Complete-market Models of Stochastic Volatility

BY MARK H.A. DAVIS

Department of Mathematics, Imperial College, London SW7 2AZ, UK

In the Black-Scholes option pricing theory, asset prices are modelled as geometric Brownian motion with a fixed volatility parameter σ , and option prices are determined as functions of the underlying asset price. Options are in principle redundant in that their exercise values can be replicated by trading in the underlying. However, it is an empirical fact that the prices of exchange-traded options do not correspond to a fixed value of σ as the theory requires. This paper proposes a modelling framework in which certain options are non-redundant: these options and the underlying are modelled as autonomous financial assets, linked only by the boundary condition at exercise. A geometric condition is given, under which a complete market is obtained in this way, giving a consistent theory under which traded options as well as the underlying asset are used as hedging instruments.

Keywords: Financial options, Black-Scholes, volatility, vega hedging, stochastic flows, Bismut formula

1. Introduction

The Black-Scholes theory is based on an asset price model which, in a risk-neutral measure Q , takes the form

$$dS_t = rS_t dt + \sigma S_t dw_t, \quad (1.1)$$

where r is the riskless rate, w_t is a Brownian motion and σ is the volatility. In this paper we are not concerned with interest-rate volatility – generally a minor factor in equity option pricing – so r will be taken as a constant (sometimes 0). We also assume the asset pays no dividends. As is well known, the price model (1.1) leads to a 5-parameter formula $C(S, K, r, \sigma, T)$ for the price of a call option. Of these parameters, (K, T) (strike and exercise time) define the option contract while (S, r) are market data, leaving the formula essentially as a map $\sigma \mapsto p = C(S, K, r, \sigma, T)$ from volatility to price. Because the call option exercise function is convex, p is an increasing function of σ , and we can compute the inverse map, the so-called *implied volatility*.

Evidence from the traded option market shows that the model (1.1) is not an accurate description of reality (see Ghysels *et al.* 1996 for a comprehensive survey and Tompkins 2001 for empirical evidence). Figure 1 shows the implied volatility for FTSE index options for a range of strike prices and maturity dates, while Figure 2 shows the evolution of at-the-money implied volatility over a 15-year period. Figure 1 shows that the log-normal distribution of S_T implied by model (1.1) cannot be correct, and Figure 2 shows that volatility is in some sense ‘stochastic’.

Figure 1. Implied volatility of FTSE100 index European options, May 2000

Studies of stochastic volatility invariably introduce more complicated models than (1.1), with a view to ‘explaining’ the features displayed in Figures 1 and 2. The overall aim is one of three things:

1. *Marking to Market*: produce a consistent valuation for an OTC (‘over-the-counter’, i.e. non-exchange traded) option given the current market data. Market data includes the prices of exchange-traded options, which are European or American puts and calls. The OTC option could be a put or call with different strike or maturity, or an ‘exotic’ option with path-dependent exercise value.
2. *Hedging*: find the hedge parameters for a portfolio of underlying assets and options on those assets.
3. *Value at Risk*: for a given portfolio, calculate the 1% or other quantile of the return distribution over a specified holding period such as 10 days (Dowd, 1998)

The third of these is a purely econometric problem and will not be considered further in this paper. The requirements for the first two tasks are very different. The first is essentially an interpolation problem. In most cases, any models that depend smoothly on their parameters and are correctly calibrated will give closely the same value for an OTC option. (Of course, a model is not needed at all for valuation of traded options.) Obtaining correct hedge parameters, on the other hand, is a much more demanding task, and these parameters are model-dependent even for traded options.

Stochastic volatility models divide into two broad classes: ‘single-factor’ models in which the original Brownian motion w_t continues to be the only source of randomness, and multi-factor models in which further Brownian motions or other random elements are introduced. While the main emphasis in this paper is on the latter, the next section discusses briefly the single-factor case.

Figure 2. ATM Implied volatility of 3m S&P options, 1988-1997

2. Stochastic volatility models

(a) Single-factor models

The simplest case here is that of *level-dependent volatility* where the price model takes the form

$$dS_t = rS_t dt + \sigma(S_t)S_t dw_t \quad (2.1)$$

where σ is a function such that $s \mapsto s\sigma(s)$ is Lipschitz continuous. (Then (2.1) has a unique solution.) Since S_t is adapted to the filtration generated by (w_t) , the market is complete and the unique price of a European option with exercise value $h(S_T)$ at time T is, as usual

$$E_Q [e^{-rT} h(S_T)] . \quad (2.2)$$

Options are redundant and the value given by (2.2) is the initial capital required to form the replicating portfolio. Special cases are the ‘constant elasticity of variance’ (CEV) model

$$dS_t = rS_t dt + \beta(S_t)^{1-\alpha} dw_t,$$

or the ‘implied tree’ models of Derman and Kani (1998) or Dupire (1994). By suitably choosing $\sigma(\cdot)$ one can obtain price distributions that match observed volatility smiles. However, these models are somewhat restricted and the implications for hedging are not at all clear. In particular, since options are redundant these models say nothing about ‘vega hedging’ (see below). They also contradict the empirical fact (Tompkins, 2001) that price and volatility are not perfectly correlated, as implied by (2.1).

A more interesting class of models arises from the following observation: it is easily checked that the 2-vector random variable

$$\begin{bmatrix} w_t \\ \int_0^t w_s ds \end{bmatrix}$$

has non-singular covariance matrix

$$\begin{bmatrix} t & \frac{1}{2}t^2 \\ \frac{1}{2}t^2 & \frac{1}{3}t^3 \end{bmatrix}$$

(the correlation coefficient is $\sqrt{3}/2 = 0.866$). Thus we can ‘manufacture’ apparently extra randomness by using the past of the Brownian motion, without losing completeness by introducing additional random variables. Hobson and Rogers (1998) have used this idea in an interesting paper where the choice of volatility is inspired by GARCH modelling. Here is another formulation, a one-factor version of the Hull-White stochastic volatility model (Hull & White, 1987). The price process is $x_0(t)$ satisfying

$$dx_0(t) = \sqrt{x_1(t)}x_0(t)dw_t \quad (2.3)$$

$$dx_1(t) = (\theta - \lambda x_1(t))dt + \gamma dw_t \quad (2.4)$$

The volatility is thus $\sqrt{x_1}$ where x_1 is an Ornstein-Uhlenbeck process driven by the same Brownian motion as x_0 . It is not immediately obvious that equations (2.3),(2.4) have a solution, since the usual Lipschitz condition is not satisfied. But in fact they do (up to the stopping time $\tau = \inf\{t : x_1(t) = 0\}$), as one can write down a solution in closed form (first of (2.4), then of (2.3)). In this model the 2-vector random variable $x(t) = (x_1(t), x_2(t))$ has a density, since the Hörmander condition (see Appendix A) is satisfied, at least for some choices of the coefficients. Indeed, (2.3),(2.4) can be written in Stratonovich form as

$$\begin{aligned} dx_0 &= -\frac{1}{2}(x_1 + \gamma x_1^{-1/2}/2)dt + x_1^{1/2}x_0 \circ dw_t \\ dx_1 &= (\theta - \lambda x_1)dt + \gamma \circ dw_t \end{aligned}$$

Expressing this in coordinate-free form as $df(x_t) = A_0 f(x_t)dt + A_1 f(x_t) \circ dw_t$, we find that the Lie bracket $[A_0, A_1]$ has coefficients

$$\begin{bmatrix} \left(\frac{1}{2}\gamma \left(1 - \frac{1}{4}\gamma x_1^{-3/2} \right) - \frac{1}{2}(\theta - \lambda x_1)x_1^{-1/2} \right) x_0 \\ \gamma \lambda \end{bmatrix}$$

The Hörmander condition is satisfied if $\text{rank}(A_1, [A_0, A_1]) = 2$ and in fact the determinant is

$$\gamma x_0 \left(\frac{1}{2}\gamma - \frac{1}{8}\gamma^2 x_1^{-3/2} - \frac{1}{2}\theta x_1^{-1/2} - \left(1 - \frac{1}{2}\lambda \right) x_1^{1/2} \right).$$

The determinant is non-zero for all $x_1 > 0$ for reasonable ranges of the coefficients θ, λ, γ , for instance the representative values $\theta = .0025, \lambda = .05, \gamma = .01$.

The above calculations show that we can produce complete-market models where the volatility is ‘stochastic’ in the sense that it is not just a function of current price. While this is satisfactory from an econometric standpoint, the trading strategies that these models imply – delta-hedging in the underlying asset – are completely unrealistic. The obvious way to hedge volatility is to use traded options, and these models give us no clue how to do so, since all options are in theory redundant.

(b) Multi-factor models

The standard way to hedge against volatility risk is ‘vega hedging’. The *vega* of an option C is $v = \partial C / \partial \sigma$, the sensitivity of the Black-Scholes value to changes

in the volatility σ . If we hold option C (say an OTC option) we could in principle hedge the volatility risk by selling v/v' units of an exchange traded option C' whose vega is v' , giving a ‘vega neutral’ portfolio $C - (v/v')C'$. Effectively we are, correctly, treating the exchange-traded option as an independent financial asset. However, the procedure is theoretically inconsistent in that the valuation method – Black-Scholes – assumes *no* variation in volatility. We are, in fact, hedging ‘outside the model’.

To get a consistent treatment, an obvious approach is to introduce models in which the volatility parameter is treated as a stochastic process, not a constant. Thus our market model, in the physical measure P , takes the form

$$\begin{aligned} dS(t) &= \mu S(t)dt + \sigma(t)S(t)dw_t \\ d\sigma(t) &= a(S(t), \sigma(t))dt + b(S(t), \sigma(t))dw_t^\sigma \end{aligned}$$

where a, b define the volatility model and w_t, w_t^σ are Brownian motions with constant correlation $Edw_t dw_t^\sigma = \rho dt$, so movements of volatility are possibly correlated with movements of underlying asset price. Well known models of this type are those of Hull and White (1987) and Heston (1993). We can write $w_t^\sigma = \rho w_t + \rho' w'_t$ where w'_t is a Brownian motion independent of w_t and $\rho' = \sqrt{1 - \rho^2}$. Measures Q equivalent to P then have densities of the form

$$\frac{dQ}{dP} = \exp \left(\int_0^T \Phi_s dw_s - \frac{1}{2} \int_0^T \Phi_s^2 ds + \int_0^T \Psi_s dw'_s - \frac{1}{2} \int_0^T \Psi_s^2 ds \right) \quad (2.5)$$

for some integrands Φ, Ψ . Taking $\Phi = (r - \mu)/\sigma$ and $\Psi = \Psi(S, \sigma)$ we find that the equations for S, σ under measure Q are

$$\begin{aligned} dS(t) &= rS(t)dt + \sigma S(t)d\tilde{w}_t \\ d\sigma(t) &= \tilde{a}(S(t), \sigma(t))dt + b(S(t), \sigma(t))d\tilde{w}_t^\sigma \end{aligned}$$

where $\tilde{w}, \tilde{w}^\sigma$ are Q -Brownian motions with $Ed\tilde{w}d\tilde{w}^\sigma = \rho dt$ and $\tilde{a}(S, \sigma) = a + b\rho\Phi + b\rho'\Psi$. Then $S(t)$ has the riskless growth rate r , but σ is not a traded asset so arbitrage considerations do not determine the drift of σ , leaving Ψ as an arbitrary choice. Suppose we now have an option written on $S(t)$ with exercise value $g(S(T))$ at time T . We *define* its value at $t < T$ to be

$$C(t, S(t), \sigma(t)) = E_Q \left[e^{-r(T-t)} g(S(T)) \middle| S(t), \sigma(t) \right].$$

C then satisfies the PDE

$$\frac{\partial C}{\partial t} + rs \frac{\partial C}{\partial s} + \tilde{a} \frac{\partial C}{\partial \sigma} + \frac{1}{2} \sigma^2 s^2 \frac{\partial^2 C}{\partial s^2} + \frac{1}{2} b^2 \frac{\partial^2 C}{\partial \sigma^2} + \rho \sigma s b \frac{\partial^2 C}{\partial s \partial \sigma} - rC = 0$$

and we find that the process $Y(t) := C(t, S(t), \sigma(t))$ satisfies

$$dY(t) = rY(t)dt + \frac{\partial C}{\partial s} \sigma S d\tilde{w} + \frac{\partial C}{\partial \sigma} b d\tilde{w}^\sigma. \quad (2.6)$$

If the map $\sigma \mapsto y = C(t, s, \sigma)$ is invertible, so that $\sigma = D(t, s, y)$ for some smooth function D , then the diffusion coefficients in (2.6) can be expressed as functions of $t, S(t), Y(t)$ and we obtain an equation of the form

$$dY(t) = rY(t)dt + F(t, S(t), Y(t))d\hat{w}_t, \quad (2.7)$$

where $\hat{w}_t$ is another Brownian motion, again correlated with $\tilde{w}_t$. $S(t)$ and $Y(t)$ are linked by the fact that, at time T , $Y(T) = g(S(T))$. We have now created a complete market model with traded assets $S(t)$, $Y(t)$ for which Q is the unique EMM. By trading these assets we can perfectly replicate any other contingent claim in the market. We have however created a whole range of such models, one for each choice of the integrand Ψ in (2.5). The choice of Ψ ultimately determines the ‘volatility structure’ F of $Y(t)$ in (2.7), which is all that is relevant for hedging. This choice is an empirical question. The relationship with implied volatility is clear: if $\text{BS}(t, S, \sigma)$ denotes the Black-Scholes price at time t with volatility parameter σ , then the implied volatility $\hat{\sigma}(t)$ must satisfy $Y(t) = \text{BS}(t, S(t), \hat{\sigma}(t))$, so each stochastic volatility model implicitly specifies a model for implied volatility.

3. A unified approach to stochastic volatility

In this section we sketch a general framework along the lines of the previous section that includes existing models as special cases. The approach is in the same general spirit as Lyons (T.J. Lyons 1997, unpublished work), Babbar (2001) and Schönbucher (1999).

(a) The general model

To keep things simple, suppose our market contains one underlying asset with price $S(t)$, two exchange-traded European call options on S_t with maturity times $T_1 \leq T_2$ and strikes K_1, K_2 , and the usual riskless account with interest rate r . $O_1(t), O_2(t)$ will denote the prices of the options at time t . Since we are interested in complete markets, essentially we are only describing the situation up to time T_1 . At some time at or before T_1 another option must appear, maturing at some later time T_3 , to complete the market after T_1 . This third option will be redundant before T_1 .

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, w_t, P)$ be the canonical Wiener space for Brownian motion w_t in R^3 . Here P will play the role of the risk-neutral measure, not the physical measure. Let $Y > 0$ be an integrable $\mathcal{F}_{T_2}$ -measurable random variable and *define*

$$S(t) = E[e^{-r(T_2-t)}Y|\mathcal{F}_t], \quad t \in [0, T_2] \quad (3.1)$$

$$O_2(t) = E[e^{-r(T_2-t)}[Y - K_2]^+|\mathcal{F}_t], \quad t \in [0, T_2] \quad (3.2)$$

$$O_1(t) = E[e^{-r(T_1-t)}[S(T_1) - K_1]^+|\mathcal{F}_t], \quad t \in [0, T_1]. \quad (3.3)$$

As in the previous section, such a model automatically specifies a model for implied volatilities $\hat{\sigma}_1, \hat{\sigma}_2$ of the two options, which (with obvious notation) must satisfy

$$O_i(t) = \text{BS}(T_i - t, S(t), \hat{\sigma}_i(t), K_i) \quad i = 1, 2.$$

To get something more explicit, we need a Markovian framework. Thus, suppose that $m : R^3 \rightarrow R^3$ and $\Sigma : R^3 \rightarrow R^3 \times R^3$ are Lipschitz continuous functions with $\Sigma(x)\Sigma^T(x)$ uniformly positive definite[†], and let ξ_t be the unique solution of the SDE

$$d\xi_t = m(\xi_t)dt + \Sigma(\xi_t)dw_t.$$

[†] Superscript ‘T’ denotes transpose.

Associated with this is the backward equation, for a function $v(t, x)$:

$$\frac{\partial v}{\partial t} + \mathcal{A}v - rv = 0, \quad (t, x) \in [0, T] \times R^3 \quad (3.4)$$

$$v(T, x) = h(x). \quad (3.5)$$

In this equation, h is given boundary data at some terminal time T and $\mathcal{A}$ is the generator of ξ_t , i.e.

$$\mathcal{A}f(x) = \nabla f(x) m(x) + \frac{1}{2} \sum_{i,j} (\Sigma(x) \Sigma^T(x))_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(x).$$

(Throughout this section, ∇v denotes the gradient of a function v in the x variables, expressed as a row vector.) By the Feynman-Kac formula the solution of (3.4), (3.5) is

$$\begin{aligned} v(t, x) &= P_{T-t} h(x) \\ &= E_{t,x} \left[e^{-r(T-t)} h(\xi_T) \right]. \end{aligned}$$

Here P_t is the semigroup of operators associated with the generator $\mathcal{A}$.

We take ξ_t as the process of ‘factors’ underlying the financial market and suppose that Y takes the form

$$Y = h_0(\xi_{T_2})$$

for some function h_0 . In view of (3.1) the price process is now

$$S_t = P_{T_2-t} h_0(\xi_t) =: v_0(t, \xi_t),$$

and from (3.2) and (3.3) the option values are given for $t \leq T_1$ by

$$O_i(t) = P_{T_i-t} h_i(\xi_t) =: v_i(t, x_t), \quad i = 1, 2, \quad (3.6)$$

where $h_2(x) = [h_0(x) - K]^+$ and $h_1(x) = [P_{T_2-T_1} h_0(x) - K_1]^+$. By the Ito formula, the discounted asset price satisfies

$$d(e^{-rt} S_t) = e^{-rt} \nabla v_0 \Sigma dw, \quad (3.7)$$

with similar expressions for the option values $O_1(t), O_2(t)$.

We would like to show that this market is complete in that any other contingent claim maturing at time $T \leq T_1$ can be hedged using a portfolio of cash, underlying S and options O_1 and O_2 . A *trading strategy* is a 3-vector $\mathcal{F}_t$ -adapted process $\alpha(t)$ (written as a row vector) such that $\int_0^{T_1} |\alpha(t)|^2 dw_t < \infty$ almost surely. $\alpha_0(t) [\alpha_1(t), \alpha_2(t)]$ represents the number of units of $S [O_1, O_2]$ invested at time t . A *self-financing portfolio* with value X_t at time t is constructed in the following way from the trading strategy α : the increment of portfolio value is

$$\begin{aligned} dX_t &= \alpha_0(t) dS_t + \alpha_1(t) dO_1(t) + \alpha_2(t) dO_2(t) \\ &\quad + (X_t - \alpha_0(t) S_t - \alpha_1(t) O_1(t) - \alpha_2(t) O_2(t)) r dt. \end{aligned} \quad (3.8)$$

The first three terms on the right are the increments in value of the holdings in underlying asset and options, and the fourth term indicates that all residual value

is held in the riskless account, where it earns interest at rate r . Let tilde denote discounted quantities: $\tilde{X}_t = e^{-rt} X_t$ etc. Applying the Ito formula to (3.8) and using (3.7) we obtain

$$\begin{aligned} d\tilde{X}_t &= \alpha_0(t)d\tilde{S}_t + \alpha_1(t)d\tilde{O}_1(t) + \alpha_2(t)d\tilde{O}_2(t) \\ &= e^{-rt} \left(\sum_{i=0}^2 \alpha_i(t) \nabla v_i \right) \Sigma dw. \end{aligned} \quad (3.9)$$

Proposition 3.1. *Suppose that the matrix*

$$G(t, x) = \begin{bmatrix} \nabla v_0(t, x) \\ \nabla v_1(t, x) \\ \nabla v_2(t, x) \end{bmatrix} \quad (3.10)$$

is nonsingular for all $(t, x) \in [0, T] \times R^3$. Then we have a complete market model and the hedge ratios for hedging any other contingent claim are given by (3.12) below.

Proof. Let H be the exercise value of a contingent claim exercised at time $T < T_1$, i.e. H is an $\mathcal{F}_T$ -measurable random variable with $E[H^2] < \infty$. By the martingale representation theorem for Brownian motion (Rogers & Williams 2000, Theorem IV.36.1), there is an integrand χ_t such that $E \int_0^T |\chi_t|^2 dt < \infty$ and

$$e^{-rT} H = E[e^{-rT} H] + \int_0^T \chi_t dw_t. \quad (3.11)$$

Assuming G is non-singular, we can define a trading strategy α_t by

$$\alpha_t = e^{rt} \chi_t \Sigma^{-1}(\xi_t) G^{-1}(t, \xi_t). \quad (3.12)$$

Then $e^{-rt} \alpha G \Sigma = \chi$, and we see from (3.9) and (3.11) that $H = X_T$ a.s. if $X_0 = E[e^{-rT} H]$. Thus arbitrary contingent claims can be replicated and the market is complete. $\square$

As a special case, suppose that $H = h(\xi_T)$ and define

$$v(t, x) = P_{T-t} h(x), \quad t \leq T. \quad (3.13)$$

By the Ito formula and (3.4), (3.5)

$$e^{-rt} v(t, \xi_t) = e^{-rt} \nabla v \Sigma dw,$$

and we see that the replicating strategy is given by

$$\alpha(t) = \nabla v(t, \xi_t) G^{-1}(t, \xi_t). \quad (3.14)$$

In order to implement these trading strategies, we need to know the value of the factor process ξ_t , but this is not directly observed: the market data consists of the traded asset prices $(S_t, O_1(t), O_2(t))$ and we must recover the state vector ξ_t from these. The condition given in proposition 3.1 ensures local invertibility. Global invertibility generally follows from monotonicity properties, as in the example of the

Underlying spot	100
Option strike K_1	96
Option maturity T_1	1
Implied vol	20.085%
Option price	10.00
Delta	0.6193
Vega	38.10
Riskless rate r	0
Dividend yield	0

Figure 3. Data for example (b)

next section. This question has been considered by Bajeux-Besnainou and Rochet (1996).

The advantage of our approach is that there is no calibration, since market option prices are inputs to the model, and no complicated conditions to avoid arbitrage (as in Schönbucher, 1999), as the model is automatically arbitrage-free. On the other hand the implied model for implied volatility is rather indirect, leaving us with the problem of determining good classes of factor processes ξ_t to capture the volatility structures we need. In the following section we present some quick calculations using the Hull-White volatility model (Hull & White, 1987).

(b) *An example*

Here we will assume there is just one exchange-traded option, maturing at time T_1 . The parameter values are as shown in Table 3; in particular, purely for ease of exposition, the riskless rate r is zero. The factor process $\xi(t) = (\xi_1(t), \xi_2(t))$ is

$$\begin{aligned} d\xi_1(t) &= \sqrt{\xi_2(t)}\xi_1(t)dw_1(t) \\ d\xi_2(t) &= \lambda(\sigma_0^2 - \xi_2(t))dt + \gamma dw_2(t), \end{aligned}$$

where w_1, w_2 are independent Brownian motions. Thus ξ_1 has ‘volatility’ $\sqrt{\xi_2}$ where ξ_2 is an Ornstein-Uhlenbeck process independent of w_1 . The mean reversion rate of ξ_2 is λ and the mean reversion level is σ_0^2 . Think of σ_0 as the long-run average volatility. These are the same equations as those used by Hull and White (1987), but we use them in a somewhat different way. First, define

$$\text{BS}(S, K, a) = SN(d_1) - KN(d_2)$$

where

$$\begin{aligned} d_1 &= \frac{1}{a} \log(S/K) + \frac{1}{2}a \\ d_2 &= d_1 - a. \end{aligned}$$

We take $Y = \xi_1(T_1)$; then since $\xi_1(t)$ is a martingale, $S_t = \xi_1(t)$, $t \leq T_1$. Since $\xi_2(\cdot)$ and $w_1(\cdot)$ are independent we can calculate the option value as

$$\begin{aligned} O_1(t) &= E\{[\xi_1(T_1) - K_1]^+ | \mathcal{F}_t\} \\ &= E_{t,x}\{E_{t,x}([\xi_1(T_1) - K_1]^+ | \xi_2(s), t \leq s \leq T_1)\} \Big|_{x=\xi(t)} \\ &= \int_{-\infty}^{\infty} \text{BS}(\xi_1(t), K_1, a) \phi_A(a) da, \end{aligned} \quad (3.15)$$

where ϕ_A is the density function of the random variable

$$A = \int_t^{T_1} \xi_2(s) ds.$$

(Of course, there is positive probability that $A < 0$; we take $\text{BS}(S, K, a)$ to be equal to the intrinsic value when $a \leq 0$.) A short computation shows that

$$\int_0^T \xi_2(t) dt = b(T, \lambda) \xi_2(0) + c(T, \lambda, \sigma_0) + Z,$$

where

$$\begin{aligned} b(T, \lambda) &= \frac{1}{\lambda} (1 - e^{-\lambda T}) \\ c(T, \lambda, \sigma_0) &= \sigma_0^2 (T - b(T, \lambda)) \end{aligned}$$

and Z is a zero-mean gaussian random variable with standard deviation

$$\eta(T, \lambda, \gamma) = \frac{\gamma}{\lambda} \sqrt{T - 2b(T, \lambda) + b(T, 2\lambda)}.$$

In view of these expressions and the fact that the ξ_t equations are time-invariant, we can express the option value (3.15) as (with $\tau = T - t$)

$$O_1(t) = \int_{-\infty}^{\infty} \text{BS}[\xi_1(t), K_1, b(\tau, \lambda) \xi_2(t) + c(\tau, \lambda, \sigma_0) + \eta(\tau, \lambda, \gamma)z] \sqrt{\frac{1}{2\pi}} e^{-z^2/2} dz. \quad (3.16)$$

This is the convenient representation noted by Hull and White (1987). In this case the inverse problem is readily solved: given $S_t = s_1$ and $O_1(t) = o_1$, the corresponding values of the factor process are $\xi_1(t) = s_1$ and $\xi_2(t) = x_0$, where x_0 is the value of $\xi_2(t)$ such that (3.16) is satisfied when the left-hand side is equal to o_1 . Since the right-hand side is monotone increasing in $\xi_2(t)$, this value is easily found by one-dimensional search. Figure 4 shows the values of $\sqrt{\xi_2(0)}$ corresponding to different values of γ, σ_0 . The Black-Scholes implied volatility is shown for comparison in the right-hand column.

The option O we wish to hedge has strike $K = 110$ and matures at $T = 0.5$. Taking the volatility as the implied volatility of O_1 , the Black-Scholes value of O is 2.231 and the delta and vega are $\Delta = 0.2742$, $v = 23.56$. Thus – referring to Figure 3 – the standard vega hedge at time 0 has $\alpha_1 = 23.56/38.10 = 0.618$ units of O_1 , leaving a residual delta of -0.109.

To calculate the hedge corresponding to our stochastic volatility model we simply apply the 2×2 version of formula (3.14), computing the gradients by finite

Figure 4. Values of $\sqrt{\xi_2(0)}$ for different values of 'vol of vol' γ , with $\sigma_0 = 15\%$, $\sigma_0 = 25\%$

Figure 5. Hedge parameters corresponding to different levels of 'vol of vol' γ

differences. The resulting initial hedge parameters $\alpha_0(0), \alpha_1(0)$ are shown in Figure 5 for the same range of γ, σ_0 as in Figure 4. Recall that in this model these are the true hedge parameters for a perfectly-replicating portfolio. The standard vega hedge is shown on the right, for comparison.

The hedge parameters vary surprisingly little over the different model parameters. The truth is that this class of models doesn't have much pizzazz. With only one traded option we are not capturing any 'smile' effect, and the independence of the two underlying Brownian motions is computationally convenient but hardly realistic. The example however gives us a 'proof of principle': the method has been completely implemented and the hedge parameters computed. With more realistic models the procedure would be exactly the same, but with efficient PDE solvers replacing the one-dimensional integration (3.16).

(c) *The non-singularity condition*

An important question is to find conditions on the model and the option contracts under which the non-singularity condition of proposition 3.1 is satisfied. There are no general results in this direction in the option pricing literature except for the 2-factor case, which has been more or less completely resolved by Romano and Touzi (1997) following earlier work by El Karoui *et al.* (1998), Bajeux-Besnainou & Rochet (1996) and Bergman *et al.* (1996). The basic insight is that option values are, for very general models, convex functions of the underlying asset value and hence monotonically increasing functions of volatility. Romano and Touzi (1997) work with a model

$$\begin{aligned}\frac{dS_t}{S_t} &= m(S_t, Y_t)dt + \sigma(Y_t)(\rho'_t dw_t^1 + \rho_t dw_t^2) \\ dY_t &= \eta(S_t, Y_t)dt + \gamma(S_t, Y_t)dw_t^2\end{aligned}$$

where $\rho_t - \rho(S_t, Y_t)$ and $\rho'_t = \sqrt{1 - \rho_t^2}$. Then, under technical conditions, any option with a convex exercise value completes the market. This is shown to be equivalent to saying that $(\partial U / \partial y)(t, s, y) \neq 0$ where U is the option value. This is the scalar version of our condition.

When more than one option is required to complete the market the situation is more delicate, and in fact leads to a problem of independent mathematical interest. Let us consider an n -dimensional case where the factor process is a non-degenerate diffusion $(\xi_t, 0 \leq t \leq T)$ in R^n , the exercise value functions $h_i(x)$ are such that $E|h_i(\xi_T)| < \infty$, $i = 1, \dots, n$, and the exercise time is T for each option. Define functions v_i by

$$v_i(t, x) = E_{t,x}[h_i(\xi_T)], \quad (3.17)$$

and let $G(t, x)$ be the $n \times n$ matrix

$$G(t, x) = \begin{bmatrix} \nabla v_1(t, x) \\ \vdots \\ \nabla v_n(t, x) \end{bmatrix}, \quad (3.18)$$

where ∇ denotes the gradient in the x variables. Consider first the Brownian case.

Proposition 3.2. *Suppose ξ_t is Brownian motion in R^n . Then $G(t, x)$ defined by (3.18) is non-singular if and only if there exists no non-zero vector $\alpha \in R^n$ such that*

$$H_\alpha \perp \mathcal{L}\{\xi_T^1, \dots, \xi_T^n\}. \quad (3.19)$$

Here $\mathcal{L}\{\dots\}$ denotes the linear subspace spanned by the indicated random variables in $L_2(\Omega, \mathcal{F}_T, P_{t,x})$, and

$$H_\alpha = \sum_{k=1}^n \alpha_k h_k(\xi_T).$$

Proof. For $0 \leq \tau \leq T$ we have

$$v_i(T - \tau, x) = \frac{1}{(2\pi\tau)^{n/2}} \int_{-\infty}^{\infty} h_i(y) e^{-|y-x|^2/2\tau} dy,$$

so that

$$\begin{aligned}\frac{\partial v_i}{\partial x_j}(t, x) &= \frac{1}{\tau} \frac{1}{(2\pi\tau)^{n/2}} \int_{-\infty}^{\infty} h_i(y)(y_j - x_j) e^{-|y-x|^2/2\tau} dy \\ &= \frac{1}{\tau} E_{t,x}[h_i(\xi_T)(\xi_T^j - x_j)] \\ &= \frac{1}{\tau} \text{cov}_{t,x}(h_i(\xi_T), \xi_T^j).\end{aligned}$$

Thus

$$\nabla v_i(t, x) = \frac{1}{\tau} \text{cov}(H_i, \xi_T) \quad (3.20)$$

where $H_i = h_i(\xi_T)$ and $\text{cov}(\cdots)$ denotes the obvious componentwise covariance. Defining H_α as in (3.19) we therefore have

$$\tau G\alpha = \sum_{i=1}^n \alpha_i \text{cov}(H_i, \xi_T) = \text{cov}(H_\alpha, \xi_T),$$

so that G is singular when $H_\alpha = 0$ or H_α is, for some $\alpha \neq 0$, orthogonal to the linear span $\mathcal{L}\{\xi_T^1, \dots, \xi_T^n\}$. $\square$

Since H_α is a function of ξ_T , proposition 3.2 shows that G is in some sense ‘generically’ non-singular. However, it is singular at all (t, x) if either the functions h_i are linearly dependent (so $H_\alpha \equiv 0$ for some α), or the h_i functions do not depend on all coordinates of x . For example, if $h_i(x) = \tilde{h}_i(x_2, \dots, x_n)$, $i = 1, \dots, n$ for some functions $\tilde{h}_i$, then $\text{cov}(H_i, \xi_T^1) = 0$, so that $\text{rank}(G) \leq n-1$. A more subtle example, based on the idea that Z and $Z^2 - 1$ are uncorrelated for $Z \sim N(0, 1)$, is this. Let $h_i(x) = x_1^2 + 2cx_1$ for some constant c . Then

$$E_{t,x}[h_1(\xi_T)(\xi_T^1 - x_1)] = 2(T-t)(x_1 + c),$$

while plainly

$$E_{t,x}[h_1(\xi_T)(\xi_T^k - x_k)] = 0, \quad k > 1.$$

Thus G is singular on the subspace $\{x : x_1 + c = 0\}$.

When ξ_t is a non-degenerate diffusion process, a generalization of (3.20) can be obtained by using a version of the ‘Bismut formula’ (Bismut 1984; Elworthy & Li 1994) which we will describe following the approach to stochastic flow theory of Elliott & Kohlmann (1989). We suppose that the process $\xi_{s,t}$ satisfies the stochastic differential equation

$$d\xi_{s,t}(x) = m(\xi_{s,t}(x))dt + \sum_{i=1}^n \sigma_i(\xi_{s,t}(x))dw_t^i, \quad \xi_{s,s} = x \quad (3.21)$$

where $w_t = (w_t^1, \dots, w_t^n)$ is Brownian motion in R^n and m, σ_i are smooth functions with bounded derivatives. It is well known (see Rogers and Williams 2000, section V.13) that under these conditions with probability one the map $x \mapsto \xi_{s,t}(x)$ is smooth, and the Jacobian $D_{s,t} = \partial \xi_{s,t}(x) / \partial x$ satisfies

$$dD_{s,t} = \frac{\partial m}{\partial x} D_{s,t} dt + \sum_{i=1}^n \frac{\partial \sigma_i}{\partial x} D_{s,t} dw_t^i, \quad D_{s,s} = I.$$

For a smooth function h , the Clark-Haussmann-Ocone formula gives the following stochastic integral representation of the random variable $h(\xi_{s,T}(x))$ (Davis 1980, Elliott & Kohlmann 1989)

$$h(\xi_{s,T}(x)) = E_{s,x}[h] + \int_s^T E[\nabla h(\xi_{s,T}(x))D_{s,T}|\mathcal{F}_t]D_{s,t}^{-1}\Sigma(\xi_{s,t}(x))dw_t \quad (3.22)$$

(Σ is the matrix whose i 'th column is σ_i .) From (3.22) we immediately obtain the following 'integration by parts' formula: for a vector process u such that $E_{s,x} \int_s^T |u_t|^2 dt < \infty$ we have

$$\sum_1^n E \left[h(\xi_{s,T}(x)) \int_s^T u_t^i dw_t^i \right] = \int_s^T E [\nabla h(\xi_{s,T}(x))D_{s,T}D_{s,t}^{-1}\Sigma(\xi_{s,t}(x))u_t dt] . \quad (3.23)$$

If U_t is an $n \times n$ matrix-valued process each of whose components satisfies the integrability condition then

$$E \left[h(\xi_{s,T}(x)) \int_s^T dw_t^T U_t \right] = \int_s^T E [\nabla h(\xi_{s,T}(x))D_{s,T}D_{s,t}^{-1}\Sigma(\xi_{s,t}(x))U_t dt] . \quad (3.24)$$

Indeed, (3.24) is an equality between row vectors in which the j 'th column is (3.23) with u_t equal to the j 'th column of U_t .

Define $v(s, x) = E[h(\xi_{s,T}(x))]$; then $\nabla v(s, x) = E[\nabla h(\xi_{s,T}(x))D_{s,T}]$, and taking $U_t = \Sigma^{-1}D_{s,t}$, so that $D_{s,t}^{-1}\Sigma U = I$, we obtain from (3.24) the following version of the Bismut formula (Bismut 1984):

$$\nabla v(s, x) = \frac{1}{T-s} E \left[h(\xi_{s,T}(x)) \int_s^T dw_t^T \Sigma^{-1}(\xi_{s,t}(x))D_{s,t} \right] \quad (3.25)$$

We can now state the generalization of Proposition 3.2.

Proposition 3.3. *Suppose $\xi_{s,t}$ satisfies (3.21) and let v_i and G be defined by (3.17) and (3.18) respectively. For fixed (s, x) define random variables $Y_1, \dots, Y_n$ by*

$$Y_j = \sum_{i=1}^n \int_s^T (\Sigma^{-1}(\xi_{s,t}(x))D_{s,t})_{ij} dw_t^i.$$

Then $G(s, x)$ is non-singular if and only if there exists no non-zero vector $\alpha \in R^n$ such that

$$H_\alpha \perp \mathcal{L}\{Y_1, \dots, Y_n\} \quad (3.26)$$

in $L_2(\Omega, \mathcal{F}_T, P_{s,x})$, where

$$H_\alpha = \sum_{k=1}^n \alpha_k h_k(\xi_{s,T}(x)).$$

Proof. From (3.25), for $\alpha \in R^n$

$$\begin{aligned} (T-s)\alpha G(s, x) &= \sum_{i=1}^n \alpha_i E \left[h(\xi_{s,T}(x)) \int_s^T dw_t^T \Sigma^{-1}(\xi_{s,t}(x))D_{s,t} \right] \\ &= (\text{cov}(H_\alpha, Y_1), \dots, \text{cov}(H_\alpha, Y_n)). \end{aligned}$$

The result follows. □

Elworthy & Li (1994) give the Bismut formula in a geometric setting. Singularity of G can then be interpreted in terms of vanishing of the first component of a certain Wiener chaos expansion of H_α . The implications of this will be explored in later work.

Thanks to Robert Tompkins for discussions on stochastic volatility, to participants at the Financial Options Research Centre conference, Warwick University, September 2002, for helpful comments and suggestions, and to David Elworthy and Alexander Grigor'yan for pointing out interesting directions in connection with the non-singularity problem.

Appendix A. The Hörmander theorem

Full details of the following can be found in section IV.38 of Rogers and Williams (2000), or section 2.3 of Nualart (1995).

For two continuous semimartingales X, Y the Stratonovich integral is defined as

$$\int_0^t Y \circ dX = \int_0^t Y dX + \frac{1}{2} \langle Y, X \rangle_t,$$

where $\langle Y, X \rangle$ denotes the joint quadratic variation of X, Y . If X, Y are Brownian integrals $X = \int \phi dw, Y = \int \psi dw$ then $\langle X, Y \rangle = \int \phi \psi dt$. The Ito formula expressed in terms of Stratonovich integrals coincides with the ordinary Newton-Leibnitz formula.

If we have a stochastic differential equation for $\xi_t \in R^n$ written in Stratonovich form

$$d\xi_t = m(\xi_t)dt + \sum_1^d \sigma_i(\xi_t) \circ dw_i(t) \quad (A1)$$

then for any smooth, real-valued function f ,

$$df(\xi_t) = A_0 f(\xi_t)dt + \sum_1^d A_i f(\xi_t) \circ dw_i(t)$$

where A_j are the vector fields

$$A_0 f(x) = \sum_1^n m^k(x) \frac{\partial f}{\partial x_k}, \quad A_i f(x) = \sum_1^n \sigma_i^k(x) \frac{\partial f}{\partial x_k}, i = 1 \dots d.$$

The *Lie bracket* of two vector fields A_i, A_j is the vector field $[A_i, A_j] = A_i A_j - A_j A_i$.

Theorem 3.4. *Suppose that the coefficients of the SDE (A1) are infinitely differentiable with bounded derivatives of all orders, and that the vector space spanned by the vector fields*

$$A_1, \dots, A_d, [A_i, A_j], 0 \leq i, j \leq d, [A_i, [A_j, A_k]], 0 \leq i, j, k \leq d, \dots$$

at the initial point x_0 is equal to R^n . Then for any $t > 0$ the random vector ξ_t has a density that is absolutely continuous with respect to Lebesgue measure.

References

- Babbar, K. 2001 PhD Thesis, Imperial College London.
- Bajeux-Besnainou, I. and Rochet, J.-C. 1996 Dynamic spanning: are options an appropriate instrument?, *Mathematical Finance* **6**, 1–16
- Barndorff-Nielsen, O.E and Shephard, N. 2001 Non-gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics (with discussion), *JRSS(B)* **63**, 167–241.
- Bergman, Y.Z., Grundy, R.D. and Wiener, Z. 1996 General properties of option prices, *Journal of Finance* **51**, 1573–1610.
- Bismut, J.-M. 1984 *Large deviations and the Malliavin calculus*, Boston: Birkhäuser.
- Black, F. and Scholes, M. 1973 The pricing of options and corporate liabilities, *J. Political Economy* **81**, 637–654.
- Davis, M.H.A. 1980 Functionals of diffusion processes as stochastic integrals, *Math. Proc. Camb. Phil. Soc.* **87**, 157–166
- Davis, M. 2001 Mathematics of financial markets. In *Mathematics Unlimited: 2001 and Beyond* (eds B. Engquist and W. Schmid), Heidelberg: Springer-Verlag.
- Derman, E and Kani, I. 1998 Stochastic implied trees: arbitrage pricing with stochastic term and strike structure of volatility, *Int. J. Theor. and Appl. Finance* **1** 61–110.
- Dowd, K. 1998 *Beyond value at risk: the new science of risk management*, Chichester: Wiley.
- Dupire, B. 1994 Pricing with a smile, *Risk* **7**, 18–20.
- El Karoui, N., Jeanblanc-Picqué, M and Shreve, S.E. 1998 Robustness of the Black and Scholes formula, *Mathematical Finance* **8**, 93–126.
- Elliott, R.J. and Kohlmann, M. 1989 Integration by parts, homogeneous chaos expansions and smooth densities, *Ann. Prob.* **17**, 194–207.
- Elworthy, K.D. and Li, X.-M. 1994 Formulae for the derivatives of heat semigroups, *J. Functional Anal.* **125**, 252–286.
- Fouque, J.-P., Papanicolaou, G. and Sircar, K.R. 2000 *Derivatives in financial markets with stochastic volatility*. Cambridge University Press.
- Ghysels, E., Harvey, A.C. and Renault, E. 1996 Stochastic volatility. In *Handbook of statistics* Vol. 14 (eds G.S. Maddala and C.R. Rao), Amsterdam: Elsevier.
- Heston, S.L. 1993 A closed-form solution for options with stochastic volatility with applications to bond and currency options, *Rev. Fin. Studies* **6**, 327–343.
- Hobson, D.G. and Rogers, L.C.G. 1998 Complete models with stochastic volatility, *Mathematical Finance* **8**, 27–48.
- Hull, J.C. 2000 *Options, Futures and Other Derivatives*, 4th ed., Upper Saddle River NJ: Prentice Hall.
- Hull, J.C. and White, A. 1987 The pricing of options on assets with stochastic volatility, *J. Finance* **42**, 281–300.
- Nualart, D. 1995 *The Malliavin calculus and related topics*, Heidelberg: Springer-Verlag.
- Rogers, L.C.G. and Williams, D. 2000 *Diffusions, Markov processes and martingales*, Cambridge University Press.
- Romano, M. and Touzi, N. 1997 Contingent claims and market completeness in a stochastic volatility model, *Mathematical Finance* **7**, 399–410.
- Schönbucher, P.J. 1999 A market model for stochastic implied volatility, *Phil. Trans. R. Soc. Lond. A* **357**, 2071–2092.
- R. Tompkins 2001 Stock index futures markets: stochastic volatility models and smiles, *Journal of Futures Markets* **21**, 43–78.