The ℓ -modular local Langlands correspondence and local constants

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Abstract

Let F be a non-archimedean local field of residual characteristic p , $\ell \neq p$ be a prime number, and W_F the Weil group of F . We classify equivalence classes of W_F -semisimple Deligne ℓ -modular representations of W_F in terms of irreducible ℓ -modular representations of W_F , and extend constructions of Artin–Deligne local constants to this setting. Finally, we define a variant of the ℓ -modular local Langlands correspondence which satisfies a preservation of local constants statement for pairs of generic representations.

1 Introduction

Let F denote a non-archimedean local field of residual cardinality q and residual characteristic p . Let ℓ be a prime different to p . We consider only smooth representations of locally profinite groups, and call them ℓ -adic when they act on $\overline{\mathbb{Q}_\ell}$ -vector spaces, and ℓ -modular when they act on $\overline{\mathbb{F}_\ell}$ -vector spaces. Let W_F denote the Weil group of F .

The local Langlands correspondence LLC for $\mathrm{GL}_n(F)$ is a canonical bijection between the set of isomorphism classes of ℓ -adic irreducible representations of $\mathrm{GL}_n(F)$ and the set of isomorphism classes of ℓ -adic n -dimensional W_F -semisimple Deligne representations, generalizing the Artin reciprocity map of local class field theory. An important property of LLC is that the Rankin–Selberg local constants of a pair of irreducible ℓ -adic representations of $\mathrm{GL}_n(F)$ and $\mathrm{GL}_m(F)$, and the Artin–Deligne local constants of the corresponding tensor product of Deligne representations of W_F are equal, and moreover this condition characterizes LLC completely.

In [19], Vignéras develops the theory of ℓ -modular representations of reductive p -adic groups. For general linear groups, this culminates in the ℓ -modular local Langlands correspondence [24]: a bijection between the set of isomorphism classes of ℓ -modular irreducible representations of $\mathrm{GL}_n(F)$ and the set of isomorphism classes of ℓ -modular n -dimensional W_F -semisimple Deligne representations with nilpotent Deligne operator. Vignéras characterizes her correspondence by compatibility with LLC and congruences, although not naively (see Section 6.2 where we recall Vignéras’ results precisely).

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The theory of Rankin–Selberg local constants of Jacquet, Piatetski-Shapiro and Shalika has a natural extension to ℓ -modular generic representations of $\mathrm{GL}_n(F)$ and $\mathrm{GL}_m(F)$, [12]. However, via the ℓ -modular local Langlands correspondence these factors do not agree with the factors of Artin–Deligne.

In this work, we classify the ℓ -modular indecomposable W_F -semisimple Deligne representations, extend the definitions of Artin–Deligne factors to this setting, and define an ℓ -modular local Langlands correspondence where in the generic case, the Rankin–Selberg factors of representations on one side equal the Artin–Deligne factors of the corresponding representations on the other.

We now recall our definitions and conventions, and state our results precisely. Let R be an algebraically closed field of characteristic ℓ different from p . Let $\nu : W_F \rightarrow R^\times$ be the unique character trivial on the inertia subgroup of W_F and sending a geometric Frobenius element to q^{-1} . Fix a nontrivial character $\psi : F \rightarrow R^\times$.

A *Deligne R -representation* of W_F is a pair (Φ, U) with Φ a finite dimensional smooth representation of W_F acting on an R -vector space, and $U \in \mathrm{Hom}_{W_F}(\nu\Phi, \Phi)$ the associated Deligne operator. We say a Deligne R -representation (Φ, U) of W_F is *nilpotent* if U is a nilpotent endomorphism of the vector space of Φ , and *W_F -semisimple* if Φ is a semisimple representation of W_F . We write $(\Phi, U)^\vee$ for the *dual* of (Φ, U) , see Definition 4.3. All Deligne $\overline{\mathbb{Q}}_\ell$ -representations of W_F are nilpotent, however as we shall soon see this is not the case for Deligne $\overline{\mathbb{F}}_\ell$ -representations.

We say that indecomposable Deligne R -representations $(\Phi, U), (\Phi', U')$ of W_F are *equivalent*, and write $(\Phi, U) \sim (\Phi', U')$, if there exists $\lambda \in R^\times$ such that $(\Phi, \lambda U) \simeq (\Phi', U')$, and we extend this definition and notation to general Deligne R -representations of W_F thanks to the Krull–Schmidt theorem (see Definition 4.6 and Remark 4.7).

Before stating our classification results, we need to define two fundamental examples. For Ψ an irreducible R -representation W_F , we denote by $\mathbb{Z}_\Psi$ the set $\{\nu^k \Psi / \simeq, k \in \mathbb{Z}\}$ of isomorphism classes.

- (1) Let Ψ be an irreducible $\overline{\mathbb{F}}_\ell$ -representation of W_F , then there is a minimal positive integer $o(\Psi)$ such that we have an isomorphism $\nu^{o(\Psi)} \Psi$ to Ψ . Let I be such an isomorphism. Define a Deligne $\overline{\mathbb{F}}_\ell$ -representation

$$\begin{aligned} \mathcal{C}(\Psi, I) &= (\Phi(\Psi), \mathcal{C}_I) \\ \Phi(\Psi) &= \bigoplus_{k=0}^{o(\Psi)-1} \nu^k \Psi \\ \mathcal{C}_I(x_0, \dots, x_{o(\Psi)-1}) &= (I(x_{o(\Psi)-1}), x_0, \dots, x_{o(\Psi)-2}). \end{aligned}$$

In Proposition 4.18, we show that $\mathcal{C}(\Psi, I)$ is irreducible. The equivalence class of $\mathcal{C}(\Psi, I)$ depends only on $\mathbb{Z}_\Psi$, and we denote it by $\mathcal{C}(\mathbb{Z}_\Psi)$.

- (2) For a positive integer r , we define a W_F -semisimple nilpotent Deligne $\overline{\mathbb{F}}_\ell$ -representation

$$\begin{aligned} [0, r-1] &= (\Phi(r), N(r)) \\ \Phi(r) &= \bigoplus_{k=0}^{r-1} \nu^k, \\ N(r)(x_0, \dots, x_{r-1}) &= (0, x_0, \dots, x_{r-2}). \end{aligned}$$

In Lemma 4.27, we show that $[0, r-1]$ is indecomposable.

We prove the following classification:

Main Theorem 1 (Theorems 4.24 and 4.35). Let (Φ, U) be an indecomposable W_F -semisimple Deligne $\overline{\mathbb{F}}_\ell$ -representation.

(1) There exists an irreducible $\overline{\mathbb{F}}_\ell$ -representation Ψ of W_F and a positive integer r such that either:

(a)

$$(\Phi, U) \sim [0, r-1] \otimes \Psi;$$

(b) there exists an isomorphism $I : \nu^{o(\Psi)}\Psi \rightarrow \Psi$ such that

$$(\Phi, U) \sim [0, r-1] \otimes \mathcal{C}(\Psi, I).$$

(2) Under the notation of the last part, (Φ, U) is irreducible if and only if $r = 1$.

(3) In case (1a), the isomorphism class of $[0, r-1] \otimes \Psi$ coincides with its equivalence class, and determines the isomorphism class of Ψ and r uniquely.

(4) In case (1b), the equivalence class of $[0, r-1] \otimes \mathcal{C}(\Psi, I)$ only depends on r and $\mathbb{Z}_\Psi$, and we denote it by $[0, r-1] \otimes \mathcal{C}(\mathbb{Z}_\Psi)$. The equivalence class determines $\mathbb{Z}_\Psi$ and r uniquely.

Let (Φ, U) be a W_F -semisimple Deligne R -representation of W_F . For an indeterminant X , we put

$$L(X, (\Phi, U)) = \det((\text{Id} - X\Phi(\text{Frob}))|_{\text{Ker}(U)^{I_F}})^{-1},$$

it is an Euler factor. Using results of [8], for ψ a nontrivial character of F , we associate to (Φ, U) a local γ -factor $\gamma(X, (\Phi, U), \psi)$, which does not see the operator U , and put

$$\epsilon(X, (\Phi, U), \psi) = \gamma(X, (\Phi, U), \psi) \frac{L(X, (\Phi, U))}{L(q^{-1}X^{-1}, (\Phi, U)^\vee)}.$$

Of course, these definitions coincide with the standard ones when $R = \overline{\mathbb{Q}}_\ell$, as well as in the case when $R = \overline{\mathbb{F}}_\ell$ and (Φ, U) is of the form $(\Phi, 0)$, see Section 5.

Following the $\overline{\mathbb{Q}}_\ell$ -case, we wish to define Artin–Deligne local constants of pairs of W_F -semisimple Deligne $\overline{\mathbb{F}}_\ell$ -representations of W_F via their tensor product. However, there are two immediate obstacles:

- (1) The tensor product of semisimple representations of W_F is not necessarily semisimple, see Example 3.11 (3).
- (2) The tensor product does not preserve equivalence, see Example 4.11.

Both of these problems have natural solutions and we define the semisimple tensor product $\otimes_{\text{ss}}$ of equivalence classes of W_F -semisimple Deligne $\overline{\mathbb{F}}_\ell$ -representations of W_F in Section 4.4.

Finally, we move on to our results on the ℓ -modular local Langlands correspondence. We call an isomorphism class of W_F -semisimple nilpotent Deligne $\overline{\mathbb{F}}_\ell$ -representations a *V-parameter* and, for a positive integer n , denote by $\mathbf{V}$ the bijection of [24]:

$$\left\{ \begin{array}{c} \text{Irreducible } \ell\text{-modular representations} \\ \text{of } \text{GL}_n(F) \text{ up to isomorphism} \end{array} \right\} \xrightarrow{\mathbf{V}} \left\{ \begin{array}{c} \text{V-parameters of dimension } n \end{array} \right\}$$

For ψ a nontrivial character of F and π, π' generic $\overline{\mathbb{F}_\ell}$ -representations of $\mathrm{GL}_n(F), \mathrm{GL}_m(F)$ respectively we denote by

$$L(X, \pi, \pi'), \quad \epsilon(X, \pi, \pi', \psi), \quad \gamma(X, \pi, \pi', \psi),$$

the local constants defined in [12].

The motivation for our next results is that the correspondence V does not preserve local constants of pairs of generic representations, for example it is not true that

$$L(X, \pi, \pi') = L(X, V(\pi) \otimes_{\mathrm{ss}} V(\pi')),$$

see Example 6.12. In Definition 6.7, we define an injective map:

$$\mathrm{CV} : \left\{ \text{V-parameters} \right\} \longrightarrow \left\{ \begin{array}{c} \text{Equivalence classes of } W_F\text{-semisimple} \\ \text{Deligne } \overline{\mathbb{F}_\ell}\text{-representations of } W_F \end{array} \right\}$$

which is not the natural inclusion, we call an element in its image a *C-parameter*. Explicitly, a V-parameter (Φ, U) can be written as:

$$(\Phi, U) = \bigoplus_{i \geq 1} \bigoplus_{k=0}^{o(\Psi)-1} a_{i,k} [0, i-1] \otimes \nu^k \Psi,$$

with all $a_{i,k}$ except possibly a finite number equal to zero, and we define $\mathrm{CV}(\Phi, U)$ by, for each i , replacing every instance of a complete cycle $\bigoplus_{k=0}^{o(\Psi)-1} [0, i-1] \otimes \nu^k \Psi$ with $[0, i-1] \otimes \mathcal{C}(\mathbb{Z}_\Psi)$:

$$\mathrm{CV}(\Phi, U) = \bigoplus_{i \geq 1} \bigoplus_{k=0}^{o(\Psi)-1} c_{i,k} [0, i-1] \otimes \nu^k \Psi \oplus \bigoplus_{i \geq 1} b_i [0, i-1] \otimes \mathcal{C}(\mathbb{Z}_\Psi),$$

where $b_i = \min\{a_{i,k} : 0 \leq k \leq o(\Psi) - 1\}$ and $c_{i,k} = a_{i,k} - b_i$ (see Section 6.3 for further details).

We can now state our main result, the first three properties of which are immediate consequence of their analogues for V and the definition of CV :

Main Theorem 2. [Lemma 6.11 and Theorem 6.13] For positive integers n , the bijections $C = \mathrm{CV} \circ V$:

$$\left\{ \begin{array}{c} \text{Irreducible } \ell\text{-modular representations} \\ \text{of } \mathrm{GL}_n(F) \text{ up to isomorphism} \end{array} \right\} \xrightarrow{C} \left\{ \text{C-parameters of dimension } n \right\}$$

satisfy the following properties:

Let π be an irreducible ℓ -modular representation of $\mathrm{GL}_n(F)$.

(1) For all characters $\chi : \mathrm{GL}_n(F) \rightarrow \overline{\mathbb{F}_\ell}^\times$,

$$C(\chi\pi) = \chi C(\pi).$$

(2) Letting c_π denote the central character of π , then via the Artin reciprocity map of local class field theory

$$c_\pi = \det(C(\pi)).$$

(3) Commutation with (smooth) duals:

$$C(\pi^\vee) = C(\pi)^\vee.$$

(4) If π is generic, for all positive integers m and all generic ℓ -modular representations π' of $\mathrm{GL}_m(F)$,

$$\begin{aligned} L(X, \pi, \pi') &= L(X, C(\pi) \otimes_{\mathrm{ss}} C(\pi')), \\ \gamma(X, \pi, \pi', \psi) &= \gamma(X, C(\pi) \otimes_{\mathrm{ss}} C(\pi'), \psi), \\ \epsilon(X, \pi, \pi', \psi) &= \epsilon(X, C(\pi) \otimes_{\mathrm{ss}} C(\pi'), \psi). \end{aligned}$$

In Example 6.10, we give examples of C-parameters of special classes of representations.

As a corollary of our results, if one defines the L, ϵ, γ -factors of a V-parameter Φ to be the local constants of the corresponding C-parameter $\mathrm{CV}(\Phi)$, then we have a *preservation of local factors* for generic representations result for V. However, as this construction goes via the associated C-parameter it feels more natural to us to state the result in terms of C.

It is tempting to say that C should preserve local factors of pairs of irreducible ℓ -modular representations of $\mathrm{GL}_n(F)$ beyond the generic setting, for example the *Godement–Jacquet local factors* defined in [13]. This holds for $n = 2$ for the factors of [ibid.], thanks to the explicit computations of Mínguez in this case.

Vignéras’ correspondence is characterized by a non-naive, but precise compatibility with reduction modulo ℓ . At present there could be other correspondences to C which satisfy preservation of local factors, and even after fixing the set of C-parameters it is not clear whether or not the list of properties of Theorem 2 characterize the correspondence. We leave this question for future work.

Finally, we note other complementary results in the literature. Dat has partially realized Vignéras’ correspondence (denoted V above) in cohomology in [5, 6]. Starting with Vignéras’ correspondence in the supercuspidal case, Emerton and Helm [9] have introduced a *modified mod ℓ local Langlands correspondence*; this has a different flavour: it is a map from (isomorphism classes of) n -dimensional continuous representations of the absolute Galois group of F to a class of modular representations of $\mathrm{GL}_n(F)$ called *essentially AIG* (which are not necessarily irreducible). It would be interesting to further elucidate the connections between these two correspondences beyond the supercuspidal case, and to understand the structure of the modular essentially AIG representations which currently are only fully understood when $n = 2$ [11].

We now describe the structure of the paper: Section 2 introduces the most general notation of the paper. In Section 3 we recall elements from the representation theory of W_F , paying special attention to the tensor product. The first key section is Section 4, where we introduce the definition of a Deligne representation and prove Main Theorem 1. In Section 5, we extend the definitions of Artin–Deligne local constants to our setting of V and C-parameters. The second key section is the final section which recalls the V-correspondence, defines the C-correspondence, and proves Main Theorem 2.

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2 General Notation

Let F be a non-archimedean local field with finite residue field k_F of characteristic p and cardinality q . For a positive integer n , we denote by F_n the unique (up to isomorphism) unramified extension of F of degree n .

Let ℓ be a prime number different from p . We fix an algebraic closure $\overline{\mathbb{Q}_\ell}$ of the ℓ -adic numbers $\mathbb{Q}_\ell$. We denote by $\overline{\mathbb{Z}_\ell}$ the ring of integers of $\overline{\mathbb{Q}_\ell}$ and by $\mathfrak{m}$ its maximal ideal. We put $\overline{\mathbb{F}_\ell} = \overline{\mathbb{Z}_\ell}/\mathfrak{m}$, it is an algebraic closure of the finite field $\mathbb{F}_\ell$ with ℓ elements.

We consider smooth representations of locally profinite groups on $\overline{\mathbb{Q}_\ell}$ and $\overline{\mathbb{F}_\ell}$ -vector spaces, and the connections between them. Henceforth, all representations considered are implicitly assumed to be smooth and, for convenience of stating results which apply to both cases $\overline{\mathbb{Q}_\ell}$ and $\overline{\mathbb{F}_\ell}$, we let R denote either field. We abbreviate “representation on an R -vector space” to *R -representation*. When $R = \overline{\mathbb{F}_\ell}$ we say we are in the *ℓ -modular case*, and when $R = \overline{\mathbb{Q}_\ell}$ we say we are in the *ℓ -adic case*.

Let H be a locally profinite group. Let $\text{Rep}(H, R)$ denote the abelian category of R -representations of H , and $\text{Irr}(H, R)$ denote the set of isomorphism classes of irreducible R -representations of H . Let π, π' be R -representations of H . We denote by $\pi^\vee$ the (smooth) contragredient of π , and put

$$\text{End}_H(\pi) = \text{Hom}_H(\pi, \pi), \text{ and } \text{Iso}_H(\pi, \pi') = \{\phi \in \text{Hom}_H(\pi, \pi') : \phi \text{ is bijective}\}.$$

Let K be a closed subgroup of H . We fix a square root of q in $\overline{\mathbb{Z}_\ell}$ and take its image in $\overline{\mathbb{F}_\ell}$ so we have a fixed square root of q in R , which is compatible with reduction modulo ℓ . We denote by

$$\text{Ind}_K^H : \text{Rep}(K, R) \rightarrow \text{Rep}(H, R)$$

the usual normalized (with respect to this square root of q) induction functor.

Strictly speaking an R -representation consists of a pair (Φ, V) with V an R -vector space and

$$\Phi : H \rightarrow \text{GL}_R(V)$$

a group homomorphism. However, as we have done already, we will often denote the pair (Φ, V) just by Φ , and in this case we will use V_Φ , or even Φ , to denote the underlying vector space upon which H acts via Φ . We also often make no distinction between an R -representation and its isomorphism class. Finally, we will write $\Phi \in \text{Rep}(H, R)$ for Φ is an object in the category $\text{Rep}(H, R)$, i.e. an R -representation of H .

3 Representations of W_F

3.1 Notation

We refer to [3, Chapter 7] for the definitions and facts stated here concerning Weil groups. We fix a separable algebraic closure $\overline{F}$ of F , and will suppose that all finite extensions we consider

are contained in $\overline{F}$. For a finite extension E/F we let $G_E = \text{Gal}(\overline{F}/E)$ denote the *absolute Galois group* of E ; I_E denote the *inertia subgroup* of G_E ; P_E denote the *wild inertia subgroup*, it is the pro- p Sylow subgroup of I_E ; W_E denote the *Weil group* of E . We fix a geometric Frobenius element Frob in W_F . We have

$$W_F = I_F \rtimes \text{Frob}^{\mathbb{Z}},$$

and, in particular, W_F is unimodular as I_F is compact.

If E is a finite extension of F , and π is a representation of W_F , we denote by π_E the restriction of π to W_E . An R -representation of W_F is called *unramified* if it is trivial on I_F . If such a representation is irreducible, then it is necessarily a character as W_F/I_F is abelian. In fact, if Ψ is an irreducible representation of W_F such that Ψ^{I_F} (the I_F -fixed subspace of Ψ) is nonzero, then Ψ^{I_F} is a nonzero W_F -invariant subspace of Ψ , hence $\Psi = \Psi^{I_F}$ is an unramified character. We denote by ν the unramified character of W_F which satisfies $\nu(\text{Frob}) = q^{-1}$, and by $X^u(W_F)$ the group of unramified characters of W_F .

An R -irreducible representation of W_F is called *tamely ramified* if it is trivial on P_F . We denote by $\text{Irr}^{\text{tr}}(W_F, R)$ the set of isomorphism classes of tamely ramified irreducible representations of W_F .

Let $\Psi \in \text{Irr}(W_F, R)$, we put

$$\mathbb{Z}_{\Psi} = \{\nu^k \Psi / \simeq, k \in \mathbb{Z}\}$$

and call such a set of isomorphism classes an *irreducible line*. When $R = \overline{\mathbb{F}}_{\ell}$, the set $\mathbb{Z}_{\Psi}$ is finite, and we denote by $o(\Psi)$ its cardinality

$$o(\Psi) = |\mathbb{Z}_{\Psi}|.$$

The integer $o(\Psi)$ clearly divides the order of ν which is the order of q modulo ℓ . In particular $o(\Psi)$ divides $\ell - 1$ and is hence prime to ℓ . We say that Ψ is *banal* if $o(\Psi) > 1$, that is $\Psi \neq \Psi\nu$.

3.2 Irreducible representations of W_F

We now recall a description of the irreducible R -representations of W_F as induced representation which is well suited to studying congruences between $\mathbb{Q}_{\ell}$ -representations.

Let E be a finite Galois extension of F and $\pi \in \text{Irr}(W_E, R)$. For $\sigma \in \text{Gal}(E/F) \simeq W_F/W_E$, we denote by π^{σ} the R -representation of W_E defined on the same underlying space as π by

$$\pi^{\sigma}(w) = \pi(\sigma w \sigma^{-1}),$$

for all $w \in W_E$.

We say that π is $\text{Gal}(E/F)$ -regular if the set

$$\{\pi^{\sigma} : \sigma \in \text{Gal}(E/F)\} \subseteq \text{Irr}(W_E, R)$$

is of cardinality $|\text{Gal}(E/F)|$.

Lemma 3.1. Let E be a finite Galois extension of F and τ an R -representation of W_F . Then $\text{Ind}_{W_E}^{W_F}(\tau)$ is irreducible if and only if τ is irreducible and $\text{Gal}(E/F)$ -regular.

Proof. If $\text{Ind}_{W_E}^{W_F}(\tau)$ is irreducible then τ is as well, by exactness of the induction functor. Moreover its endomorphism algebra is isomorphic to R by Schur's lemma. However by Frobenius reciprocity and Mackey theory we have:

$$\begin{aligned} \text{End}_{W_F}(\text{Ind}_{W_E}^{W_F}(\tau)) &\simeq \text{Hom}_{W_E}(\text{Ind}_{W_E}^{W_F}(\tau)|_{W_E}, \tau) \\ &\simeq \text{Hom}_{W_E}\left(\bigoplus_{\sigma \in \text{Gal}(E/F)} \tau^\sigma, \tau\right) \\ &\simeq \prod_{\sigma \in \text{Gal}(E/F)} \text{Hom}_{W_E}(\tau^\sigma, \tau). \end{aligned}$$

This latter ring is isomorphic to R if and only if τ is regular.

Conversely, if τ is irreducible and $\text{Gal}(E/F)$ -regular, let W be a nonzero W_F -stable subspace of $\text{Ind}_{W_E}^{W_F}(\tau)$. Then it is a W_E -stable subspace of $\bigoplus_{\sigma \in \text{Gal}(E/F)} \tau^\sigma$ by Mackey theory. In particular as the regularity assumption implies that this direct sum is the decomposition into isotypic components of $\text{Ind}_{W_E}^{W_F}(\tau)|_{W_E}$, the space W contains τ^σ for some σ . As it is stable under W_F , it contains all τ^σ , hence it is equal to $\text{Ind}_{W_E}^{W_F}(\tau)$. $\square$

We recall the following well known description of the irreducible R -representations of W_F . It is a consequence of [4, Lemma 2.3, Proposition 2.1] in the ℓ -adic case, and of [25, 2.6] in the ℓ -modular case.

Theorem 3.2. (1) Let $\Psi \in \text{Irr}^{\text{tr}}(W_F, R)$ of dimension n , then there is a $\text{Gal}(F_n/F)$ -regular tamely ramified character χ of W_{F_n} such that

$$\Psi = \text{Ind}_{W_{F_n}}^{W_F}(\chi),$$

the character χ is unique up to conjugation by $\text{Gal}(F_n/F)$.

(2) Let $\Psi \in \text{Irr}(W_F, R)$, then there exist a (finite) tamely ramified extension E of F , $\Psi^{\text{tr}} \in \text{Irr}^{\text{tr}}(W_E, R)$, and $\tau \in \text{Irr}(W_E, R)$ which restricts irreducibly to P_F , such that

$$\Psi = \text{Ind}_{W_E}^{W_F}(\Psi^{\text{tr}} \otimes \tau),$$

the representation $\Psi^{\text{tr}} \otimes \tau$ is unique up to conjugation by $\text{Gal}(E/F)$. Moreover for fixed Ψ and τ , the representation Ψ^{tr} is unique.

For $\Psi \in \text{Irr}(W_F, R)$, we denote by $R(\Psi)$ the group of unramified characters of W_F fixing Ψ :

$$R(\Psi) = \{\mu \in X^u(W_F), \mu\Psi = \Psi\}.$$

Corollary 3.3. Let $\Psi \in \text{Irr}(W_F, R)$ and write $\Psi = \text{Ind}_{W_E}^{W_F}(\Psi^{\text{tr}} \otimes \tau)$ and $\Psi^{\text{tr}} = \text{Ind}_{W_{E_n}}^{W_E}(\chi)$, for E/F tamely ramified, a positive integer n , and χ a $\text{Gal}(E_n/E)$ -regular tamely ramified character of W_{E_n} as in Theorem 3.2. Put $r = [E_n : F]$, and $e = e(E_n/F)$ the ramification index of E_n/F . Then the map $\mu \mapsto \mu(\text{Frob})$ is an isomorphism between $R(\Psi)$ and the group of r/e -th roots of unity in $R^\times$.

Proof. The integer e is also the ramification index of E/F and we set $t = [E : F]$. One can take $\text{Frob}_E = \text{Frob}^{t/e}$ (where Frob_E stands for a geometric Frobenius element in W_E). Then an unramified character $\mu : W_F \rightarrow R^\times$ belongs to $R(\Psi)$ if and only if

$$\text{Ind}_{W_E}^{W_F}(\mu \Psi^{\text{tr}} \otimes \tau) = \text{Ind}_{W_E}^{W_F}(\Psi^{\text{tr}} \otimes \tau),$$

which is equivalent to

$$\mu_E \Psi^{\text{tr}} = \Psi^{\text{tr}}.$$

This is in turn equivalent to $\mu_{E_n} \chi$ is conjugate to χ by $\text{Gal}(E_n/E)$, i.e. μ_{E_n} is of the form χ^σ / χ for $\sigma \in \text{Gal}(E_n/E)$. Now $\text{Frob}_{E_n}^\sigma / \text{Frob}_{E_n} \in I_{E_n}$, hence μ_{E_n} is trivial on Frob_{E_n} , hence trivial as it is unramified. As one can take $\text{Frob}_{E_n} = (\text{Frob}_E)^n = \text{Frob}^{nt/e}$:

$$\mu \in R(\Psi) \Leftrightarrow \mu_{E_n} = 1 \Leftrightarrow \mu_{E_n}(\text{Frob}_{E_n}) = 1 \Leftrightarrow \mu(\text{Frob})^{nt/e} = 1 \Leftrightarrow \mu(\text{Frob})^{r/e} = 1.$$

□

As an immediate consequence we have:

Corollary 3.4. With the same notation as Corollary 3.3, we obtain that $o(\Psi) = o(\nu^{r/e}) = o(\nu_E^n)$.

3.3 Lattices and reduction modulo ℓ

An ℓ -adic finite dimensional representation Θ of W_F is called *integral* if there is a $\overline{\mathbb{Z}_\ell}[W_F]$ -stable lattice in V . In general, when we say *a lattice* in the space of an integral representation of W_F , we are always referring to a W_F -stable lattice. We denote by $\text{Irr}(W_F, \overline{\mathbb{Q}_\ell})_e$ the set of isomorphism classes of integral irreducible representations of W_F .

By [3, Proposition 28.6], if $\Theta \in \text{Irr}(W_F, \overline{\mathbb{Q}_\ell})$, then there is an unramified character μ of W_F such that $\mu\Theta$ extends to an element of $\text{Irr}(G_F, \overline{\mathbb{Q}_\ell})$. As the group G_F is profinite, the representation $\mu\Theta$ is automatically integral, and we deduce that Θ is integral if and only if μ takes values in $\overline{\mathbb{Z}_\ell}^\times$. As explained in [20, 1.8], this happens if and only if $\det(\Theta)$ takes values in $\overline{\mathbb{Z}_\ell}^\times$, hence one deduces the well known fact:

Lemma 3.5 ([20, Lemma 1.9]). The representation $\Theta \in \text{Irr}(W_F, \overline{\mathbb{Q}_\ell})$ is integral if and only if $\det(\Theta)$ is integral.

Let $\Theta \in \text{Irr}(W_F, \overline{\mathbb{Q}_\ell})_e$ and choose a lattice L in V_Θ . In this setting, we have a *Brauer–Nesbitt principle*:

Lemma 3.6 ([20, 1.10]). The semisimplification of the $\overline{\mathbb{F}_\ell}$ -representation $L/\mathfrak{m}L$ is independent of the choice of L .

(In fact, we did not need to assume Θ was irreducible, finite length would have been enough.) We let $r_\ell(\Theta) = (L/\mathfrak{m}L)_{\text{ss}}$ denote this semisimple $\overline{\mathbb{F}_\ell}$ -representation. If $r_\ell(\Theta)$ is irreducible, we let

$$\overline{\Theta} = r_\ell(\Theta),$$

and say that Θ *lifts* $\overline{\Theta}$.

Suppose now $\Psi \in \text{Irr}(W_F, \overline{\mathbb{F}_\ell})$ (because we treat ℓ -adic and ℓ -modular representations uniformly, in both cases we will often use Ψ for an element of $\text{Irr}(W_F, \overline{\mathbb{F}_\ell})$, but when needed for reduction modulo ℓ , we will use Θ in the ℓ -adic case as we did above). In this case, one can always lift Ψ to an irreducible representation in $\text{Irr}(W_F, \overline{\mathbb{Q}_\ell})_e$, this follows from the description of $\text{Irr}(W_F, R)$ given in Theorem 3.2, and is explained in [25, 2.6]. While not unique in general, when we wish to choose a lift of Ψ to $\text{Irr}(W_F, \overline{\mathbb{Q}_\ell})_e$ we will denote it by $\tilde{\Psi}$.

On the other hand, it is not true that the reduction modulo ℓ of a representation $\Theta \in \text{Irr}(W_F, \overline{\mathbb{Q}_\ell})_e$ is irreducible in general. However it is well understood, and described in [20, 1.16]. We first fix

some notation. Take $\Theta \in \text{Irr}(W_F, \overline{\mathbb{Q}_\ell})_e$ and let β be an integral unramified character of W_F such that $\beta\Theta$ extends to G_F . Write $\Theta = \text{Ind}_{W_E}^{W_F}(\Theta^{\text{tr}} \otimes \tau)$ with $\Theta^{\text{tr}} = \text{Ind}_{W_{E_n}}^{W_E}(\chi)$ for some positive n and some $\text{Gal}(E_n/E)$ -regular tamely ramified character χ of W_{E_n} , as in Theorem 3.2. Both Θ^{tr} and χ are integral, and $\chi' = \beta_{E_n}\chi$ is a character of finite order, which can uniquely be written $\chi' = \alpha_{E_n}\chi'_\ell$ for α a $\text{Gal}(E_d/E)$ -regular character of W_{E_d} where d is a divisor of n , and χ'_ℓ is of order a power of ℓ . The representation $\text{Ind}_{W_{E_d}}^{W_E}(\overline{\alpha})$ is an irreducible representation of W_E , so

$$\Psi(\alpha, \beta) = \overline{\beta_E}^{-1} \text{Ind}_{W_{E_d}}^{W_E}(\overline{\alpha}) = \text{Ind}_{W_{E_d}}^{W_E}(\overline{\beta_{E_d}^{-1}\alpha})$$

is too. The representation $r_\ell(\tau)$ is also irreducible, and we thus write it $\overline{\tau}$. The quotient $m = n/d$ is either equal to 1, or of the form $o(\nu_E^d)\ell^a$ for an integer $a \geq 0$.

Proposition 3.7. Let $\Theta \in \text{Irr}(W_F, \overline{\mathbb{Q}_\ell})_e$, then under the notation above one has

$$r_\ell(\Theta) = \bigoplus_{k=0}^{m-1} \nu^k \text{Ind}_{W_E}^{W_F}(\Psi(\alpha, \beta) \otimes \overline{\tau})$$

where $\text{Ind}_{W_E}^{W_F}(\Psi(\alpha, \beta) \otimes \overline{\tau})$ is irreducible.

According to Corollary 3.4, we can restate Proposition 3.7 in the following less precise but simpler form:

Proposition 3.8. Let $\Theta \in \text{Irr}(W_F, \overline{\mathbb{Q}_\ell})_e$, then either $r_\ell(\Theta)$ is irreducible, or it is of the form

$$r_\ell(\Theta) = \ell^a \left(\bigoplus_{k=0}^{o(\Psi)-1} \nu^k \Psi \right)$$

for $a \geq 0$ and $\Psi \in \text{Irr}(W_F, \overline{\mathbb{F}_\ell})$.

Proof. In Proposition 3.7, the integer m is of the form $o(\nu_E^d)\ell^a$ when not equal to 1, but $\Psi = \text{Ind}_{W_E}^{W_F}(\Psi(\alpha, \beta) \otimes \overline{\tau})$ satisfies $o(\Psi) = o(\nu_E^d)$ thanks to Corollary 3.4. The result is now a consequence of the definition of $o(\Psi)$. $\square$

We will denote by $\text{St}_a(\mathbb{Z}_\Psi)$ the representation $\ell^a \left(\bigoplus_{k=0}^{o(\Psi)-1} \nu^k \Psi \right)$ of the above proposition.

We deduce the following corollary we shall use later:

Corollary 3.9. Let $\Theta \in \text{Irr}(W_F, \overline{\mathbb{Q}_\ell})_e$. Then the following are equivalent:

- (1) $r_\ell(\Theta)^{I_F} \neq 0$.
- (2) There is a positive integer n , equal to 1 or otherwise of the form $n = o(\nu)\ell^r$ with $r \geq 0$, and an integral character χ of W_{E_n} with $\overline{\chi} = \mu_{E_n}$ for μ an unramified character of W_F , such that

$$\Theta = \text{Ind}_{W_{F_n}}^{W_F}(\chi).$$

When they are satisfied we have

$$r_\ell(\Theta)^{I_F} = r_\ell(\Theta) = \bigoplus_{k=0}^{n-1} \nu^k \mu = \ell^r \left(\bigoplus_{k=0}^{o(\nu)-1} \nu^k \mu \right).$$

Proof. If $\Theta = \text{Ind}_{W_{F_n}}^{W_F}(\chi)$ with χ as in the statement, then

$$r_\ell(\Theta)^{I_F} = r_\ell(\Theta) = \sum_{k=0}^{n-1} \nu^k \mu$$

thanks to Proposition 3.7. Conversely, suppose that $r_\ell(\Theta)^{I_F} \neq 0$. Write $\Theta = \text{Ind}_{W_E}^{W_F}(\Theta^{\text{tr}} \otimes \tau)$ and $\Theta^{\text{tr}} = \text{Ind}_{W_{E_n}}^{W_E}(\chi)$ as discussed before Proposition 3.7. From the same proposition, we have

$$r_\ell(\Theta) = \bigoplus_{k=0}^{m-1} \nu^k \text{Ind}_{W_E}^{W_F}(\Psi(\alpha, \beta) \otimes \bar{\tau})$$

with all of the $\nu^k \text{Ind}_{W_E}^{W_F}(\Psi(\alpha, \beta) \otimes \bar{\tau})$ irreducible. However one of them has nonzero I_F -fixed subspace by assumption, so they must all be unramified characters. The fact that they are characters implies that $F = E = E_d$, i.e. $d = 1$, and that $\chi = (\beta^{-1}\alpha)_{E_n} \chi'_\ell$, so that $\bar{\chi} = (\overline{\beta^{-1}\alpha})_{E_n}$. Moreover setting $\mu = \overline{\beta^{-1}\alpha}$, the character μ_{E_n} must be unramified, hence μ as well, as $I_{E_n} = I_F$. This proves the converse implication. $\square$

3.4 Tensor products of ℓ -modular representations of W_F

We now study $\Psi \otimes \Psi'$ when Ψ and Ψ' belong to $\text{Irr}(W_F, \overline{\mathbb{F}_\ell})$. First we observe some fundamental differences with the case of characteristic zero, where such a tensor product is semisimple: this will have consequences later in the paper, we will have to define a semisimple tensor product for W_F -semisimple Deligne representations. In fact when $R = \overline{\mathbb{Q}_\ell}$, by [3, Proposition 28.7], a finite dimensional representation Φ of W_F is semisimple if and only if it is *Frobenius semisimple*, i.e. if and only if $\Phi(\text{Frob})$ is semisimple. As an immediate consequence, the tensor product of two semisimple $\overline{\mathbb{Q}_\ell}$ -representations of W_F is itself semisimple. In the ℓ -modular case the situation is totally different: in general, Frobenius semisimple does not imply semisimple, neither does semisimple imply Frobenius semisimple, and the tensor product does not preserve semisimplicity! We give a list of examples illustrating these differences, we use the following lemma:

Lemma 3.10. Let E/F be a quadratic extension and $\ell = 2$. The $\overline{\mathbb{F}_\ell}$ -representation $\text{Ind}_{W_E}^{W_F}(\mathbf{1}_{W_E})$ is uniserial of length 2, with both subquotients isomorphic to $\mathbf{1}_{W_F}$.

Proof. It is two-dimensional, and contains $\mathbf{1}_{W_F}$ as a submodule with multiplicity one by Frobenius reciprocity law. The quotient $\text{Ind}_{W_E}^{W_F}(\mathbf{1}_{W_E})/\mathbf{1}_{W_F}$ is a character μ , so μ^{-1} is a submodule of $\text{Ind}_{W_E}^{W_F}(\mathbf{1})$ which is self-dual. Hence by Frobenius reciprocity again $\mu_E = \mathbf{1}$ so $\mu^2 = \mathbf{1}$, but the condition $\ell = 2$ implies that μ is then trivial. All in all, $\text{Ind}_{W_E}^{W_F}(\mathbf{1}_{W_E})$ is indecomposable with socle and cosocle $\mathbf{1}_{W_F}$. $\square$

Of course, one can concoct similar examples for any prime ℓ using degree ℓ extensions.

Example 3.11. For these examples we take $\ell = 2$. In all our examples we choose a separable quadratic extension E/F with Galois involution σ ; a character $\chi : W_E \rightarrow \overline{\mathbb{F}_2}^\times$, and consider the 2-dimensional $\overline{\mathbb{F}_2}$ -representation induced from χ

$$\Phi = \text{Ind}_{W_E}^{W_F}(\chi).$$

We fix a geometric Frobenius element Frob_E in W_E .

By the Artin reciprocity isomorphism of local class field theory, we can identify the topologically abelianized quotient W_E^{ab} of W_E with the group $E^\times$, and similarly W_F^{ab} with $F^\times$. Via this isomorphism the Frobenius elements $\text{Frob}, \text{Frob}_E$ corresponds to uniformizers ϖ, ϖ_E of F and E respectively. We also recall that the restriction functor from W_F^{ab} to W_E^{ab} (relative to the natural inclusion W_E^{ab} into W_F^{ab}) corresponds to the norm $N_{E/F} : E^\times \rightarrow F^\times$. In the second example, we use this identification to go from characters of W_F and W_E to characters of $F^\times$ and $E^\times$ respectively.

- (1) *Frobenius semisimple does not imply semisimple*: We take E/F ramified and $\chi = \mathbf{1}_{W_E}$. Then $\text{Frob}_E = \text{Frob}$ acts trivially on Φ as $\Phi_E = \mathbf{1}_{W_E} \oplus \mathbf{1}_{W_E}$, however Φ is not semisimple according to Lemma 3.10.
- (2) *Semisimple does not imply Frobenius semisimple*: take $F = \mathbb{Q}_2$ and $E = F_2$ the unique (up to isomorphism) quadratic unramified extension. Write

$$E^\times \simeq \mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z}) \times (1 + 2\mathbb{Z}_2),$$

and, by abuse of notation, we identify $E^\times$ with this product. Notice that the norm N_{k_E/k_F} is trivial as a character of $k_E^\times$ because $k_F^\times = \{1\}$, in particular

$$N_{E/F}(\mathbb{Z}/3\mathbb{Z}) = \{1\}.$$

We take χ a non trivial character of $\mathbb{Z}/3\mathbb{Z}$ with values in $\overline{\mathbb{F}_2}$ and extend it trivially on $\mathbb{Z}$ and extend it on $(1 + 2\mathbb{Z}_2)$ to a character χ of $E^\times$. Then χ does not factor through $N_{E/F}$, but it satisfies $\chi(\varpi_E) = 1$. The corresponding character χ of W_E is thus $\text{Gal}(E/F)$ -regular, but satisfies $\chi(\text{Frob}_E) = 1$. Hence the representation $\text{Ind}_{W_E}^{W_F}(\chi)$ is an irreducible representation of W_F . If Frob acted via a scalar on this representation, then $\text{Ind}_{W_E}^{W_F}(\chi)$ would be an irreducible representation of $I_F = I_E$ (E/F is unramified). This is absurd as its restriction to W_E is a direct sum of two lines. However Frob_E acts trivially on $\text{Ind}_{W_E}^{W_F}(\chi)_E = \chi \oplus \chi^\sigma$, and moreover one can take $\text{Frob}_E = \text{Frob}^2$.

We conclude that Φ is irreducible hence semisimple, $\Phi(\text{Frob})$ is not a scalar hence not Id , but $\Phi(\text{Frob})^2 = \text{Id}$, hence $\Phi(\text{Frob})$ is not semisimple because $\ell = 2$.

- (3) *$\otimes$ does not preserve semi-simplicity*: take χ to be $\text{Gal}(E/F)$ -regular, hence Φ and $\Phi^\vee$ are irreducible. The following decomposition of their tensor product, which follows from Mackey theory,

$$\Phi \otimes \Phi^\vee = \text{Ind}_{W_E}^{W_F}(\mathbf{1}_{W_E}) \oplus \text{Ind}_{W_E}^{W_F}(\chi^{-1}\chi^\sigma)$$

and Lemma 3.10 show that it is not semisimple.

Remark 3.12. By the main result of [17], the tensor product of two semisimple $\overline{\mathbb{F}_\ell}$ -representations of respective dimensions d and d' is semisimple if $d+d'-2 < \ell$. In Example 3.11 (3) $d = d' = \ell = 2$ which just falls out of the range of Serre's result, so fortunately does not provide a counterexample!

If Φ and Φ' are two semisimple representations of W_F , we denote by

$$\Phi \otimes_{\text{ss}} \Phi' = (\Phi \otimes \Phi')_{\text{ss}}$$

the semisimplification of their tensor product.

To end this section, we describe $(\Psi \otimes \Psi')^{I_F}$ when Ψ and Ψ' are two irreducible banal $\overline{\mathbb{F}_\ell}$ -representations of W_F . It is the central result of this section.

Proposition 3.13. Let Ψ and Ψ' be two banal irreducible $\overline{\mathbb{F}_\ell}$ -representations of W_F , and let $\tilde{\Psi}$ and $\tilde{\Psi}'$ be ℓ -adic lifts of Ψ and Ψ' respectively. Choose lattices L in $V_{\tilde{\Psi}}$ and L' in $V_{\tilde{\Psi}'}$, and put $M = L \otimes_{\overline{\mathbb{Z}_\ell}} L'$. For any $\overline{\mathbb{Q}_\ell}$ -subspace W of $V_{\tilde{\Psi}} \otimes V_{\tilde{\Psi}'}$, set $\overline{W} = (W \cap M) \otimes_{\overline{\mathbb{Z}_\ell}} \overline{\mathbb{F}_\ell}$. Then

$$\overline{(V_{\tilde{\Psi}} \otimes V_{\tilde{\Psi}'})^{I_F}} \simeq (V_{\Psi} \otimes V_{\Psi'})^{I_F} \quad \text{and} \quad ((V_{\Psi} \otimes V_{\Psi'})^{I_F})_{\text{ss}} = (V_{\Psi} \otimes_{\text{ss}} V_{\Psi'})^{I_F}.$$

Proof. We write

$$\tilde{\Psi} \otimes \tilde{\Psi}' = \bigoplus_i \mu_i \oplus \bigoplus_k \Psi_k$$

as a (finite) sum of unramified characters μ_i and ramified irreducible representations Ψ_k . In particular,

$$(\tilde{\Psi} \otimes \tilde{\Psi}')^{I_F} = \bigoplus_i \mu_i.$$

The inclusion

$$\overline{(V_{\tilde{\Psi}} \otimes V_{\tilde{\Psi}'})^{I_F}} \subseteq (V_{\Psi} \otimes V_{\Psi'})^{I_F}$$

is clear. Suppose that the inclusion was strict, this would imply that one of the Ψ_k 's is such that $\overline{(V_{\Psi_k})^{I_F}}$ is nonzero. For the sake of contradiction we suppose that it is the case. According to Corollary 3.9, setting either $E = F$ or $E = F_{o(\nu)\ell^m}$ for $m \geq 0$, one has

$$\Psi_k = \text{Ind}_{W_E}^{W_F}(\chi)$$

for χ an integral $\text{Gal}(E/F)$ -regular character of $E^\times$ such that $r_\ell(\chi) = \mu_E$ for μ an unramified character of W_F .

Now by definition, the space $\text{Hom}_{W_F}(\tilde{\Psi} \otimes \tilde{\Psi}', \text{Ind}_{W_E}^{W_F}(\chi))$ is nonzero. Take U a nonzero element in this space, as $\text{Ind}_{W_E}^{W_F}(\chi)$ is irreducible U is surjective. By Mackey theory

$$\text{Ind}_{W_E}^{W_F}(\chi)|_{W_E} = \bigoplus_{s \in W_F/W_E} s.D$$

where D is a line over $\overline{\mathbb{Q}_\ell}$ on which W_E acts as χ , and W_E more generally acts on $s.D$ as χ^s . Denote by p the projection on D with respect to $\bigoplus_{s \neq 1} s.D$, then $p(U(M)) = O_D$ is a lattice in D by [19, 9.3, (vi)], and

$$N = \bigoplus_{s \in W_F/W_E} s.O_D = \text{Ind}_{W_E}^{W_F}(O_D)$$

is a W_F -stable lattice in $\text{Ind}_{W_E}^{W_F}(\chi)$. Notice that for $s \in W_F/W_E$, the map $p_s = s \circ p \circ s^{-1}$ the projection on D with respect to $\bigoplus_{s' \neq s} s'.D$. We have

$$p_s(U(M)) = s(p(U(s^{-1}.M))) = s(p(U(M))) = s.O_D,$$

hence

$$U(M) = \bigoplus_{s \in W_F/W_E} p_s(U(M)) = \bigoplus_{s \in W_F/W_E} s.O_D = N.$$

Tensoring by $\overline{\mathbb{F}_\ell}$, we obtain a surjective W_F -intertwining operator $\overline{U}$ from $\Psi \otimes \Psi'$ onto

$$\text{Ind}_{W_E}^{W_F}(r_\ell(\chi)) = \text{Ind}_{W_E}^{W_F}(\mu_E),$$

i.e. $\text{Ind}_{W_E}^{W_F}(\mu_E)$ is a quotient of $\Psi \otimes \Psi'$. Dualizing, we obtain that

$$\text{Ind}_{W_E}^{W_F}(\mu_E^{-1})$$

is a submodule of $\Psi^\vee \otimes \Psi'^\vee$. By Frobenius reciprocity, we also have for any $k \in \mathbb{Z}$:

$$\begin{aligned} \mathrm{Hom}_{W_F}(\mu^{-1}\nu^{-k}, \mathrm{Ind}_{W_E}^{W_F}(\mu_E^{-1})) &\simeq \mathrm{Hom}_{W_E}((\mu^{-1}\nu^{-k})_E, \mu_E^{-1}) \\ &\simeq \mathrm{Hom}_{W_E}(\mu_E^{-1}, \mu_E^{-1}) \simeq \overline{\mathbb{F}_\ell}. \end{aligned}$$

Hence for any k , $\mu^{-1}\nu^{-k}$ is a submodule of $\mathrm{Ind}_{W_E}^{W_F}(\mu_E^{-1})$ which is itself a submodule of $\Psi^\vee \otimes \Psi'^\vee$. Dualizing again, we deduce that $\mu\nu^k$ is a quotient of $\Psi \otimes \Psi'$ for all k . However this would imply that $\Psi'^\vee \simeq \mu\nu^k\Psi$ for all k , and this would in particular imply that $\nu\Psi \simeq \Psi$ which is absurd by our banality assumption. Hence we just proved that

$$\overline{(V_{\tilde{\Psi}} \otimes V_{\tilde{\Psi}'})}^{I_F} \simeq (V_\Psi \otimes V_{\Psi'})^{I_F}.$$

In fact, we proved that $\overline{\Psi_k}^{I_F} = \{0\}$ for all Ψ_k . However $r_\ell(\Psi_k)^{I_F} = \{0\}$ if and only if $\overline{\Psi_k}^{I_F} = \{0\}$ according to Corollary 3.9, hence the isomorphism

$$[(V_\Psi \otimes V_{\Psi'})^{I_F}]_{\mathrm{ss}} \simeq (V_\Psi \otimes_{\mathrm{ss}} V_{\Psi'})^{I_F}.$$

□

4 Deligne representations

In this section, we classify W_F -semisimple Deligne representations (see Definition 4.1), in terms of irreducible representations of W_F , up to a certain equivalence relation (Definition 4.6). The purpose of doing this is that in the ℓ -modular case, we will parametrize (see Section 6.3) ℓ -modular irreducible representations of $\mathrm{GL}_n(F)$ by equivalence classes of ℓ -modular n -dimensional semisimple Deligne representations. The classification of W_F -semisimple Deligne representations we obtain in the ℓ -adic setting is well known, see [7, Proposition 3.1.3] and [16, Sections 3, 5].

4.1 Definitions, notation and basic properties

Here is our definition of Deligne representations, which as we shall see, specializes to the usual definition when $R = \overline{\mathbb{Q}_\ell}$.

Definition 4.1. A *Deligne R -representation* of W_F is a pair (Φ, U) where Φ is a finite dimensional R -representation of W_F , and $U \in \mathrm{Hom}_{W_F}(\nu\Phi, \Phi)$. Moreover, we say that (Φ, U) is a *W_F -semisimple Deligne R -representation* if Φ is semisimple as an R -representation of W_F .

Let $(\Phi, U), (\Phi', U')$ be Deligne R -representations of W_F . We write

$$\begin{aligned} \mathrm{Hom}_D((\Phi, U), (\Phi', U')) &= \{f \in \mathrm{Hom}_{W_F}(\Phi, \Phi') : f \circ U = U' \circ f\}, \\ \mathrm{End}_D(\Phi, U) &= \mathrm{Hom}_D((\Phi, U), (\Phi, U)), \\ \mathrm{Iso}_D((\Phi, U), (\Phi', U')) &= \mathrm{Hom}_D((\Phi, U), (\Phi', U')) \cap \mathrm{Iso}_{W_F}(\Phi, \Phi'). \end{aligned}$$

If this latter space is non-empty, we say that (Φ, U) and (Φ', U') are isomorphic. A *Deligne subrepresentation* of (Φ, U) is a subspace of V_Φ which is stable under $\Phi(W_F)$ and U , considered as a Deligne R -representation.

We will sometimes write $V_{(\Phi, U)}$ for V_Φ , and also $d_{(\Phi, U)}$ or d_Φ for $\dim_R(V_\Phi)$.

For U an endomorphism of a finite dimensional R -vector space, we denote by $\mathrm{Spec}(U)$ the set of its eigenvalues.

Remark 4.2. (1) Take Φ a finite dimensional representation of W_F , and $U \in \text{End}_{W_F}(V_\Phi)$ with Jordan decomposition $S + N$ (S semisimple and N nilpotent). Then $U \in \text{Hom}_{W_F}(\nu\Phi, \Phi)$ if and only if S and N are in $\text{Hom}_{W_F}(\nu\Phi, \Phi)$.

(2) If $R = \overline{\mathbb{Q}_\ell}$, and $U \in \text{Hom}_{W_F}(\nu\Phi, \Phi)$, then $U = N$ is nilpotent as $\text{Spec}(U)$ is stable under multiplication by q .

The Jordan decomposition of U will always be denoted by $S + N$, unless explicitly stated.

As we only consider Deligne R -representations of W_F , we will suppress the “of W_F ” in our notation.

Definition 4.3. Let $(\Phi, U), (\Phi', U')$ be Deligne R -representations and let $U = S + N$.

(1) The *direct sum* of (Φ, U) and (Φ', U') is defined by

$$(\Phi, U) \oplus (\Phi', U') = (\Phi \oplus \Phi', U \oplus U').$$

(2) The *dual* of (Φ, U) is defined by

$$(\Phi, U)^\vee = (\Phi^\vee, S^\vee - N^\vee).$$

(3) The *tensor product* of (Φ, U) and (Φ', U') is defined by

$$(\Phi, U) \otimes (\Phi', U') = (\Phi \otimes \Phi', U \otimes \text{Id} \oplus \text{Id} \otimes U').$$

Notation 4.4. We introduce the following notation:

- $\text{Rep}_{\text{ss}}(\mathbf{D}, R)$ for the set of isomorphism classes of W_F -semisimple Deligne R -representations.
- $\text{Indec}_{\text{ss}}(\mathbf{D}, R)$ for the isomorphism classes of Deligne R -representations in $\text{Rep}_{\text{ss}}(\mathbf{D}, R)$ which are indecomposable under direct sum of Deligne R -representations.
- $\text{Irr}_{\text{ss}}(\mathbf{D}, R)$ for the isomorphism classes of irreducible Deligne R -representations in $\text{Rep}_{\text{ss}}(\mathbf{D}, R)$. (Note that $(\Phi, U) \in \text{Irr}_{\text{ss}}(\mathbf{D}, R)$ does not imply $\Phi \in \text{Irr}(W_F, R)$.)
- $\text{Nilp}_{\text{ss}}(\mathbf{D}, R)$ for the isomorphism classes of Deligne R -representations $(\Phi, U) \in \text{Rep}_{\text{ss}}(\mathbf{D}, R)$ with $U = N$ nilpotent. In particular, by Remark 4.2 (2), $\text{Rep}_{\text{ss}}(\mathbf{D}, \overline{\mathbb{Q}_\ell}) = \text{Nilp}_{\text{ss}}(\mathbf{D}, \overline{\mathbb{Q}_\ell})$.

Clearly, $\text{Rep}_{\text{ss}}(\mathbf{D}, R)$ is stable under direct sum and duals:

Lemma 4.5. For $(\Phi, U), (\Phi', U') \in \text{Rep}_{\text{ss}}(\mathbf{D}, R)$, then

$$(\Phi, U) \oplus (\Phi', U'), (\Phi, U)^\vee \in \text{Rep}_{\text{ss}}(\mathbf{D}, R).$$

In contrast, the set $\text{Rep}_{\text{ss}}(\mathbf{D}, R)$ is not stable under the tensor product of Deligne R -representations, according to Example 3.11, (3). To remedy this problem, we will introduce a *semisimple* tensor product $\otimes_{\text{ss}}$ in Section 4.4. Of course, whenever the tensor product is W_F -semisimple we will have $\otimes_{\text{ss}} = \otimes$, which will be the case when the characteristic is not too small in relation to the dimensions of the representations by Remark 3.12.

Now we introduce an equivalence relation $\sim$ on $\text{Rep}_{\text{ss}}(\mathbf{D}, R)$. We do this as we shall later parametrize irreducible representations of the group $\text{GL}_n(F)$ by equivalence classes rather than isomorphism classes of Deligne R -representations.

Definition 4.6. The definition is in two steps:

(1) Deligne R -representations $(\Phi, U), (\Phi', U') \in \text{Indec}_{\text{ss}}(\mathbf{D}, R)$ are *equivalent*, denoted

$$(\Phi, U) \sim (\Phi', U'),$$

if there exists $\lambda \in R^\times$ such that

$$(\Phi', U') = (\Phi, \lambda U).$$

(2) In the general case, $(\Phi, U), (\Phi', U') \in \text{Rep}_{\text{ss}}(\mathbf{D}, R)$ are *equivalent*, still denoted

$$(\Phi, U) \sim (\Phi', U'),$$

if one can decompose $(\Phi', U') = \bigoplus_{i=1}^r (\Phi_i, U_i)$ and $(\Phi, U) = \bigoplus_{i=1}^r (\Phi_i, U_i)$ such that $(\Phi_i, U_i) \sim (\Phi_i, U_i)$ in $\text{Indec}_{\text{ss}}(\mathbf{D}, R)$.

Remark 4.7. To see that Definition 4.6 defines an equivalence relation we use that the decomposition of a W_F -semisimple Deligne R -representation into indecomposable (necessarily W_F -semisimple) Deligne R -representations is unique. This will be a consequence of our classification of indecomposable W_F -semisimple Deligne representations: we show as a consequence of Propositions 4.28 and 4.30, and Theorems 4.29 that the endomorphism ring of such a representation is local, which is what is needed in the proof of the Krull–Schmidt theorem.

Notation 4.8. We introduce square brackets to denote equivalence classes:

- For $(\Phi, U) \in \text{Rep}_{\text{ss}}(\mathbf{D}, R)$, we let $[\Phi, U]$ denote its equivalence class.
- $[\text{Rep}_{\text{ss}}(\mathbf{D}, R)] = \text{Rep}_{\text{ss}}(\mathbf{D}, R) / \sim$.
- $[\text{Irr}_{\text{ss}}(\mathbf{D}, R)] = \text{Irr}_{\text{ss}}(\mathbf{D}, R) / \sim$.
- $[\text{Indec}_{\text{ss}}(\mathbf{D}, R)] = \text{Indec}_{\text{ss}}(\mathbf{D}, R) / \sim$.
- $[\text{Nilp}_{\text{ss}}(\mathbf{D}, R)] = \text{Nilp}_{\text{ss}}(\mathbf{D}, R) / \sim$.

Let $[\Phi, U], [\Phi', U'] \in [\text{Rep}_{\text{ss}}(\mathbf{D}, R)]$, then the equivalence class of the direct sum and dual Deligne R -representations are independent of choice of representatives, so we let

$$\begin{aligned} [\Phi, U] \oplus [\Phi', U'] &= [(\Phi, U) \oplus (\Phi', U')] \\ [\Phi, U]^\vee &= [(\Phi, U)^\vee], \end{aligned}$$

giving a well defined direct sum and dual on $[\text{Rep}_{\text{ss}}(\mathbf{D}, R)]$.

In the ℓ -adic case, we recall that $\text{Rep}_{\text{ss}}(\mathbf{D}, \overline{\mathbb{Q}_\ell}) = \text{Nilp}_{\text{ss}}(\mathbf{D}, \overline{\mathbb{Q}_\ell})$, hence the following proposition shows that in this case one gains nothing by introducing the equivalence relation $\sim$.

Proposition 4.9. The canonical surjection from $\text{Nilp}_{\text{ss}}(\mathbf{D}, R)$ onto $[\text{Nilp}_{\text{ss}}(\mathbf{D}, R)]$ is the identity, we write:

$$[\text{Nilp}_{\text{ss}}(\mathbf{D}, R)] = \text{Nilp}_{\text{ss}}(\mathbf{D}, R).$$

Proof. Take (Φ, N) a Deligne representation with N nilpotent, it is enough to prove that (Φ, N) and $(\Phi, \lambda N)$ are isomorphic if $\lambda \in R^\times$. Let r be the nilpotency index of N . As the iterated kernels $\text{Ker}(N^k)$ are all W_F -stable, we can construct a W_F -stable complement Q_{r-1} of $\text{Ker}(N^{r-1})$ in $\text{Ker}(N^r) = V_\Phi$. Then $N(Q_{r-1})$ is also W_F -stable, and

$$N(Q_{r-1}) \cap \text{Ker}(N^{r-2}) = \{0\}.$$

Hence $N(Q_{r-1}) \oplus \text{Ker}(N^{r-2})$ admits a W_F -stable complement U_{r-1} in $\text{Ker}(N^{r-1})$, and

$$Q_{r-2} = N(Q_{r-1}) \oplus U_{r-1}$$

is a W_F -stable complement of $\text{Ker}(N^{r-2})$ in $\text{Ker}(N^{r-1})$ such that $N(Q_{r-1}) \subseteq Q_{r-2}$. Continuing in this fashion, we obtain for $k = 0, \dots, r-1$, a W_F -stable complement Q_k of $\text{Ker}(N^k)$ in $\text{Ker}(N^{k+1})$ such that $N(Q_k) \subseteq Q_{k-1}$ for $k \geq 1$, and

$$V_\Phi = \bigoplus_{k=0}^{r-1} Q_k.$$

We define $P \in \text{GL}(V_\Phi)$ to be equal to $\lambda^i \text{Id}_{Q_i}$ on Q_i . Then P commutes with $\Phi(w)$ for all $w \in W_F$, and as is checked on each Q_i , one has $P\lambda N = NP$. Hence P defines the required isomorphism. $\square$

Corollary 4.10. We have $[\text{Rep}_{\text{ss}}(\text{D}, \overline{\mathbb{Q}_\ell})] = \text{Nilp}_{\text{ss}}(\text{D}, \overline{\mathbb{Q}_\ell})$.

In particular, for $[\Phi, U], [\Phi', U']$ in $[\text{Rep}_{\text{ss}}(\text{D}, \overline{\mathbb{Q}_\ell})]$, the equivalence class $[(\Phi, U) \otimes (\Phi', U')]$ is independent of the choice of representatives for $[\Phi, U], [\Phi', U']$ (we can in fact remove the brackets everywhere so the assertion is obvious). This is not true anymore when $R = \overline{\mathbb{F}_\ell}$, first we already saw in Example 3.11 that W_F -semisimplicity is not preserved, but even if it is, $[\Phi, U] \otimes [\Phi', U']$ is still not well-defined in general. However there is also a natural solution to this problem as we shall see in Section 4.4. We give an example:

Example 4.11. Take $\ell \neq 2$, $\Phi = \mathbf{1} \in \text{Irr}(W_F, \overline{\mathbb{F}_\ell})$ and suppose that $q \equiv 1[\ell]$. Then $(\mathbf{1}, \text{Id}) \sim (\mathbf{1}, -\text{Id})$, but

$$(\mathbf{1}, \text{Id}) \otimes (\mathbf{1}, \text{Id}) = (\mathbf{1}, 2\text{Id}), \text{ whereas } (\mathbf{1}, \text{Id}) \otimes (\mathbf{1}, -\text{Id}) = (\mathbf{1}, 0),$$

which are inequivalent. However, for all pairs $(\lambda, \mu) \in (\overline{\mathbb{F}_\ell}^\times)^2$ outside the hyperplane of $(\overline{\mathbb{F}_\ell})^2$ defined by $\mu + \lambda = 0$, we have $[(\mathbf{1}, \lambda \text{Id}) \otimes (\mathbf{1}, \mu \text{Id})] = [\mathbf{1}, \text{Id}]$, and we shall set

$$[\mathbf{1}, \text{Id}] \otimes [\mathbf{1}, \text{Id}] = [\mathbf{1}, \text{Id}].$$

There are two cases where the tensor product behaves well which will prove fundamental later:

Lemma 4.12. Let $(\Phi, U), (\Phi', U') \in \text{Rep}_{\text{ss}}(\text{D}, \overline{\mathbb{F}_\ell})$.

- (1) If Φ is a direct sum of characters, then $(\Phi, U) \otimes (\Phi', U')$ is W_F -semisimple.
- (2) If moreover $U = N$ is nilpotent, and if $(\Phi_1, U_1) \sim (\Phi, U)$ and $(\Phi'_1, U'_1) \sim (\Phi', U')$, then

$$(\Phi_1, U_1) \otimes (\Phi'_1, U'_1) \sim (\Phi, U) \otimes (\Phi', U').$$

Proof. Tensoring by a character preserves simplicity hence (1). For (2), it is enough to treat the case where (Φ', U') is indecomposable. Take $(\Phi_1, U_1) \sim (\Phi, U)$ and $(\Phi'_1, U'_1) \sim (\Phi', U')$, then $(\Phi_1, U_1) = (\Phi, U)$ as U is nilpotent, and $(\Phi'_1, U'_1) = (\Phi', \lambda U')$. This implies

$$(\Phi_1, U_1) \otimes (\Phi'_1, U'_1) = (\Phi, U) \otimes (\Phi', \lambda U') \sim (\Phi, \lambda^{-1}U) \otimes (\Phi', U') = (\Phi, U) \otimes (\Phi', U')$$

because U is nilpotent. $\square$

In the situation of the lemma we define

$$[\Phi, U] \otimes [\Phi', U'] = [(\Phi, U) \otimes (\Phi', U')].$$

We introduce the following definition, which will be convenient when dealing with indecomposable objects in $\text{Rep}_{\text{ss}}(\mathbf{D}, R)$.

Definition 4.13. Let $\Psi \in \text{Irr}(\mathbf{W}_F, R)$ and $(\Phi, U) \in \text{Rep}_{\text{ss}}(\mathbf{D}, R)$. We say that (Φ, U) is *supported on $\mathbb{Z}_\Psi$* if $\Phi = \bigoplus_i \Psi_i$, for $\Psi_i \in \mathbb{Z}_\Psi$.

Let $(\Phi, U) \in \text{Rep}_{\text{ss}}(\mathbf{D}, R)$ and $\Psi \in \text{Irr}(\mathbf{W}_F, R)$. We write $\Phi(i, \Psi)$ for its $\nu^i \Psi$ isotypic component which only depends on the class $i \pmod{o(\Psi)}$. As U belongs to $\text{Hom}_{\mathbf{W}_F}(\nu \Phi, \Phi)$, the $\mathbf{W}_F$ -subrepresentation $\Phi(\mathbb{Z}_\Psi) = \bigoplus_{i=1}^{o(\Psi)-1} \Phi(i, \Psi)$ is stable under U . As there is a finite subset S of $\text{Irr}(\mathbf{W}_F, R)$ such that $\Phi = \bigoplus_{\Psi \in S} \Phi(\mathbb{Z}_\Psi)$, we deduce the following lemma:

Lemma 4.14. If $(\Phi, U) \in \text{Indec}_{\text{ss}}(\mathbf{D}, R)$, then there is $\Psi \in \text{Irr}(\mathbf{W}_F, R)$ such that (Φ, U) is supported on $\mathbb{Z}_\Psi$.

4.2 Irreducible $\mathbf{W}_F$ -semisimple Deligne representations

We start with the following simple observation.

Lemma 4.15. If $(\Phi, U) \in \text{Irr}_{\text{ss}}(\mathbf{D}, R)$, then either U is bijective, or U is zero.

Proof. Clear, as $\text{Ker}(U)$ is a (Φ, U) -stable subspace. □

An irreducible Deligne representation $(\Phi, 0)$ is nothing more than an irreducible representation of $\mathbf{W}_F$. In particular, when $R = \overline{\mathbb{Q}_\ell}$, the map $\Phi \mapsto (\Phi, 0)$ is a bijection between $\text{Irr}(\mathbf{W}_F, \overline{\mathbb{Q}_\ell})$ and $\text{Irr}_{\text{ss}}(\mathbf{D}, \overline{\mathbb{Q}_\ell})$ by Remark 4.2. When $R = \overline{\mathbb{F}_\ell}$, we record this as a lemma:

Lemma 4.16. The map $\Phi \mapsto (\Phi, 0)$ is a bijection between $\text{Irr}(\mathbf{W}_F, \overline{\mathbb{F}_\ell})$ and $\text{Irr}_{\text{ss}}(\mathbf{D}, \overline{\mathbb{F}_\ell}) \cap \text{Nilp}_{\text{ss}}(\mathbf{D}, \overline{\mathbb{F}_\ell})$.

Notation 4.17. By abuse of notation, we set $\Psi = (\Psi, 0) = [\Psi, 0]$.

We now consider the case U bijective, hence $R = \overline{\mathbb{F}_\ell}$. We first start with an example:

Proposition 4.18. Let $\Psi \in \text{Irr}(\mathbf{W}_F, \overline{\mathbb{F}_\ell})$ and $I \in \text{Iso}_{\mathbf{W}_F}(\nu^{o(\Psi)} \Psi, \Psi)$, and write $\mathcal{C}(\Psi, I) = (\Phi(\Psi), \mathcal{C}_I)$ for the pair

$$\begin{aligned} \Phi(\Psi) &= \bigoplus_{k=0}^{o(\Psi)-1} \nu^k \Psi, \\ \mathcal{C}_I(x_0, \dots, x_{o(\Psi)-1}) &= (I(x_{o(\Psi)-1}), x_0, \dots, x_{o(\Psi)-2}). \end{aligned}$$

- (1) We have $\mathcal{C}(\Psi, I) \in \text{Irr}_{\text{ss}}(\mathbf{D}, \overline{\mathbb{F}_\ell})$.
- (2) For all $k \in \mathbb{Z}$, $\mathcal{C}(\Psi, I) = \mathcal{C}(\nu^k \Psi, I)$.
- (3) The equivalence class $[\mathcal{C}(\Psi, I)]$ is independent of $I \in \text{Iso}_{\mathbf{W}_F}(\nu^{o(\Psi)} \Psi, \Psi)$.

Proof. (1) The endomorphism $\mathcal{C}_I$ belongs to $\text{Hom}_{W_F}(\nu\Phi, \Phi)$ by definition, hence $\mathcal{C}(\Psi, I) \in \text{Rep}_{\text{ss}}(\mathbb{D}, \overline{\mathbb{F}}_\ell)$. Whenever W is a W_F -stable subspace of $V = V_{\Phi(\Psi)}$, we set $W(i) = W(i, \Psi)$. Let W be a nonzero Deligne subrepresentation of V . As it is W_F -stable, we have

$$W = \bigoplus_{i=0}^{o(\Psi)-1} W(i),$$

and as $W \neq 0$, there is an $i \in \{0, \dots, o(\Psi) - 1\}$ such that $W(i) \neq 0$. Hence $W(i) = V(i) = \nu^i \Psi$ by irreducibility of $V(i)$. But then $\mathcal{C}_I^k(W(i)) = \mathcal{C}_I^k(V(i)) = V(i + k[o(\Psi)]) \subseteq W$ for all $k \in \mathbb{Z}$, and $W = V$.

- (2) By definition $\mathcal{C}_I \in \text{Iso}_{W_F}(\Phi(\Psi), \nu^{-1}\Phi(\Psi))$ and commutes with $\mathcal{C}_I$.
- (3) Write $\mathcal{C}(\Psi, I) = (\Phi(\Psi), \mathcal{C}_I)$ and take $\lambda \in \overline{\mathbb{F}}_\ell^\times$. Set $V = V_\Phi$. Take $u \in \overline{\mathbb{F}}_\ell$ such that $u^{o(\Psi)} = \lambda$. Define $A \in \text{End}(V)$ by the formula

$$A(x_0, x_1, x_2, \dots, x_{o(\Psi)-1}) = (x_0, ux_1, \dots, u^{o(\Psi)-2}x_{o(\Psi)-2}, u^{o(\Psi)-1}x_{o(\Psi)-1}).$$

Then

$$\begin{aligned} AC_{\lambda I}(x_0, x_1, \dots, x_{o(\Psi)-2}, x_{o(\Psi)-1}) &= A(\lambda I(x_{o(\Psi)-1}), x_0, x_1, \dots, x_{o(\Psi)-2}) \\ &= (\lambda I(x_{o(\Psi)-1}), ux_0, u^2x_1, \dots, u^{o(\Psi)-1}x_{o(\Psi)-2}) \\ &= u(u^{o(\Psi)-1}I(x_{o(\Psi)-1}), x_0, ux_1, \dots, u^{o(\Psi)-2}x_{o(\Psi)-2}) \\ &= u\mathcal{C}_I A(x_0, x_1, \dots, x_{o(\Psi)-2}, x_{o(\Psi)-1}), \end{aligned}$$

i.e.

$$AC_{\lambda I} = u\mathcal{C}_I A.$$

As A commutes with $\Phi(w)$ for any $w \in W_F$, it defines an isomorphism between the Deligne representations $(\Phi(\Psi), \mathcal{C}_{\lambda I})$ and $(\Phi(\Psi), u\mathcal{C}_I)$, hence the result. □

Notation 4.19. Under the notation of Proposition 4.18 and by Parts (2), (3), it thus makes sense to set

$$\mathcal{C}(\mathbb{Z}_\Psi, I) = \mathcal{C}(\Psi, I), \quad \text{and} \quad \mathcal{C}(\mathbb{Z}_\Psi) = [\mathcal{C}(\mathbb{Z}_\Psi, I)].$$

We now want to show that all elements of $\text{Irr}_{\text{ss}}(\mathbb{D}, \overline{\mathbb{F}}_\ell)$ with bijective Deligne operator are in such a class.

Let (Φ, U) be an irreducible W_F -semisimple Deligne representation with U bijective, and set $V = V_\Phi$. By Lemma 4.14, we know that (Φ, U) is supported on an irreducible line $\mathbb{Z}_\Psi$. Again we set $W(i) = W(i, \Psi)$ for any W_F -stable subspace W of V . We will now gather some information on the structure of (Φ, U) . The first basic observation is that the relation

$$\nu(w)U\Phi(w) = \Phi(w)U,$$

for $w \in W_F$, can be rewritten as

$$(U\nu(w)\Phi(w)U^{-1})U = \Phi(w)U,$$

which shows that U sends $V(i)$ to $V(i + 1)$, in a necessarily bijective manner. We deduce the following lemma:

Lemma 4.20. We have $V(i) = U^i(V(0))$, for $i = 0, \dots, o(\Psi) - 1$. In particular all the W_F -isotypic components $V(i)$ have the same dimension for $i = 0, \dots, o(\Psi) - 1$.

We now give a useful description the endomorphism ring of (Φ, U) . In order to do so, we fix an isomorphism of W_F -modules

$$J : V(0) \xrightarrow{\sim} V(0)$$

from $\nu^{o(\Psi)}\Phi|_{V(0)}$ to $\Phi|_{V(0)}$. Notice that as $\Phi|_{V(0)}$ is the direct sum of m -copies of Ψ , for some $m \geq 1$, then

$$A_0 = \text{End}_{W_F}(\Phi|_{V(0)})$$

is isomorphic to $\text{Mat}(m, R)$. This implies that all automorphisms of A_0 are inner, in particular as A_0 is also equal (not just isomorphic) to $\text{End}_{W_F}(\nu^{o(\Psi)}\Phi|_{V(0)})$, there is $P \in A_0^\times$ such that

$$J^{-1}AJ = P^{-1}AP,$$

for all $A \in A_0$. We fix such a P and put

$$Y = (U^{o(\Psi)}|_{V(0)})JP^{-1} \in A_0.$$

Proposition 4.21. Under the notation above, the endomorphism algebra $\text{End}_{\mathbb{D}}(V)$ is isomorphic to the algebra $C_{A_0}(Y)$ of elements of A_0 commuting with Y . An isomorphism is given by

$$\begin{aligned} C_{A_0}(Y) &\rightarrow \text{End}_{\mathbb{D}}(V) \\ L_0 &\mapsto L = L_0 \oplus \dots \oplus L_{o(\Psi)-1}, \end{aligned}$$

where $L_i = U^i L_0 U^{-i} \in \text{End}_{W_F}(\Phi|_{V(i)})$.

Proof. Take $L \in \text{End}_{\mathbb{D}}(V)$, then L stabilizes each $V(i)$ as it commutes with the action of W_F , and we denote by L_i the restriction $L|_{V(i)}$. Because L commutes with U we have

$$L_i = U^i L_0 U^{-i} \in \text{End}_{W_F}(\Phi|_{V(i)}),$$

and moreover L_0 must commute with $U_0 = (U^{o(\Psi)})|_{V(0)}$. Notice that $U_0 \in \text{End}_{\overline{\mathbb{F}_\ell}}(V(0))$ but has no reason to belong to A_0 .

Conversely given $L_0 \in A_0$ commuting with U_0 , the map

$$L = L_0 \oplus \dots \oplus L_{o(\Psi)-1}$$

with $L_i = U^i L_0 U^{-i}$ belongs to $\text{End}_{\mathbb{D}}(V)$. Hence we have an isomorphism from the subalgebra

$$B_0 = \{L_0 \in A_0, L_0 U_0 = U_0 L_0\}$$

of A_0 to $\text{End}_{\mathbb{D}}(V)$ defined by $L_0 \mapsto L$. Now the map $X = U_0 J$ belongs to A_0 , hence the relation

$$L_0 U_0 = U_0 L_0$$

is equivalent to

$$\begin{aligned} L_0 U_0 = U_0 J J^{-1} L_0 &\Leftrightarrow L_0 X J^{-1} = X J^{-1} L_0 \Leftrightarrow L_0 X = X J^{-1} L_0 J \Leftrightarrow L_0 X = X P^{-1} L_0 P \\ &\Leftrightarrow L_0 X P^{-1} = X P^{-1} L_0. \end{aligned}$$

This ends the proof. □

Let F be a field, $X \in \text{Mat}(n, F)$, and set

$$C_{\text{Mat}(n, F)}(X) = \{Y \in \text{Mat}(n, F), YX = XY\},$$

the subalgebra of matrices commuting with X . For use in the next proof, we recall the following lemma from basic linear algebra:

Lemma 4.22. The algebra $C_{\text{Mat}(n, F)}(X)$ has dimension 1 if and only if $n = 1$.

Proof. Suppose that $C_{\text{Mat}(n, F)}(X)$ has dimension one, then in particular $F[B] \simeq F[X]/(f_B)$ where f_B is the minimal polynomial of B must be one dimensional. Hence f_B has degree 1 and B is a multiple of I_n . The statement follows. $\square$

A corollary of Proposition 4.21 and Lemma 4.22 is the following:

Corollary 4.23. We have $\Phi(i) \simeq \nu^i \Psi$ for $i = 0, \dots, o(\Psi) - 1$.

Proof. By Schur's lemma, the ring $\text{End}_{\mathbb{D}}(V_{\Phi})$ is one dimensional, hence by Proposition 4.21, the algebra $C_{A_0}(Y)$ is one-dimensional. However, as $A_0 \simeq \text{Mat}(m, \overline{\mathbb{F}}_{\ell})$, if

$$\Phi = \underbrace{\Psi \oplus \dots \oplus \Psi}_{m \times},$$

Lemma 4.22 implies that $m = 1$ i.e. $\Phi(0) \simeq \Psi$, and $\Phi(i) \simeq \nu^i \Psi$ by Lemma 4.20. $\square$

We obtain the main result of this section.

Theorem 4.24. Take $(\Phi, U) \in \text{Irr}_{\text{ss}}(\mathbb{D}, \overline{\mathbb{F}}_{\ell})$ with U bijective, then there is a unique irreducible line $\mathbb{Z}_{\Psi}$ of W_F such that

$$[\Phi, U] = \mathcal{C}(\mathbb{Z}_{\Psi}).$$

Proof. We already explained that (Φ, U) is supported on an irreducible line $\mathbb{Z}_{\Psi}$, which is necessarily unique by uniqueness of the decomposition of Φ into a direct sum of elements in $\text{Irr}(W_F, \overline{\mathbb{F}}_{\ell})$. Set $V = V_{\Phi}$ and $V(i)$ its $\nu^i \Psi$ -isotypic component. Take $I \in \text{Iso}_{W_F}(\nu \Psi, \Psi)$ and consider $\mathcal{C}(\Psi, I) = (\Phi(\Psi), \mathcal{C}_I)$. By Corollary 4.23, $V(i) \simeq \nu^i \Psi$. Let J_0 be an isomorphism between $V(0)$ and Ψ , then for each i between 0 and $o(\Psi) - 1$, the map $J_i = \mathcal{C}_I^i J_0 U^{-i}$ is a linear bijection from $V(i)$ to $\nu^i \Psi$ commuting with the action of W_F . Hence the map $J = \bigoplus_{i=0}^{o(\Psi)-1} J_i$ is a linear bijection from V to $\Phi(\Psi)$ commuting with the action of W_F . It intertwines the action of U and $\mathcal{C}_I$ on each $V(i)$: for $i = 0, \dots, o(\Psi) - 2$, we have

$$JU|_{V(i)} = \mathcal{C}_I J|_{V(i)} = \mathcal{C}_{\mu I} J|_{V(i)}$$

for any $\mu \in \overline{\mathbb{F}}_{\ell}^{\times}$. On $V(o(\Psi) - 1)$, by Schur's lemma, there is $\lambda \in \overline{\mathbb{F}}_{\ell}^{\times}$ such that

$$JU|_{V(o(\Psi)-1)} = \lambda \mathcal{C}_I J|_{V(o(\Psi)-1)} = \mathcal{C}_{\lambda I} J|_{V(o(\Psi)-1)}.$$

This shows that J is an isomorphism between (Φ, U) and $\mathcal{C}(\Psi, \lambda I)$. $\square$

4.3 Indecomposable W_F -semisimple Deligne representations

Take $(\Phi, U) \in \text{Indec}_{\text{ss}}(\mathbf{D}, R)$, in particular it is supported on $\mathbb{Z}_\Psi$ for Ψ an irreducible representation of W_F . As before we write $V = V_\Phi$ and $U = S + N$, and we already observed that $U = N$ when $R = \overline{\mathbb{Q}_\ell}$. We consider $R = \overline{\mathbb{F}_\ell}$ for a moment, the Deligne relation satisfied by U implies that the nonzero eigenvalues of S can be partitioned into orbits of $q^\mathbb{Z} = q^{\mathbb{Z}/o(\nu)\mathbb{Z}} \simeq \mathbb{Z}/o(\nu)\mathbb{Z}$ acting by multiplication.

Lemma 4.25. For $(\Phi, U) \in \text{Indec}_{\text{ss}}(\mathbf{D}, \overline{\mathbb{F}_\ell})$, the eigenvalues of U lie in a single orbit under this action of $\mathbb{Z}/o(\nu)\mathbb{Z}$, in particular U is either bijective or nilpotent.

Proof. We write $U = S + N$, and as before we have

$$\text{Spec}(S) = \{0\} \sqcup O_1 \sqcup \cdots \sqcup O_s$$

as the union of the orbit of 0 and the s orbits of nonzero eigenvalues. We set $V_0 = \text{Ker}(S)$, and $V_i = \bigoplus_{\lambda \in O_i} V_\lambda$ so that $V = \bigoplus_{i=0}^s V_i$ is a decomposition into W_F -stable summands. The summands are stable under U as U commutes with S . Hence the decomposition $V = \bigoplus_{i=0}^s V_i$ is a direct sum of Deligne representations. As (Φ, U) is indecomposable, we must have $V = V_i$ for some i . $\square$

We continue with our coefficient ring R denoting either $\overline{\mathbb{Q}_\ell}$ or $\overline{\mathbb{F}_\ell}$. Let us first consider the case U nilpotent, and start with a classical example.

Definition 4.26. Take $m \geq 1$, and denote by $[0, r-1]$ the W_F -semisimple Deligne representation $(\Phi(r), N(r))$ where

$$\Phi(r) = \bigoplus_{k=0}^{r-1} \nu^k, \quad \text{and} \quad N(r)(x_0, \dots, x_{r-1}) = (0, x_0, \dots, x_{r-2}).$$

For $a \leq b$ in $\mathbb{Z}$, we set $[a, b] = \nu^a[0, b-a]$. By convention if $b < a$, we set $[a, b]$ to be the null vector space.

Lemma 4.27. We have $[0, r-1] \in \text{Indec}_{\text{ss}}(\mathbf{D}, R)$.

Proof. In fact it is already indecomposable as an $R[N(r)]$ -module. Indeed, the endomorphism algebra $\text{End}_{R[N(r)]}(V_{[0, r-1]})$ is equal to $R[N(r)]$ because $N(r)$ is a cyclic endomorphism of $V_{[0, r-1]}$. Now the ring $R[N(r)] = R[X]/(X^r)$ is local, so the $R[N(r)]$ -module $V_{[0, r-1]}$ is indecomposable. $\square$

More generally, we have:

Proposition 4.28. Take $\Psi \in \text{Irr}(W_F, R) = \text{Irr}_{\text{ss}}(\mathbf{D}, R) \cap \text{Nilp}_{ss}(\mathbf{D}, R)$ and $r \geq 1$, then

$$[0, r-1] \otimes \Psi \in \text{Indec}_{\text{ss}}(\mathbf{D}, R).$$

Moreover, take $N(r)$ as in Definition 4.26. If $R = \overline{\mathbb{F}_\ell}$, and $I \in \text{Iso}_{W_F}(\nu^{o(\Psi)}\Psi, \Psi)$, then

$$\text{End}_{\mathbf{D}}([0, r-1] \otimes \Psi) = \overline{\mathbb{F}_\ell}[N(r)^{o(\Psi)} \otimes I^{-1}].$$

If $R = \overline{\mathbb{Q}_\ell}$, then

$$\text{End}_{\mathbf{D}}([0, r-1] \otimes \Psi) \simeq \overline{\mathbb{Q}_\ell}.$$

In both cases they are local rings.

Proof. First, by Lemma 4.12, $[0, r-1] \otimes \Psi \in \text{Rep}_{\text{ss}}(\mathbf{D}, R)$. Let's compute the endomorphism algebra of $[0, r-1] \otimes \Psi$. We do the case $R = \overline{\mathbb{F}_\ell}$, the case $R = \overline{\mathbb{Q}_\ell}$ being similar. One has

$$\text{End}_{\overline{\mathbb{F}_\ell}}([0, r-1] \otimes \Psi) = \text{End}_{\overline{\mathbb{F}_\ell}}([0, r-1]) \otimes \text{End}_{\overline{\mathbb{F}_\ell}}(\Psi).$$

The Deligne operator $N(\Psi, r)$ of $[0, r-1] \otimes \Psi \in \text{Indec}_{\text{ss}}(\mathbf{D}, R)$ is $N(r) \otimes \text{Id}$, hence

$$\text{End}_{\overline{\mathbb{F}_\ell}[N(\Psi, r)]}([0, r-1] \otimes \Psi) = \text{End}_{\overline{\mathbb{F}_\ell}[N(r)]}([0, r-1]) \otimes \text{End}_{\overline{\mathbb{F}_\ell}}(\Psi) = \overline{\mathbb{F}_\ell}[N(r)] \otimes \text{End}_{\overline{\mathbb{F}_\ell}}(\Psi).$$

Fixing $I \in \text{Hom}(\nu^{o(\Psi)}\Psi, \Psi)$, and using the basis $\text{Id}, N(r), \dots, N(r)^{r-1}$ of $\overline{\mathbb{F}_\ell}[N]$, one checks that the subalgebra of $\overline{\mathbb{F}_\ell}[N(r)] \otimes \text{End}_{\overline{\mathbb{F}_\ell}}(\Psi)$ commuting with the action of W_F is $\mathbb{F}_l[N(r)^{o(\Psi)} \otimes I^{-1}]$, i.e.

$$\text{End}_{\mathbf{D}}([0, r-1] \otimes \Psi) = \overline{\mathbb{F}_\ell}[N(r)^{o(\Psi)} \otimes I^{-1}].$$

When $R = \overline{\mathbb{Q}_\ell}$ we find

$$\text{End}_{\mathbf{D}}([0, r-1] \otimes \Psi) \simeq \overline{\mathbb{Q}_\ell}.$$

In both cases, these algebras are local as they are of the form $R[X]/(X^l)$ for $l \geq 0$. $\square$

Now we check that these are the only indecomposable W_F -semisimple Deligne representations with nilpotent Deligne operator. If N is a nilpotent endomorphism of an R -vector space, we denote by $\text{ind}(N)$ its nilpotency index.

Theorem 4.29. Let $(\Phi, N) \in \text{Indec}_{\text{ss}}(\mathbf{D}, R)$ with N nilpotent, then there is a unique $\Psi \in \text{Irr}(W_F, R)$ and a unique $r \geq 1$ such that

$$[\Phi, N] = (\Phi, N) = [0, r-1] \otimes \Psi = [0, r-1] \otimes [\Psi].$$

Proof. The uniqueness of Ψ (up to isomorphism or equivalence as these notions agree for nilpotent Deligne representations) and r are clear. Set $r = \text{ind}(N)$ and $V = V_\Phi$. As in the proof of Proposition 4.9, we construct W_F -stable subspaces S_i such that $V = \bigoplus_{i=0}^{r-1} S_i$ and $N(S_i) \subseteq S_{i-1}$. Suppose one S_{r-i} has length ≥ 2 as a W_F -module and take $i = i_0$ minimal for this property. Write

$$S_{r-i_0} = \Psi_{r-i_0} \oplus U_{r-i_0}$$

with $\Psi_{r-i_0} \in \text{Irr}(W_F, R)$ and U_{r-i_0} stable under W_F , then take a complement Z_{r-i_0-1} of

$$N(\Psi_{r-i_0}) \oplus N(U_{r-i_0})$$

in S_{r-i_0-1} , set

$$U_{r-i_0-1} = N(U_{r-i_0}) \oplus Z_{r-i_0-1}$$

and continue. We construct a nontrivial (Φ, N) -stable decomposition

$$[S_{r-1} \oplus \dots \oplus S_{r-(i_0-1)} \oplus \Psi_{r-i_0} \oplus \dots \oplus N^{r-i_0}(\Psi_{r-i_0})] \oplus [U_{r-i_0} \oplus U_{r-i_0-1} \oplus \dots \oplus U_0]$$

of V , a contradiction. Hence each S_{r-i} is an irreducible representation of W_F . It then suffices to choose $\Psi = S_{r-1}$. $\square$

It remains to consider the indecomposable W_F -semisimple Deligne representations (Φ, U) with U invertible, in particular $R = \overline{\mathbb{F}_\ell}$. We begin with an example.

Proposition 4.30. Let $\Psi \in \text{Irr}(W_F, \overline{\mathbb{F}_\ell})$, $I \in \text{Iso}_{W_F}(\nu^{o(\Psi)}\Psi, \Psi)$, and $r \geq 1$, then

$$[0, r-1] \otimes \mathcal{C}(\mathbb{Z}_\Psi, I) \in \text{Indec}_{\text{ss}}(\text{D}, \overline{\mathbb{F}_\ell}),$$

and its Deligne operator $U(r, I)$ is bijective with Jordan decomposition $U(r, I) = S(r, I) + N(r, I)$

$$S(r, I) = \text{Id} \otimes \mathcal{C}_I, \quad \text{and} \quad N(r, I) = N(r) \otimes \text{Id}.$$

Its endomorphism algebra is the local ring

$$\text{End}_{\text{D}}([0, r-1] \otimes \mathcal{C}(\mathbb{Z}_\Psi, I)) = \overline{\mathbb{F}_\ell}[N(r) \otimes \mathcal{C}_I^{-1}] = \overline{\mathbb{F}_\ell}[N(r, I) \circ S(r, I)^{-1}].$$

Proof. First notice that $[0, r-1] \otimes \mathcal{C}(\mathbb{Z}_\Psi, I)$ is indeed W_F -semisimple thanks to Lemma 4.12. Set $S = S(r, I) = \text{Id} \otimes \mathcal{C}_I$ and $N = N(r, I) = N(r) \otimes \text{Id}$. Then S and N clearly commute, N is nilpotent and S is semisimple because $\mathcal{C}_I$ is $(\mathcal{C}_I^{o(\nu)})$ is a nonzero scalar μ , and $X^{o(\nu)} - \mu$ has simple roots because $o(\nu)$ is prime to ℓ . Thus $U = S + N$ is bijective because S is, hence it remains to check that $[0, r-1] \otimes \mathcal{C}(\mathbb{Z}_\Psi, I)$ is indecomposable, and we do this by looking at the endomorphism algebra again. We recall that the commutant of U is that of S intersected with that of N . The commutant of N in $\text{End}_{\overline{\mathbb{F}_\ell}}([0, r-1] \otimes \mathcal{C}(\mathbb{Z}_\Psi, I))$ is $\overline{\mathbb{F}_\ell}[N] \otimes \text{End}_{\overline{\mathbb{F}_\ell}}(\mathcal{C}(\mathbb{Z}_\Psi, I))$. Hence we need to look at the commutant of the joint action of W_F and S inside $\overline{\mathbb{F}_\ell}[N] \otimes \text{End}_{\overline{\mathbb{F}_\ell}}(\mathcal{C}(\mathbb{Z}_\Psi, I))$. Writing an element in this commutant under the form $\sum_{k=0}^{r-1} N^k \otimes A_k$, the endomorphism A_k must belong to $\text{Hom}_{W_F}(\mathcal{C}(\mathbb{Z}_\Psi, I), \nu^k \mathcal{C}(\mathbb{Z}_\Psi, I))$ and must commute with $\mathcal{C}_I$, i.e.

$$\mathcal{C}_I^k A_k \in \text{End}_{\text{D}}(\mathcal{C}(\mathbb{Z}_\Psi, I)) \simeq \overline{\mathbb{F}_\ell},$$

the latter isomorphism by Schur's lemma. Hence our element is of the form $\sum_{k=0}^{r-1} \lambda_k N(r)^k \otimes \mathcal{C}_I^{-k}$, and this implies that

$$\text{End}_{\text{D}}([0, r-1] \otimes \mathcal{C}(\mathbb{Z}_\Psi, I)) = \overline{\mathbb{F}_\ell}[N(r) \otimes \mathcal{C}_I^{-1}] = \overline{\mathbb{F}_\ell}[N(r, I) \circ S(r, I)^{-1}].$$

As $N(r) \otimes \mathcal{C}_I^{-1}$ is nilpotent, this ring is local. □

We notice as in Proposition 4.18 (3), that the equivalence class of such a Deligne representation is independent of I . It is a consequence of Lemma 4.12 and Proposition 4.18 (3):

Lemma 4.31. We have

$$[[0, r-1] \otimes \mathcal{C}(\mathbb{Z}_\Psi, I)] = [0, r-1] \otimes [\mathcal{C}(\mathbb{Z}_\Psi, I)] = [0, r-1] \otimes \mathcal{C}(\mathbb{Z}_\Psi).$$

Now we move on to the description of $[\text{Indec}_{\text{ss}}(\text{D}, \overline{\mathbb{F}_\ell})]$. We know by Lemma 4.14 that $(\Phi, U) \in \text{Indec}_{\text{ss}}(\text{D}, \overline{\mathbb{F}_\ell})$ is always supported on an irreducible line. *For the remainder of this section, we take $(\Phi, U) \in \text{Indec}_{\text{ss}}(\text{D}, \overline{\mathbb{F}_\ell})$ supported on the irreducible line $\mathbb{Z}_\Psi$ with $U = S + N$ invertible.*

Lemma 4.32. The Deligne representation (Φ, S) is semisimple as a Deligne representation.

Proof. Let W be a (Φ, S) -stable subspace of $V = V_\Phi$, and V_λ be the eigenspace of S associated to an eigenvalue λ . As W is S -stable, it is the direct sum of the eigenspaces

$$W = \bigoplus_{\lambda \in \text{Spec}(S)} W_\lambda$$

where $W_\lambda = W \cap V_\lambda$. Moreover as W is W_F -stable, all the eigenspaces W_λ for λ in an orbit $O_\mu = \{q^{\mathbb{Z}}\mu\}$ have the same dimension, as $\Phi(\text{Frob})$ sends W_λ to $W_{q^{-1}\lambda}$. Hence the direct sum $W(\mu) = \bigoplus_{\lambda \in O_\mu} W_\lambda$ is W_F -stable thanks to the Weil-Deligne relation, and we write

$$W = \bigoplus_{i=1}^r W(\mu_i).$$

For each μ_i , take a complement W'_{μ_i} of W_{μ_i} inside V_{μ_i} , so that $W'_{q^{-k}\mu_i} = \Phi(\text{Frob})^k(W'_{\mu_i})$ is a complement of $W_{q^{-k}\mu_i}$ inside $V_{q^{-k}\mu_i}$. Then we set

$$W'(\mu_i) = \bigoplus_{\lambda \in O_{\mu_i}} W'_\lambda,$$

and we set

$$W' = \bigoplus_{i=1}^r W'(\mu_i).$$

As each $W'(\mu_i)$ is W_F -stable, so is W' , and as W' is a direct sum of subspaces of the eigenspaces V_λ , the space W' is also S -stable. To conclude, notice that $V = W \oplus W'$ by construction. $\square$

Now we notice that (Φ, S) is isotypic, i.e. it is the direct sum of isomorphic (not only equivalent) irreducible Weil-Deligne representations:

Lemma 4.33. There is $I \in \text{Iso}_{W_F}(\nu^{o(\Psi)}\Psi, \Psi)$ such that (Φ, S) is isotypic of type $\mathcal{C}(\mathbb{Z}_\Psi, I)$.

Proof. Set $N_S = NS^{-1} \in \text{End}_{W_F}(\Phi)$, and as N commutes with S , in fact $N_S \in \text{End}_{\mathbb{D}}((\Phi, S))$. Now (Φ, S) is the direct sum of its isotypic components of type $\mathcal{C}(\mathbb{Z}_\Psi, I_k)$ for

$$I_k \in \text{Iso}_{W_F}(\nu^{o(\Psi)}\Psi, \Psi)$$

thanks to Lemma 4.32 and Theorem 4.24. As N_S belongs to $\text{End}_{\mathbb{D}}((\Phi, S))$, it stabilizes each of these and so does S by definition of the isotypic components, so they are in fact stable under $U = S(\text{Id} + N_S)$. As (Φ, U) is indecomposable there is only one of them. $\square$

Using this lemma, we can obtain more information about the structure of indecomposable Deligne representations:

Put $N_S = NS^{-1}$ as in the proof of Lemma 4.33, and take the model $\mathcal{C}(\Psi, I) = (\Phi(\Psi), \mathcal{C}_I)$ for the type $\mathcal{C}(\mathbb{Z}_\Psi, I)$ of the isotypic Deligne representation (Φ, S) . Define

$$H = \text{Hom}_{\mathbb{D}}(\mathcal{C}(\Psi, I), (\Phi, S)).$$

We then define a Deligne representation structure $(\Phi_0, U_0 = S_0 + N_0)$ on $H \otimes_{\overline{\mathbb{F}}_\ell} V_{\Phi(\Psi)}$ as follows:

$$\Phi_0(w) = \text{Id} \otimes \Phi(\Psi)(w)$$

for $w \in W_F$,

$$S_0 = \text{Id} \otimes \mathcal{C}_I$$

and

$$N_0(h \otimes v) = Nh\mathcal{C}_I^{-1} \otimes \mathcal{C}_I \cdot v = N_S h \otimes \mathcal{C}_I \cdot v.$$

Proposition 4.34. The Deligne representations (Φ, U) and (Φ_0, U_0) are isomorphic. In particular setting $m = \text{ind}(N) = \text{ind}(N_S)$, we have $m = d_\Phi/d_{\Psi o}(\Psi)$.

Proof. We introduce the map $\alpha : H \otimes_{\overline{\mathbb{F}_\ell}} V_{\mathcal{C}(\Psi, I)} \rightarrow V$ defined by $\alpha(\psi \otimes v) = \psi(v)$. One checks that α intertwines the action of W_F , of N and N_0 and of S and S_0 . It is surjective by Lemma 4.33. By Schur's lemma, H has dimension $d_\Phi/d_\Psi o(\Psi)$, hence both spaces have the same dimension, so α is bijective. $\square$

We have now reached the following classification:

Theorem 4.35. Take $(\Phi, U) \in \text{Indec}_{\text{ss}}(\mathbf{D}, \overline{\mathbb{F}_\ell})$ with U bijective, there is a unique irreducible line $\mathbb{Z}_\Psi$ and a unique $r \geq 1$ such that

$$[\Phi, U] = [0, r-1] \otimes \mathcal{C}(\mathbb{Z}_\Psi).$$

Proof. Set $r = d_\Phi/d_\Psi o(\Psi)$. We define (Φ_0, U_0) as in Proposition 4.34, it is enough to show that

$$[\Phi_0, U_0] = [0, r-1] \otimes \mathcal{C}(\mathbb{Z}_\Psi).$$

Write $V_{[0, r-1]} = \text{Vect}_{\overline{\mathbb{F}_\ell}}(e_0, \dots, e_{r-1})$ with $e_i = N(r)^i(e_0)$, and W_F acting on $S_i = \text{Vect}_{\overline{\mathbb{F}_\ell}}(e_i)$ as ν^i . Similarly write $H = \text{Vect}_{\overline{\mathbb{F}_\ell}}(h_0, \dots, h_{r-1})$ with $h_i = N_S^i(h_0)$. Denote by B the $\overline{\mathbb{F}_\ell}$ -linear isomorphism sending $V_{[0, r-1]} \otimes V_\Phi$ to $H \otimes V_\Phi$ defined by

$$B\left(\sum_{i=0}^{r-1} e_i \otimes v_i\right) = \sum_{i=1}^{r-1} h_i \otimes \mathcal{C}_I^i v_i.$$

Then

$$B \in \text{Iso}_{\mathbf{D}}([0, r-1] \otimes \mathcal{C}(\Psi, I), (\Phi_0, U_0)).$$

The uniqueness of r and $\mathbb{Z}_\Psi$ are immediate. $\square$

4.4 Tensor product for W_F -semisimple Deligne representations

We already saw the two problems of the tensor product in our setting: by Example 3.11 (3) it does not preserve W_F -semisimplicity, and by Example 4.11 even when it does it does not preserve equivalence classes. In this section we take care of these two problems. These problems occur only when $R = \overline{\mathbb{F}_\ell}$ (see the beginning of Section 3.4 and Remark 4.12), so for this section $R = \overline{\mathbb{F}_\ell}$.

First we define the W_F -semisimplification $(\Phi, S)_{\text{ss}} = (\Phi_{\text{ss}}, S_{\text{ss}})$ of (Φ, S) when S is semisimple.

Proposition 4.36. Let (Φ, S) be a Deligne $\overline{\mathbb{F}_\ell}$ -representation with S semisimple.

(1) There is a filtration

$$\{0\} = V_0 \subseteq V_1 \subseteq \dots \subseteq V_n = V_\Phi$$

of V_Φ by (Φ, S) -stable subspaces such that the induced Deligne $\overline{\mathbb{F}_\ell}$ -representation (Φ_i, S_i) on V_i/V_{i-1} is in $\text{Irr}_{\text{ss}}(\mathbf{D}, \overline{\mathbb{F}_\ell})$.

(2) The W_F -semisimple Deligne $\overline{\mathbb{F}_\ell}$ -representation

$$(\Phi, S)_{\text{ss}} = (\Phi_{\text{ss}}, S_{\text{ss}}) = \bigoplus_{i=1}^n (\Phi_i, S_i)$$

is independent of the filtration, and S_{ss} and S are semisimple with the same characteristic polynomial.

Proof. By induction, it is enough to show that (Φ, S) contains an irreducible W_F -semisimple Deligne subrepresentation. The uniqueness follows from standard facts on Jordan-Hölder composition series. Take Ψ an irreducible W_F -subrepresentation of Φ . Then $\sum_{k \geq 0} \mathcal{C}^k(\Psi)$ is a W_F -semisimple Deligne subrepresentation of (Φ, S) . It thus contains an indecomposable W_F -semisimple Deligne subrepresentation $(\Phi_0, S|_{\Phi_0})$. But $S|_{\Phi_0}$ being semisimple, there is Ψ an irreducible representation of W_F such that $(\Phi_0, S|_{\Phi_0})$ is either of the form $\Psi = (\Psi, 0)$ or of the form $\mathcal{C}(\Psi, I)$ for I an isomorphism between $\nu^{o(\Psi)}\Psi$ and Ψ , thanks to Proposition 4.29 and Theorem 4.35. $\square$

We can now define the operation $\otimes_{\text{ss}}$ in $\text{Rep}_{\text{ss}}(\mathbf{D}, \mathbb{F}_l)$.

Definition 4.37. For $i = 1, 2$, let $(\Phi_i, U_i) \in \text{Indec}_{\text{ss}}(\mathbf{D}, \overline{\mathbb{F}}_\ell)$, and write

$$(\Phi_1, U_1) = [0, r_1 - 1] \otimes (\Phi_1, C_1) \text{ and } (\Phi_2, U_2) = [0, r_2 - 1] \otimes (\Phi_2, C_2),$$

with $(\Phi_i, C_i) \in \text{Irr}_{\text{ss}}(\mathbf{D}, \mathbb{F}_l)$ using Proposition 4.29 and Theorem 4.35. In particular, as C_1, C_2 are semisimple so is $C_1 \otimes \text{Id} \oplus \text{Id} \otimes C_2$. Hence $[(\Phi_1, C_1) \otimes (\Phi_2, C_2)]_{\text{ss}}$ is well defined according to Lemma 4.36. We then set

$$(\Phi, U) \otimes_{\text{ss}} (\Phi', U') = [0, r_1 - 1] \otimes [0, r_2 - 1] \otimes [(\Phi_1, C_1) \otimes (\Phi_2, C_2)]_{\text{ss}}.$$

We extend $\otimes_{\text{ss}}$ from $\text{Indec}_{\text{ss}}(\mathbf{D}, \overline{\mathbb{F}}_\ell)$ to $\text{Rep}_{\text{ss}}(\mathbf{D}, \overline{\mathbb{F}}_\ell)$ by distributivity with respect to $\oplus$.

Remark 4.38. The operation $\otimes_{\text{ss}}$ is bilinear by definition, and is associative and commutative, these properties inherited from $\otimes$.

We are now ready to define $[\Phi, U] \otimes [\Phi', U']$ for any pair of Deligne representations in $\text{Rep}_{\text{ss}}(\mathbf{D}, \overline{\mathbb{F}}_\ell)$. We first consider the indecomposable case. We recall that if $U = S + N$, then S and N are polynomials in U , and that the projections onto an eigenspace of S with respect to the sum of its other eigenspaces is a polynomial in S , hence in U .

Proposition 4.39. Let (Φ, U) and (Φ', U') be two W_F -semisimple indecomposable Deligne representations with U and U' bijective, then for all (λ, μ) in a subset $Z(U, U')$ of $(\overline{\mathbb{F}}_\ell^\times)^2$ consisting of all elements outside a finite number of hyperplanes of $\overline{\mathbb{F}}_\ell^2$, all representations

$$(\Phi, \lambda U) \otimes_{\text{ss}} (\Phi', \mu U')$$

are equivalent to one another, say equivalent to $(\Phi, \lambda_0 U) \otimes_{\text{ss}} (\Phi', \mu_0 U')$, and we set

$$[\Phi, U] \otimes_{\text{ss}} [\Phi', U'] = [(\Phi, \lambda_0 U) \otimes_{\text{ss}} (\Phi', \mu_0 U')].$$

Proof. We write $(\Phi, U) = [0, r_1 - 1] \otimes (\Phi_1, S_1)$ and $(\Phi', U') = [0, r_2 - 1] \otimes (\Phi_2, S_2)$, with (Φ_i, S_i) irreducible. Let's first do the irreducible case, we are going to show that

$$(\Phi_1, \lambda S_1) \otimes_{\text{ss}} (\Phi_2, \mu S_2) = ((\Phi_1, \lambda S_2) \otimes (\Phi_2, \mu S_2))_{\text{ss}}$$

is equivalent to

$$(\Phi_1, S_1) \otimes_{\text{ss}} (\Phi_2, S_2) = ((\Phi_1, S_1) \otimes (\Phi_2, S_2))_{\text{ss}}$$

for $(\lambda, \mu) \in (\overline{\mathbb{F}}_\ell^\times)^2$ outside a finite number of hyperplanes of $(\overline{\mathbb{F}}_\ell)^2$.

We let

$$\text{Spec}(S_1) - \{0\} = \{a_1, \dots, a_m\}, \quad \text{Spec}(S_2) - \{0\} = \{b_1, \dots, b_n\},$$

and put

$$Z(S_1, S_2) = \{(\lambda, \mu) \in (\overline{\mathbb{F}}_\ell^\times)^2, \forall i, j, \lambda a_i + \mu b_j \neq 0, \text{ and } \forall (i, j) \neq (k, l), \lambda a_i + \mu b_j \neq \lambda a_k + \mu b_l\}.$$

For (λ, μ) in $Z(S_1, S_2)$, all $S(\lambda, \mu) = \lambda S_1 \otimes \text{Id} + \text{Id} \otimes \mu S_2$ have the same eigenspaces in the sense that they have the same kernel, and for (λ, μ) and (λ', μ') in $Z(S_1, S_2)$, the eigenspace of $S(\lambda, \mu)$ corresponding to $\lambda a_i + \mu b_j$ is the same space as the eigenspace of $S(\lambda', \mu')$ corresponding to $\lambda' a_i + \mu' b_j$. In particular $S(\lambda, \mu)$ and $S(\lambda', \mu')$ are polynomials in one another. This implies that for (λ, μ) and (λ', μ') in $Z(S_1, S_2)$, if (Φ_*, U_*) is an irreducible subquotient of $((\Phi_1 \otimes \Phi_2), S(\lambda, \mu))$ (see Proposition 4.36) with U_* the Deligne endomorphism of V_{Φ_*} induced by $S(\lambda, \mu)$, then (Φ_*, T_*) is also an irreducible subquotient of $((\Phi_1 \otimes \Phi_2), S(\lambda', \mu'))$ with T_* the Deligne endomorphism of V_{Φ_*} induced by $S(\lambda', \mu')$. As U_* and T_* are bijective together, or zero together, Theorem 4.24 and Lemma 4.16 imply that $[\Phi_*, U_*] = [\Phi_*, T_*]$ (both representations have the same dimension and are supported on the same irreducible line). Now let's go back to the indecomposable case.

For (λ, μ) and (λ', μ') in $Z(S_1, S_2)$:

$$\begin{aligned} (\Phi, \lambda U) \otimes_{\text{ss}} (\Phi', \mu U') &= (\Phi(r_1), \lambda N(r_1)) \otimes (\Phi(r_2), \mu N(m_2)) \otimes (\Phi_1, \lambda S_1) \otimes_{\text{ss}} (\Phi_2, \mu S_2) \\ &= (\Phi(r_1), \lambda N(r_1)) \otimes (\Phi(r_2), \mu N(m_2)) \otimes (\Phi_1, \lambda S_1) \otimes_{\text{ss}} (\Phi_2, \mu S_2) \end{aligned}$$

according to Proposition 4.9. But

$$(\Phi_1, \lambda S_1) \otimes_{\text{ss}} (\Phi_2, \mu S_2) \sim (\Phi_1, \lambda' S_1) \otimes_{\text{ss}} (\Phi_2, \mu' S_2)$$

and the W_F -representation $\Phi(r_1) \otimes \Phi(r_2)$ is a direct sum of characters, hence this implies

$$(\Phi, \lambda U) \otimes_{\text{ss}} (\Phi', \mu U') \sim (\Phi(r_1), N(r_1)) \otimes (\Phi(r_2), N(m_2)) \otimes (\Phi_1, \lambda' S_2) \otimes_{\text{ss}} (\Phi_2, \mu' S_2),$$

which is itself isomorphic to

$$(\Phi(m_1), \lambda' N(m_1)) \otimes (\Phi(m_2), \mu' N(m_2)) \otimes (\Phi_1, \lambda' C_2) \otimes_{\text{ss}} (\Phi_2, \mu' S_2) = (\Phi, \lambda U) \otimes_{\text{ss}} (\Phi', \mu U')$$

thanks to Proposition 4.9. Finally one can set $Z(U, U') = Z(S_1, S_2)$ □

For the general case, as $[\Phi, U] \oplus [\Phi', U']$ is a well defined operation from

$$[\text{Rep}_{\text{ss}}(\mathbf{D}, \overline{\mathbb{F}}_\ell)] \times [\text{Rep}_{\text{ss}}(\mathbf{D}, \overline{\mathbb{F}}_\ell)]$$

to $[\text{Rep}_{\text{ss}}(\mathbf{D}, \overline{\mathbb{F}}_\ell)]$, we then extend the definition of $\otimes_{\text{ss}}$ by distributivity (bilinearity) from $[\text{Indec}_{\text{ss}}(\mathbf{D}, \overline{\mathbb{F}}_\ell)]$ to $[\text{Rep}_{\text{ss}}(\mathbf{D}, \overline{\mathbb{F}}_\ell)]$.

Remark 4.40. If (Φ, U) and (Φ', U') are in $\text{Rep}_{\text{ss}}(\mathbf{D}, \mathbb{F}_\ell)$ with U or U' both bijective, and if $(\Phi_0, U_0) \in [\Phi, U] \otimes_{\text{ss}} [\Phi', U']$, then U_0 is bijective. Indeed it suffices to check this when (Φ, U) and (Φ', U') are both indecomposable. It then follows from the proof of Theorem 4.39 that $S(\lambda, \mu)_{\text{ss}}$ is bijective for $(\lambda, \mu) \in Z(U, U')$, hence U_0 too.

4.5 Reduction modulo ℓ of Deligne representations

We now define the reduction modulo ℓ of an integral ℓ -adic Deligne representation, we do this via the following decomposition which follows from Proposition 4.29:

Definition 4.41. An element (Φ, U) of $\text{Nilp}_{\text{ss}}(\mathbf{D}, R)$ supported on an irreducible line $\mathbb{Z}_\Psi$ can be written in a unique manner as

$$(\Phi, U) = \bigoplus_{i \geq 1} \bigoplus_{k \in \mathbb{Z}} a_{i,k} [0, i-1] \otimes \nu^k \Psi$$

with almost all $a_{i,k} \in \mathbb{N}$ equal to zero. We call this decomposition the *standard form* of (Φ, U) .

We denote by $\text{Rep}_{\text{ss}}(\mathbf{D}, \overline{\mathbb{Q}_\ell})_e$ the classes of $(\Phi, U) \in \text{Rep}_{\text{ss}}(\mathbf{D}, \overline{\mathbb{Q}_\ell})_e$ with Φ integral, and set $\text{Nilp}_{\text{ss}}(\mathbf{D}, \overline{\mathbb{Q}_\ell})_e = \text{Nilp}_{\text{ss}}(\mathbf{D}, \overline{\mathbb{Q}_\ell}) \cap \text{Rep}_{\text{ss}}(\mathbf{D}, \overline{\mathbb{Q}_\ell})_e$.

Definition 4.42. Let $(\Phi, U) \in \text{Nilp}_{\text{ss}}(\mathbf{D}, \overline{\mathbb{Q}_\ell})_e$ be supported on an irreducible line $\mathbb{Z}_\Theta$, with $\Theta \in \text{Irr}(\mathbf{W}_F, \overline{\mathbb{Q}_\ell})$ necessarily integral, and with standard form

$$(\Phi, U) = \bigoplus_{i \geq 1} \bigoplus_{k \in \mathbb{Z}} a_{i,k} [0, i-1] \otimes \nu^k \Theta.$$

We define the *reduction modulo ℓ* of (Φ, U) by

$$r_\ell(\Phi, U) = \bigoplus_{i \geq 1} \bigoplus_{k \in \mathbb{Z}} a_{i,k} [0, i-1] \otimes \nu^k r_\ell(\Theta).$$

We extend this definition to all of $\text{Rep}_{\text{ss}}(\mathbf{D}, \overline{\mathbb{Q}_\ell})_e$ via the direct sum (as indecomposable representations are supported on irreducible lines).

To understand the structure of the reduction modulo ℓ of (Φ, U) above, it is useful to recall that $r_\ell(\Theta)$ is either irreducible, or of the form

$$r_\ell(\Theta) = \text{St}_a(\mathbb{Z}_\Psi)$$

for $a \geq 0$ and $\Psi \in \text{Irr}(\mathbf{W}_F, \overline{\mathbb{F}_\ell})$ thanks to Proposition 3.8.

5 Local constants of Deligne representations

In this section, we extend definitions of Artin–Deligne local constants to our setting of $\mathbf{W}_F$ -semisimple Deligne R -representations with $R = \overline{\mathbb{Q}_\ell}$ or $\overline{\mathbb{F}_\ell}$. We fix $\psi : F \rightarrow \overline{\mathbb{Q}_\ell}^\times$ a necessarily integral nontrivial character, and by abuse of notation also write ψ for $r_\ell(\psi)$.

5.1 Definition and basic properties

We begin with the Artin L -factor:

Definition 5.1. Let $(\Phi, U) \in \text{Rep}_{\text{ss}}(\mathbf{D}, R)$, we set

$$L(X, (\Phi, U)) = \det((\text{Id} - X\Phi(\text{Frob}))|_{\text{Ker}(U)^{I_F}})^{-1}.$$

In particular, $L(X, (\Phi, U)) = 1$ whenever U is bijective.

To define ϵ, γ -factors of $\mathbf{W}_F$ -semisimple Deligne R -representations, we first need to recall the well-known definitions of ϵ, γ -factors of irreducible R -representations of $\mathbf{W}_F$. Let Ψ be an irreducible R -representation of $\mathbf{W}_F$. The *epsilon factor* $\epsilon(X, \Psi, \psi)$ is defined in [8]. If Ψ is not an unramified character, we set

$$\gamma(X, \Psi, \psi) = \epsilon(X, \Psi, \psi).$$

If $\Psi = \chi$ is an unramified character, viewing χ as a character of $F^\times$ via the Artin reciprocity map of local class field theory, we take $\gamma(X, \chi, \psi)$ to be the Tate γ -factor defined in Tate's thesis [18] when $R = \overline{\mathbb{Q}_\ell}$, and in [13, 12] when $R = \overline{\mathbb{F}_\ell}$.

We can now define ϵ, γ -factors of $\mathbf{W}_F$ -semisimple Deligne R -representations:

Definition 5.2. Let $(\Phi, U) \in \text{Rep}_{\text{ss}}(D, R)$, with $\Phi = \bigoplus_{i=1}^r \Psi_i$ the underlying direct sum of irreducible R -representations of W_F .

(1) We define

$$\gamma(X, (\Phi, U), \psi) = \prod_{i=1}^r \gamma(X, \Psi_i, \psi).$$

(2) We define

$$\epsilon(X, (\Phi, U), \psi) = \gamma(X, (\Phi, U), \psi) \frac{L(X, (\Phi, U))}{L(q^{-1}X^{-1}, (\Phi, U)^\vee)}.$$

The following proposition follows from the definitions:

Proposition 5.3. The maps $(\Phi, U) \mapsto L(X, (\Phi, U))$, $(\Phi, U) \mapsto \gamma(X, (\Phi, U), \psi)$, hence the map $(\Phi, U) \mapsto \epsilon(X, (\Phi, U), \psi)$ are multiplicative with respect to direct sums.

Corollary 5.4. Two equivalent W_F -semisimple Deligne R -representations have the same constants L , γ and ϵ , in particular we can talk of the local constants L , γ and ϵ of $[\Phi, U] \in [\text{Rep}_{\text{ss}}(D, \mathbb{F}_l)]$.

Proof. By Proposition 5.3 one reduces to the indecomposable case. But if (Φ, U) is indecomposable, then $L(X, (\Phi, U))$ and $L(X, (\Phi, \lambda U))$ are equal for $\lambda \neq 0$ as $\text{Ker}(U) = \text{Ker}(\lambda U)$, and the definition of $\gamma(X, (\Phi, U), \psi)$ and $\gamma(X, (\Phi, \lambda U), \psi)$ does not see U and λU . Finally, the result for ϵ follows. $\square$

The next proposition also follows from the definitions and the known multiplicativity properties of γ -factors in the ℓ -adic case.

Proposition 5.5. (1) When $R = \overline{\mathbb{Q}_\ell}$, the factors defined in Definitions 5.1 and 5.2 are the usual L , γ and ϵ -factors from [8, 8.12], (see [10] for γ).

(2) When $R = \overline{\mathbb{F}_\ell}$, if Φ is irreducible as an $\overline{\mathbb{F}_\ell}$ -representation of W_F and $U = 0$, then

$$L(X, (\Phi, 0)) = L(X, \Phi)$$

is the usual Artin L -factor of Φ , and

$$\epsilon(X, (\Phi, 0), \psi) = \epsilon(X, \Phi, \psi)$$

is also the usual ϵ -factor defined in [8].

Proof. The second assertion follows from the definitions. Suppose that $R = \overline{\mathbb{Q}_\ell}$, the quickest way to check our the first assertion is to use [10, Lemma 4.4]. Let γ' be the γ factor defined in [ibid. (4.6)], and (Φ, U) a W_F -semisimple Deligne representation of W_F . By [ibid. Lemma 4.4], one has

$$\gamma'(X, (\Phi, U), \psi) = \gamma'(X, \Phi, \psi) = \epsilon(X, \Phi, \psi) \frac{L(q^{-1}X^{-1}, \Phi^\vee)}{L(X, \Phi)},$$

where ϵ and L are the local constants attached to representations of W_F in [8]. In particular as they are multiplicative with respect to direct sum, this implies that γ' is, hence if $\Phi = \bigoplus_{i=1}^r \Psi_i$ with Ψ_i irreducible representations of W_F , one has

$$\gamma'(X, (\Phi, U), \psi) = \prod_{i=1}^r \gamma'(X, \Psi_i, \psi).$$

So to prove that γ is equal to γ' , it suffices to check that $\gamma'(X, \Psi, \psi) = \gamma(X, \Psi, \psi)$ when Ψ is an irreducible representation of W_F . If Ψ is not an unramified character then

$$\gamma'(X, \Psi, \psi) = \epsilon(X, \Psi, \psi) = \gamma(X, \Psi, \psi).$$

If χ is an unramified character, the assertion for L follows from the definitions, hence the assertion for ϵ is equivalent to $\gamma' = \gamma$, which is by definition of ϵ in this case. $\square$

When $R = \overline{\mathbb{Q}_\ell}$, the constant $\epsilon(X, (\Phi, U), \Psi)$ is an invertible element of the ring $R[X^{\pm 1}]$. Let's check this when $R = \overline{\mathbb{F}_\ell}$. We will write $P(X) \underset{\overline{\mathbb{F}_\ell}[X^{\pm 1}]^\times}{\sim} Q(X)$ if $P, Q \in \overline{\mathbb{F}_\ell}(X)$ are equal up to an element in $\overline{\mathbb{F}_\ell}[X^{\pm 1}]^\times$.

Proposition 5.6. Let (Φ, U) be an ℓ -modular W_F -semisimple Deligne representation, then

$$\epsilon(X, (\Phi, U), \Psi) \in \overline{\mathbb{F}_\ell}[X^{\pm 1}]^\times.$$

Proof. Thanks to Proposition 5.3, and Corollary 5.4, we can suppose that $(\Phi, U) = [0, r-1] \otimes \Psi$ or $(\Phi, U) = [0, r-1] \otimes \mathcal{C}(\Psi, I)$.

Let's start with the second case. As $L(X, (\Phi, U)) = 1$ we have

$$\epsilon(X, (\Phi, U), \psi) = \gamma(X, (\Phi, U), \psi) = \prod_{k=0}^{o(\Psi)-1} \gamma(X, \nu^k \Psi, \psi)^r.$$

If Ψ is either ramified, or a non-banal character χ (non-banal meaning $o(\chi) = o(\nu) = 1$, or equivalently $q \equiv 1[\ell]$) then $\gamma(X, \nu^k \chi, \psi) = \epsilon(X, \nu^k \chi, \psi)$ and we are done.

If Ψ is a banal unramified character χ (i.e. $o = o(\nu) = o(\chi) \geq 2$) with $\chi(\varpi) = t \in \overline{\mathbb{F}_\ell}^\times$

$$\prod_{k=0}^{o-1} L(X, \nu^k \chi) = \prod_{k=0}^{o-1} (1 - tq^{-k}X)^{-1} = (1 - (tX)^o)^{-1}$$

and similarly

$$\prod_{k=0}^{o-1} L(q^{-1}X^{-1}, \nu^{-k}\chi^{-1}) = (1 - (tX)^{-o})^{-1},$$

hence

$$\epsilon(X, (\Phi, U), \psi) = \prod_{k=0}^{o-1} \epsilon(X, \nu^k \chi, \psi)^r (- (tX)^o)^r \in \overline{\mathbb{F}_\ell}[X^{\pm 1}]^\times.$$

It remains to deal with the case $(\Phi, U) = [0, r-1] \otimes \Psi$. If Ψ is not an unramified character or if Ψ is a non banal unramified character then

$$\epsilon(X, (\Phi, U), \psi) = \gamma(X, (\Phi, U), \psi) = \prod_{i=0}^{r-1} \gamma(X, \nu^i \Phi, \psi) = \prod_{i=0}^{r-1} \epsilon(X, \nu^i \Phi, \psi) \in \overline{\mathbb{F}_\ell}[X^{\pm 1}]^\times.$$

If Ψ is a banal unramified character χ ($q \not\equiv 1[\ell]$) with $\chi(\varpi) = t \in \overline{\mathbb{F}_\ell}^\times$, then

$$\epsilon(X, (\Phi, U), \psi) = \gamma(X, (\Phi, U), \psi) \frac{L(X, (\Phi, U))}{L(q^{-1}X^{-1}, (\Phi, U)^\vee)} = \gamma(X, (\Phi, U), \psi) \frac{(1 - t^{-1}q^{-1}X^{-1})}{(1 - tq^{1-r}X)}$$

However

$$\gamma(X, (\Phi, U), \psi) = \prod_{i=0}^{r-1} \gamma(X, \nu^i \chi, \psi) = \prod_{i=0}^{r-1} \epsilon(X, \nu^i \chi, \psi) \prod_{i=0}^{r-1} \frac{(1 - tq^{-i}X)}{(1 - t^{-1}q^{i-1}X^{-1})}$$

Now

$$\prod_{i=0}^{r-1} \frac{(1 - tq^{-i}X)}{(1 - t^{-1}q^{i-1}X^{-1})} = \frac{\prod_{i=0}^{r-1} (1 - tq^{-i}X)}{\prod_{i=-1}^{r-2} (1 - t^{-1}q^i X^{-1})} \underset{\mathbb{F}_\ell[X^{\pm 1}]^\times}{\sim} \frac{(1 - tq^{1-r}X)}{(1 - t^{-1}q^{-1}X^{-1})},$$

hence $\epsilon(X, (\Phi, U), \psi) \underset{\mathbb{F}_\ell[X^{\pm 1}]^\times}{\sim} 1$. $\square$

We finally prove the multiplicativity relation for the L -factors of nilpotent Deligne R -representations:

Lemma 5.7. Let Ψ and Ψ' be two irreducible R -representations of W_F , and $n \geq m \geq 1$, then:

$$L(X, ([0, n-1] \otimes \Psi) \otimes_{\text{ss}} ([0, m-1] \otimes \Psi')) = \prod_{k=0}^{m-1} L(X, \nu^{n-1} \Psi \otimes_{\text{ss}} \nu^k \Psi').$$

Proof. By definition

$$([0, n-1] \otimes \Psi) \otimes_{\text{ss}} ([0, m-1] \otimes \Psi') = [0, n-1] \otimes [0, m-1] \otimes (\Psi \otimes_{\text{ss}} \Psi').$$

Moreover, writing as usual $[0, r-1] = (\Phi(r), N(r))$, as $\Phi(n)^{I_F} = \Phi(n)$ and $\Phi(m)^{I_F} = \Phi(m)$, we deduce that

$$(\Phi(n) \otimes \Phi(m) \otimes (\Psi \otimes_{\text{ss}} \Psi'))^{I_F} = \Phi(n) \otimes \Phi(m) \otimes (\Psi \otimes_{\text{ss}} \Psi')^{I_F}.$$

Hence we can suppose that Ψ and Ψ' are both the trivial character to prove the multiplicativity relation. Writing $\Phi(l) = \bigoplus_{i=0}^{l-1} \nu^i \cdot e_i$, one can decompose $\Phi(n) \otimes \Phi(m)$ as $\bigoplus_{j=0}^{n+m-2} V_k$, where $V_k = \text{Vect}(e_i \otimes e_j)_{i+j=k}$. Write $N = N(n) \oplus \text{Id} + \text{Id} \otimes N(m)$, then $N(V_k) \subseteq V_{k+1}$ for $k < n+m-2$ and $N(V_{n+m-2}) = \{0\}$. Hence

$$\text{Ker}(N) = \bigoplus_{j=0}^{n+m-2} \text{Ker}(N) \cap V_k.$$

However one checks that $\text{Ker}(N) \cap V_k = \{0\}$ for $j < n-1$, and

$$\text{Ker}(N) \cap V_{n-1+k} = \text{Vect}\left(\sum_{i=0}^{m-1} (-1)^i e_{n+m-2-i} \otimes e_i\right)$$

for $k = 0, \dots, m-1$. Denoting K_k the space $\text{Ker}(N) \cap V_k$, we have showed that

$$\text{Ker}(N) = \bigoplus_{k=0}^{m-1} K_{n-1+k}$$

where W_F acts via ν^{n-1+k} on the line K_{n-1+k} , and the result follows. $\square$

5.2 Reduction modulo ℓ of local constants

We state two results concerning good reduction of local constants. The first is a result of [10], which states that γ -factors are compatible with reduction modulo ℓ . For $P \in \overline{\mathbb{Z}}_\ell[X]$, we denote by $r_\ell(P) = \overline{\mathbb{F}}_\ell[X]$ the polynomial obtained by applying r_ℓ to the coefficients of P . For Q in $\overline{\mathbb{Z}}_\ell[X]$, such that $r_\ell(Q) \neq 0$, we set $r_\ell(P/Q) = r_\ell(P)/r_\ell(Q)$.

Proposition 5.8. [10, Theorem 1.1, 1]: Let Φ be an integral semisimple representation of W_F , then

$$r_\ell(\gamma(X, \Phi, \psi)) = \gamma(X, r_\ell(\Phi), \psi).$$

Then we state the following immediate consequence of Proposition 3.13.

Theorem 5.9. Let Ψ and Ψ' be banal irreducible $\overline{\mathbb{F}_\ell}$ -representations of W_F , and let $\tilde{\Psi}$ and $\tilde{\Psi}'$ be irreducible ℓ -adic lifts of Ψ and Ψ' respectively, then

$$L(X, \Psi \otimes_{\text{ss}} \Psi') = r_\ell(L(X, \tilde{\Psi} \otimes \tilde{\Psi}')).$$

Proof. Because $\tilde{\Psi} \otimes \tilde{\Psi}'$ is integral,

$$\det((\text{Id} - X\Phi(\text{Frob}))|_{(V_\Psi \otimes V_{\Psi'})^{I_F}}) \in \overline{\mathbb{Z}_\ell}[X],$$

hence it makes sense to consider $r_\ell(L(X, \tilde{\Psi} \otimes \tilde{\Psi}'))$.

Moreover thanks to Proposition 3.13, we know that $\overline{(V_{\tilde{\Psi}} \otimes V_{\tilde{\Psi}'})^{I_F}} = (V_\Psi \otimes V_{\Psi'})^{I_F}$, hence

$$r_\ell(L(X, \tilde{\Psi} \otimes \tilde{\Psi}')) = \det((\text{Id} - X\Psi \otimes \Psi'(\text{Frob}))|_{(V_\Psi \otimes V_{\Psi'})^{I_F}})^{-1}.$$

Finally according to [ibid.] again, we have $((V_\Psi \otimes V_{\Psi'})^{I_F})_{\text{ss}} \simeq (V_\Psi \otimes_{\text{ss}} V_{\Psi'})^{I_F}$, hence

$$\det((\text{Id} - X\Phi(\text{Frob}))|_{(V_\Psi \otimes V_{\Psi'})^{I_F}})^{-1} = L(X, \Psi \otimes_{\text{ss}} \Psi').$$

□

6 The ℓ -modular Langlands correspondence and local constants

In [24], Vignéras defines a bijection V (Theorem 6.3) between $\coprod_{n \geq 0} \text{Irr}(G_n, \overline{\mathbb{F}_\ell})$ (see Notation 6.1) and $\text{Nilp}_{\text{ss}}(D, \overline{\mathbb{F}_\ell})$, the “semisimplification” of which is obtained by “reducing modulo ℓ ” the ℓ -adic LLC, and which moreover commutes with character twists, taking duals, and takes central characters to determinant (Lemma 6.4). The aim of this last section is to define an injection C (Definition 6.3) of $\text{Irr}(G, \overline{\mathbb{F}_\ell})$ into $[\text{Rep}_{\text{ss}}(D, \overline{\mathbb{F}_\ell})]$, which besides sharing all these properties with V , takes the local factors of pairs of generic representations defined in [12] to those of tensor products of elements in $[\text{Rep}_{D, \text{ss}}]$ (Theorem 6.13).

6.1 Representations of $\text{GL}_n(F)$

We put $G_n = \text{GL}_n(F)$ (G_0 is trivial by convention), and denote by N_n the group of unipotent upper triangular matrices in G_n . For $n \in N_n$ and $\psi : F \rightarrow R^\times$ a non-trivial character, we set

$$\theta_\psi(n) = \psi\left(\sum_{i=1}^{n-1} n_{i, i+1}\right).$$

We say that an irreducible representation π of G_n is *generic* if

$$\text{Hom}_{N_n}(\pi, \theta_\psi) \neq \{0\},$$

in which case it is well known that $\dim_R(\text{Hom}_{N_n}(\pi, \theta_\psi)) = 1$, but we do not use this *multiplicity one* property.

For $\{n_1, \dots, n_r\}$ positive integers, let $\pi_i \in \text{Rep}(G_{n_i}, R)$ and put $n = \sum_{i=1}^r n_i$. We denote by

$$\pi_1 \times \dots \times \pi_r \in \text{Rep}(G_n, R)$$

the normalized parabolic induction of the π_i 's. An irreducible R -representation of G_n is called *cuspidal* if it does not appear as a subrepresentation of a properly parabolically induced representation. It is called *supercuspidal* if it does not appear as a subquotient of a properly parabolically induced representation. By classical results (see for example [2], [27] and [22]), if the π_i 's are generic, then $\pi_1 \times \dots \times \pi_r$ has a unique generic subquotient and cuspidal representations are always generic. If ρ is cuspidal, we denote by $\text{St}(r, \rho)$ the unique generic subquotient of $\rho \times \nu\rho \times \dots \times \nu^{r-1}\rho$. By convention, $\text{St}(0, \rho)$ is the trivial representation of the trivial group G_0 .

On the other hand, in [2] when $R = \overline{\mathbb{Q}_\ell}$, and [22] or [14, Definitions 7.5] when $R = \overline{\mathbb{F}_\ell}$, to a *cuspidal segment*

$$[a, b]_\rho = (\nu^a \rho, \dots, \nu^b \rho)$$

with $a \leq b$, the authors attach a certain irreducible quotient $L([a, b]_\rho)$ and a certain irreducible submodule $Z([a, b]_\rho)$ of

$$\nu^a \rho \times \dots \times \nu^b \rho,$$

which are respectively the cosocle and socle of it when $R = \overline{\mathbb{Q}_\ell}$. The *length* of the segment $[a, b]_\rho$ is $b - a + 1$. By convention if $b < a$, then $[a, b]_\rho$ is the empty segment and $L([a, b]_\rho)$ and $Z([a, b]_\rho)$ are the trivial representation of G_0 .

Consider two cuspidal segments Δ and Δ' , we say that Δ precedes Δ' if one can extract a segment longer than both from the sequence (Δ, Δ') , in which case we set $\Delta \prec \Delta'$. We say that two cuspidal segments are *linked* if either of them precedes the other one, otherwise we say that they are *unlinked*.

We denote by $*$ the Aubert-Zelevinsky involution on $\text{Irr}(G_n, R)$ for any $n \geq 0$ (see [27, 1] when $R = \overline{\mathbb{Q}_\ell}$ and [21, 15] when $R = \overline{\mathbb{F}_\ell}$), it satisfies

$$(\pi_1 \times \pi_2)^* = \pi_1^* \times \pi_2^*$$

when $\pi_1 \times \pi_2$ is irreducible. This $*$ involution also commutes with taking duals: in the ℓ -modular case, using the notation of [15, Theorem 8], this property follows from the theorem itself and the fact that $\mathbf{D}$ commutes with taking duals. It is shown in these references (for example [15, Proposition 4.10]) that for a cuspidal segment Δ :

$$L(\Delta) = Z(\Delta)^*.$$

Notation 6.1. Let now us fix some more notation:

- $\text{Irr}(G, R) = \coprod_{n \geq 0} \text{Irr}(G_n, R)$.
- $\text{Irr}_{\text{gen}}(G_n, R)$: the set of isomorphism classes of generic representations in $\text{Irr}(G_n, R)$.
- $\text{Irr}_{\text{gen}}(G, R) = \coprod_{n \geq 0} \text{Irr}_{\text{gen}}(G_n, R)$.
- $\text{Irr}_c(G_n, R)$: the set of isomorphism classes of cuspidal representations in $\text{Irr}(G_n, R)$.
- $\text{Irr}_{\text{sc}}(G_n, R)$: the set of isomorphism classes of supercuspidal representations in $\text{Irr}(G_n, R)$.
- $\text{Irr}_c(G, R) = \coprod_{n \geq 0} \text{Irr}_c(G_n, R)$.
- $\text{Irr}_{\text{sc}}(G, R) = \coprod_{n \geq 0} \text{Irr}_{\text{sc}}(G_n, R)$.

If $\pi \in \text{Rep}(G, R)$ has a central character (for example if it belongs to $\text{Irr}(G, R)$), we denote it by c_π . If ρ is a cuspidal representation, we denote by $\mathbb{Z}_\rho$ the associated *cuspidal line*

$$\mathbb{Z}_\rho = \{\nu^k \rho, k \in \mathbb{Z}\}.$$

If ρ is supercuspidal, we say that $\pi \in \text{Irr}(G, R)$ is *supported on* $\mathbb{Z}_\rho$ if all supercuspidal representations of its supercuspidal support (which is unique by [22] or [14]) belong to $\mathbb{Z}_\rho$. The set $\mathbb{Z}_\rho$ is finite if and only if $R = \overline{\mathbb{F}_\ell}$, in which case we set

$$o(\rho) = |\mathbb{Z}_\rho|.$$

Following [14, Remarque 8.15], we say that $\pi \in \text{Irr}(G, R)$ is *banal* if the cuspidal support of π contains no cuspidal line, in particular non-banal irreducible representations exist only when $R = \overline{\mathbb{F}_\ell}$ and a cuspidal representation ρ is non-banal if and only if $o(\rho) = 1$. By [19], [22] or [14, Theorem 6.4], if a cuspidal representation ρ is banal, then it is supercuspidal.

If τ is cuspidal, then there is a non-negative integer k such that

$$\tau = \text{St}(o(\rho)\ell^k, \rho)$$

for a supercuspidal representation ρ , the cuspidal line of which is unique (ρ can be replaced by any supercuspidal representation on the same line). Therefore, in this case, we set

$$\text{St}_k(\mathbb{Z}_\rho) = \text{St}(o(\rho)\ell^k, \rho).$$

If ρ is cuspidal, then $L([0, r-1]_\rho) = \text{St}(r, \rho)$ if and only if either $r < o(\rho)$ when ρ is banal, or $r < \ell$ when ρ is non-banal ([14, Remarque 8.14]).

By [27, Theorem 9.7] when $R = \overline{\mathbb{Q}_\ell}$, and [22, Theorem V.7] or [14, Theorem 9.10] when $R = \overline{\mathbb{F}_\ell}$, a representation $\pi \in \text{Irr}_{\text{gen}}(G)$ can be written under the form of a commutative product

$$\pi = L([0, m_1-1]_{\rho_1}) \times \cdots \times L([0, m_r-1]_{\rho_r}) = \text{St}(m_1, \rho_1) \times \cdots \times \text{St}(m_r, \rho_r)$$

where, for $i \in \{1, \dots, r\}$, the cuspidal segments $[0, m_i-1]_{\rho_i}$ are unlinked and unique up to ordering.

In particular, a representation $\pi \in \text{Irr}_{\text{gen}}(G, R)$ supported on a supercuspidal line $\mathbb{Z}_\rho$, π can be written in a unique manner $\pi_b \times \pi_{\text{tnb}}$ as in [12, Proposition 2.3]. The representation π_b is a banal representation, which can be written in a unique manner as a (possibly empty) product

$$\pi_b = \prod_{i=1}^s L([c_i, d_i]_\rho) = \prod_{i=1}^s \text{St}(d_i - c_i + 1, \nu^{c_i} \rho)$$

with the segments $[c_i, d_i]_\rho$ unlinked (in particular all of the lengths $d_i - c_i + 1$ are $< o(\rho)$, hence the product is empty if ρ is non banal). The representation π_{tnb} is *totally non-banal*, i.e. no segment occurring in π_{tnb} is banal, it is a (possibly empty) product of the form

$$\pi_{\text{tnb}} = \prod_{k=0}^r L([0, a_k-1]_{\text{St}_k(\mathbb{Z}_\rho)}), \text{ for } 0 \leq a_k < \ell.$$

6.2 The V-correspondence

As in the context of representations of W_F , there is also a notion of reduction modulo ℓ for integral ℓ -adic representations of G_n . If π is a finite length representation in $\text{Rep}(G_n, \overline{\mathbb{Q}_\ell})$, one

says that it is integral if it contains a stable G_n -lattice Λ . We denote by a right index e the set of classes of integral representations inside a set of classes of ℓ -adic representations, for example $\text{Irr}(G_n, \overline{\mathbb{Q}_\ell})_e$. If π has finite length, the ℓ -modular representation $\pi \otimes_{\overline{\mathbb{F}_\ell}} \Lambda$ is of finite length: we denote by $r_\ell(\pi)$ its semisimplification which is independent of Λ , as proved in [26, Theorem 1]. By [19, II.4.12], $\text{Irr}_c(G, \overline{\mathbb{Q}_\ell})_e$ are the elements in $\text{Irr}_c(G, \overline{\mathbb{Q}_\ell})$ with integral central character. By [19, III.5.10], if $\rho \in \text{Irr}_c(G_n, \overline{\mathbb{F}_\ell})$, there is $\tilde{\rho} \in \text{Irr}_c(G_n, \overline{\mathbb{Q}_\ell})_e$ such that $r_\ell(\tilde{\rho}) = \rho$.

In [24, I.8.4], Vignéras introduces a surjection

$$J_\ell : \text{Irr}(G_n, \overline{\mathbb{Q}_\ell})_e \rightarrow \text{Irr}(G_n, \overline{\mathbb{F}_\ell}).$$

Let ψ be a (necessarily integral) character of F such that $r_\ell(\psi)$ is nontrivial. For A a finite subset of $\{1, \dots, n-1\}$, we denote by $\theta_{\psi, A}$ the (degenerate when $A \neq \emptyset$) character of N_n defined by

$$\theta_{\psi, A}(n) = \psi\left(\sum_{i \notin A} n_{i, i+1}\right).$$

Then by [27, Theorem 8.2], for $\pi \in \text{Irr}(G_n, \overline{\mathbb{Q}_\ell})_e$, there is a unique A minimal for the inclusion relation, such that π has a Whittaker model (which is unique) with respect to $\theta_{\psi, A}$ (we will say of type A). By [14, Proposition 9.19], the reduction modulo ℓ of π has a unique irreducible summand π' which has a Whittaker model with respect to $r_\ell(\theta_{\psi, A})$. The map J_ℓ is then defined by $J_\ell(\pi) = \pi'$. Let's compute some examples:

Example 6.2. (1) If $\Delta = [a, b]_\tau$ is a cuspidal ℓ -adic segment so that $\rho = r_\ell(\tau)$ is cuspidal (see [23, Section 1] for this fact), and we set $r_\ell(\Delta) = [a, b]_\rho$, then

$$J_\ell(Z(\Delta)) = Z(r_\ell(\Delta)).$$

(2) Take $\tau \in \text{Irr}_c(G, \overline{\mathbb{Q}_\ell})_e$ such that $\rho_0 = r_\ell(\tau)$ is cuspidal non-supercuspidal. Then $\rho_0 = \text{St}_r(\mathbb{Z}_\rho)$. We set $\rho_i = \text{St}_{r+i}(\mathbb{Z}_\rho)$. For $k \in \mathbb{N}$, with ℓ -adic expansion $k = a_0 + a_1\ell + \dots + a_d\ell^d$, we have

$$J_\ell(\text{St}(k, \tau)) = \text{St}(a_0, \rho_0) \times \text{St}(a_1, \rho_1) \times \dots \times \text{St}(a_d, \rho_d).$$

(3) Take $\tau \in \text{Irr}_c(G, \overline{\mathbb{Q}_\ell})_e$ such that $\rho = r_\ell(\tau)$ is supercuspidal. For $k \in \mathbb{N}$, we write the euclidean division of k by $o(\rho)$ as $k = uo(\rho) + r$. Again, we set $\rho_i = \text{St}_i(\mathbb{Z}_\rho)$ and we write the ℓ -adic expansion of u as $u = a_0 + a_1\ell + \dots + a_d\ell^d$. Then

$$J_\ell(\text{St}(k, \tau)) = \text{St}(r, \rho) \times \text{St}(a_0, \rho_0) \times \text{St}(a_1, \rho_1) \times \dots \times \text{St}(a_d, \rho_d).$$

We denote by

$$\text{LLC} : \text{Nilp}_{\text{ss}}(\mathbb{D}, \overline{\mathbb{Q}_\ell}) \rightarrow \text{Irr}(G, \overline{\mathbb{Q}_\ell}),$$

the ℓ -adic local Langlands correspondence, characterized by compatibility with the formation of local constants of pairs, this correspondence depends on our fixed choice of square root of q . We now recall the main result of [24], the *ℓ -modular local Langlands correspondence*, defined by compatibility with LLC and reduction modulo ℓ of Deligne representations (as we defined in Section 4.5), but in a non-naive way:

Theorem 6.3 ([24, Theorems 1.6 and 1.8.5]). There is a bijection

$$V : \text{Irr}(G, \overline{\mathbb{F}_\ell}) \xrightarrow{\sim} \text{Nilp}_{\text{ss}}(\mathbb{D}, \overline{\mathbb{F}_\ell}),$$

characterized by the property $V(J_\ell(\text{LLC}(\Phi)^*))^* = r_\ell(\Phi)$ for any $\Phi \in \text{Nilp}_{\text{ss}}(\mathbb{D}, \overline{\mathbb{Q}_\ell})_e$. Its restriction to $\text{Irr}_{\text{sc}}(G, \overline{\mathbb{F}_\ell})$ induces a bijection between $\text{Irr}_{\text{sc}}(G, \overline{\mathbb{F}_\ell})$ and $\text{Irr}(\mathbb{W}_F, \overline{\mathbb{F}_\ell})$.

The following immediate properties of V are clear, though not explicitly stated in [24]:

Lemma 6.4. Let $\pi \in \text{Irr}(G, \overline{\mathbb{F}}_\ell)$.

(1) For all characters $\chi : \text{GL}_n(F) \rightarrow \overline{\mathbb{F}}_\ell^\times$,

$$V(\chi\pi) = \chi V(\pi).$$

(2) Letting c_π denote the central character of π , then using local class field theory

$$c_\pi = \det(V(\pi)).$$

(3) We have

$$V(\pi^\vee) = V(\pi)^\vee.$$

(4) For $\pi' \in \text{Irr}(G, \overline{\mathbb{F}}_\ell)$ such that $\pi \times \pi'$ is irreducible, we have $V(\pi \times \pi') = V(\pi) \oplus V(\pi')$.

(5) Moreover if $\pi = \pi(\mathbb{Z}_{\rho_i}) \times \cdots \times \pi(\mathbb{Z}_{\rho_r})$ with $\pi(\mathbb{Z}_{\rho_i})$ supported on the supercuspidal line $\mathbb{Z}_{\rho_i}$ and $\mathbb{Z}_{\rho_i} \neq \mathbb{Z}_{\rho_j}$ for $i \neq j$, then

$$V(\pi) = \bigoplus_{i=1}^r V(\pi(\mathbb{Z}_{\rho_i})).$$

Proof. Both ℓ -modular and ℓ -adic Aubert-Zelevinsky involutions, J_ℓ and LLC commute with character twists, and

$$r_\ell : \text{Nilp}_{\text{ss}}(\mathbf{D}, \overline{\mathbb{Q}}_\ell)_e \rightarrow \text{Nilp}_{\text{ss}}(\mathbf{D}, \overline{\mathbb{F}}_\ell),$$

as well, hence the first statement. For $\pi \in \text{Irr}(G, \overline{\mathbb{Q}}_\ell)_e$, the central character $c_{J_\ell(\pi)}$ of $J_\ell(\pi)$ is equal to $r_\ell(c_\pi)$, both Zelevinsky involutions do not touch the central character, and LLC takes determinant to central character. The commutation with taking duals also follows from the fact that r_ℓ , both Aubert-Zelevinsky involutions, LLC and J_ℓ share this property. The next property is a consequence of the analogous property for LLC, the fact that both $*$ -involutions commute with irreducible parabolic induction, and the fact that $J_\ell(\pi_1 \times \pi_2) = J_\ell(\pi_1) \times J_\ell(\pi_2)$ when $\pi_1 \times \pi_2$ is irreducible (this last assertion follows from standard properties of degenerate Whittaker models with respect to parabolic induction). The last property follows from the previous property. $\square$

Following Theorem 6.3, we call an element (Φ, U) of $\text{Nilp}_{\text{ss}}(\mathbf{D}, \overline{\mathbb{F}}_\ell)$ a V -parameter and if $V(\pi) = (\Phi, U)$ for $\pi \in \text{Irr}(G, \overline{\mathbb{F}}_\ell)$ we say that (Φ, U) is the V -parameter of π . We end this section with some examples of V -parameters.

Example 6.5. (1) If $\rho = \text{St}_k(\mathbb{Z}_{\rho_0})$ is cuspidal non supercuspidal, with $\rho_0 \in \text{Irr}_{\text{sc}}(G)$ and set

$$\Psi_0 = V(\rho_0) \in \text{Irr}(\mathbf{W}_F, \overline{\mathbb{F}}_\ell)$$

so that $o(\rho_0) = o(\Psi_0)$. Then if $\tilde{\rho}_0$ is a (necessary cuspidal) lift of ρ_0 , and $\tilde{\Psi}_0$ is a (necessary irreducible) lift of Ψ_0 , by Theorem 6.3, we have $r_\ell(\tilde{\Psi}_0) = \Psi_0$. Now we also have

$$\rho = J_\ell(\text{St}(o(\rho_0)\ell^k, \tilde{\rho}_0)),$$

hence

$$\rho = \rho^* = J_\ell(\text{St}(o(\rho_0)\ell^k, \tilde{\rho}_0))^* = J_\ell(\mathbf{Z}([0, o(\rho_0)\ell^k - 1]_{\tilde{\rho}_0})^*)^*.$$

On the other hand

$$r_\ell(\text{LLC}(\mathbf{Z}([0, o(\rho_0)\ell^k - 1]_{\tilde{\rho}_0}))) = r_\ell(\bigoplus_{i=0}^{o(\Psi_0)\ell^k-1} \nu^i \tilde{\Psi}_0) = \ell^k (\bigoplus_{i=0}^{o(\Psi_0)-1} \nu^i \Psi_0),$$

hence

$$V(\text{St}_k(\mathbb{Z}_{\rho_0})) = \text{St}_k(\mathbb{Z}_{\Psi_0}).$$

(2) If ρ is supercuspidal, and $\Psi = V(\rho) \in \text{Irr}(W_F, \overline{\mathbb{F}_\ell})$, then

$$V(L([a, b]_\rho)) = [a, b] \otimes \Psi.$$

Indeed start with an ℓ -adic lift $\tilde{\rho}$ of ρ with Langlands parameter $\tilde{\Psi}$ so that $\Psi = r_\ell(\tilde{\Psi})$. Then

$$J_\ell(L([a, b]_{\tilde{\rho}})^*)^* = J_\ell(Z([a, b]_{\tilde{\rho}}))^* = Z([a, b]_\rho)^* = L([a, b]_\rho)$$

but on the other hand

$$r_\ell(L([a, b]_{\tilde{\rho}})) = r_\ell([a, b] \otimes \tilde{\Psi}) = [a, b] \otimes \Psi.$$

Similarly if $\rho = \text{St}_k(\rho_0)$ is cuspidal non supercuspidal with $V(\rho_0) = \Psi_0$, we find

$$V(L([a, b]_\rho)) = [a, b] \otimes \text{St}_k(\mathbb{Z}_{\Psi_0}).$$

(3) Take $\pi \in \text{Irr}_{\text{gen}}(G, \overline{\mathbb{F}_\ell})$, supported on the supercuspidal line $\mathbb{Z}_\rho$, and that $V(\rho) = \Psi$. We write $\pi = \pi_b \times \pi_{\text{tnb}}$ where π_b is banal and π_{tnb} is totally non-banal. Write

$$\pi_b = \prod_{i \geq 1} \prod_{k=0}^{o(\Psi)-1} L([0, i-1]_{\nu^k \rho})^{c_{i,k}},$$

where the occurring segments are unlinked, hence in particular for each fixed i , there is a k such that $c_{i,k} = 0$. Write

$$\pi_{\text{tnb}} = \prod_{k \geq 0} L([0, a_k - 1]_{\text{St}_k(\rho)})$$

with $0 \leq a_k < \ell$. Then using a generic *standard lift* of π as in [12, Definition 2.24], one checks that $V(\pi) = V(\pi_b) \oplus V(\pi_{\text{tnb}})$, with

$$\begin{aligned} V(\pi_b) &= \bigoplus_{i \geq 1} \bigoplus_{k=0}^{o(\Psi)-1} c_{i,k} [0, i-1] \otimes \nu^k \Psi, \\ V(\pi_{\text{tnb}}) &= \bigoplus_{j \geq 0} [0, a_j - 1] \otimes \text{St}_j(\mathbb{Z}_\Psi). \end{aligned}$$

6.3 The C-correspondence

Let (Φ, U) be a V-parameter, supported on an irreducible line $\mathbb{Z}_\Psi$, with standard form

$$(\Phi, U) = \bigoplus_{i \geq 0} \bigoplus_{k=0}^{o(\Psi)-1} a_{i,k} [0, i-1] \otimes \nu^k \Psi.$$

Definition 6.6. We say that (Φ, U) is *acyclic* if for each fixed i , there is $0 \leq k \leq o(\Psi) - 1$ such that $a_{i,k} = 0$. We say that (Φ, U) is *cyclic* if for each fixed i , the coefficient $a_{i,k}$ is independent of k .

Take a general V-parameter (Φ, U) supported on an irreducible line $\mathbb{Z}_\Psi$ with standard form written as above, and set

$$b_i = \min_k a_{i,k}, \text{ and } c_{i,k} = a_{i,k} - b_i.$$

It can then be rewritten as

$$(\Phi, U) = (\Phi, U)_{\text{acyc}} \oplus (\Phi, U)_{\text{cyc}},$$

with

$$\begin{aligned}
(\Phi, U)_{\text{acyc}} &= \bigoplus_{i \geq 1} \bigoplus_{k=0}^{o(\Psi)-1} c_{i,k}[0, i-1] \otimes \nu^k \Psi \\
(\Phi, U)_{\text{cyc}} &= \bigoplus_{j \geq 1} b_j[0, j-1] \otimes \bigoplus_{k=0}^{o(\Psi)-1} \nu^k \Psi.
\end{aligned}$$

Notice that for each i , we have $c_{i,k} = 0$ for at least one k so it makes sense to call $(\Phi, U)_{\text{acyc}}$ the *acyclic part* of Φ , and $(\Phi, U)_{\text{cyc}}$ is cyclic and we call it the *cyclic part* of Φ . Conversely, if a V -parameter supported on $\mathbb{Z}_\Psi$ is written as the sum of an acyclic and a cyclic parameter,

$$\left(\bigoplus_{i \geq 1} \bigoplus_{k=0}^{o(\Psi)-1} c_{i,k}[0, i-1] \otimes \nu^k \Psi \right) \oplus \left(\bigoplus_{j \geq 1} b_j[0, j-1] \otimes \bigoplus_{k=0}^{o(\Psi)-1} \nu^k \Psi \right),$$

then its standard form is equal to

$$\bigoplus_{i \geq 1} \bigoplus_{k=0}^{o(\Psi_0)-1} a_{i,k}[0, i-1] \otimes \nu^k \Psi$$

with $a_{i,k} = c_{i,k} + b_i$. Thus the decomposition of (Φ, U) as the direct sum of a cyclic and acyclic parameter is unique, and it is

$$(\Phi, U) = (\Phi, U)_{\text{acyc}} \oplus (\Phi, U)_{\text{cyc}}.$$

We now define an injection CV of $\text{Nilp}_{\text{ss}}(\mathbf{D}, \overline{\mathbb{F}_\ell})$ into $[\text{Rep}_{\text{ss}}(\mathbf{D}, \overline{\mathbb{F}_\ell})]$, which is not the natural inclusion.

Definition 6.7. Take $(\Phi, U) = (\Phi, U)_{\text{acyc}} \oplus (\Phi, U)_{\text{cyc}} \in \text{Nilp}_{\text{ss}}(\mathbf{D}, \overline{\mathbb{F}_\ell})$ supported on an irreducible line $\mathbb{Z}_\Psi$, and write

$$(\Phi, U)_{\text{cyc}} = \bigoplus_{j \geq 1} b_j[0, j-1] \otimes \bigoplus_{k=0}^{o(\Psi)-1} \nu^k \Psi.$$

We set

$$\text{CV}((\Phi, U)_{\text{cyc}}) = \bigoplus_{j \geq 1} b_j[0, j-1] \otimes \mathcal{C}(\mathbb{Z}_\Psi) \in [\text{Rep}_{\text{ss}}(\mathbf{D}, \overline{\mathbb{F}_\ell})],$$

and then

$$\text{CV}(\Phi, U) = (\Phi, U)_{\text{acyc}} \oplus \text{CV}((\Phi, U)_{\text{cyc}}) \in [\text{Rep}_{\text{ss}}(\mathbf{D}, \overline{\mathbb{F}_\ell})].$$

Finally, if $(\Phi, U) = \bigoplus_j (\Phi(\mathbb{Z}_{\Psi_j}), U_j) \in \text{Nilp}_{\text{ss}}(\mathbf{D}, \overline{\mathbb{F}_\ell})$ where $\Phi(\mathbb{Z}_{\Psi_j})$ is supported on the irreducible line $\mathbb{Z}_{\Psi_j}$, and $\mathbb{Z}_{\Psi_k} \neq \mathbb{Z}_{\Psi_l}$ for $k \neq l$, we set

$$\text{CV}(\Phi, U) = \bigoplus_j \text{CV}(\Phi(\mathbb{Z}_{\Psi_j}), U_j).$$

We have the following immediate lemma.

Lemma 6.8. The map $\text{CV} : \text{Nilp}_{\text{ss}}(\mathbf{D}, \overline{\mathbb{F}_\ell}) \rightarrow [\text{Rep}_{\text{ss}}(\mathbf{D}, \overline{\mathbb{F}_\ell})]$ is injective.

Combining V and CV , we make the following definition:

Definition 6.9. Let $C : \text{Irr}(G, \overline{\mathbb{F}_\ell}) \rightarrow [\text{Rep}_{\text{ss}}(D, \overline{\mathbb{F}_\ell})]$ denote the injection defined by

$$C(\pi) = \text{CV}(\text{V}(\pi)),$$

for $\pi \in \text{Irr}(G)$.

We do the C -correspondence version of Example 6.5. We use the notation

$$\mathcal{C}_k(\mathbb{Z}_\Psi) := \ell^k \mathcal{C}(\mathbb{Z}_\Psi)$$

for $\Psi \in \text{Irr}(\text{W}_F, \overline{\mathbb{F}_\ell})$ (in particular $\mathcal{C}_0(\mathbb{Z}_\Psi) = \mathcal{C}(\mathbb{Z}_\Psi)$).

Example 6.10. (1) If $\pi \in \text{Irr}_{\text{sc}}(G)$ and $\text{V}(\pi) = \Psi$, then

$$C(\pi) = \begin{cases} \Psi & \text{if } \pi \text{ is banal,} \\ \mathcal{C}(\mathbb{Z}_\Psi) = \mathcal{C}(\{\Psi\}) & \text{if } \pi \text{ is non-banal.} \end{cases}$$

(2) If $\rho = \text{St}_k(\rho_0)$ is cuspidal non-supercuspidal with $\rho_0 \in \text{Irr}_{\text{sc}}(G, \overline{\mathbb{F}_\ell})$ and $\Psi_0 = \text{V}(\rho_0) \in \text{Irr}(\text{W}_F, \overline{\mathbb{F}_\ell})$, then

$$C(\rho) = \mathcal{C}_k(\mathbb{Z}_{\Psi_0}).$$

(3) If ρ is supercuspidal, and $\Psi = \text{V}(\rho) \in \text{Irr}(\text{W}_F, \overline{\mathbb{F}_\ell})$, then

$$C(\text{L}([a, b]_\rho)) = \begin{cases} [a, b] \otimes \Psi & \text{if } \rho \text{ is banal,} \\ [a, b] \otimes \mathcal{C}(\mathbb{Z}_\Psi) & \text{if } \rho \text{ is non-banal.} \end{cases}$$

More generally, for $k \geq 0$,

$$C(\text{L}([a, b]_{\text{St}_k(\rho)})) = [a, b] \otimes \mathcal{C}_k(\mathbb{Z}_\Psi).$$

(4) Take $\pi = \pi_b \times \pi_{\text{tnb}} \in \text{Irr}_{\text{gen}}(G, \overline{\mathbb{F}_\ell})$, supported on the supercuspidal line $\mathbb{Z}_\rho$ with $\text{V}(\rho) = \Psi$. Write

$$\pi_b = \prod_{i \geq 1} \prod_{k=0}^{o(\Psi)-1} \text{L}([0, i-1]_{\nu^k \rho})^{c_{i,k}},$$

where the occuring segments are unlinked, hence in particular for each fixed i , there is a k such that $c_{i,k} = 0$. Write

$$\pi_{\text{tnb}} = \prod_{k \geq 0} \text{L}([0, a_k - 1]_{\text{St}_k(\rho)})$$

with $0 \leq a_k < \ell$. Then

$$\text{V}(\pi)_{\text{acyc}} = \text{V}(\pi_b), \text{ and } \text{V}(\pi)_{\text{cyc}} = \text{V}(\pi_{\text{tnb}}).$$

Hence

$$C(\pi) = C(\pi_b) \oplus C(\pi_{\text{tnb}})$$

with

$$\begin{aligned} C(\pi_b) &= \bigoplus_{i \geq 1} \bigoplus_{k=0}^{o(\Psi)-1} c_{i,k} [0, i-1] \otimes \nu^k \Psi, \\ C(\pi_{\text{tnb}}) &= \bigoplus_{j \geq 0} [0, a_j - 1] \otimes \mathcal{C}_j(\mathbb{Z}_\Psi). \end{aligned}$$

6.4 Preservation of local constants

It follows immediately from its definition that CV commutes with taking duals, direct sums, twisting by characters, and does not change the determinant. Hence the C -correspondence shares with the V -correspondence the properties of Lemma 6.4:

Lemma 6.11. Let $\pi \in \text{Irr}(G, \overline{\mathbb{F}_\ell})$.

(1) For all characters $\chi : \text{GL}_n(F) \rightarrow \overline{\mathbb{F}_\ell}^\times$,

$$C(\chi\pi) = \chi C(\pi).$$

(2) Letting c_π denote the central character of π , then using local class field theory

$$c_\pi = \det(C(\pi)).$$

(3) We have

$$C(\pi^\vee) = C(\pi)^\vee.$$

(4) For $\pi' \in \text{Irr}(G, \overline{\mathbb{F}_\ell})$ such that $\pi \times \pi'$ is irreducible, we have $C(\pi \times \pi') = C(\pi) \oplus C(\pi')$.

(5) Moreover if $\pi = \pi(\mathbb{Z}_{\rho_i}) \times \cdots \times \pi(\mathbb{Z}_{\rho_r})$ with $\pi(\mathbb{Z}_{\rho_i})$ supported on the supercuspidal line $\mathbb{Z}_{\rho_i}$ and $\mathbb{Z}_{\rho_i} \neq \mathbb{Z}_{\rho_j}$ for $i \neq j$, then

$$C(\pi) = \bigoplus_{i=1}^r C(\pi(\mathbb{Z}_{\rho_i})).$$

Hence for the moment we have not lost anything in introducing C , except the (not so direct) compatibility with reduction modulo ℓ of the V -correspondence. Of course, we can keep track of this property as the map CV is injective. On the other hand we have not gained anything yet. However there is one important property that the V correspondence does not share with the LLC, which is the preservation of local constants. For the last sentence to make sense, one must have a definition of local constants for elements in $\text{Irr}(G, \overline{\mathbb{F}_\ell})$. Indeed there is one: for standard local constants, they have been defined in [13] à la *Godement-Jacquet*, and for local constants of pairs, they have been defined in [12] for pairs of generic representations by the *Rankin-Selberg* method of Jacquet, Piatetski-Shapiro and Shalika. It should be true that $L^{\text{RS}}(X, \pi, \mathbf{1}) = L^{\text{GJ}}(X, \pi)$, and similarly for γ - and ϵ -factors for $\pi \in \text{Irr}_{\text{gen}}(G)$, but we did not check this. In what follows we will only consider the Rankin-Selberg local constants defined in [12], and we will drop the RS exponent. We claim that C preserves local constants of pairs, whereas V does not. Let us give a basic example where we consider the L -factor only.

Example 6.12. Consider the cuspidal representation $\rho = \text{St}(o(\nu), \mathbf{1})$ of G_n for $n = o(\nu)$, then

$$V(\rho) = \bigoplus_{k=0}^{o(\nu)-1} \nu^k, \quad \text{whereas} \quad C(\rho) = \mathcal{C}(\mathbb{Z}_1).$$

According to [12, Theorem 4.9], we have

$$L(X, \pi) = L(X, \pi, \mathbf{1}) = 1.$$

By definition of the Deligne L -factor, we also have

$$L(X, \mathcal{C}(\mathbb{Z}_1)) = 1$$

because the Deligne operator associated to $\mathcal{C}(\mathbb{Z}_1)$ is bijective. However,

$$L(X, \bigoplus_{k=0}^{o(\nu)-1} \nu^k) = \prod_{k=0}^{o(\nu)-1} \frac{1}{1 - q^{-k}X}.$$

We can finally prove the central result of this paper. We fix a nontrivial character $\psi : F \rightarrow \overline{\mathbb{F}_\ell}^\times$ and $\tilde{\psi} : F \rightarrow \overline{\mathbb{Z}_\ell}^\times$ a lift of ψ .

Theorem 6.13. For $\pi, \pi' \in \text{Irr}_{\text{gen}}(G, \overline{\mathbb{F}_\ell})$, we have

$$\begin{aligned} L(X, C(\pi) \otimes_{\text{ss}} C(\pi')) &= L(X, \pi, \pi'), \\ \gamma(X, C(\pi) \otimes_{\text{ss}} C(\pi'), \psi) &= \gamma(X, \pi, \pi', \psi), \\ \epsilon(X, C(\pi) \otimes_{\text{ss}} C(\pi'), \psi) &= \epsilon(X, \pi, \pi', \psi). \end{aligned}$$

Proof. Observe that the equality for ϵ -factors follows from the first two, by definition. Let's prove the statement for γ -factors first. Let π and π' belong to $\text{Irr}_{\text{gen}}(G)$ and choose $\tilde{\pi}$ and $\tilde{\pi}'$ two ℓ -adic generic representations such that $\pi = J_\ell(\tilde{\pi})$ and $\pi' = J_\ell(\tilde{\pi}')$ (for example *standard lifts* as in [12, Definition 2.24]). Then according to [12, Theorem 3.13], one has

$$\gamma(X, \pi, \pi', \psi) = r_\ell(\gamma(X, \tilde{\pi}, \tilde{\pi}', \tilde{\psi})).$$

Set $\tilde{\Phi}$ and $\tilde{\Phi}'$ be the semisimple representations of W_F corresponding to the supercuspidal support of $\tilde{\pi}$ and $\tilde{\pi}'$ via LLC (i.e. $\text{LLC}(\tilde{\pi}) = (\tilde{\Phi}, \star)$ and $\text{LLC}(\tilde{\pi}') = (\tilde{\Phi}', \star)$). The LLC and the standard properties of γ -factors tell us that

$$\gamma(X, \tilde{\pi}, \tilde{\pi}', \tilde{\psi}) = \gamma(X, \tilde{\Phi} \otimes \tilde{\Phi}', \tilde{\psi}).$$

Set $r_\ell(\tilde{\Phi}) = \Phi$ and $r_\ell(\tilde{\Phi}') = \Phi'$, hence $r_\ell(\tilde{\Phi} \otimes \tilde{\Phi}') = \Phi \otimes_{\text{ss}} \Phi'$. According to Theorem 6.3, or more simply [24, Theorem 1.6] which states that the semisimple LLC commutes with reduction modulo ℓ , we deduce that Φ corresponds to the supercuspidal support of π , whereas Φ' corresponds to the supercuspidal support of π' , i.e. $C(\pi) = [\Phi, \star]$ and $C(\pi') = [\Phi', \star]$, so that

$$\gamma(X, C(\pi) \otimes_{\text{ss}} C(\pi'), \psi) = \gamma(X, \Phi \otimes_{\text{ss}} \Phi', \psi).$$

However

$$\gamma(X, \Phi \otimes_{\text{ss}} \Phi', \psi) = r_\ell(\gamma(X, \tilde{\Phi} \otimes \tilde{\Phi}', \tilde{\psi}))$$

according to Proposition 5.8. This ends the proof of the assertion on γ -factors.

It remains to prove that on L -factors. As before, we write $\pi = \pi_b \times \pi_{\text{tnb}}$ and $\pi' = \pi'_b \times \pi'_{\text{tnb}}$ for their decompositions into banal and totally non-banal representations. By [12, Theorem 4.19] tells us that

$$L(X, \pi, \pi') = L(X, \pi_b, \pi'_b).$$

Set $[\Phi_b, U_b] = C(\pi_b)$, $[\Phi_{\text{tnb}}, U_{\text{tnb}}] = C(\pi_{\text{tnb}})$, $[\Phi'_b, U'_b] = C(\pi'_b)$, $[\Phi'_{\text{tnb}}, U'_{\text{tnb}}] = C(\pi'_{\text{tnb}})$, in particular U_{tnb} and U'_{tnb} are bijective whereas U_b and U'_b are nilpotent as in Example 6.10 (4). One has

$$C(\pi) \otimes_{\text{ss}} C(\pi') = (C(\pi_b) \otimes_{\text{ss}} C(\pi'_b)) \oplus (C(\pi_b) \otimes_{\text{ss}} C(\pi'_{\text{tnb}})) \oplus (C(\pi_{\text{tnb}}) \otimes_{\text{ss}} C(\pi'_b)) \oplus (C(\pi_{\text{tnb}}) \otimes_{\text{ss}} C(\pi'_{\text{tnb}})).$$

If one writes any of the latter three direct summands under the form $[\Phi, U]$, then U is bijective (see in particular Remark 4.40). This implies that

$$L(X, C(\pi) \otimes_{\text{ss}} C(\pi')) = L(X, C(\pi_b) \otimes_{\text{ss}} C(\pi'_b)).$$

Hence it remains to prove the equality:

$$L(X, C(\pi_b) \otimes_{\text{ss}} C(\pi'_b)) = L(X, \pi_b, \pi'_b)$$

Now [12, Theorem 4.12 and Corollary 4.20], the multiplicativity of Deligne L -factors with respect to direct sums, and the multiplicativity relation of Lemma 5.7 show that it is enough to prove it

for $\pi_b = \rho$ and $\pi'_b = \rho'$ banal supercuspidal representations. Take $\tilde{\rho}$ (resp. $\tilde{\rho}'$) a cuspidal lift of ρ (resp. ρ'), so that $\tilde{\Psi} = \text{LLC}(\tilde{\rho})$ (resp. $\tilde{\Psi}' = \text{LLC}(\tilde{\rho}')$) is an irreducible lift of $\Psi = \text{C}(\rho)$ (resp. $\Psi' = \text{LLC}(\rho')$). Then

$$L(X, \rho, \rho') = r_\ell(L(X, \tilde{\rho}, \tilde{\rho}'))$$

by [12, Theorem 4.18], and

$$L(X, \Psi \otimes_{\text{ss}} \Psi') = r_\ell(L(X, \tilde{\Psi} \otimes \tilde{\Psi}'))$$

thanks to Theorem 5.9. Moreover,

$$L(X, \tilde{\rho}, \tilde{\rho}') = L(X, \tilde{\Psi} \otimes \tilde{\Psi}')$$

by the ℓ -adic LLC, and we deduce the desired equality:

$$L(X, \rho, \rho') = L(X, \Psi \otimes_{\text{ss}} \Psi').$$

□

Remark 6.14. As we said we leave the equality $L^{\text{GJ}} = L^{\text{RS}}$ on generic representations for future work. We also believe C sends the Godement-Jacquet local constants of Mínguez on $\text{Irr}(G, \overline{\mathbb{F}}_\ell)$ to the standard local constants on $[\text{Rep}_{\text{ss}}(D, \overline{\mathbb{F}}_\ell)]$. For G_2 it can be checked as a consequence of the calculations carried out in [13], we leave the general case for a further investigation.

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