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An analytical framework to extend the general structural stability analysis by considering certain inelastic effects—theory and application to delaminated composites

A. Köllner^{a,*}, C. Völlmecke^a

^a*Technische Universität Berlin, School V, Institute of Mechanics, Stability and Failure of Functionally Optimized Structures Group, Einsteinufer 5, 10587 Berlin, Germany.*

Abstract

An analytical framework which incorporates damage propagation/growth into the general structural stability analysis is presented. Therefore, the conventional total potential energy approach is extended by introducing an *extended total potential energy-like functional* capable of describing inelastic processes in which equilibrium holds between available and the required force for producing a change in structure. The work deals with systems which are described by I generalized coordinates and K damage parameters. The damage parameters are found to be functions of I generalized coordinates and M load parameters. The underlying variational principle for inelastic solids may be solved using discrete formulations or approximate methods such as a RAYLEIGH-RITZ formulation. This leads to a set of non-linear algebraic equations, comprising post-critical equilibrium paths and damage propagation. In order to verify the framework, it is applied to the well-known problem in which a delaminated composite strut/plate is subjected to an in-plane compressive load.

Keywords: Extended total potential energy, postbuckling, composites, plate, delamination growth

1. Introduction

Structural stability problems are often analyzed using the total potential energy (TPE) approach, as documented in established textbooks such as [1] and [30]. The description of a system's response by a set of generalized coordinates (GCs) q_i often allows the modelling of various stability phenomena by means of just a few degrees of freedom; for instance, see [13, 31] regarding stability phenomena in sandwich structures and [7, 32] for plated structures. However, applying the structural stability analysis framework, as described in [1] and [30], exhibits one major limitation—its restriction to conservative processes. Hence, changes in structure, *e.g.* damage propagation, may not be considered. However,

*Corresponding author

Email address: anton.koellner@tu-berlin.de (A. Köllner)

stability phenomena are often linked to material damage or failure. This also counts for a main area of the structural stability analysis which is the subject of interest of the current work—the buckling and postbuckling behaviour of slender structures such as struts, plates and shells. Moreover, application to composite materials is also sought after since such materials are increasingly used nowadays, specifically once lightweight constructions are desired. Hence, damage mechanics relevant to composites are required to be considered when such structures are subjected to compressive loads. Such phenomena include for instance, matrix cracking, fibre failure, fibre kinking, delamination initiation and delamination growth, *e.g.* see [14, 17, 18, 28, 33].

A well-known example which requires both structural stability analysis and damage mechanics is the (post-)buckling response of composite plates containing an initial delamination. This problem has been studied by many researchers since the 1980s, starting with the work from Chai *et al.* [3, 4]. However, all pre-existing (semi-)analytical models are limited to stationary delaminations, *i.e.* non-growing delaminations. Postbuckling solutions are provided for stationary delaminations, and damage analysis (*e.g.* examining the strain energy release rate) reveals if growth occurs and whether such growth is stable or unstable. However, post-critical responses beyond the state where delamination growth occurs are yet solely modelled using finite element (FE) simulations (*e.g.* see [20]). In the current work, with the aid of the framework proposed, structural stability analysis and damage propagation are combined into one single formulation. Thus, post-critical responses beyond the state where delamination growth occurs can be determined.

Conditions for (in-)stability for inelastic solids based on variational principles and energy criteria are well documented within the literature. Starting with [10] and [11], it is shown that equilibrium equations and their stability are determined by a variational principle derived from the underlying general boundary value problem. Extensions to these studies are made in [22] and [23], allowing for a more general formulation and application to deformation paths (relevant addition to equilibrium solutions if path-dependent solids are subject of interest).

In the work presented, such a variational principle as well as a method to derive strain energy-like potentials [19, 26, 27] are applied to the structural stability analysis formalism. Active damage parameters are described as functions of I generalized coordinates and M load parameters. Hence, the total work of deformation (consisting of the strain energy and energy associated with K inelastic effects) may be described by a potential which allows the derivation of an *extended total potential energy* (ETPE) functional. By describing the system using a discrete formulation or implementing an approximate method, *e.g.* a RAYLEIGH-RITZ formulation, the solution of the underlying variational principle yields the deformation paths for the irreversible process. With the solution obtained, the evolution of the damage parameters is then readily determined. Moreover, the stability of the equilibrium solutions may be evaluated using the governing functional. The outcome of the current work is an analytical framework which enables the modelling of stability related phenomena of structures by means of just a few GCs. Thus, efficient simulations of pre-damaged slender structures, comprising post-critical deformation paths and possible damage propagation, are enabled. This depicts an extension to the conventional structural stability analysis.

2. Analytical framework

2.1. Total potential energy principle and its limitation

The total potential energy (Π) of a structural system described by I generalized coordinates (q_i) is defined as [1, 30],

$$\Pi(q_i, A_m) = W(q_i) - A_m \alpha_m(q_i), \quad (1)$$

with $i = 1, 2, \dots, I$, $m = 1, 2, \dots, M$ and W being the strain energy. The conjugate variables prescribed loads (A_m) and their respective displacements (α_m) describe the work done by external forces. If not explicitly stated otherwise, the summation convention is employed in which a repeated index is to be summed over its entire range. The prescribed loads A_m and the conjugate displacements α_m may also be understood as independent generalized forces and generalized displacements respectively. Applying the well-known variational principle, $\delta\Pi(q_i) = 0$, yields the equilibrium solutions, *i.e.* the deformation paths, $A_m(q_i)$.

In Eq. (1), the current state of damage may be considered by a set of damage parameters ξ_k . However, the assumption that all damage parameters do not change must hold (reversible processes). Thus, the total potential energy (TPE) given in Eq. (1) may be expressed as

$$\Pi(q_i, \xi_k, A_m) = W(q_i, \xi_k) - A_m \alpha_m(q_i, \xi_k), \quad \text{for } \xi_k = \text{const.}, \quad (2)$$

in which the dependency in regards to the current state of damage is accounted for (with $k = 1, 2, \dots, K$). Thus, only the deformation behaviour under a given constant state of damage may be modelled which limits the applicability of the TPE principle.

2.2. Extended total potential energy principle

An analytical framework which enables the modelling of the (post-)buckling responses by a set of GCs (q_i), without the limitation of a non-evolving state of damage (ξ_k are not assumed constant anymore), requires the derivation of an extended total potential energy-like functional, $\Pi^*(q_i)$. In order to derive such a functional, the method of deriving strain energy-like potentials [27] is applied to systems described by I GCs. Subsequently, the variational principle for inelastic solids (see *e.g.* [10, 11, 22]) may be applied yielding the deformation path for an evolving state of damage.

2.2.1. Potential character of the total work of deformation and the extended total potential energy

The sufficient condition for the total work of deformation being a potential is derived next. Therefore, in the spirit of [27], the response of the system is described in terms of generalized displacements (α_m) and generalized forces (A_m). Consequently, it is necessary to differentiate between prescribed displacements (displacement controlled setups) and prescribed loads (load controlled setups). For prescribed displacements, the governing functional is the strain energy which is a function (W_{rev}) of the independent generalized displacements (α_m) and the current state of damage defined by K damage parameters (ξ_k), *i.e.* $W_{\text{rev}} = W_{\text{rev}}(\alpha_m, \xi_k)$. On the other hand, once prescribed loads are given, the problem is described by the total potential energy which is a function of the independent generalized forces (A_m) and the damage parameters (ξ_k), *i.e.* $V = V(A_m, \xi_k)$.

Prescribed displacements

In order to describe the total work of deformation W_{tot} as a potential, *i.e.* $A_m = \partial W_{\text{tot}} / \partial \alpha_m$ and consequently $W_{\text{tot}} = \int A_m d\alpha_m$, the strain energy is examined for a change in structure (*e.g.* increase in damage). The total derivative of the strain energy yields

$$dW_{\text{rev}} = \left. \frac{\partial W_{\text{rev}}}{\partial \alpha_m} \right|_{\xi_k} d\alpha_m + \left. \frac{\partial W_{\text{rev}}}{\partial \xi_k} \right|_{\alpha_m} d\xi_k \quad \text{with} \quad f_k = -\frac{\partial W_{\text{rev}}}{\partial \xi_k}, \quad (3)$$

where f_k is the thermodynamic force [25] associated to the k th damage parameter, *i.e.* the force available for producing a change in the structure [27]. The total work is equal to the sum of elastic energy (W_{rev}) plus energy associated with dissipation (W_d), thus

$$W_{\text{tot}}(\alpha_m, \xi_k) = W_{\text{rev}}(\alpha_m, \xi_k) + W_d(\xi_k). \quad (4)$$

The dissipative energy is assumed to be a state function depending on the damage parameters only [27]. The force required for a change in structure (g_k), *i.e.* the change in dissipative energy with respect to the damage parameter (which may be a material parameter), reads

$$g_k = \frac{\partial W_d}{\partial \xi_k}. \quad (5)$$

If the force available for a change in structure (f_k) is equal to the force required (g_k), Eqs. (3)₂ and (5) yield

$$f_k(\alpha_m, \xi_k) = g_k(\xi_k) \implies -\frac{\partial W_{\text{rev}}}{\partial \xi_k} = \frac{\partial W_d}{\partial \xi_k}. \quad (6)$$

Eq. (6) outlines the sufficient condition to describe the total work of deformation as a potential. From the set of equations in Eq. (6)₂ the damage parameters ξ_k may be derived in terms of independent generalized displacements α_m , *i.e.* $\xi_k = \xi_k(\alpha_m)$. Hence, Eq. (6)₂ may be also understood as the evolution law for the damage parameters ξ_k . It is assumed that Eq. (6)₂ provides a unique solution for the damage parameters ξ_k , differentiable in α_m [27]. By inserting $\xi_k(\alpha_m)$ in Eq. (4), W_{tot} is found as a function of the independent generalized displacements, $W_{\text{tot}} = W_{\text{tot}}(\alpha_m)$, which describes a potential of the generalized forces. It should be noted that Eqs. (3)₂ and (6) apply for systems where the boundary is subjected to prescribed displacements.

Prescribed loads

For load controlled setups (generalized forces serve as independent variables), the thermodynamic force is equal to the change of the total potential energy, $V = V(A_m, \xi_k)$, with respect to the damage parameters ξ_k . This may be noted by examining the total derivative of V ,

$$dV = \left. \frac{\partial V}{\partial A_m} \right|_{\xi_k} dA_m + \left. \frac{\partial V}{\partial \xi_k} \right|_{A_m} d\xi_k \quad \text{with} \quad f_k(A_m, \xi_k) = -\frac{\partial V}{\partial \xi_k}. \quad (7)$$

As stated in Eq. (6), for processes in which the force available (f_k) for producing a change in structure equals the force required (g_k) follows

$$f_k(A_m, \xi_k) = g_k(\xi_k) \implies -\frac{\partial V}{\partial \xi_k} = \frac{\partial W_d}{\partial \xi_k}, \quad (8)$$

from which the damage parameters may be found as functions of independent generalized forces, *i.e.* $\xi_k = \xi_k(A_m)$. Hence, Eq. (8)₂ serves as the evolution law for ξ_k and is the sufficient condition for deriving an extended total potential energy function, $V^* = W_{\text{tot}} - A_m \alpha_m$, such that $\partial V^* / \partial A_m = -\alpha_m$.

2.2.2. Variational principle using the ETPE

For processes in which the condition of equilibrium between f_k and g_k holds, a variational principle in form of $\delta \Pi^*(q_i) = 0$ may be derived; Π^* is the extended total potential energy functional comprising the total work of deformation and the work done by external forces. Such a variational principle is similar to the principles described in [10, 11, 22] for continuous problems,

$$\delta J(u_i) = \int_B \delta w_t(\nabla u_i) dV - \int_{S_t} t_i \delta u_i dS_t - \int_B \rho b_i \delta u_i dV = 0, \quad (9)$$

in which $\int w_t dV$ is the total work of deformation W_t (reversible plus irreversible energy contributions) being also a potential. Eq. (9) yields the EULER-LAGRANGE equations of systems under quasi-static loading.

In the context of the current work, such a formulation is transformed into a discrete description using I GCs. Thus, the variational principle using the ETPE reads

$$\delta \Pi^*(q_i) = \delta (W_t(q_i) - A_m \alpha_m(q_i)) = 0. \quad (10)$$

It should be noted that in Eq. (10) the damage parameters are already considered and replaced by functions of the generalized forces A_m and the generalized coordinates q_i , *i.e.* $\xi_k = \xi_k(q_i, A_m)$ using the condition $f_k = g_k$. The deformation state described by I generalized coordinates is considered and therefore present in the description of the damage parameters.

In comparison with Eq. (2) (TPE-principle), the dissipative energy (W_d) is added and the damage parameters are replaced. Thus, the ETPE can be written as

$$\begin{aligned} \Pi^*(q_i, A_m) &= W(q_i, \xi_k(q_i, A_m)) + W_d(\xi_k(q_i, A_m)) - A_m \alpha_m(q_i, \xi_k(q_i, A_m)) \\ &= W_t(q_i, A_m) - A_m \alpha_m(q_i, A_m). \end{aligned} \quad (11)$$

in which the total work of deformation (W_t) is the sum of the reversible strain energy (W) and the dissipative energy (W_d) associated with a change in the damage parameters.

The derivation of the damage parameters is performed using Eq. (8)₂ and is explained in section 3. By solving Eq. (10), the deformation behaviour of the structural system is obtained in terms of $A_m(q_i)$ starting from the deformation state where damage propagation is initiated. Inserting the solution into the K damage parameters, $\xi_k(q_i, A_m(q_i))$, provides information about the damage propagation within the structure.

Unlike the variational principles described in [5, 8] which give the macroscopic force balance and the yield criterion, the formulation proposed in the current work yields the deformation behaviour (equilibrium equations) only. It is assumed that during the irreversible process loading will follow the condition $f_k = g_k$, which embodies the yield criterion for the proposed formalism. This is in contrast to [5] in which the yield criterion is an inequality. Hence, owing to the condition $f_k = g_k$, no unilateral constraint is considered within the proposed principle. However, the requirement of $f_k = g_k$ allows the damage parameters to be represented as functions of the load parameters as well as the generalized coordinates. Such functions replace the damage parameters in the energy formulation. Thus, the variational problem is one in which only the deformation state is perturbed. For example, the difference in comparison with [5, 8] can be observed by examining the total work of deformation, as done in [5], and assuming for demonstration purposes a one degree of freedom problem with a single damage parameter (ξ). Thus, following [5], both the deformation state and damage state are perturbed, yielding a minimization problem which reads

$$W_t(q, \xi) \leq W_t(q + \eta, \xi + \psi), \quad (12)$$

in which η and ψ are independent perturbations of the deformation and damage state respectively. However, in the framework proposed, the damage parameter is obtained *a priori* by means of the deformation state which results in

$$W_t(q, \xi) \leq W_t(q + \eta, \xi(q + \eta)) \text{ with } \xi = \xi(q) \text{ for } f_k = b_k, \quad (13)$$

which holds starting from the deformation state where damage propagation is initiated, and the solution is a minimizer of the total work of deformation.

The solution must also comply with the second law of thermodynamics, *i.e.* healing of the structure is not allowed, so $\dot{\xi}_k \geq 0$ or in a quasi-static loading regime $\xi_k^i \geq \xi_k^{i-1}$ for i loading steps. Owing to the condition $f_k = g_k$, this unilateral constraint is not included *a priori* in the formalism. However, as the damage parameters $\xi_k = \xi_k(q_i, A_m)$ have been predetermined, the variational principle provides one solution path, $A_m(q_i)$. Depending on the applied loading in the solving algorithm (*e.g.* loading or unloading), starting from the deformation state where the condition $f_k = g_k$ is fulfilled first, the path $A_m(q_i)$ complies with the second law of thermodynamics or violates it. Fig. 1 illustrates that matter with the aid of the well-known double cantilever beam (DCB) test.

Figure 1: DCB test with the characteristic load-deflection response.

At the deformation state (α^0, A^0) growth of the damage parameter is initiated and the solution of the variational principle would provide the deformation path starting from (α^0, A^0) . The loading path which fulfills the condition of non-healing damage (solid line in Fig. 1) is the actual deformation path of the system. This procedure is readily implemented in a respective solving algorithm and therefore seen as advantageous for the framework proposed.

3. Structural stability analysis with damage propagation

It is assumed that the given structure contains K pre-existing damage states, so damage initiation is not considered within the current work. In multi-layered composite structures, such pre-existing damage may be, for instance, a transverse matrix crack or a delamination where ξ_k would be the crack area or the delamination area, respectively.

Initially, the structure is treated as a conservative system, *i.e.* the damage parameters are constant. Considering a problem definition in which independent generalized forces act on the boundary, the deformation paths, $A_m(q_i)$, are determined for the given (constant) state of damage using the conventional total potential energy formulation [1, 30], see section 2.1.

In the next step, the state of deformation is determined in which the force available for producing a change in structure with respect to the k th damage parameter (f_k) equals the force required (g_k), *i.e.* a change in the respective ξ_k would occur and the conventional formulation (Eq. (1)) does not hold anymore. This deformation state is defined by a set of GCs (q_i^0) and the load parameters (A_m^0), and will be referred to as “damage state” rather than “critical state” to avoid confusion with the terminology of structural stability analysis. To derive the evolution of the respective (active) damage parameters ξ_k , Eq. (8)₂ is used. If displacements were prescribed, the governing functional would be the strain energy and Eq. (6)₂ would be used instead. In general, it is not possible to obtain an explicit solution from Eqs. (8)₂ or (6)₂, however, rewriting Eqs. (8)₂ such that

$$f_k - g_k = -\frac{\partial \Pi}{\partial \xi_k} - \frac{\partial W_d}{\partial \xi_k} = 0, \quad (14)$$

yields functions D_k , thus

$$D_k(q_i, A_m, \xi_k) \equiv 0, \quad (15)$$

from which $\xi_k(q_i, A_m)$ is implicitly given assuming that a unique solution of $\xi_k(q_i, A_m)$ exists. Thus, Eq. (15) may be rewritten as

$$D_k(q_i, A_m, \xi_k(q_i, A_m)) \equiv 0. \quad (16)$$

In order to obtain an explicit form of $\xi_k(q_i, A_m)$ a TAYLOR series approximation around the “damage state”, *i.e.* q_i^0, A_m^0 , is employed, thus

$$\begin{aligned}
 \xi_k(q_i, A_m) = & \xi_k^0 + \left. \frac{\partial \xi_k}{\partial q_i} \right|_{q_i^0, A_m^0} (q_i - q_i^0) + \left. \frac{\partial \xi_k}{\partial A_m} \right|_{q_i^0, A_m^0} (A_m - A_m^0) + \\
 & + \frac{1}{2} \left. \frac{\partial^2 \xi_k}{\partial q_i \partial q_j} \right|_{q_i^0, A_m^0} (q_i - q_i^0)(q_j - q_j^0) + \\
 & + \frac{1}{2} \left. \frac{\partial^2 \xi_k}{\partial A_m \partial A_n} \right|_{q_i^0, A_m^0} (A_m - A_m^0)(A_n - A_n^0) + \\
 & + \left. \frac{\partial^2 \xi_k}{\partial q_i \partial A_m} \right|_{q_i^0, A_m^0} (q_i - q_i^0)(A_m - A_m^0) + \mathcal{O}(3).
 \end{aligned} \tag{17}$$

For the sake of clarity, the TAYLOR series in Eq. (17) is provided up to the 2nd order only. However, depending on the given problem, higher order terms may, of course, be considered in the formulation which also results in higher computational demand. The derivatives of ξ_k with respect to q_i and A_m are obtained, with the aid of Eq. (16), by implicit differentiation. Thus, differentiating Eq. (16) with respect to q_i and A_m and applying the chain rule yields the derivatives of ξ_k used in Eq. (17), following some rearranging. For instance, the first order derivative with respect to q_i reads

$$\frac{dD_k}{dq_i} = \frac{\partial D_k}{\partial q_i} + \frac{\partial D_k}{\partial \xi_k} \frac{\partial \xi_k}{\partial q_i} \equiv 0 \implies \frac{\partial \xi_k}{\partial q_i} = - \frac{\frac{\partial D_k}{\partial q_i}}{\frac{\partial D_k}{\partial \xi_k}}. \tag{18}$$

Subsequently, with the damage parameters given by Eq. (17), the extended total potential energy formulation (ETPE) is applied (see Eq. (10)) starting from the “damage state” (q_i^0, A_m^0) in which the active damage parameters are associated with the respective dissipative energy contributions. Hence, the ETPE principle is obtained in form of

$$\begin{aligned}
 \delta \Pi^*(q_i) = & \delta \left(W(q_i, \xi_k(q_i, A_m)) + W_d(\xi_k(q_i, A_m)) - A_m \alpha_m(q_i, \xi_k(q_i, A_m)) \right) \\
 = & \delta (W_t - A_m \alpha_m) = 0.
 \end{aligned} \tag{19}$$

The solution of Eq. (19) is the deformation path $A_m(q_i)$ for inelastic processes in which $f_k = g_k$ holds. The damage parameters ξ_k are obtained by inserting the solution from Eq. (19) into Eq. (17).

As the evolution of the damage parameters are approximated, the solution obtained from Eq. (19) is examined if the requirement of $f_k = g_k$ is fulfilled. Once this is violated, the procedure restarts with a “new damage state” (q_i^0, A_m^0) which describes the state of deformation for which the requirement $f_k = g_k$ was fulfilled last. Hence, an iterative scheme is implemented which approximates the damage evolution and solves the ETPE principle consecutively.

In summary, using the conventional total potential energy formulation up to the “damage state” and subsequently extending the functional to the ETPE formulation enables the determination of the deformation and stability behaviour of structural systems during an entire loading process depending on I GCs only.

4. Application to a delaminated composite strut/plate

The problem of a delaminated composite plate which is subjected to an in-plane compressive load is extensively discussed within the literature (*e.g.* see [3, 4, 15, 16, 21, 29]). Several studies (*e.g.* see [2, 3, 4, 15, 29]) investigate by means of an analytical formulation if delamination growth occurs within the postbuckling range and whether such growth would be stable or unstable. This provides information about the qualitative behaviour of possible delamination growth, however, the actual deformation path and hence the post-critical behaviour while delamination grows is not provided. To date this has been solely done by finite element simulations [20]. In order to demonstrate the method proposed in the previous section, delamination growth in a composite plate described by a simple one-dimensional model is studied. Therefore, we will refer to the structure as a *strut* rather than a plate. Findings with respect to the post-critical behaviour for a stationary delamination are compared for verification purposes of the conservative model to [29], and the post-critical response during delamination growth are presented and compared to a finite element (FE) simulation using the commercial code Abaqus [9]. In the following sections, the terminology “conservative”, as done before, is used to refer to a non-dissipative deformation process (stationary delamination).

4.1. One-dimensional model of a composite strut

It is assumed that the strut is subjected to a compressive load in form of an independent generalized force ($m = 1$, $A_1 = P$). For comparison purposes, the case of clamped boundary conditions will be studied even though the model may be easily reformulated for a simply supported case. For the geometric model shown in Fig. 2, the description provided in Refs. [12] and [15] is taken as a benchmark model.

Figure 2: One-dimensional model of a composite strut with a through-the-width delamination.

A central delamination with the length L is assigned to the strut. Thus, as shown in Fig. 2, the strut may be subdivided into four parts. Each part is given its own coordinate system. Parts ① and ② are the upper and lower sublaminates respectively. Parts ③ and ④ describe the undelaminated region of the strut. Subsequently, owing to the symmetry, part ③ will be considered for the undelaminated region only. The depth of the delamination is described

by the parameter a . The GCs q_i ($i = 1, \dots, 4$) fully describe the system. The generalized coordinates q_1 and q_2 are the amplitudes of the upper and lower sublaminates respectively (with respect to the neutral axis, see dashed line in Fig. 2). The generalized coordinate q_3 is the rotation (also the slope) of the plane at the interface between delaminated and undelaminated part, and q_4 (not shown in Fig. 2) describes the total end-shortening of the delaminated parts. In-plane stretching of the undelaminated region is omitted as its influence on the (post-)buckling response is assumed small [12] and it avoids the introduction of another degree of freedom.

The total potential energy is derived using the CLASSICAL LAMINATE THEORY [24] in which coupling effects due to asymmetric lay-ups caused by the delamination are considered by incorporating the coupling stiffness matrix, B_{ij} . A RAYLEIGH-RITZ formulation is employed approximating the displacements (buckling displacement of each part, ①-④) using the GCs, thus

$$\begin{aligned} w_i &= q_i \sin^2 \left(\frac{\pi x_i}{L} \right) + C_0 x_i^3 + C_1 x_i^2 + C_2 x_i + C_3 \quad \text{with } i = 1, 2 \quad \text{and} \\ C_0 &= \frac{q_3}{2L^3} \left(\frac{L_{\text{tot}}}{\pi} \tan \left(\frac{\pi SL}{L_{\text{tot}}} \right) - \frac{L_{\text{tot}}}{\pi \cos \left(\frac{\pi SL}{L_{\text{tot}}} \right)} \sin \left(\frac{\pi(SL + L)}{L_{\text{tot}}} \right) \right), \\ C_1 &= \frac{1}{2} \left(-\frac{2q_3}{L} - 3C_0 L \right), \\ C_2 &= q_3, \quad C_3 = \frac{q_3 L_{\text{tot}}}{2\pi} \tan \left(\frac{\pi SL}{L_{\text{tot}}} \right), \\ w_3 &= q_3 \frac{L_{\text{tot}}}{2\pi \cos \left(\frac{\pi SL}{L_{\text{tot}}} \right) \sin \left(\frac{\pi SL}{L_{\text{tot}}} \right)} \sin^2 \left(\frac{\pi x_3}{L_{\text{tot}}} \right), \end{aligned} \quad (20)$$

in which the coefficients of the polynomial ($C_{0..3}$) are determined such that the geometric boundary and the continuity conditions for the buckling displacements,

$$\begin{aligned} w_3(x_3 = 0) &= 0, & w_3(x_3 = L_{\text{tot}}) &= 0, \\ w_3(x_3 = SL) &= w_i(x_i = 0), & w_i(x_i = L) &= w_3(x_3 = SL + L), \\ w'_3(x_3 = 0) &= 0, & w'_3(x_3 = L_{\text{tot}}) &= 0, \\ w'_3(x_3 = SL) &= w'_i(x_i = 0), & w'_i(x_i = L) &= w'_3(x_3 = SL + L), \end{aligned} \quad (21)$$

with $i = 1, 2$, are enforced. In Eq. (21), a $(')$ denotes a differentiation with respect to the x coordinate of the respective part.

A non-linear kinematic approach, as documented in Ref. [12], is used to describe the axial shortening of the delaminated region ($u^{①}$, $u^{②}$),

$$\begin{aligned} u^{①} &= q_4 - \frac{1}{2} \int_0^L \left(\frac{\partial w_1}{\partial x_1} \right)^2 dx_1 - 2 \left(\frac{1-a}{2} \right) tq_3, \\ u^{②} &= q_4 - \frac{1}{2} \int_0^L \left(\frac{\partial w_2}{\partial x_2} \right)^2 dx_2 + 2 \left(\frac{a}{2} \right) tq_3, \end{aligned} \quad (22)$$

from which, assuming evenly distributed strain [12], the respective in-plane strains ($\varepsilon_0^{①}$, $\varepsilon_0^{②}$) are obtained by simple division with the respective length of the delaminated parts.

The reversible strain energy W combines energy contributions from stretching and bending deformations for each part of the strut, thus

$$\begin{aligned} W = & \frac{1}{2}b \left(2D_{\text{eff}}^{(3)} \int_0^{SL} w_3''^2 dx_3 + \right. \\ & + \int_0^L \left(D_{\text{eff}}^{(1)} w_1''^2 + A_{\text{eff}}^{(1)} \varepsilon_0^{(1)2} + 2B_{\text{eff}}^{(1)} w_1'' \varepsilon_0^{(1)} \right) dx_1 + \\ & \left. + \int_0^L \left(D_{\text{eff}}^{(2)} w_2''^2 + A_{\text{eff}}^{(2)} \varepsilon_0^{(2)2} + 2B_{\text{eff}}^{(2)} w_2'' \varepsilon_0^{(2)} \right) dx_2 \right). \end{aligned} \quad (23)$$

in which, as described before, stretching contributions for the undelaminated part ③ are not considered and b denotes the width of the strut. In Eq. (23), entries responsible for bending-twisting (D_{16}, D_{26}), bending-bending (D_{12}) as well as stretching-shearing (A_{16}, A_{26}) and stretching-stretching (A_{12}) are considered and incorporated by calculating effective parameters for in-plane (A_{eff}), coupling (B_{eff}) and bending (D_{eff}) stiffness of the respective parts. This is done assuming that the forces n_{yy} and n_{xy} as well as the moments m_{yy} and m_{xy} are zero which allows resulting strains ($\varepsilon_{yy}, 2\varepsilon_{xy}$) and curvatures (κ_{yy}, κ_{xy}) to be determined. Such strains and curvatures are subsequently replaced in the relevant expressions for n_{xx} or m_{xx} .

The only damage parameter, in this example, is the delamination length, *i.e.* $k = 1, \xi_1 = L$. Following the TPE principle, as described in section 2.1, the total potential energy can be written as

$$\Pi(q_i, P) = W(q_i) - P\mathcal{E}(q_i) \quad \text{with} \quad i = 1, \dots, 4, \quad (24)$$

where $\mathcal{E}$ is the conjugate generalized displacement (end-shortening of the strut, $\alpha_1 = \mathcal{E}$) to the independent generalized force ($A_1 = P$) and writes as

$$\mathcal{E} = q_4 + \int_0^{SL} w_3'{}^2 dx_3, \quad (25)$$

in which the shortening of the undelaminated part results entirely from the buckling displacement [12] (as its in-plane contributions are neglected).

The deformation paths $P(q_i)$ for the “pre-damage” behaviour (conservative system) are found by solving the non-linear set of algebraic equations ($\partial\Pi/\partial q_i = 0$) resulting from the condition that the first variation of the total potential energy vanishes, *i.e.* $\delta\Pi = 0$, if the system is in an equilibrium state. The calculation was carried out numerically using the software AUTO-07P [6], implementing a continuous NEWTON method.

According to sections 2.2 and 3, the energy release rate G is effectively the thermodynamic force available for delamination growth. The force required for a change in structure is the critical energy release rate G_c which is a material parameter depending on the mode mixture [20]. However, for reasons of simplicity, G_c is assumed constant in the current example. The energy release rate is calculated as

$$f_1 = G = -\frac{1}{b} \frac{\partial\Pi}{\partial L}, \quad (26)$$

and the condition for the existence of an extended total potential energy $f_1 = g_1$ is

$$G = -\frac{1}{b} \frac{\partial\Pi(q_i, P, L)}{\partial L} = G_c \quad \text{yielding} \quad D(q_i, P, L(q_i, P)) = G - G_c \equiv 0. \quad (27)$$

With the aid of Eq. (27)₁ and the solution for the “pre-damage” behaviour $P(q_i)$, the “damage state” (q_i^0, P^0) is identified from where the conventional total potential energy formulation loses its validity. As described in section 3, with the aid of Eq. (27)₂, the delamination length is described as a function of q_i and P , *i.e.* $L = L(q_i, P)$, using a TAYLOR series approximation around the “damage state” (q_i^0, P^0) up to the 2nd order.

The *extended total potential energy formulation* is found by adding the dissipative energy term associated with delamination growth,

$$W_d = W_d(L) = G_c(L - L^0)b, \quad (28)$$

to Eq. (24) in which L^0 is the initial delamination length and b is the width of the delamination area (assumed to be constant). As it is commonly done within the literature [3, 4, 29], a GRIFFITH-type (quasi-brittle) damage process is assumed for delaminated composite plates. In contrast to a purely tensile loading, the condition of $G = G_c$ does not necessarily lead to a catastrophic failure of the strut [3, 4, 29] in the postbuckling regime, which will be explained in section 4.2.2.

Subsequently, the damage parameter L is replaced in Eqs. (24) and (28). Hence, the variational principle using the ETPE can be written as

$$\delta\Pi^*(q_i) = \delta\left(W(q_i, L(q_i, P)) + W_d(L(q_i, P)) - P\mathcal{E}(q_i, L(q_i, P))\right) = 0, \quad (29)$$

from which the solution in terms of $P(q_i)$ starting at the point (q_i^0, P^0) is obtained numerically by applying the NEWTON method. As described in section 3, an iterative scheme is implemented deriving a new expression for the delamination length once the condition of $G = G_c$ is violated due to the approximation by the TAYLOR series.

4.2. Results

4.2.1. Verification of the model for a stationary delamination

First, in order to verify the model proposed in section 4.1, results obtained for the postbuckling response (without considering possible delamination growth yet) are compared to findings documented in [29]. Fig. 3 shows the postbuckling response in terms of normalized load versus normalized midpoint deflections, $P_{\text{norm}}(w_{\text{norm}})$, for a multi-layered cross-ply composite plate ($[0^\circ/(90^\circ/0^\circ)_7]$) with a delamination at depth $a = 4/15$ (*cf.* Fig. 2).

As in [29], an imperfection (assumed to be caused by the delamination) is considered in form of an initial midpoint deflection. The imperfection is added to the energy formulation by assigning q_1 an initial value of $q_1^0 = t/1000$ and subtracting its respective energy contributions. The normalization of the load in Fig. 3 is carried out with respect to the EULER buckling load for an undelaminated strut with 0° degree layers only. The midpoint deflections are normalized to the total thickness of the strut¹, $w_{\text{norm}} = w_{1,2}(x_{1,2} = L/2)/t$.

The results shown in Fig. 3 are in very good agreement. The initial postbuckling solution almost coincides. Slight differences can be seen at larger deflections

¹For the purpose of comparison, the coordinate system of [29] is used in Fig. 3. Thus, the signs of deflection are opposed to Fig. 2.

Figure 3: Postbuckling response in form of normalized load P_{norm} vs. normalized midpoint deflections w_{norm} with comparison with [29].

with deviations smaller than 4%. Based on such findings, the model proposed appears to describe the buckling and postbuckling behaviour sufficiently well in the conservative range.

4.2.2. Results with delamination growth

For the current analysis, the dimensions and material parameters of the plate documented in [29] are used. Both are listed in Table 1 in which h describes the thickness of a single layer and t the total thickness of the plate.

Dimensions		Material Parameters	
L_{tot}	96.52 mm	E_{11}	137.90 GPa
b	12.70 mm	E_{22}	8.98 GPa
t	1.337 mm	G_{12}	7.20 GPa
a	4/15	ν_{12}	0.30
h	0.0889 mm	G_c^I	190 Nm/m ²

Table 1: Dimensions and material parameters taken from [29].

Energy release rate

Fig. 4 shows the normalized energy release rate, $G_{\text{norm}} = G/G_c^I$, with respect to the normalized delamination length, $L_{\text{norm}} = L/L_{\text{tot}}$, (used instead of A_{Del} as the width b is constant) and increasing levels of constant end-shortening. The energy release rate is normalized with respect to the mode I critical energy release rate G_c^I provided in Table 1 where $G_{\text{norm}} = 1$ indicates $G = G_c$ which is a conservative measure. It is generally advisable to evaluate the energy release

rate in terms of end-shortening $\mathcal{E}$ rather than forces as weakly stable or even unstable buckling phenomena can be analyzed.

Figure 4: Normalized energy release rate G_{norm} vs. normalized delamination length L_{norm} for increasing load levels in form of normalized end-shortening $\mathcal{E}_{\text{norm}}$; delamination depth $a = 4/15$.

Fig. 4 depicts the typical behaviour of the energy release rate for a composite plate with a through-the-width delamination. Similar results are documented in several studies, for instance [3, 4, 29]. The solid lines in Fig. 4 describe the energy release rate for respective constant values of end-shortening. The smallest value ($\mathcal{E}_{\text{norm}} = 2.3$) indicates the lowest level of “load input” for which delamination growth occurs. Thus, for an initial delamination length of $L_{\text{norm}} = 0.21$, growth is generated at $\mathcal{E}_{\text{norm}} = 2.3$. An initial delamination length of $L_{\text{norm}} = 0.45$ requires the highest value of end-shortening ($\mathcal{E}_{\text{norm}} = 9.72$) to cause growth. A vertical dotted line is added to Fig. 4 corresponding to a delamination length of $L_{\text{norm}} = 0.45$. The left vertical dotted line in Fig. 4 indicates an arbitrary chosen initial delamination length of a strut ($L_{\text{norm}} = 0.36$) which is studied next.

At an end-shortening of $\mathcal{E}_{\text{norm}} = 6.58$, the energy release rate equals the critical value ($G = G_c$, $G_{\text{norm}} = 1$) for an initial delamination length of $L_{\text{norm}} = 0.36$. Once the critical energy release rate is reached, delamination growth occurs. Thus, for the current state of loading (quasi-static process) associated with the critical energy release rate G_c , the delamination length increases from L to $L + \Delta L$. For the new configuration, *i.e.* the constantly kept state of loading and a delamination length of $L + \Delta L$, the energy release rate changes from G_c ($G_{\text{norm}} = 1$) to another value, which is referred to as G_{new} . If the energy release rate decreases ($G_c > G_{\text{new}}$), growth stops and the process is termed stable. If $G_{\text{new}} > G_c$, unstable growth occurs causing catastrophic failure of the strut. If stable delamination growth occurs, further loading may be applied until G_c is reached again. Thus, the postbuckling response under stable delamination growth follows the condition of $G = G_c$ (horizontal dashed line in Fig. 4) which also dictates the load that may be applied to the system

and embodies the requirement for deriving the ETPE ($f_k = g_k$) used within the framework proposed in section 3.

Thus, by analysing Fig. 4, regions of stable or unstable growth for initial delamination lengths are obtained. For delamination lengths of $0.21 \leq L_{\text{norm}} < 0.45$ stable delamination growth occurs. Delamination lengths larger than $L_{\text{norm}} = 0.45$ but smaller than $L_{\text{norm}} = 0.73$ cause unstable growth. Stable growth occurs again if the delamination length is larger than $L_{\text{norm}} = 0.73$.

Postbuckling

The postbuckling response for the delaminated strut analyzed in Fig. 4 is subsequently examined. Unlike in section 4.2.1, normalization is carried out with respect to the EULER buckling load for the undelaminated strut investigated herein (cross-ply lay-up). An initial delamination length of $L_{\text{norm}} = 0.36$ ($L = 35$ mm) is assigned in between the eleventh and twelfth layer ($a = 4/15$). From Fig. 4 follows that for such an initial delamination length growth will be stable until a delamination length of $L_{\text{norm}} = 0.45$ is reached (in between left and right vertical dotted line in Fig. 4).

In Figs. 5 and 6, the postbuckling behaviour is studied by examining the load P_{norm} against the midpoint deflection of the sublaminates w_{norm} . The results are compared to findings from a finite element simulation using Abaqus [9]. The strut is built-up by shell elements (type S4R) with an element size of 0.2 mm by 0.2 mm and a total of 62790 nodes. Delamination propagation is modelled using the virtual crack closure technique.

Figure 5: Normalized load P_{norm} vs. normalized midpoint deflections w_{norm} for an initial delamination length of $L_{\text{norm}} = 0.36$ and a delamination depth of $a = 4/15$.

Fig. 5 shows a characteristic postbuckling response as long as the delamination does not grow (up to “•”). A local buckling response is shown at around $P_{\text{norm}} = 0.45$ where the upper delaminated part (thinner and less stiff than the lower part) buckles whereas the lower part stays unaffected. Once the global

buckling response is triggered the lower sublaminate deflects in the negative direction causing an opening mode buckling response [12, 32]. The global buckling response (buckling of the undelaminated and lower sublaminate) causes the midpoint deflection of the upper laminate also into the negative direction (with respect to the coordinate system of the undelaminated part in Fig. 2). However, the structure remains in an opening mode response as the delamination opens further, even though both parts show a negative deflection.

As documented in Fig. 5, the FE simulation shows the same qualitative behaviour compared to the model whereas quantitatively the respective local buckling load as well as the global buckling differ by approximately 5%. This appears plausible as the model does not incorporate shear deformations and neglects in-plane contributions from the undelaminated region. A changeover in the buckling response can be seen once delamination growth occurs. This is illustrated in Fig. 6 which takes a closer look at the buckling response around the deformation state where delamination growth is caused (“•”).

Figure 6: Closer look at the normalized load P_{norm} vs. normalized midpoint deflections w_{norm} for an initial delamination length of $L_{\text{norm}} = 0.36$ and a delamination depth of $a = 4/15$.

Fig. 6 shows that once the delamination grows both sublaminates deflect in opposite directions associated with a slightly decreasing load. The same qualitative behaviour is obtained by the FE simulation with the load being approximately 4% smaller. The dashed line shows the deformation path as long as delamination growth is stable. At “♦”, the delamination reaches a length of $L_{\text{norm}} = 0.45$ at which (with the corresponding end-shortening, see Fig. 7) unstable growth is triggered causing failure of the strut.

Fig. 7 shows the postbuckling response in the form of the normalized compressive load P_{norm} versus normalized end-shortening $\mathcal{E}_{\text{norm}}$. Initially, a typical “conservative” buckling response is shown where the postbuckling stiffness of the strut remains unaffected during local buckling and almost reaches zero at global buckling illustrating the characteristic weakly stable behaviour [30]. However, in contrast to other (semi-)analytical models, the deformation behaviour is also determined once delamination growth occurs. As predicted by Fig. 4, delamina-

Figure 7: Normalized load P_{norm} vs. normalized end-shortening $\mathcal{E}_{\text{norm}}$ for an initial delamination length of $L_{\text{norm}} = 0.36$ and a delamination depth of $a = 4/15$.

tion growth is initiated at $\mathcal{E}_{\text{norm}} = 6.58$ indicated by “•” in Fig. 7. The dashed line following “•” describes stable damage growth as it was predicted in Fig 4. Once damage growth is caused a slight drop in load occurs. In the range from $\mathcal{E}_{\text{norm}} = 6.58$ (•) to $\mathcal{E}_{\text{norm}} = 9.72$ (◊), the strut loses approximately 3.3% of the maximum load. Up to an end-shortening of $\mathcal{E}_{\text{norm}} = 9.72$, corresponding to a delamination length of $L_{\text{norm}} = 0.45$, the delamination propagation is stable, *i.e.* the strut does not fail. However, at an end-shortening of $\mathcal{E}_{\text{norm}} = 9.72$ (◊) the delamination grows unstably. Thus, for this configuration, sudden failure of the strut occurs once end-shortening reaches $\mathcal{E}_{\text{norm}} = 9.72$ which may be understood as the “failure displacement” of the system.

Regarding stability analysis the initiation of delamination growth is important. For a load-controlled problem, “•” exhibits a limit point from which the system will lose its stability causing a dynamic snap and ultimately stability failure. If the problem was displacement-controlled (end-shortening), the loss of structural stability would coincide with the initiation of unstable delamination growth at “◊”.

Another configuration with a larger initial delamination length ($L_{\text{norm}} = 0.67$) but the same depth ($a = 4/15$) is studied in Figs. 8 and 9. From Fig. 4, it follows that once growth is initiated, it is unstable. However, interestingly Fig. 4 also depicts that such growth would stop at larger delamination length and henceforth proceed in a stable regime (this is similar to the case shown for $\mathcal{E}_{\text{norm}} = 6.58$ in Fig. 4 at $L_{\text{norm}} = 0.58$). Thus, contrary to the previously discussed case, post-critical behaviour causing initially an unstable delamination growth but no complete failure is documented in Figs. 8 and 9.

Fig. 8 shows the normalized compressive load P_{norm} against normalized end-shortening $\mathcal{E}_{\text{norm}}$. As before, “•” indicates where delamination growth occurs, *i.e.* at $\mathcal{E}_{\text{norm}} = 5.75$. As discussed, unstable delamination growth is caused at “•”. This is associated to a sudden increase in delamination from $L_{\text{norm}} = 0.67$

Figure 8: Normalized load P_{norm} vs. normalized end-shortening $\mathcal{E}_{\text{norm}}$ for an initial delamination length of $L_{\text{norm}} = 0.67$ and a delamination depth of $a = 4/15$.

to 0.77 and a drop in load of approximately 12% which is indicated by the vertical dotted line from “•” to “□” in Fig. 8. The model proposed provides the solution for the condition $G = G_c$ (dashed line in Fig. 8) which is violated for unstable growth. Thus, the structure would follow the vertical dotted line and would not undergo the snap-back response indicated in Fig. 8. At “□”, growth would follow a stable regime (corresponding to increasing end-shortening) until “◇” which is illustrated by the dashed line. At “◇”, the delaminated parts of the strut are completely separated.

The comparison with the FE simulation shows a very good agreement with the findings using the analytical framework. Slight differences in load of approximately 1% are documented. The end-shortening at which growth occurs shows slight differences between the models. However, the characteristic drop in load is also illustrated by the FE results and the points of complete separation almost coincide.

In a load-controlled configuration, “•” would correspond to the failure load of the system causing stability failure. The deformation path shown in Fig. 8 could be followed in a displacement controlled setup. However, regarding the quasi-brittle damage behaviour assumed and present in such composites, the system would most likely not recover to a stable regime once unstable growth is initiated.

The corresponding load against midpoint deflection behaviour is shown in Fig. 9. The local buckling as well as the global buckling response are in a very good agreement with the findings from the FE simulation. The larger delamination length almost diminishes the shear deformation effects which are not considered in the analytical formulation. In contrast with Figs. 5 and 6, both sublaminates deflect in opposite direction, without that the undelaminated part pulls both sublaminates into the negative direction. The deflection at which growth occurs (“•”) as well as the point of complete separation (“◇”) almost

coincide between the model and the FE results.

Figure 9: Normalized load P_{norm} vs. normalized midpoint deflection w_{norm} for an initial delamination length of $L_{\text{norm}} = 0.67$ and a delamination depth of $a = 4/15$.

5. Discussion

The modelling of the postbuckling behaviour by means of just a few generalized coordinates, without the limitation to stationary delaminations, indicates the capability of the presented framework. Deformation paths, stability analysis and damage growth characteristics are analyzed by a set of GCs which is not possible using the conventional structural stability analysis approach [1, 30]. However, describing the ETPE by GCs only, restricts the framework. Deformation processes to be investigated must allow for a potential formulation of the total work of deformation (and hence the ETPE). This is in general not always feasible, however, many processes are known (*e.g.* metal-like inelasticity, micro-cracking in ceramics, see [27]) for which such a potential may be derived. Especially regarding composites, studies in [19, 26] have shown that a potential formulation for damaging processes can be found (*e.g.* for matrix cracking). Furthermore, the analysis of the energy release rate within the current work has demonstrated that a potential formulation for delamination growth in layered composites holds.

The requirement of $f_k = g_k$, which for the given problem reads as $G = G_c$, also dictates the variational principle described in section 2.2. As a consequence, damage parameters within the formulation are replaced by functions of applied loads and generalized coordinates and are therefore not directly present within the governing functional. Hence, the commonly comprised unilateral constraints within the variational description of fracture (Kuhn-Tucker conditions) do not exist in the framework. Thus, the variational principle is of the type as proposed in [10, 22] (for the continuous case) instead of the principles described in [5, 8].

The framework proposed provides a solution path starting from the deformation state where damage growth is initiated. As shown in Fig. 1, depending on the loading regime, the second law of thermodynamics is enforced or violated. The “loading direction” which complies with the second law of thermodynamics provides the actual solution of the deformation process. The choice of direction (which may be an obvious one) is readily implemented within the solving algorithm.

The results provided in section 4.2.2 are in very good agreement with findings from a finite element simulation. The model used to describe the buckling and postbuckling phenomena by means of four GCs incorporates several simplifying assumptions such as omitting out-of-plane shear deformations and the in-plane stiffness of the undelaminated part. Such simplifications affect the response of struts with smaller delamination lengths (Figs. 5 and 6) more than with larger delamination lengths (Figs. 8 and 9). This appears plausible and was expected due to the larger length-to-thickness ratio of the respective delaminated parts. In Fig. 9, the corresponding in-plane displacement of the undelaminated part has been added to the solution in order to enable the comparison with the FE simulation. As described in section 4.2.2, mode mixture is not considered and the critical energy release rate of mode I is taken as G_c . However, for “deep” delaminations mode I is dominant, for which the simplification of $G_c = G_c^I$ in the results presented is an acceptable choice.

Even if not specifically investigated within the current work, savings of computational cost is tremendous using the framework proposed. The commercial FE code (specifically in the damaging process) requires hours for solving the presented problem (Intel® i7, 3.4 GHz 16 GB RAM), whereas the implementation of the framework solves the four degree of freedom model within a few minutes.

6. Summary and Conclusion

A novel *extended total potential energy* (ETPE) formulation based on the conventional total potential energy (TPE) approach using generalized coordinates in order to describe the deformation behaviour of structural systems has been presented. The ETPE is capable of modelling deformation processes relevant to structural stability (*e.g.* (post-)buckling responses), without the restriction to conservative processes. Inelastic deformation processes in which the force available for a change in structure equals the force required for these changes can be described, since such processes allow for the derivation of an extended total potential energy.

In the framework presented, deformation processes of structures which possess an initial state of damage are modelled using the TPE approach up to the state of deformation which causes a change in damage within the structure. Starting from this deformation state, the ETPE principle is implemented in which the governing functional is one of I generalized coordinates only, yielding the consecutive deformation path of the structure. The damage parameters are replaced *a priori* by functions of loading parameters and generalized coordinates using the equality of force available and force required for producing a change in structure. This is the requirement for the ETPE principle, and it describes the evolution law of the respective damage parameters. Hence, with the varia-

tional principle, $\delta\Pi^* = 0$, the deformation path is determined from which the corresponding growth of the damage parameters is subsequently computed.

The framework has been applied to the problem of a composite plate/strut with a through-the-width delamination which is subjected to a compressive in-plane load. Postbuckling responses beyond the state where delamination growth occurs are found. Moreover, quantitative results for propagating delaminations during postbuckling are obtained. Thus, the framework provides information beyond the well-known results of energy release rate versus delamination area and postbuckling responses for stationary delamination lengths. For composite plates/struts, such information is not provided by semi-analytical models described within the literature. Hence, information about additional load bearing capacities of systems, as well as damage growth and failure characteristics (stability and/or material failure), are provided by applying the framework. Moreover, the capability of implementing the framework with just a few generalized coordinates allows for fast solutions with demonstrably good accuracy, while demanding low computational costs.

In the given example, the (post-)buckling response with delamination growth has been successfully described by four generalized coordinates only. Hence, deformation behaviour, stability analysis and damage propagation have been determined by solving a set of just four non-linear algebraic equations. Results for the conservative range have been verified by comparison with findings documented within the literature. Deformation paths for the entire process (reversible/irreversible) have been compared to a finite element simulation. The results are in good agreement and quantitative differences obtained (up to 5%) are directly linked to the simplifications made in the model description of the composite plate/strut.

In summary, the presented analytical framework embodies an extension of the conventional structural stability analysis, comprising all advantages in modelling the deformation behaviour by means of generalized coordinates, without the limitation to conservative processes.

References

- [1] Bazant, Z. P., Cedolin, L., 2010. *Stability of Structures: elastic, inelastic, fracture and damage theories*. World Scientific.
- [2] Butler, R., Rhead, A. T., Liu, W., Kontis, N., 2012. Compressive strength of delaminated aerospace composites. *Phil. Trans. R. Soc. Lond. A* 370, 1759–1779.
- [3] Chai, H., Babcock, C. D., 1985. Two-dimensional modelling of compressive failure in delaminated laminates. *J. Compos. Mater.* 19, 67–98.
- [4] Chai, H., Babcock, C. D., Knauss, W. G., 1981. One dimensional modelling of failure in laminated plates by delamination buckling. *Int. J. Solids Structures* 17 (11), 1069–1083.
- [5] Del Piero, G., 2013. A variational approach to fracture and other inelastic phenomena. *J. Elast.* 112 (1), 3–77.
- [6] Doedel, E. J., Oldeman, B. E., 2012. *Auto-07p: Continuation and bifurcation software for ordinary differential equations*. User's Manual.

- [7] Everall, P. R., Hunt, G. W., 2000. Mode jumping in the buckling of struts and plates: a comparative study. *Int. J. Non-Linear Mech.* 35, 1067–1079.
- [8] Francfort, G. A., Marigo, J.-J., 1998. Revisiting brittle fracture as an energy minimization problem. *J. Mech. Phys. Solids* 46 (8), 1319–1342.
- [9] Hibbett, Karlsson, Sorensen, 1998. ABAQUS/standard: User's Manual. Vol. 1. Hibbett, Karlsson & Sorensen.
- [10] Hill, R., 1958. A general theory of uniqueness and stability in elastic-plastic solids. *J. Mech. Phys. Solids* 6, 236–246.
- [11] Hill, R., 1962. Uniqueness criteria and extremum principles in self-adjoint problems of continuum mechanics. *J. Mech. Phys. Solids* 10, 185–194.
- [12] Hunt, G. W., Hu, B., Butler, R., Almond, D. P., Wright, J. E., 2004. Nonlinear modeling of delaminated struts. *AIAA Journal* 42 (11), 2364–2372.
- [13] Hunt, G. W., Wadee, M. A., 1998. Localization and mode interaction in sandwich structures. *Proc. R. Soc. Lond. A* 454, 1197–1216.
- [14] Jelf, P. M., Fleck, N. A., 1992. Compression failure mechanisms in unidirectional composites. *J. Compos. Mater.* 26 (18), 2706–2726.
- [15] Köllner, A., Jungnickel, R., Völlmecke, C., 2016. Delamination growth in buckled composite struts. *Int. J. Fract.* 202 (2), 261–269.
- [16] Köllner, A., Völlmecke, C., 2017. Buckling and postbuckling behaviour of delaminated composite struts. *Int. J. Comput. Meth. Eng. Sci. Mech.* DOI 10.1080/15502287.2016.1276340.
- [17] Kutlu, Z., Chang, F.-K., 1992. Modeling compression failure of laminated composites containing multiple through-the-width delaminations. *J. Compos. Mater.* 26 (3), 350–387.
- [18] Kyriakides, S., Arseculeratne, R., Perry, E. J., Liechti, K., 1995. On the compressive failure of fiber reinforced composites. *Int. J. Solids Structures* 32 (6-7), 689–738.
- [19] Lamborn, M. J., Schapery, R. A., 1993. An investigation of the existence of a work potential for fiber-reinforced plastic. *J. Compos. Mater.* 27 (4), 352–382.
- [20] Nilsson, K.-F., Asp, L. E., Alpman, J. E., Nystedt, L., 2001. Delamination buckling and growth for delaminations at different depths in a slender composite panel. *Int. J. Solids Structures* 38, 3039–3071.
- [21] Ovesy, H. R., Kharazi, M., 2011. Stability analysis of composite plates with through-the-width delamination. *J. Eng. Mech. ASCE* 137 (2), 87–100.
- [22] Petryk, H., 1991. The energy criteria of instability in time-independent inelastic solids. *Arch. Mech.* 43 (4), 519–545.
- [23] Petryk, H., 2005. Thermodynamic conditions for stability in materials with independent dissipation. *Phil. Trans. R. Soc. A* 363, 2479–2515.

- [24] Reddy, J. N., 2004. Mechanics of laminated composite plates and shells: theory and analysis. CRC press.
- [25] Rice, J. R., 1971. Inelastic constitutive relations for solids: an internal-variable theory and its application to metal plasticity. *J. Mech. Phys. Solids* 19, 433–455.
- [26] Schapery, R. A., 1987. Deformation and fracture characterization of inelastic composite materials using potentials. *Polym. Eng. Sci.* 27 (1), 63–76.
- [27] Schapery, R. A., 1990. A theory of mechanical behavior of elastic media with growing damage other changes in structure. *J. Mech. Phys. Solids* 38 (2), 215–253.
- [28] Schultheisz, C. R., Waas, A. M., 1996. Compressive failure of composites, part I: Testing and micromechanical theories. *Prog. Aerospace Sci.* 32, 1–42.
- [29] Sheinman, I., Kardomateas, G. A., Pelegri, A. A., 1998. Delamination growth during pre- and post-buckling phases of delaminated composite laminates. *Int. J. Solids Structures* 35 (1-2), 19–31.
- [30] Thompson, J. M. T., Hunt, G. W., 1973. A general theory of elastic stability. John Wiley & Sons.
- [31] Wadee, M. A., 2002. Localized buckling in sandwich struts with pre-existing delaminations and geometrical imperfections. *J. Mech. Phys. Solids* 50, 1767–1787.
- [32] Wadee, M. A., Völlmecke, C., 2011. Semi-analytical modelling of buckling driven delamination in uniaxially compressed damaged plates. *IMA J. Appl. Math.* 76, 120–145.
- [33] Wadee, M. A., Völlmecke, C., Haley, J. F., Yiatros, S., 2012. Geometric modelling of kink banding in laminated structures. *Phil. Trans. R. Soc. A* 370, 1827–1849.