian inference procedure. Many millions of Monte Carlo samples could be required to accurately determine these parameters, and so using the expensive DBEM model for this sampling is not practical. Therefore, relatively cheap stochastic Kriging models are used in this work to conduct the Monte Carlo sampling.

Based on observations made by Aliabadi [24], the size of the elements added to each crack tip during the incremental crack growth procedure should be of a size similar to that of the existing crack tip element in order to provide high levels of accuracy. Based on this, the number of increments needed to grow a crack to some final size can be determined. However, since in this work the initial crack size is varied according to the EIFS distribution, the size of each increment will also vary and therefore the number of increments needed to grow a crack to a final size (always 20mm in this work) will vary. In this work the lengths of the elements were chosen to be as similar as possible to provide a stable solution. Therefore, each element was given a length of around 1 mm. The initial crack size varied around 5 mm, with a minimum of 2 mm and a maximum of 8 mm, therefore the number of elements composing each surface of the initial crack varied from 2 to 8 elements. The number of increments needed to grow the cracks to 20 mm was found to vary between 12 and 18, with an average of 15 increments. Due to this variation, the relationship in the DBEM model between the initial crack size a_0 and the final crack size a to the effective stress intensity factor K_{eff} at a will not be perfectly smooth and will have a degree of roughness. For this reason, it would not be suitable to use regular Kriging since it would interpolate between the rough data points. Therefore, stochastic Kriging is used in this work because it does not interpolate data points and instead smoothly follows the general trend of the data points.

Since the initial cracks in the structure are small relative to the dimensions of the structure, they are assumed to be straight. However, as the cracks grow it is unlikely that the cracks would grow in straight lines. Therefore, the initial size of a crack influences the direction of the growth of that crack. Therefore, if the value of K_{eff} for a crack of length a is to be accurately determined, information regarding the initial size a_0 needs to be provided

when creating the stochastic Kriging models:

$$K_{eff}(a_0, a) = f^{sur}(a_0, a) \quad (39)$$

where f^{sur} is a surrogate model and $K_{eff}(a_0, a)$ is the surrogate model approximation of the effective stress intensity factor evaluated at the tip of a crack of length a that was grown from an initial size of a_0 . As stated in section 2.1, a range of values for $\hat{\mu}$ and $\hat{\sigma}$ need to be tested, and the range of a_0 in equation (39) should take this into account. Therefore, a_0 can take a range between 2 mm and 8 mm. Also, since the cracks are being grown to 20 mm, a is given a range around this value. In this work, a can take a range of values from 2 mm to 30 mm. The applied loads t and p are not included as inputs to the surrogate model since linear elastic fracture mechanics assumes that the stress intensity factor is directly proportional to the applied loads. All of the surrogate models were designed at $t = 10$ MPa and $p = 1$ MPa. As stated before, it is assumed that $t = 10p$ at all times. Therefore, to evaluate the surrogate models for $t = 11$ MPa, the output of the surrogate models would need to be multiplied by a factor of 1.1.

If there is a single crack in the structure, then it can be located at any of the four corners. Likewise, if there are two cracks in the structure, then there will be two corners which will have one crack each. Stochastic Kriging models were created for every possible combination of cracks in the structure, a total of 16 combinations. However, due to the symmetry of the plate and the loading conditions, many of these combinations are functionally identical and so only four combinations need to be considered. Therefore, only four stochastic Kriging models needed to be created, one for when there is only one crack in the structure, and three for when there are two cracks in the structure.

It is difficult to determine the optimal number and location of design points when creating surrogate models. Therefore, an output-based sequential design strategy is used in this work for creating the four stochastic Kriging models. The Delaunay triangulation sequential sampling technique is used to determine the optimal location of new design points when creating the stochastic Kriging models. In 2D, Delaunay triangulation is the triangulation of a set n points such that there are no elements of n within the circumcircle of any triangle [35]. In practice, this means that new samples are chosen in locations far away from previous samples. This ensures that the design space is filled with samples in a very uniform manner. Each surrogate model is continually refined as more design points are added and the Root-Relative Squared Error (RRSE) of the model is estimated using cross-validation. If this RRSE value is less than some threshold RRSE value, then the creation of the surrogate model is considered complete and the sequential design procedure is stopped. The procedure used to create the stochastic Kriging models can be seen in Figure 11. Initially, a Latin hypercube design of 10 design points is created which includes design points at the corners of the design space. A threshold RRSE value of 5% was set.

Figure 11: Flowchart for the creation of the stochastic Kriging models.

Details of the design of the four surrogate models can be seen in Table 3. The notation "13/24" indicates that the surrogate model was designed for the case where the two cracks are at corners 1 and 3, or the case where the two cracks are at corners 2 and 4. Due to symmetry, these cases are functionally identical. It can be seen that all of the surrogate models reached the threshold RRSE value of 5%. There was some variation in the number of design points needed to reach this threshold. This can be explained by the fact that occasionally during the sequential design procedure, adding a new design point did not actually improve the RRSE^{CV} value of the surrogate model, and in some cases it actually worsened it. However, despite this, model 12/34 demonstrates similar accuracy to the other models based on its RRSE^{CV} value.

Table 3: Design data for the 4 surrogate models. Superscript CV denotes error values calculated using cross-validation.

Model	Number of design points	CPU time to train (min)	RRSE^{CV} (%)
1/2/3/4	157	30.7	4.99
12/34	181	69.9	4.98
13/24	104	38.9	4.91
14/23	124	47.9	4.99

Stochastic Kriging models do not interpolate data points and therefore there will be some error when run with their training data. Therefore, the level of roughness in the training data, and hence in the DBEM model, can be interpreted by looking at the error statistics of the stochastic Kriging models when run with their training data. This data is presented in Table 4 for the four models. Looking at this data, it can be seen that the error statistics are very small. For example, the R^2 values are very high at over 99.90%. This indicates that there is very little roughness and hence very small error in the training data. This small level of roughness can also be seen visually in Figure 12.

Table 4: Error values for each model when evaluating K_{eff} with its design dataset. Superscript D denotes error values calculated using the design dataset.

Model ^D	RRSE ^D (%)	MAPE ^D (%)	RMSE ^D (MPa $\sqrt{m}$)	(R^2) ^D (%)
1/2/3/4	3.23	0.38	0.032	99.90
12/34	3.19	0.34	0.027	99.90
13/24	2.39	0.31	0.027	99.94
14/23	2.54	0.30	0.025	99.94

Once the four surrogate models were created, they were each run with a test dataset composed of 400 points not used in the creation of the surrogate models. Figure 12 shows a surface plot for the output of the stochastic surrogate model for a single crack in the structure (model 1/2/3/4). The black dots show the K_{eff} values from the test dataset. Detailed error statistics for all of the surrogate models can be seen in Table 5. They include the Root Relative Squared Error (RRSE), the Mean Absolute Percentage Error (MAPE), the Mean Absolute Error (MAE), and the Root Mean Square Error (RMSE). These very low errors suggest that the surrogate models are very accurate at estimating K_{eff} values for the built-up structure used in this numerical example. The high accuracy of the surrogate models can also be seen visually in Figure 12.

Figure 12: Surface plot showing the K_{eff} values given by the surrogate model 1/2/3/4 over a range of values for a_0 and a . Also shown are the 400 K_{eff} values from the test dataset as black dots.

Table 5: Error values for each model when evaluating K_{eff} with its test dataset. Superscript T denotes error values calculated using the test dataset.

Model	RRSE ^T (%)	MAPE ^T (%)	MAE ^T (MPa $\sqrt{m}$)	RMSE ^T (MPa $\sqrt{m}$)
1/2/3/4	4.42	0.495	0.031	0.044
12/34	4.61	0.488	0.030	0.040
13/24	3.79	0.470	0.030	0.044
14/23	3.82	0.463	0.029	0.039

Using the Paris law (equation 28) it is possible to use the K_{eff} values from the surrogate models to estimate the number of cycles required to grow from an initial crack size a_0 to a final size a . These N values were obtained

for the mean values of the variables seen in Table 2. Test datasets for N obtained from DBEM were also created for each surrogate model. Figure 13 shows a surface plot for the N values obtained using the outputs of the surrogate model 1/2/3/4. The black dots indicate N values from its corresponding test dataset. It can be seen that the surrogate model and its test dataset compare well. However, there seems to be a notable amount of difference between the two when $a_0 = 2$ mm. As stated in section 2.1, as part of the Bayesian inference the cracks will be grown to 20 mm. Therefore, for $a_0 = 2$ mm, only $a = 20$ mm matters, and in this case the error is about 7.41 %. This represents quite a large error and is due to exponent m in the Paris law amplifying the error in the K_{eff} values from the surrogate model. However, $a_0 = 2$ is at the very edge of the domain. It can be seen in Figure 13 that when other values of a_0 between 2 mm and 8 mm are used with $a = 20$ mm, the difference is much less - the average error in N is only 1.63%. Detailed error statistics averaged over the full domain of a_0 and a values can be seen in Table 6. As expected, the error values are higher than those seen in Table 5 for K_{eff} . However, they are still quite low with a MAPE values of around only 2-3%.

Figure 13: Surface plot showing the N values calculated using the K_{eff} values from surrogate model 1/2/3/4 over a range of values for a_0 and a . Also shown are the 400 N values from the test dataset as black dots.

Table 6: Error values for each model when evaluating N . Superscript T denotes error values calculated using the test dataset.

Model	RRSE ^T (%)	MAPE ^T (%)	MAE ^T ($\times 10^5$ cycles)	RMSE ^T ($\times 10^5$ cycles)
1/2/3/4	8.82	2.57	0.056	0.101
12/34	7.13	2.28	0.049	0.087
13/24	4.82	2.05	0.036	0.053
14/23	5.92	2.11	0.044	0.068

3.3. Results and Discussion

The Bayesian inference methodology described in section 2.1 is employed here for inferring the EIFSD based on simulated inspection data for the assembled plate structure seen in Figure 5. As stated before, Bayesian inference requires an initial guess (prior distribution) for the distribution under investigation. It is assumed that an engineer with good engineering judgement would have a good idea of the EIFSD for the particular

type of crack in this structure and would use this judgement when deciding on the initial guess for the prior distribution of the EIFSD. To test the robustness of the proposed Bayesian inference methodology, three different prior distributions are investigated:

- A 'poor' prior distribution. This represents the case where the engineer does not have a good idea of the EIFSD. This is a lognormal distribution with a mean of 3 mm and a standard deviation of 1.5 mm.
- An 'adequate' prior distribution. This represents the case where the engineer has a slightly better idea of the EIFSD than the poor case. This is a lognormal distribution with a mean of 8 mm and a standard deviation of 0.8 mm.
- A 'good' prior distribution. This represents the case where the engineer has a good idea of the EIFSD. This is a lognormal distribution with a mean of 6 mm and a standard deviation of 0.6 mm.

It can be seen that the 'good' prior distribution above is the closest to the actual EIFSD which has a mean of 5 mm and a standard deviation of 0.5 mm, while the 'adequate' and 'poor' prior distributions are further away from the actual EIFSD. For each prior distribution, the posterior EIFSD obtained from Bayesian inference should converge towards the actual EIFSD as more inspection data becomes available.

Three cases were investigated in this work to test the robustness of the proposed methodology for the inference of the EIFSD for built-up structures. Each case uses a different level of uncertainty from Table 2 and a different prior distribution:

- **Case 1:** Low uncertainty and the 'good' prior distribution.
- **Case 2:** Medium uncertainty and the 'adequate' prior distribution.
- **Case 3:** High uncertainty and the 'poor' prior distribution.

Figure 14 shows the posterior distributions from Bayesian inference for the three cases after the data from 50, 100, 300, and 500 inspections have been used. It can be seen that the posterior EIFSD converges towards the actual EIFSD for each of the three levels of uncertainty. However, as the uncertainty is increased, more inspection data is required for convergence. For case 1 the posterior EIFSD from Bayesian inference converges to the actual EIFSD very quickly in around 50 inspections, but for case 3 over 300 inspections are required. GIF files showing the convergence of the EIFSD from Bayesian updating for the three levels of uncertainty can be found as supplementary files attached to online version of this paper.

(a)

(b)

(c)

Figure 14: Posterior distributions for the EIFS after 50, 100, 300, and 500 inspections. Also shown is the actual EIFSD and the initial guess for the EIFSD. Three levels of uncertainty are investigated: (a) Case 1, (b) Case 2, (c) Case 3

Figure 15 and Tables 7 and 8 show the convergence of the mean and the standard deviation of the posterior EIFSDs for the three cases. It can be seen that the number of inspections required for the mean to converge within ± 0.1 mm of the actual mean (5 mm) was less than 100 for the three cases. Case 1 required only around 30 inspections, case 2 required 58 inspections, and case 3 required 68 inspections. While cases 1 and 2 remained steady, case 3 seemed to diverge slightly by a value of about 0.05 mm from the other two cases after around 350 inspections. The convergence of the standard deviation for the three cases showed more variation. Case 1 converged to within ± 0.1 mm of the actual standard deviation (0.5 mm) after only 43 inspections, case 2

converged within 121 inspections, and case 3 converged within 441 inspections. This variation can be explained by the fact that when going from: Case 1 $\rightarrow$ Case 2 $\rightarrow$ Case 3, the level of uncertainty increases and the quality of the prior distribution decreases. Both of these factors cause the rate of converge of the mean and standard deviation to slow. Despite the high level of uncertainty associated with case 3, as can be seen in Figure 10, it did eventually converge. This demonstrates that there is a high level of robustness in the proposed methodology for the inference of the EIFSD for built-up structures.

(a)

(b)

Figure 15: Convergence of the EIFSD (a) mean and (b) standard deviation estimates from Bayesian updating for the three cases.

Table 7: Convergence of the mean (mm) from Bayesian updating for the three cases. The mean of the actual EIFSD is 5.00 mm.

	Number of inspections			
Case	50	100	300	500
Case 1	5.08	5.02	4.95	4.95
Case 2	5.14	5.02	4.95	4.95
Case 3	5.20	5.02	4.95	4.88

Table 8: Convergence of the standard deviation (mm) from Bayesian updating for the three cases. The standard deviation of the actual EIFSD is 0.500 mm.

Case	Number of inspections			
	50	100	300	500
Case 1	0.414	0.459	0.480	0.481
Case 2	0.319	0.399	0.436	0.451
Case 3	0.253	0.318	0.375	0.426

3.4. Computational Effort

A total of 1×10^6 Monte Carlo simulations were required to estimate the two parameters α_{ij} and β_{ij} in equation (2). This was done for every combination of $\hat{\mu}_{ij}$ and $\hat{\sigma}_{ij}$ for a total of $(P^{int})^2$ or 60^2 combinations. The stochastic Kriging models described in section 3.2 were used to carry out these Monte Carlo simulations. A comparison of the average CPU time to complete one Monte Carlo simulation using the stochastic Kriging models and the DBEM model can be seen in Table 9. It can be seen that the stochastic Kriging models are significantly cheaper computationally than the DBEM model. Given the values presented in this Table, it is estimated that to complete the required 1×10^6 Monte Carlo simulations, the Stochastic Kriging models would require 27 s while the DBEM model would require 2.46×10^7 s, or 285 days. Despite the inherent benefits of the DBEM, the time required by the DBEM is impractical and is due to the complexity of the built-up structure used in this example. If the time required to train the Stochastic Kriging models (see Table 3) is taken into account, the total time to complete the required 1×10^6 Monte Carlo simulations is 193 minutes for the stochastic Kriging models. This is still significantly less than the time required by the DBEM given above and demonstrates the importance of using the stochastic Kriging models in the EIFSD inference procedure.

Table 9: Average CPU time to complete one Monte Carlo simulation using the stochastic Kriging models and the DBEM model.

Model	CPU time (s) per simulation
Stochastic Kriging models	2.7×10^{-5}
DBEM model	24.6

3.5. Reliability Analysis with the Inferred EIFSD

The inferred EIFSD can be used to estimate the fatigue life of the structure by modelling the growth of cracks in the structure from the EIFSD to some critical crack size a_{crit} . If a crack is not detected before it reaches the critical crack size then the structure could risk failing. Therefore, inspection intervals should be timed in such a way as to ensure that cracks are detected before they reach this size. The goal of this section is to provide a simple showcase of how the EIFSD could potentially be used to help determine inspection intervals and improve aircraft safety. Typically, the process of determining inspection intervals should also consider additional factors such as the skills and equipment of the maintenance team, age of the aircraft, the cost of inspections, as well as other factors.

If N_{ins} is the number of cycles at which an inspection is carried out, the following limit state function can then be derived:

$$g(\mathbf{Z}) = N_{crit}(\mathbf{X}) - N_{ins} \quad (40)$$

where N_{crit} is the number of cycles required to grow a crack from the EIFSD to a_{crit} , and $\mathbf{X}$ is the set of all of the variables that influence N_{crit} , therefore $\mathbf{X} = \{\theta, C, m, t_{max}, p_{max}, R\}$, where θ is a random variable representing

the inferred EIFSD. a_{crit} and N_{ins} were arbitrarily chosen to be 30 mm and 250×10^5 cycles respectively. N_{crit} was calculated using the stochastic Kriging models introduced in section 3.2. MCS, FORM, and SORM were used to analyse reliability. Figure 16 shows the proposed reliability-EIFS approach for optimising inspection intervals with MCS. The probability of failure P_F is the proportion of cracks grown from the EIFSD that reach a_{crit} before being detected by the inspection undertaken at the number of cycles N_{ins} .

The three levels of uncertainty shown in Table 2 are also used here for C , m , t_{max} , p_{max} , and R . As seen in section 3.3, the posterior estimates from Bayesian inference converged towards the actual EIFSD for each of the three uncertainty levels. Therefore, θ represents the actual EIFS with a mean of 5 mm and standard deviation of 0.5 mm.

Figure 16: Reliability analysis with the inferred EIFSD.

Table 10 shows the probabilities of failure P_F and reliability indices β obtained from MCS, FORM, and SORM for the three levels of uncertainty. For MCS, 1×10^6 simulations were used. It can be seen that increasing the level of uncertainty leads to higher values of P_F , which is intuitive. It can also be seen that the values of P_F and β obtained from MCS, FORM, and SORM agree very well with each other. On average, SORM agrees better with MCS than FORM does, this is due to the fact that SORM approximates the limit state function in equation (40) using a second-order Taylor expansion while FORM only uses a first-order Taylor expansion.

Table 10: Probabilities of failure and reliability indices obtained from MCS, FORM, and SORM for the three levels of uncertainty.

Level of uncertainty	Probability of failure P_F (%)			Reliability index β		
	MCS	FORM	SORM	MCS	FORM	SORM
Low	0.2836	0.2813	0.2835	2.7661	2.7688	2.7662
Medium	2.868	2.788	2.820	1.9006	1.9129	1.9080
High	8.116	7.972	7.996	1.3973	1.4069	1.4053

By using the probability of failure from this reliability-EIFS approach, inspection intervals can be optimised. This example presented a case where there is only one inspection before the crack grows to a_{crit} . In reality there will be multiple inspections of differing man-hour requirements. The procedure described here can adapted to include the possibility of multiple inspections.

4. Conclusions

In this work a new methodology for the statistical inference of the Equivalent Initial Flaw Size Distribution (EIFSD) for built-up structures with Bayesian updating has been developed for the first time. Bayesian updating was employed to allow the inferred EIFSD to be updated using new information provided by routine inspections of an aluminium stiffened panel from a fleet of the same aircraft. By using fleet-wide inspection data, aircraft with low flight-hours can benefit from data obtained from aircraft with high flight-hours. Bayesian updating is especially useful when available data is limited, as it allows the engineer to include subjective engineering judgement in the updating procedure in the form of a prior distribution. To reduce the computational cost associated with modelling the stiffened panel, and to reflect the stochastic nature of fatigue crack growth, stochastic (or regression) Kriging was used as part of the Bayesian inference procedure. Outputs from a Dual Boundary Element Method (DBEM) model of the stiffened panel were used to train the stochastic Kriging models. The DBEM enabled the automatic calculation of crack growth direction, and required no remeshing during the crack growth procedure. The use of the DBEM therefore allowed the growth of multiple cracks to be modelled in a very efficient manner.

The EIFSD was inferred for three arbitrary levels of uncertainty. It was found that the mean of the EIFSD could be inferred accurately within ± 0.1 mm with the data from only 30 inspections in the case of low uncertainty, 58 for medium uncertainty, and 68 for high uncertainty. Also, the standard deviation could be inferred accurately within ± 0.1 mm with the data from only 43 inspections in the case of low uncertainty, 121 for medium uncertainty, and 441 for high uncertainty. Therefore, the proposed methodology was able to provide convergence even with significant levels of uncertainty. This demonstrates the high-level of robustness and efficiency associated with the proposed methodology for the inference of the EIFSD for built-up structures.

As presented at the end of the paper, once an EIFSD has been inferred using the proposed methodology it can then be applied to optimise the intervals between inspections in a combined reliability-EIFS approach. Offering the capability to reduce the costs associated with inspections without compromising on safety.

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