Accepted Manuscript

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PII: S0263-8223(17)34182-X
DOI: <https://doi.org/10.1016/j.compstruct.2018.03.010>
Reference: COST 9457

To appear in: *Composite Structures*

Received Date: 13 December 2017
Revised Date: 27 February 2018
Accepted Date: 7 March 2018

Please cite this article as: Köllner, A., Völlmecke, C., Post-buckling behaviour and delamination growth characteristics of delaminated composite plates, *Composite Structures* (2018), doi: <https://doi.org/10.1016/j.compstruct.2018.03.010>

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Post-buckling behaviour and delamination growth characteristics of delaminated composite plates

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Abstract

The problem of a delaminated composite plate subjected to in-plane compressive loading is investigated by employing a novel analytical framework previously developed by the authors. The framework is capable of modelling the post-buckling behaviour considering damage growth by using a set of generalized coordinates only. Therefore, in order to model the post-buckling responses of delaminated composite plates a RAYLEIGH–RITZ formulation is employed. Thus, the post-buckling behaviour as well as the delamination growth characteristics are determined by solving a set of non-linear algebraic equations only. For the cases investigated, the study reveals that delamination growth is associated with the global buckling response. So long as stable delamination growth is present, the post-buckling response remains also stable. However, unstable delamination growth may be caused which would occur unexpectedly yielding sudden failure of the structure. This underlines the importance of considering delamination growth when studying the structural stability behaviour of these structures.

Keywords: Composite plates, Delamination, Post-buckling, Growth, Stability

1. Introduction

The buckling and post-buckling behaviour of composite plates describes a main area of ongoing structural stability research. Particularly, the use of layered composites structures requires the consideration of material damage and failure. The layered construction and the heterogeneity give rise to various damage mechanisms and hence inelastic deformations, which might have an influence on the stability behaviour. In this study, the structural stability behaviour of layered composite plates containing an initial defect in the form of a delamination is investigated.

This problem—a delaminated plate loaded under axial compression—has received much attention since the 1980s starting with the pioneering work of CHAI and co-workers [6, 7]. Since then a large number of studies has investigated the problem mainly focusing on:

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- the buckling and post-buckling behaviour for the case of non-growing delaminations (*e.g.* [12, 17, 24, 25, 34, 35]) and
- the calculation of the energy release rate along the delamination tip (*e.g.* [4, 5, 6, 7, 13, 20, 21, 22]).

Hitherto, the structural stability behaviour once the delamination grows has been barely investigated. In [20, 21, 22], the post-buckling behaviour considering propagating delaminations is described for a plate with clamped loading edges and free conditions on the remaining edges. Thus, the structural response is similar to a strut showing a weakly stable behaviour in the global buckling regime [17]. For these plates, delamination growth is associated with the onset of the global buckling response. Post-buckling responses for a plate being supported on all edges and containing an embedded delamination can be found in [9, 10, 30, 33], however only [30] considers a non-rectangular delamination shape in the form of an ellipse. In an experimental study [8], such an elliptical delamination shape was observed following impact scenarios.

In summary, information is sparse within the literature regarding the post-buckling behaviour of elliptically delaminated composite plates. Furthermore, the problem of a composite plate with an embedded delamination loaded under axial compression is mainly studied by employing the finite element method. Thus, there is a lack of semi-analytical modelling approaches which consider delamination growth.

The current work investigates the post-buckling behaviour of elliptically delaminated composite plates. Therefore, a semi-analytical model which employs the analytical framework documented in [14, 16] is developed. The modelling approach is based on the fact that during stable delamination growth equality holds in between the forces available for producing a change in structure, *i.e.* the energy release rate, and the forces required for such a change, *i.e.* the critical energy release rate. This has been shown in [15, 16] for the problem of delaminated composite struts as well as in [13] for delaminated composite plates and is addressed in Section 2 summarizing characteristics of the behaviour of the energy release rate relevant for the model development.

In order to provide a coherent derivation of the semi-analytical modelling approach, following the remarks on the behaviour of the energy release rate, the analytical framework (*cf.* [16]) is briefly reviewed. Then, the geometric model is presented followed by the energy formalism. The choice of generalized coordinates required to describe the displacement field adequately is discussed before results for a unidirectional and a cross-ply laminate are presented. Conclusions are drawn regarding the interaction of structural stability and delamination growth characteristics of delaminated composite plates.

2. Relevant aspects of the behaviour of the energy release rate in delaminated composites

Adapting results provided in [13], Fig. 1 shows the behaviour of the energy release rate—normalized to the critical energy release rate—along the boundary of the delamination for increasing levels of load input starting at the load causing delamination growth (ε_0^I). A cross-ply $[(0/90)_s/90_{14}/0_{14}]$ square plate subjected to a unidirectional in-plane compressive load with an embedded circular delamination in between the fourth and fifth layer has been studied.

The arc length is normalized against the length of a quarter of the boundary of the delamination where the values zero and unity are associated with the vertex transverse to the loading direction and longitudinal to the loading direction respectively (*cf.* the points $(0, b)$ and $(l, 0)$ respectively in Fig. 3). It can be seen that at the part of the boundary experiencing growth, the energy release rate remains equal to the critical energy release rate during delamination growth for increasing levels of load input (ε_0^{II} to ε_0^{IV}). Thus, Fig. 1 visualizes the aforementioned behaviour that during

stable delamination growth the energy release rate remains equal to the critical energy release rate.

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Figure 1: Energy release rate against arc length of the boundary of the delamination, $\varepsilon_0^I < \varepsilon_0^{II} < \varepsilon_0^{III} < \varepsilon_0^{IV}$; in normalized quantities; adapted from [13].

The condition of equality in between the energy release rate and the critical energy release rate is violated during unstable delamination growth. Whether stable or unstable delamination growth is present can be determined by studying the behaviour of the energy release rate against the delamination area for increasing values of load input as done in [15, 16] for delaminated composite struts and in [6] for delaminated plates which is shown in Fig. 2.

Whenever for a given state of loading the energy release rate decreases with increasing delamination area, growth is termed stable. Otherwise, it is unstable. Note that in Fig. 2 the energy release rate is not normalized against the critical energy release rate (*cf.* [6]), thus unity on the y -axis is not necessarily associated with the threshold required for delamination growth. Moreover, delamination growth is modelled globally in [6], *i.e.* by increasing axes of the ellipse, and mode mixture is not considered. Thus, G_b describes the energy release rate for growth transverse to the loading direction (solid lines in Fig. 2) and G_l (dashed lines) the energy release rate for growth longitudinal to the loading direction (*cf.* width (b) and length (l) direction in Fig. 3, respectively). Thus, information can be gained from Fig. 2 whether growth occurs in the transverse or longitudinal direction as well as whether growth is stable or unstable.

A special case of delamination growth is visualized in Fig. 2 by the dot-dashed line which indicates the deformation states for which $G_b = G_l$, *i.e.* growth in both directions occurs. Note that at such a deformation state growth is not determined by the curves G_b and G_l since both correspond to one axis of the ellipse being fixed. As discussed in [6], the new delamination shape is determined from the maximum energy release condition, thus—following Fig. 2—the dot-dashed line dictates the delamination growth contour. In [6], it is shown that this delamination growth is unstable since the energy release rate of the path $G_b = G_l$ is monotonically increasing with growing axis of the elliptical delamination.

Figure 2: Energy release rate against delamination aspect ratio, $\varepsilon_0^I < \varepsilon_0^{II} < \varepsilon_0^{III}$; G_b – energy release rate for growth transverse to the loading direction, G_l – longitudinal to the loading direction; adapted from [6].

3. Review of the previously developed analytical framework for a structural stability analysis considering damage growth

The section summarizes main aspects of the analytical framework previously developed by the authors [14, 16]. The framework considers mechanical systems which can be described by a set of generalized coordinates q_i ($i = 1, 2, \dots, I$), a set of damage parameters, *i.e.* internal state variables [27], ξ_k ($k = 1, 2, \dots, K$) and a set of loading parameters λ_m ($m = 1, 2, \dots, M$). A deformation process is strictly separated into a conservative part, *i.e.* the current state of damage remains constant, and a non-conservative part, *i.e.* the state of damage changes from one loading step to another.

3.1. Conservative part of the deformation process

The total potential energy (Π) principle [2, 32] is employed when the damage parameters remain constant, thus

$$\Pi(q_i, A_m, \xi_k) = W(q_i, \xi_k) - A_m \alpha_m(q_i, \xi_k) \quad \text{with} \quad \xi_k = \text{const.} \quad (1)$$

and

$$\delta \Pi(q_i) = 0, \quad (2)$$

assuming that independent generalized forces are given ($\lambda_m = A_m$). The generalized displacements are denoted by α_m . In Eq. (1), the strain energy is described by W and the second term constitutes the work done by the independent generalized forces. If the system is subjected to independent

generalized displacements only, then the total potential energy is equal to the strain energy and $\lambda_m = \alpha_m$. Eq. (2) yields the deformation path during the conservative part of the deformation process in terms of $q_i(\lambda_m)$.

The damage parameters remain constant so long as the thermodynamic forces f_k associated with the respective damage parameters ξ_k do not reach the thresholds required to cause damage growth which are denoted by g_k . The thermodynamic forces are determined by evaluating the change of the total potential energy with respect to the damage parameters, thus

$$f_k = -\frac{\partial \Pi}{\partial \xi_k}, \quad (3)$$

whereas the thresholds are given by the change of the dissipative energy (W_d) with respect to the k th damage parameter, *i.e.*

$$g_k = \frac{\partial W_d}{\partial \xi_k}. \quad (4)$$

Thus, the total potential energy principle, *cf.* Eq. (2), is employed so long as

$$f_k < g_k. \quad (5)$$

3.2. Non-conservative part of the deformation process

Whenever the thermodynamic forces f_k reach the respective thresholds required, *i.e.*

$$f_k \geq g_k, \quad (6)$$

damage growth is caused. As has been shown in [15, 16], during stable damage growth equality in Eq. (6) holds, such that for these processes Eq. (6) can be rewritten in terms of

$$f_k - g_k = D_k(q_i, \lambda_m, \xi_k) = 0. \quad (7)$$

From Eq. (7)—for stable damage growth—the damage parameters are implicitly given as functions of the generalized coordinates and the loading parameters assuming that a unique solution exists [16, 28], thus

$$D_k(q_i, A_m, \xi_k(q_i, \lambda_m)) \equiv 0. \quad (8)$$

The damage parameters are obtained in explicit form by employing a TAYLOR series approximation around the deformation state causing damage growth (*cf.* [15, 16]).

With the damage parameters obtained in terms of

$$\xi_k = \xi_k(q_i, \lambda_m) \quad (9)$$

an extended total potential energy Π^* may be derived which constitutes the total work of deformation W_t in a displacement-controlled configuration ($\lambda_m = \alpha_m$), *i.e.*

$$\Pi^* = W_t = W_t(q_i, \alpha_m, \xi_k(q_i, \alpha_m)), \quad (10)$$

and which comprises the total work of deformation and the work done by the applied forces in a

load-controlled configuration ($\lambda_m = A_m$), *i.e.*

$$\Pi^* = \Pi^*(q_i, A_m, \xi_k(q_i, A_m)) = W_t - A_m \alpha_m. \quad (11)$$

The variational principle

$$\delta \Pi^*(q_i) = 0 \quad (12)$$

yields the deformation path $q_i(\lambda_m)$ considering damage growth starting from the deformation state at which growth is initiated. The behaviour of the damage parameters can then be determined by inserting the deformation path obtained into Eq. (9).

4. Model description

In the current work, a multi-layered composite plate with an embedded elliptical delamination is studied. Fig. 3 shows the geometric model of the plate. As can be seen, the plate is subdivided into three parts, two sublaminates and one undelaminated region. Parts ① and ② describe the upper and lower sublaminates respectively. The undelaminated part of the plate is denoted by ③. The delamination is visualized in Fig. 3 by a grey shaded area.

Figure 3: Geometric model of a composite plate with an embedded elliptical delamination.

An elliptical delamination is chosen since conclusive experimental evidence is available for such a shape (*cf.* [8]). On the other hand, information about the post-buckling behaviour of elliptically delaminated composite plates is comparably sparse within the literature (*cf.* Section 1). The elliptical delamination is defined by the measures l and b describing the length and width of the ellipse (semi major and semi minor axis) respectively. The overall dimensions of the plate are denoted by $2L \times 2B \times t$ (length $\times$ width $\times$ thickness). The depth of the delamination is described by the parameter a .

A uniaxial loading is applied to the plate in the x -direction in the form of an applied strain such that the boundaries at (L, y) and $(-L, y)$ are subjected to the displacements $\pm \varepsilon_0 L$ respectively (*cf.* Fig. 3), *i.e.* a displacement-controlled configuration is studied. The plate is taken to be clamped at the boundaries with in-plane displacements being restrained except for the applied compressive shortening. These boundary conditions are described in the experimental test standard [1].

The CLASSICAL LAMINATE THEORY [26] is employed. Plate dimensions are chosen such that the effect of shear deformations is assumed to be small. Therefore, out-of-plane shear contributions are omitted in the description of the displacement field. The boundary and continuity conditions for the displacement field $\mathbf{u} = \{u(x, y), v(x, y), w(x, y)\}^T$ can be expressed as

$$\begin{aligned}
 &\text{in-plane, } u: \\
 &u_3(\pm L, y) = \pm \varepsilon_0 L, & u_3(x, \pm B) = \varepsilon_0 x, \\
 &u_3(\Gamma) = u_i(\Gamma) - u_i^{\text{rot}}, \\
 &\text{in-plane, } v: \\
 &v_3(\pm L, y) = 0, & v_3(x, \pm B) = 0, \\
 &v_3(\Gamma) = v_i(\Gamma) - v_i^{\text{rot}}, \\
 &\text{out-of-plane, } w: \\
 &w_3(\pm L, y) = 0, & w_3(x, \pm B) = 0, \\
 &\nabla_j w_3(\pm L, y) = 0, & \nabla_j w_3(x, \pm B) = 0, \\
 &w_3(\Gamma) = w_i(\Gamma), & \nabla_j w_3(\Gamma) = \nabla_j w_i(\Gamma),
 \end{aligned} \tag{13}$$

where the subscript at the displacement field entries ("3" and $i = 1, 2$) refers to the respective part of the plate, $\nabla_j = \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\}^T$ and Γ describes the boundary of the ellipse, *i.e.*

$$\Gamma(x, y) = \left(\frac{x}{L} \right)^2 + \left(\frac{y}{B} \right)^2 - 1 = 0. \tag{14}$$

The post-buckling behaviour is modelled with the aid of a RAYLEIGH–RITZ formulation employing continuous mode-forms in order to approximate the displacement field of the plate. Owing to the description of the boundary of the delamination by Eq. (14), polynomial shape functions are employed in order to satisfy the geometric boundary and continuity conditions provided in Eq. (13). Those functions can be expressed in terms of a series which is shown in Eq. (16) employing Eq. (14) and the function defining the boundary of the plate,

$$\Gamma_P = \left(\left(\frac{x}{L} \right)^2 - 1 \right) \left(\left(\frac{y}{B} \right)^2 - 1 \right) = 0, \tag{15}$$

thus

part ③:

$$\begin{aligned}
 u_3(x, y) &= \varepsilon_0 x + (\Gamma_P) \left(\frac{x}{L} \right) \sum_{m=1}^M \sum_{n=1}^N \left(q_{mn}^{u_3} \left(\frac{x}{L} \right)^{2(m-1)} \left(\frac{y}{B} \right)^{2(n-1)} \right), \\
 v_3(x, y) &= (\Gamma_P) \left(\frac{y}{B} \right) \sum_{m=1}^M \sum_{n=1}^N \left(q_{mn}^{v_3} \left(\frac{x}{L} \right)^{2(m-1)} \left(\frac{y}{B} \right)^{2(n-1)} \right), \\
 w_3(x, y) &= (\Gamma_P)^2 \sum_{m=1}^M \sum_{n=1}^N \left(q_{mn}^{w_3} \left(\frac{x}{L} \right)^{2(m-1)} \left(\frac{y}{B} \right)^{2(n-1)} \right),
 \end{aligned}$$

part ④: (16)

$$\begin{aligned}
 u_i(x, y) &= u_3 + (\Gamma) \left(\frac{x}{l} \right) \sum_{m=1}^M \sum_{n=1}^N \left(q_{mn}^{u_i} \left(\frac{x}{l} \right)^{2(m-1)} \left(\frac{y}{b} \right)^{2(n-1)} \right) + u_i^{\text{rot}}, \\
 v_i(x, y) &= v_3 + (\Gamma) \left(\frac{y}{b} \right) \sum_{m=1}^M \sum_{n=1}^N \left(q_{mn}^{v_i} \left(\frac{x}{l} \right)^{2(m-1)} \left(\frac{y}{b} \right)^{2(n-1)} \right) + v_i^{\text{rot}}, \\
 w_i(x, y) &= w_3 + (\Gamma)^2 \sum_{m=1}^M \sum_{n=1}^N \left(q_{mn}^{w_i} \left(\frac{x}{l} \right)^{2(m-1)} \left(\frac{y}{b} \right)^{2(n-1)} \right),
 \end{aligned}$$

with $i = 1, 2$, where u_i^{rot} and v_i^{rot} describe contributions to the in-plane displacements of the sublaminates resulting from the rotation of the interface between the sublaminates and the undelaminated part considering the offset of the midplanes. Such contributions can be approximated by

$$\begin{aligned}
 u_i^{\text{rot}} &= h_i \left(-\frac{\partial w_3}{\partial x} \Big|_{\Gamma} \right), \\
 v_i^{\text{rot}} &= h_i \left(-\frac{\partial w_3}{\partial y} \Big|_{\Gamma} \right) \quad \text{with} \quad h_i = \left\{ \begin{array}{l} \frac{(1-a)t}{2} \\ -\frac{at}{2} \end{array} \right\},
 \end{aligned} \tag{17}$$

where i indicates the respective sublaminate and the offsets of the midplanes are denoted by h_i . It should be noted that the following symmetries are employed in the displacement functions (Eq. (16)),

$$\begin{aligned}
 u_i(x, y) &= -u_i(-x, -y), & u_i(x, y) &= -u_i(-x, y) = u_i(x, -y), \\
 v_i(x, y) &= -v_i(-x, -y), & v_i(x, y) &= -v_i(-x, y) = v_i(x, -y), \\
 w_i(x, y) &= w_i(-x, -y), & w_i(x, y) &= w_i(-x, y) = w_i(x, -y),
 \end{aligned} \tag{18}$$

with $i = 1, 2, 3$. Those symmetries are present for typical laminates, such as unidirectional, cross-ply and quasi-isotropic layups, using the coordinate system shown in Fig. 3, whereby the symmetry of the out-of-plane displacement is also associated with studying square plates ($L = B$, cf. Fig. 3), as it is done in the present work.

In Eq. (16), any chosen M and N satisfies the geometric boundary conditions mandatory for the RAYLEIGH–RITZ formulation. However, with increasing M and N the accuracy of the approximation of the displacement field improves which, on the other hand, requires higher computational cost.

With the displacement field being described by a set of generalized coordinates, the analytical framework reviewed in Section 3 can be applied. This is done in the subsequent section presenting the energy formalism.

5. Energy formalism

The modelling approach considers the non-linear terms in the GREEN–LAGRANGE strain tensor associated with the out-of-plane displacements, *i.e.* the VON KÁRMÁN strains [26]. The strain energy density w_s can be written as

$$w_s = \frac{1}{2} \bar{Q}_{IJ} \varepsilon_I \varepsilon_J, \quad \text{with} \quad I, J = 1, 2, 6, \tag{19}$$

in which the plane stress assumption is considered, $\bar{Q}_{IJ}$ is the reduced transformed stiffness matrix and ε_I comprises in-plane contributions ($\{\varepsilon_{(0)}\}$) and strains associated with bending deformations ($z\{\kappa\}$),¹ *i.e.*

$$\{\varepsilon\} = \{\varepsilon_{(0)}\} + z\{\kappa\} = \begin{Bmatrix} \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \\ \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \end{Bmatrix} + z \begin{Bmatrix} -\frac{\partial^2 w}{\partial x^2} \\ -\frac{\partial^2 w}{\partial y^2} \\ -2\frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix}. \quad (20)$$

Integrating Eq. (19) over the volume and employing Eq. (20) and the CLASSICAL LAMINATE THEORY yields the strain energy,

$$W = \frac{1}{2} \int_x \int_y (\varepsilon_I^0 A_{IJ} \varepsilon_J^0 + 2\varepsilon_I^0 B_{IJ} \kappa_J + \kappa_I D_{IJ} \kappa_J) dy dx, \quad (21)$$

where A_{IJ} , B_{IJ} and D_{IJ} are the in-plane, coupling and bending stiffness matrix respectively. Owing to the subdivision of the plate into three parts, the strain energy of each part is determined with Eq. (21) and subsequently summed up.

Since a displacement-controlled configuration is studied, the strain energy in Eq. (21) is the governing functional. Thus, by employing the RAYLEIGH-RITZ method using the displacement field defined in Eq. (16), a set of non-linear algebraic equations is obtained by applying the variational principle, *i.e.*

$$\delta\Pi = \delta W(q_i) = 0 \implies \frac{\partial W}{\partial q_i} = 0, \quad (22)$$

where all generalized coordinates used in Eq. (16) are comprised by the set q_i . Eq. (22) yields the deformation path for the case of stationary delaminations in terms of $q_i(\varepsilon_0)$ since ε_0 is the loading parameter in the current model description.

The model considers an imperfection in the form of an initial out-of-plane deflection which is assumed to be caused by the pre-existing delamination. The amplitude of the imperfection is taken as $t/1000$ [29]. The energy contributions associated with the imperfection are deducted from the total potential energy.

Following the analytical framework developed in [14, 16] and reviewed in Section 3, with the aid of the equilibrium path $q_i(\varepsilon_0)$ obtained by solving Eq. (22), the thermodynamic force associated with delamination growth, thus the energy release rate, is determined next.

Even though a single damage parameter ξ , *i.e.* the delamination area A_{ell} ,

$$\xi = A_{\text{ell}} = \pi lb, \quad (23)$$

is present in the current application example, owing to the model description, delamination growth into two directions can be investigated. The force available for producing delamination growth in the width direction (G_b) and in the length direction (G_l) of the ellipse can be calculated

¹The in-plane strains are denoted by $\{\varepsilon_{(0)}\}$ in order to avoid confusion with the applied strain ε_0 used in the model description.

as

$$G_b = -\frac{1}{\pi l} \frac{\partial W}{\partial b} \quad \text{and} \quad G_l = -\frac{1}{\pi b} \frac{\partial W}{\partial l} \quad (24)$$

respectively.

A quasi-brittle fracture behaviour [6, 18, 19] is considered, thus growth into the length direction occurs, whenever

$$G_l \geq G_c, \quad (25)$$

and growth into the width direction, whenever

$$G_b \geq G_c. \quad (26)$$

It should be noted that simultaneous growth into both directions is also possible, whenever for a current state of loading

$$\left. \begin{matrix} G_l \\ G_b \end{matrix} \right\} \geq G_c. \quad (27)$$

Subsequently, the extended total potential energy principle (*cf.* Eq. (12)) is applied in order to determine the post-buckling responses beyond the deformation state causing delamination growth. As has been discussed in Section 2, during stable delamination growth the condition

$$G = G_c \quad (28)$$

holds, which can be rewritten in terms of the current model description, such that

$$G_b = G_c \quad \text{or} \quad G_l = G_c. \quad (29)$$

Eq. (29) is the requirement for the existence of an extended total potential energy, thus the total work of deformation being a potential of the generalized forces (*cf.* [16]).

Since the width and length of the ellipse cannot be explicitly obtained from Eq. (29), it is rewritten such that

$$G_b - G_c = D_b(q_i, \varepsilon_0, b) = 0 \quad \text{and} \quad G_l - G_c = D_l(q_i, \varepsilon_0, l) = 0, \quad (30)$$

from where the width and length of the ellipse are implicitly given by the functions D_b and D_l respectively. It should be noted that the indicated dependencies of D_b on the width b and D_l on the length l only account for the possible directions of growth and do not delineate two distinct damage parameters.

An explicit form of the width b and the length l of the ellipse is obtained by a TAYLOR series approximation around the damage state (q_i^0, ε_0^0) , *i.e.* the deformation state at which delamination growth is initiated. Thus, the delamination width and length are obtained in terms of

$$b = b(q_i, \varepsilon_0) \quad \text{and} \quad l = l(q_i, \varepsilon_0) \quad (31)$$

respectively, depending on whether growth in the width or length direction is initiated. In the current application example, the TAYLOR series is truncated after the second order terms.

For growth in the width direction, the extended total potential energy can be derived by inserting Eq. (31)₁ into the strain energy of the plate and adding the dissipative energy associated

with delamination growth,

$$W_d = G_c(A_{\text{ell}} - A_{\text{ell}}^0) \quad (32)$$

which can be rewritten regarding growth in the width direction, *i.e.*

$$W_d = G_c \pi l (b - b^0), \quad (33)$$

where A_{ell}^0 and b^0 denote the initial delamination area and the initial delamination width respectively. Thus, the extended total potential energy, *i.e.* the total work of deformation, during delamination growth in the width direction reads

$$W_{\text{tot}} = W(q_i, \varepsilon_0, b(q_i, \varepsilon_0)) + W_d(b(q_i, \varepsilon_0)). \quad (34)$$

For growth in the length direction, Eq. (31)₂ is used instead for replacing the delamination length in Eqs. (32) and (34) while keeping the delamination width b constant.

The total work of deformation given by Eq. (34) (or the respective form for growth in the length direction) is a potential of the generalized forces and the governing functional of the deformation process during delamination growth [16]. Thus, the variational principle

$$\delta W_{\text{tot}}(q_i) = 0 \quad (35)$$

is applied yielding the equilibrium path in terms of $q_i(\varepsilon_0)$ starting from the damage state (q_i^0, ε_0^0) . It should be stressed that for each loading step the energy release rates for growth in the length and width direction need to be determined in order to trace the growth direction accurately. Furthermore, owing to the TAYLOR series approximation of the delamination length and width, the respective delamination parameter has to be recalculated once the condition of $G = G_c$ is violated.²

With the variational principles in Eqs. (22) and (35) an entire loading process starting from an unloaded configuration up to the failure displacement (stability and/or material failure) can be modelled.

Before results for characteristic post-buckling responses of multi-layered composite plates with embedded elliptical delaminations are presented, the adequate choice of the order of the displacement functions shown in Eq. (16) is addressed next. This is done in order to enable an efficient modelling approach for describing post-buckling responses of such structures. Efficiency is understood as an optimal choice in between accuracy of the approximation and computational cost.

6. Order of the displacement functions

Regarding an adequate choice of the displacement functions, *i.e.* determining the order of the polynomials provided in Eq. (16), plates exhibiting the following features are considered:

- square dimensions, *i.e.* $L = B$ (*cf.* Fig.3),
- a length/width to thickness ratio of greater than 40,
- elliptical delaminations (including circular delaminations) and

²This strictly refers to a violation due to the approximation of the damage parameter.

- a delamination depth of less than 0.2 (normalized to the total thickness of the plate), *i.e.* shallow delaminations.

All plates investigated in the current work are required to comply with the aforementioned criteria. The length to thickness ratio is taken such that shear effects are small. Elliptical delaminations are studied owing to the conclusive experimental evidence provided in [8]. Shallow delaminations are investigated which are defined such that $a \leq 0.2$ (*cf.* Fig.3).

The displacement functions investigated are evaluated by means of the prediction of the buckling and post-buckling response for the case of a stationary delamination. The displacement functions are determined by performing two steps. First, the order of the polynomials is continuously increased. Second, generalized coordinates remaining negligibly small are omitted. Furthermore, it is well-documented and therefore considered that the in-plane displacements require higher order approximations than the out-of-plane displacement [23, 34, 35].

The outcome of the evaluation is presented in Fig. 4 showing a post-buckling response for the case of a stationary delamination in terms of normalized compressive applied strain against normalized midpoint deflection. The applied strain is normalized against the buckling strain of an undelaminated plate which has been determined by a linear analysis employing a RAYLEIGH–RITZ formulation in which only the out-of-plane displacement is considered. The midpoint deflection is normalized with respect to the total thickness of the plate.

A plate with the dimensions $150 \text{ mm} \times 150 \text{ mm} \times 3.115 \text{ mm}$ and an elliptical delamination with $l = 25 \text{ mm}$ and $b = 50 \text{ mm}$ is taken as an example. The plate has a unidirectional layup of 35 layers. The material parameters are provided in Table 1. The delamination is in between the 32nd and 33rd layer ($a = 3/35$).

Figure 4: Post-buckling response in terms of applied normalized strain ($\varepsilon_{\text{norm}}$) against normalized midpoint deflections (w_{norm}) for different numbers of degrees of freedom (DOF).

The cases shown in Fig. 4 are compared with findings obtained from a finite element simulation using ABAQUS [31] (“FEM” in Fig. 4) and refer to Eq. (16) as follows:

- nine degrees of freedom (9 DOF, and thus nine generalized coordinates) – first order polynomials for the out-of-plane and in-plane displacements,
- 27 degrees of freedom (27 DOF) – second order polynomials for the out-of-plane and in-plane displacements,
- 34 degrees of freedom (34 DOF) – second order polynomials for the out-of-plane and third order polynomials for the in-plane displacements; omitting vanishingly small coefficients ($|q_i| \leq 10^{-4}$),
- 49 degrees of freedom (49 DOF) – second order polynomials for the out-of-plane and third order polynomials for the in-plane displacements,
- 78 degrees of freedom (78 DOF) – third order polynomials for the out-of-plane and fourth order polynomials for the in-plane displacements.

As can be seen in Fig. 4, the case of 9 DOF does not yield adequate results. A significant improvement is documented in between the cases of 9 DOF and 27 DOF. The post-buckling response improves further specifically for normalized compressive strains greater than 1 for the case of 34 DOF. The cases of 34 and 49 DOF are barely distinguishable with the thicker lower sublamine showing marginally larger out-of-plane deflections (softer response). The case of 78 DOF exhibits an improvement of approximately 2% compared with the case of 34 DOF.

Based on the findings, the case of 34 DOF is chosen to model the post-buckling behaviour of delaminated composite plates. Regarding Eq. (16), all degrees of freedom of the local displacement field contributions of the thicker sublamine are vanishingly small and therefore omitted. The case of 34 DOF covers all buckling phenomena (critical and post-critical responses) adequately and yields quantitative results which are deemed sufficiently, *i.e.* being in a margin of 5% to the reference solution (the finite element simulation). On the other hand, computational cost is significantly lower in comparison with the cases 49 and 78 DOF.

Concluding, an efficient modelling approach, being understood as an optimal choice in between accuracy and computational cost, is provided by the case of 34 DOF which is adopted henceforth.

7. Results

With the aid of the geometric model (*cf.* Section 4) and the analytical framework (*cf.* Section 3) the post-buckling responses of delaminated plates with a unidirectional ($[0^\circ]_{35}$) and a cross-ply ($[0^\circ/(90^\circ/0)_{17}]$) layup are studied.

The dimensions of the plate are taken as $150 \text{ mm} \times 150 \text{ mm} \times 3.1115 \text{ mm}$ ($2L \times 2B \times t$). A delamination is assigned in between the 32nd and 33rd layer, thus $a = 3/35$ (*cf.* Fig. 3). The dimensions of the plate follow case studies from the literature (*e.g.* [22]). The material parameters of a unidirectional ply and the plate dimensions are listed in Table 1.

The results are compared with findings from finite element simulations using ABAQUS. The finite element model consists of two layers possessing the layup of the upper and lower sublamine, respectively. The layers are built-up by S4R shell elements. The two layers are bonded with each other in the undelaminated region and disbonded in the region of the delamination. The virtual crack closure technique [18, 19], as implemented in ABAQUS [31], was employed where growth is allowed to propagate in the plane of the delamination. The mesh is refined around the delamination tip with an element size of $0.5 \text{ mm} \times 0.5 \text{ mm}$. An element size of $1.5 \text{ mm} \times 1.5 \text{ mm}$ is assigned to the rest of the model. A small imperfection load at the centre of the plate is used to enable the tracing of the post-buckling path.

First, post-buckling responses of the unidirectional laminate are studied. An elliptical delamination with $l_{\text{norm}} = 1/3$ and $b_{\text{norm}} = 2/3$ is investigated where the lengths are normalized against the respective dimension of the plate, *i.e.*: $l_{\text{norm}} = l/L$ and $b_{\text{norm}} = b/B$.

Dimensions		Material Parameters	
L	75.00 mm	E_{11}	137.90 GPa
B	75.00 mm	E_{22}	8.98 GPa
t	3.1115 mm	G_{12}	7.20 GPa
a	3/35	ν_{12}	0.30
h	0.0889 mm	G_c^I	190 Nm/m ²

Table 1: Dimensions and material parameters of the plate. Material parameters are taken from [16, 29].

Fig. 5 comprises the structural stability and the material damaging behaviour of the system by delineating the deformation paths in terms of normalized compressive applied strain ($\varepsilon_{\text{norm}}$) *vs.* normalized midpoint deflections (w_{norm}) in Fig. 5a and by visualizing the delamination growth contours calculated with the aid of the current model and with the finite element simulation in Fig. 5b and Fig. 5c respectively. In this application example (Fig. 5), the condition of $G_c = G_c^I$ is employed in the analytical model and the finite element simulation.

Characteristic deformation states which are analysed with regards to the delamination growth behaviour are highlighted in Fig. 5 by Roman numerals for the current analytical model and by Arabic numerals for the finite element simulation (denoted by FEM in Fig. 5a).

As described in Section 6, the applied strain is normalized against the buckling strain of an undelaminated plate and the midpoint deflection is normalized with respect to the total thickness of the plate.

Fig. 5a shows that, initially, the upper less stiff sublaminate mainly experiences out-of-plane deflection (local response), whereas the lower more stiff sublaminate slightly deflects in the opposite direction. Thus, the delaminated composite plate exhibits an opening-mode buckling response [11, 33]. Once the global buckling response is triggered, the thicker more stiff sublaminate pulls the upper sublaminate into the negative direction. However, the buckling response remains in the opening-mode.

The post-buckling behaviour determined by the analytical model is in very good agreement with the finite element model. The critical behaviour as well as the initial post-buckling response coincide. In the post-buckling range, small deviations of approximately 4% are present in between the analytical model and the FEM.

The onset of delamination growth is indicated in Fig. 5a by ① (red dot) for the analytical model and by ① and ② (blue circles) for the FEM. Two deformation states, ① and ②, are used for the FEM in order to emphasize the difference in predicting delamination growth compared with the current model description ①, which is discussed next.

In the FEM, the initiation of growth is given by disbonding of a single node as shown by ① in Fig. 5c, whereas the analytical model—owing to the model description—considers the onset of growth by an entire disbonding of the boundary (① in Fig. 5b). After growth by disbonding of a single node is initiated, until the deformation state indicated by ② is reached only further disbonding of the nodes along the initial boundary occurs during the subsequent post-buckling path. Thus, the FEM generates a delamination growth contour (① in Fig. 5c) which is similar to the one of the analytical model. Therefore, the second growth contour ② is provided in Fig. 5c which serves for the comparison with the analytical model as ① and ② constitute the deformation states at which growth is generated beyond the initial boundary of the delamination.

Comparisons between the deformation states indicated by ① and ② in Fig. 5a associated

Figure 5: Post-buckling response of a $[0_{35}^{\circ}]$ plate with an elliptical delamination ($l_{\text{norm}} = 1/3$ and $b_{\text{norm}} = 2/3$) at depth $a = 3/35$, $G_c = G_c^I$; (a) normalized compressive applied strain ($\varepsilon_{\text{norm}}$) against normalized midpoint deflections (w_{norm}); (b) delamination growth contours of the current model (Roman numerals); (c) delamination growth contours of the FEM (Arabic numerals).

with the growth contours ① and ① in Figs. 5b and 5c respectively show good agreement in which the quantitative values for applied strain and midpoint deflection deviate by approximately 12%.

Subsequently, both models predict the same behaviour where growth occurs in the width (b) direction and the post-buckling path remains thoroughly stable. At the deformation states denoted by ② for the analytical model and by ② for the FEM in Fig. 5a, the maximum load bearable by the system is reached. For those deformation states, the growth profiles ② and ② in Figs. 5b and 5c respectively almost coincide.

At the deformation state denoted by ② in Fig. 5a, the energy release rate for growth in the length direction of the ellipse reaches the critical energy release rate, thus $G_l = G_b = G_c$. As discussed in Section 2 and in [6, 14], if $G_l = G_b = G_c$, unstable delamination growth is caused. Thus, at ② sudden failure of the system occurs which is indicated by the red symbol “ $\diamond$ ” in Fig. 5a. The applied strain at “ $\diamond$ ” can therefore be seen as the failure load of the system. As a consequence, the delamination grows instantaneously from ② to ③ in Fig. 5b which visualizes the material failure.³

This behaviour is verified by the FEM where failure is indicated by the blue “ $\diamond$ ” in Fig. 5a which is associated with an instantaneous growth of the delamination from ② to ③ in Fig. 5c.

As mentioned before, the findings provided in Fig. 5 are associated with the condition that $G_c = G_c^I$ which may be regarded as a conservative measure. However, mode mixture is, in general, relevant for the case of plates with embedded delaminations.

Therefore, the effect of mode mixture is addressed with the aid of another example (Fig. 6) in which a delamination of $l_{\text{norm}} = 0.20$ and $b_{\text{norm}} = 0.53$ is assigned to the plate with the unidirectional layup. This example also serves to further clarify the effect of the global description of the delamination, *i.e.* by the parameters l and b , on predicting the onset of delamination growth.

Since the analytical model does not consider mode mixture, owing to the geometry of the delamination ($l < b$), it is assumed that growth in the length direction is dominated by mode II, thus $G_c^I = G_c^{II}$, and growth in the width direction by mode I, thus $G_c^b = G_c^I$. The smaller length of the ellipse (l) yields larger in-plane strains in the loading direction (*cf.* x -axis in Fig. 3, *i.e.* ε_{xx}) due to the post-buckling deformation, thus favouring the sliding mode II at the longitudinal boundary. This effect is deemed smaller for the width direction (*cf.* y -axis in Fig. 3, *i.e.* ε_{yy}), so that the opening mode I is assumed to be dominant. This assumption may only serve as a rough approximation which, however, is qualitatively similar to the observations made by finite element simulations documented in [22]. In [22], it is shown that for the case of shallow delaminations growth in the width (b) direction is associated with mode mixture where G_c is closer to G_c^I than G_c^{II} and growth in the length (l) direction relates purely to mode II.

In the FEM in Fig. 6, mode mixture is considered in terms of the BENZEGGAGH–KENANE (BK) criterion [3], as implemented in ABAQUS.⁴

Fig. 6 shows the post-buckling response (Fig. 6a) in terms of normalized compressive applied strain ($\varepsilon_{\text{norm}}$) *vs.* normalized midpoint deflections (w_{norm}) as well as the delamination growth contours of the analytical model (Fig. 6b) and the finite element simulation (Fig. 6c).

The qualitative post-buckling behaviour is similar to the case studied in Fig. 5. However, owing to the smaller delamination area—specifically the smaller delamination length—the critical load is increased, the local buckling response is smaller than in Fig. 5a and the global buckling response is more dominant. The post-buckling response determined with the aid of the model

³The solving algorithm is aborted at a delamination size in which l or b exceeds 0.95.

⁴The parameters used are: $G_c^I = 0.19 \text{ N/mm}$, $G_c^{II} = 0.63 \text{ N/mm}$, $G_c^{III} = 0.63 \text{ N/mm}$, mixture parameter $\eta = 1.75$.

Figure 6: Post-buckling response of a $[0_{35}^{\circ}]$ plate with an elliptical delamination ($l_{\text{norm}} = 0.20$, $b_{\text{norm}} = 0.53$) at depth $a = 3/35$, $G_c \neq G_c^I$; (a) normalized compressive applied strain ($\varepsilon_{\text{norm}}$) against normalized midpoint deflections (w_{norm}); (b) delamination growth contours of the current model (Roman numerals); (c) delamination growth contours of the FEM (Arabic numerals).

description is in very good agreement with the finite element simulation (“FEM” in Fig. 6a).

The effect of the decreased length of the delamination is clearly visible when studying the deformation state causing delamination growth which is highlighted in Fig. 6a by ① (red dot) and by ① (blue circle) for the analytical model and the FEM respectively. The deformation states causing growth almost coincide for the analytical model and the FEM. The smaller length of the elliptical delamination yields that the onset of delamination growth is associated with a larger disbond along the boundary compared with the case studied in Fig. 5. This is visualized by the delamination contours ① and ① in Figs. 6b and 6c respectively.

As can be seen in Fig. 6a, the onset of growth does not change the stability of the deformation process, thus further loading can be applied during delamination growth.

The influence of the mode mixture on the growth behaviour as well as the post-buckling response is documented by the growth profiles ② and ③ as well as ② and ③ in Figs. 6b and 6c respectively in conjunction with the post-buckling path during growth in Fig. 6a. In both models, growth occurs in the width direction. However, the analytical model assumes that growth follows $G_c^b = G_c^I$ along the entire boundary of the delamination. In the FEM, growth is only governed by mode I at the vertex of the delamination $(0, b)$. Mode mixture is present in the vicinity of the vertex $(0, b)$ and growth is governed by mode II in the longitudinal direction. Thus, growth proceeds significantly slower in comparison with the analytical model and is more localized around the vertex of the ellipse $(0, b)$. This is visualized by the delamination contours associated with ② and ② in Figs. 6b and 6c respectively. At the deformation states indicated with ② and ② in Fig. 6, the analytical model predicts more than double the magnitude of growth compared with the FEM.

Furthermore, mode mixture and the local description of damage growth affect the qualitative behaviour of delamination growth. As mentioned before, whenever the condition of $G_l = G_b = G_c$ is fulfilled, unstable growth is triggered. This condition refers to a global description of the delamination in terms of the parameters l and b . Thus, in the analytical model the condition for unstable growth reads $G_l = G_c^{II}$ and $G_b = G_c^I$ which is not fulfilled during the post-buckling response in Fig. 6a. In the FEM, the direction of growth as well as the mode partition are locally evaluated along the boundary of the delamination. As a consequence, unstable growth is caused in the FEM at the blue symbol “◊” causing failure of the system.

Further information can be gained by studying the post-buckling response in terms of normalized compressive force (P_{norm}) against normalized end-shortening ($\mathcal{E}_{\text{norm}}$), as shown in Fig. 7.

The force associated with the applied strain is calculated by integrating the force resultant n_{xx} at the boundary (L, y) over the width of the plate, *i.e.*

$$P = \int_{-B}^B n_{xx} \Big|_{x=L} dy. \quad (36)$$

The end-shortening is calculated by simple multiplication of the applied strain with the entire length of the plate.

As can be seen in Fig. 7, the local buckling response of shallow delaminations barely affects the compressive stiffness of the system. Once global buckling occurs, a change in the compressive stiffness of the system is documented where a large ratio of the compressive stiffness is retained in the post-buckling range. This is a characteristic response for fully clamped plates. Thus, visually, delamination growth is barely detectable from Fig. 7. As a consequence, unexpected failure may occur at the deformation state indicated by the symbol “◊”.

Compared with the FEM, the analytical model shows approximately 5% larger forces in the post-buckling regime after the global buckling occurred. The post-buckling stiffness as well as the

Figure 7: Normalized compressive force (P_{norm}) against normalized end-shortening ($\mathcal{E}_{\text{norm}}$) of a $[0_{35}^{\circ}]$ laminate with an elliptical delamination ($l_{\text{norm}} = 0.20$, $b_{\text{norm}} = 0.53$) at the depth $a = 3/35$, $G_c \neq G_c^1$.

prediction of the onset of delamination growth are similar for both models. During delamination growth, the differences in the quantitative and qualitative damage growth behaviour due to the mode mixture, as described in Fig. 6, causes the FEM to fail by unstable growth at the blue symbol “ $\diamond$ ”, whereas further loading can be applied in the analytical model which predicts failure by delamination growth through the entire width of the plate at the red symbol “ $\diamond$ ”.

Next, the post-buckling behaviour of a cross-ply laminate ($[0^{\circ}/(90^{\circ}/0)_{17}]$) is compared with the response of the unidirectional layup ($[0_{35}^{\circ}]$). This provides insight into the influence of the stacking sequence on the initiation of delamination growth, *i.e.* the resistance against damage growth, and the subsequent post-buckling behaviour during delamination growth.

Therefore, a delamination with a normalized length (l_{norm}) of 0.2 and a width (b_{norm}) of 0.267 is assigned to the plate, *i.e.* the aspect ratio (b/l) is $4/3$. The delamination depth remains unchanged compared with the cases studied for the unidirectional layup ($a = 3/35$). Fig. 8 shows the normalized compressive applied strain ($\varepsilon_{\text{norm}}$) against the normalized midpoint deflections (w_{norm}) for each laminate. It should be stressed that, in order to analyse the effect of the stacking sequence on the delamination growth, both responses are normalized against the critical strain for the respective undelaminated plate.

First, Fig. 8 shows for the case of stationary delaminations that the unidirectional laminate provides a larger resistance against buckling delineated by a higher critical load, smaller out-of-plane deflections during the local response as well as a “sharper” transition into the global buckling response.

The deformation state causing delamination growth is indicated in Fig. 8 by the red (cross-ply) and blue (unidirectional) symbol “ $\bullet$ ”. It can be seen that for the cross-ply laminate growth is generated with the onset of the global buckling response. On the other hand, the unidirectional laminate experiences delamination growth considerably later during global buckling. Thus, for

Figure 8: Normalized compressive applied strain (ϵ_{norm}) against normalized midpoint deflection (w_{norm}) of a $[0^\circ/(90^\circ/0^\circ)_{17}]$ laminate and a $[0^\circ_{35}]$ laminate; elliptical delamination with $l_{\text{norm}} = 0.20$ and $b_{\text{norm}} = 0.267$ at the depth $a = 3/35$.

the case considered in Fig. 8, the unidirectional laminate exhibits a higher resistance against delamination growth than the cross-ply laminate. This is further underlined by the fact that the cross-ply laminate already fails by a complete separation along the width of the plate, indicated by the red symbol “ $\diamond$ ”, shortly after growth is initiated for the unidirectional laminate.

Significantly higher loads beyond the range shown in Fig. 8 can be withstood by the unidirectional laminate which is, for illustration purposes, visualized in Fig. 9 which shows the post-buckling responses in terms of normalized compressive force (P_{norm}) against normalized end-shortening (ϵ_{norm}). In order to enable a comparison, the response of the unidirectional laminate is, in contrast with Fig. 8, visualized by a dashed line for a non-growing delamination and by a dotted line for a propagating delamination.

Fig. 9 shows that delamination growth for the cross-ply laminate is generated when the system starts to lose its linear behaviour visually, thus at the onset of the global buckling response. During the ensuing post-buckling response with increasing delamination size, an effect of growth on the post-buckling stiffness is barely detectable.

Comparing the compressive force causing growth for both laminates a difference of approximately 15% is documented in Fig. 9. Furthermore, whereas the cross-ply laminate experiences growth at smaller forces than the critical buckling load of an undelaminated plate, the unidirectional layup can be loaded slightly above the critical point without causing delamination growth.

In addition, Fig. 9 shows that significantly larger forces are required to cause a complete separation (red and blue symbols “ $\diamond$ ”) along the width of the unidirectional plate than for the cross-ply laminate. This also underlines the higher resistance against delamination growth of the unidirectional layup for the case studied in Figs. 8 and 9.

Figure 9: Normalized compressive force (P_{norm}) against normalized end-shortening (E_{norm}) of a $[0^\circ/(90^\circ/0^\circ)_{17}]$ laminate and a $[0^\circ_{35}]$ laminate; elliptical delamination with $l_{\text{norm}} = 0.20$ and $b_{\text{norm}} = 0.267$ at the depth $a = 3/35$.

8. Conclusions

The recently published novel analytical framework [16] has been successfully applied to the problem of multi-layered delaminated composite plates subjected to compressive in-plane loading. Characteristic post-buckling responses of unidirectional and cross-ply laminates are obtained. In the following, conclusions are drawn regarding the semi-analytical modelling approach and the post-buckling responses obtained.

8.1. Semi-analytical modelling

With the condition for the existence of an extended total potential energy being fulfilled for stable delamination growth, *i.e.* the equality $f_k = g_k$ holds (*cf.* Fig. 1, Section 3 and [16]), the analytical framework for a structural stability analysis considering damage growth could be applied to the given problem. Owing to the analytical framework and the model description, the problem has been solved semi-analytically.

The semi-analytical modelling approach enables the prediction of the post-buckling behaviour of delaminated multi-layered composite plates by means of 34 generalized coordinates. While this constitutes considerably more generalized coordinates than for the problem of a delaminated strut [15, 16], the amount of generalized coordinates appears small in comparison with semi-analytical models aiming at the case of stationary delaminations and the behaviour of the energy release rate.⁵

⁵For instance, more than 300 generalized coordinates are employed in [23], and more than 30 in [34] for thin-film buckling only.

The choice of the displacement functions made yields adequate predictions of the post-buckling behaviour with expected confined deviations. Such deviations associated with the order of the approximation, as determined in the preceded analysis in Section 6, are considered as expedient in order to determine an efficient model description, *i.e.* requiring the least amount of computational cost while yielding adequate buckling responses.

The description of the damage parameter, *i.e.* the delamination area, in terms of the semi major and semi minor axis of the ellipse appears beneficial regarding the approximation of the displacement field, but causes restrictions with regards to the modelling of delamination growth. Such restrictions can be summarized as:

- delamination growth can be modelled in the width and length direction of the ellipse,
- growth in either direction can only be predicted by a complete disbonding of the boundary and
- mode mixture, which would require an analysis along the boundary, cannot be considered.

Studying elliptical delaminations, the influence of the first two bullets diminishes the smaller the dimensions of the ellipse are as well as with increasing aspect ratios (b/l), as delineated in Section 7. Neglecting mode mixture and assuming the conservative measure for delamination growth $G_c = G_c^I$ yields very good agreement of the qualitative and quantitative growth behaviour in comparison with the FEM. Characteristic deformation states causing unstable delamination growth and thus sudden failure are determined. The implementation of a rough approximation of mode mixture such that growth in the width direction is governed by mode I and growth in the length direction by mode II results in the adequate prediction of the growth direction but overestimates growth and omits deformation states causing unstable growth.

The aforementioned issues are strictly associated with the model description and thus independent from the analytical framework. Considering the mandatory restrictions the model description yields results which capture the post-buckling behaviour and damage growth characteristics adequately.

The current semi-analytical approach enables an efficient modelling of the post-buckling behaviour up to the deformation state causing failure (material and/or stability) which provides important insight into the structural stability of delaminated composite plates.

8.2. Post-buckling responses

With the aid of the results provided in Section 7, the following conclusions regarding the post-buckling behaviour considering delamination growth can be drawn.

- For the plates investigated (unidirectional and cross-ply laminates with elliptical shallow delaminations), the onset of delamination growth does not alter the stability of the system, thus the deformation process remains stable.
- Delamination growth is triggered with the initiation of global buckling or shortly afterwards.
- For the cases studied (shallow delaminations), delamination growth commences transverse to the loading direction.
- During an initial period of delamination growth, further loading may be applied to the system and the post-buckling stiffness barely reduces.
- Unstable delamination growth, thus failure of the system, occurs, whenever the energy release rate for growth in the width direction and in the length direction reach the respective critical energy release rates for a given state of loading.

Regarding the structural stability analysis, considering delamination growth in a buckling analysis is crucial, since failure may occur unexpected once growth reaches a certain magnitude. Up to such a deformation state, growth may be barely detectable tracing the post-buckling paths such as compressive load against midpoint deflections and compressive force against end-shortening. With the semi-analytical modelling approach presented in this work such a structural stability analysis is enabled.

The post-buckling responses have been verified by comparison with finite element simulations using ABAQUS. Despite the vast difference in degrees of freedom, the approximation of the displacement field by means of 34 generalized coordinates yields almost no deviations for the critical and the initial post-critical response compared with the FEM. Deviations of approximately 5% regarding the applied loads during the post-buckling response succeeding global buckling are documented.

The issue of mode mixture and the description of growth by means of the delamination length l and width b (*cf.* Fig. 3) has been addressed in Section 8.1. Both affect the post-buckling responses once growth is considered. The influence of describing the delamination by l and b diminishes with increasing aspect ratios (b/l) and decreasing dimensions of the delamination.

In summary, the semi-analytical model presented in the current work constitutes a highly efficient engineering tool which provides insight into the interaction of structural stability and damage growth such that the post-buckling behaviour beyond deformation states causing delamination growth can be analysed.

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