A variational formulation of anisotropic geometric evolution equations in higher dimensions

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Abstract We present a novel variational formulation of fully anisotropic motion by surface diffusion and mean curvature flow in $\mathbb{R}^d$, $d \geq 2$. This new formulation leads to an unconditionally stable, fully discrete, parametric finite element approximation in the case $d = 2$ or 3 . The resulting scheme has very good properties with respect to the distribution of mesh points and, if applicable, volume conservation. This is demonstrated by several numerical experiments for $d = 3$, including regularized crystalline mean curvature flow and regularized crystalline surface diffusion.

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1 Introduction

The numerical approximation of solutions to geometric evolution equations, which govern the motion of a hypersurface, is a notoriously difficult task. The underlying partial differential equations are highly nonlinear and the meshes used to approximate the evolving surfaces often tend to deteriorate during the evolution. The most prominent example of a geometric evolution equation for hypersurfaces is the mean curvature flow $\mathcal{V} = \kappa$, where $\mathcal{V}$ is the normal velocity and κ is the mean curvature, i.e. the sum of the principal curvatures. The mean curvature flow is the L^2 -gradient

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flow of the surface area functional. The H^{-1} -gradient flow of the area functional gives rise to surface diffusion

$$\mathcal{V} = -\Delta_s \mathcal{K},$$

where Δ_s is the Laplace–Beltrami operator. This fourth order geometric evolution law plays a prominent role in materials science with important applications in epitaxial growth, see e.g. [21]. In this paper we introduce and analyse numerical schemes for anisotropic versions of mean curvature flow, surface diffusion and other related anisotropic evolution equations. Although some schemes for such evolution equations for hypersurfaces in $\mathbb{R}^3$ already exist, see e.g. [12, 22], our scheme is the first one for which one can prove a stability bound. This leads to superior properties in practice, in particular there are no restrictions on the time step size and the associated meshes will generically not deteriorate.

Anisotropic geometric evolution laws are typically based on the anisotropic surface energy

$$|\Gamma|_\gamma := \int_\Gamma \gamma(\vec{\nu}) \, d\mathcal{H}^{d-1}, \quad (1.1)$$

where $\Gamma \subset \mathbb{R}^d$, $d \geq 2$, is a closed compact and orientable hypersurface with outer unit normal $\vec{\nu}$, $\gamma: \mathbb{R}^d \setminus \{\vec{0}\} \rightarrow \mathbb{R}_{>0}$ is a given anisotropy function, and $\mathcal{H}^{d-1}$ is the $(d-1)$ -dimensional Hausdorff measure in $\mathbb{R}^d$. More generally, the anisotropic energy density γ may depend on space as well as on other local parameters, see e.g. [26, 9] for more details on possible anisotropy functions. However, in this paper we restrict ourselves to surface energies of the form (1.1), and we further assume that the function γ is absolutely homogeneous of degree one, i.e.

$$\gamma(\lambda \vec{p}) = |\lambda| \gamma(\vec{p}) \quad \forall \vec{p} \in \mathbb{R}^d, \forall \lambda \in \mathbb{R} \quad \Rightarrow \quad \gamma'(\vec{p}) \cdot \vec{p} = \gamma(\vec{p}) \quad \forall \vec{p} \in \mathbb{R}^d \setminus \{\vec{0}\}, \quad (1.2)$$

where γ' is the gradient of γ . In the isotropic case we have that $\gamma(\vec{p}) = |\vec{p}|$ and so $\gamma(\vec{\nu}) = 1$, which means that $|\Gamma|_\gamma$ reduces to $|\Gamma|$, the surface area of Γ . If γ is not constant on the unit sphere, then the surface energy density in (1.1) will depend on the local orientation of the surface Γ . Such surface energies frequently appear in applications, e.g. in materials science and in image processing. We refer to the papers [31, 23, 16, 30, 29] for more details on anisotropy in materials science and geometry.

To compute the L^2 -gradient flow of the anisotropic energy (1.1) one needs to compute its first variation. On introducing the Cahn–Hoffmann vector, see [10],

$$\vec{\nu}_\gamma := \gamma'(\vec{\nu}),$$

we define the weighted mean curvature as

$$\mathcal{K}_\gamma := -\nabla_s \cdot \vec{\nu}_\gamma. \quad (1.3)$$

Here ∇_s is the surface (tangential) gradient and $\nabla_s \cdot$ is the surface (tangential) divergence on Γ , so that $\Delta_s = \nabla_s \cdot \nabla_s$. The first variation of (1.1) is then given by the following lemma.

Lemma 1.1 *Let $\Gamma(\delta) := \{\vec{z} + \delta g(\vec{z}) \vec{\nu} : \vec{z} \in \Gamma\}$, for a smooth function $g: \mathbb{R}^d \rightarrow \mathbb{R}$. Then it holds that*

$$\frac{d}{d\delta} |\Gamma(\delta)|_\gamma \big|_{\delta=0} = - \int_\Gamma \mathcal{K}_\gamma g \, d\mathcal{H}^{d-1}. \quad (1.4)$$

For a proof we refer to [13, Lemma 8.2], but we remark that these authors use a different sign convention. Anisotropic versions of mean curvature flow and surface diffusion are now given by

$$(a) \quad \mathcal{V} = \beta(\vec{v}) \kappa_\gamma, \quad \text{and} \quad (b) \quad \mathcal{V} = -\nabla_s \cdot (\beta(\vec{v}) \nabla_s \kappa_\gamma), \quad (1.5)$$

where $\beta : S^{d-1} \rightarrow \mathbb{R}_{>0}$, a kinetic coefficient, is assumed to be a smooth and positive function. For a derivation of these laws we refer to [1, 20, 30, 29]. It can be easily established that the anisotropic surface diffusion law preserves the volume enclosed by Γ . Both laws in (1.5) decrease the anisotropic surface energy $|\Gamma|_\gamma$, which follows for surface diffusion using the Gauss theorem on manifolds as follows

$$\frac{d}{dt} |\Gamma(t)|_\gamma = - \int_\Gamma \kappa_\gamma \mathcal{V} \, d\mathcal{H}^{d-1} = - \int_\Gamma \beta(\vec{v}) (\nabla_s \kappa_\gamma)^2 \, d\mathcal{H}^{d-1} \leq 0. \quad (1.6)$$

It is also possible to introduce a second order evolution law which preserves volume and decreases the anisotropic energy. This law is the conserved anisotropic mean curvature flow and it is given by the nonlocal evolution law

$$\mathcal{V} = \beta(\vec{v}) \kappa_\gamma - \frac{\int_\Gamma \beta(\vec{v}) \kappa_\gamma \, d\mathcal{H}^{d-1}}{\int_\Gamma 1 \, d\mathcal{H}^{d-1}}. \quad (1.7)$$

In addition, let us remark that the numerical method we are going to introduce can be used to study nonlinear relations between $\mathcal{V}$ and κ_γ of the form

$$\mathcal{V} = \beta(\vec{v}) f(\kappa_\gamma), \quad (1.8)$$

where f is a strictly monotonically increasing continuous function. Of particular interest is the case $f(r) = -r^{-1}$, i.e. the inverse anisotropic mean curvature flow, see [5] for the relevant details in the isotropic case.

For analytical results on the anisotropic mean curvature flow we refer to the recent book [18] by Giga, and the references therein. For crystalline surface energies nondifferentiable surface energy densities have to be used, see [27, 1, 11]; and also in this case so called crystalline evolution equations, which are nonlocal in nature, can be derived. As we can approximate crystalline anisotropies very accurately by smooth anisotropies, see Figures 1.1–1.5 below, we will only consider the smooth (regularized) case in what follows.

A variational discretization of equations involving mean curvature was given by Dziuk ([14]). He used the identity

$$\vec{\kappa} = \kappa \vec{v} = \Delta_s \vec{x} = \Delta_s \vec{\text{id}} \quad (1.9)$$

to come up with a variational discretization of κ involving only first order derivatives of the identity function $\vec{\text{id}}$ on Γ , or equivalently, of the parameterization $\vec{x} : \Omega \times [0, T] \rightarrow \mathbb{R}^d$ of Γ , where $\Omega \subset \mathbb{R}^d$ is a suitable reference manifold and $T > 0$ is a positive time. Observe that in (1.9) we use a slight abuse of notation, so that the equation can be interpreted to hold either on Γ or on Ω . Our idea is to introduce an anisotropic version of the identity (1.9). The main observation is that (1.9) remains true if we replace the standard Euclidean inner product in $\mathbb{R}^d$ by an inner product

$$(\vec{u}, \vec{v})_{\tilde{G}} := \vec{u} \cdot \tilde{G} \vec{v} \quad \forall \vec{u}, \vec{v} \in \mathbb{R}^d, \quad (1.10)$$

where $\tilde{G} \in \mathbb{R}^{d \times d}$ is symmetric and positive definite. One only has to replace the mean curvature vector $\vec{\kappa}$ and the Laplace–Beltrami operator Δ_s by versions which are appropriate for this new inner product. In fact one just has to consider the canonical Laplace–Beltrami operator on Γ with respect to the Riemannian metric given by the new inner product. It only remains to identify the corresponding surface energy density. In fact it turns out that we have to define the surface area element induced by this new inner product as the surface energy density γ . In Lemma 2.1 below we show that

$$\gamma(\vec{v}) = \sqrt{\vec{v} \cdot G \vec{v}}, \quad (1.11)$$

where G and $\tilde{G}$ are related by $\tilde{G} = [\det G]^{\frac{1}{d-1}} G^{-1}$. One can then deduce that

$$\kappa_\gamma \vec{v} = \gamma(\vec{v}) \tilde{G} \Delta_s^{\tilde{G}} \vec{\text{id}},$$

where $\Delta_s^{\tilde{G}} = \nabla_s^{\tilde{G}} \cdot \nabla_s^{\tilde{G}}$ is the Laplace–Beltrami operator induced by the $\tilde{G}$ inner product (1.10), with $\nabla_s^{\tilde{G}}$ and $\nabla_s^{\tilde{G}}$ the associated tangential divergence and tangential gradient, respectively. A suitable generalized Gauss theorem on manifolds allows one to introduce a weak formulation of $\gamma(\vec{v}) \tilde{G} \Delta_s^{\tilde{G}} \vec{\text{id}}$. Unfortunately simple anisotropies of the form (1.11) only lead to ellipsoidal Wulff shapes, see below, and of course we would like to handle more general situations.

Therefore we consider the following class of surface energy densities, which are given as an l^r -norm of the above anisotropies, i.e. they are assumed to be of the form

$$\gamma(\vec{p}) = \left(\sum_{\ell=1}^L [\gamma_\ell(\vec{p})]^r \right)^{\frac{1}{r}}, \quad \gamma_\ell(\vec{p}) := \sqrt{\vec{p} \cdot G_\ell \vec{p}}, \quad (1.12)$$

so that

$$\gamma'(\vec{p}) = [\gamma(\vec{p})]^{1-r} \sum_{\ell=1}^L [\gamma_\ell(\vec{p})]^{r-1} \gamma'_\ell(\vec{p}), \quad (1.13)$$

where $r \in [1, \infty)$ and $G_\ell \in \mathbb{R}^{d \times d}$, $\ell = 1 \rightarrow L$, are symmetric and positive definite. We will see later on that these anisotropies can be used to describe a wide class of possible physical situations. While it should be noted, that γ given by (1.12) defines a norm on $\mathbb{R}^d$ (norms being the “natural” anisotropy densities), and hence more general anisotropy densities cannot be modelled by this choice, we are satisfied that any given norm can be approximated by (1.12) with suitably chosen $\{G_\ell\}_{\ell=1}^L$ and $r \in [1, \infty)$.

Let us now introduce the variational form which we are going to derive and use in the following sections. The evolution equations (1.5), for anisotropies as in (1.12), can be reformulated as

$$(a) \quad \vec{x}_t \cdot \vec{v} = \beta(\vec{v}) \kappa_\gamma \quad \text{and} \quad (b) \quad \vec{x}_t \cdot \vec{v} = -\nabla_s \cdot (\beta(\vec{v}) \nabla_s \kappa_\gamma), \quad (1.14)$$

together with, see Theorem 2.1 below,

$$\kappa_\gamma \vec{v} = \sum_{\ell=1}^L \gamma_\ell(\vec{v}) \tilde{G}_\ell \nabla_s^{\tilde{G}_\ell} \cdot \left[\left[\frac{\gamma_\ell(\vec{v})}{\gamma(\vec{v})} \right]^{r-1} \nabla_s^{\tilde{G}_\ell} \vec{\text{id}} \right]. \quad (1.15)$$

If we test (1.14) and (1.15) with suitable test functions φ and $\vec{\varphi}$ we obtain, on using a generalized Gauss theorem on manifolds, compare Lemma 2.6 and Theorem 2.1 below, the weak formulations

$$\int_{\Gamma} \vec{x}_t \cdot \vec{v} \varphi \, d\mathcal{H}^{d-1} = \begin{cases} \int_{\Gamma} \beta(\vec{v}) \kappa_{\gamma} \varphi \, d\mathcal{H}^{d-1} \\ \int_{\Gamma} \beta(\vec{v}) \nabla_s \kappa_{\gamma} \cdot \nabla_s \varphi \, d\mathcal{H}^{d-1} \end{cases} \quad (1.16a)$$

$$\text{and } \int_{\Gamma} \kappa_{\gamma} \vec{v} \cdot \vec{\varphi} \, d\mathcal{H}^{d-1} = - \sum_{\ell=1}^L \int_{\Gamma} \left[\frac{\gamma_{\ell}(\vec{v})}{\gamma(\vec{v})} \right]^{r-1} (\nabla_s^{\tilde{G}_{\ell}} \text{id}, \nabla_s^{\tilde{G}_{\ell}} \vec{\varphi})_{\tilde{G}_{\ell}} \gamma_{\ell}(\vec{v}) \, d\mathcal{H}^{d-1}, \quad (1.16b)$$

where the tensor inner products $(\nabla_s^{\tilde{G}} \vec{u}, \nabla_s^{\tilde{G}} \vec{v})_{\tilde{G}}$ are induced by (1.10), see (2.8) below. In the above weak formulation only derivatives up to first order appear and hence the equations can be discretized with the help of continuous, piecewise affine linear finite elements, see Section 3 below. Let us now discuss a second possible ansatz which is based on the identity, see e.g. [13, p. 194],

$$\kappa_{\gamma} \vec{v} = -\nabla_s \cdot (\vec{v} [\gamma'(\vec{v})]^T) + \nabla_s \cdot (\gamma(\vec{v}) \nabla_s \text{id}) - \gamma(\vec{v}) \Delta_s \text{id}. \quad (1.17)$$

Applying this characterization of $\kappa_{\gamma} \vec{v}$, we can rewrite (1.16b) as

$$\int_{\Gamma} \kappa_{\gamma} \vec{v} \cdot \vec{\varphi} \, d\mathcal{H}^{d-1} = \int_{\Gamma} [\vec{v} [\gamma'(\vec{v})]^T] \cdot \nabla_s \vec{\varphi} \, d\mathcal{H}^{d-1} - \int_{\Gamma} \gamma(\vec{v}) \nabla_s \text{id} \cdot \nabla_s \vec{\varphi} \, d\mathcal{H}^{d-1}. \quad (1.18)$$

The advantage of (1.16a), (1.18) is, that it can be used for arbitrary differentiable anisotropies, but the disadvantage is that it does not seem to be possible to derive a stability bound for discretizations using this formulation; see Remark 3.1 below.

We therefore prefer the ansatz (1.16a,b), and we will restrict ourselves to anisotropies of the form (1.12) from now on in this paper. Below we highlight a few features of discretizations which are based on the formulation (1.16a,b).

- With an argument similar to the isotropic case it is possible to derive a stability bound for a fully discrete finite element approximation of (1.5) for $d = 2$ or 3 , something that is new in the literature.
- The resulting scheme has very good mesh properties, i.e. mesh points tend to be very well distributed over the discrete surface. One reason for this is that the scheme allows for tangential movement, which is in contrast to most other parametric approaches. For the isotropic scheme in two spatial dimensions, mesh points generically tend to be equidistributed, see [6], and for the isotropic case in three spatial dimensions the good behaviour of the mesh can be partially explained with the help of discrete conformal mappings, see [4]. Also in the anisotropic case the meshes produced by our scheme tend to behave very well and only rarely do we encounter mesh distortions. In this context we stress that the parametric approximation of geometric flows typically leads to mesh distortions even for isotropic surface energies, and this is particularly true for fourth order equations. Other approaches deal with such mesh distortions by redistributing the mesh several times during the evolution either with the help of local criteria, see

[2], or by using a reparameterization which, in case that the surface is topologically equivalent to a sphere, is derived by a conformal map to the sphere, see [15]. We remark that our scheme generically does not need such remeshing during the evolution.

- For anisotropic surface diffusion and anisotropic conserved mean curvature flow the algorithm has very good properties with respect to volume conservation.
- Although the underlying analysis justifying our approach is rather involved, see Sections 2 and 3 below, the resulting discrete system is easy to solve, as it is only slightly different from that in the isotropic case.
- With the anisotropy formulation introduced in this paper, it is easy to define and implement anisotropies taking given crystal symmetries into account.

Let us illustrate the last issue in more detail. For a given anisotropy γ one defines its dual function

$$\gamma^*(\vec{q}) = \sup_{\vec{p} \in \mathbb{R}^d \setminus \{\vec{0}\}} \frac{\vec{p} \cdot \vec{q}}{\gamma(\vec{p})},$$

its Frank diagram

$$\mathcal{F} := \{\vec{p} \in \mathbb{R}^d : \gamma(\vec{p}) \leq 1\}$$

and the corresponding Wulff shape, [31],

$$\mathcal{W} := \{\vec{q} \in \mathbb{R}^d : \gamma^*(\vec{q}) \leq 1\},$$

which can be used to visualize the given anisotropy, see [20]. The Cauchy–Schwarz and Hölder inequalities imply, for anisotropies of the form (1.12), that

$$\vec{p} \cdot \gamma'(\vec{q}) \leq [\gamma(\vec{q})]^{1-r} \sum_{\ell=1}^L [\gamma_\ell(\vec{q})]^{r-1} \gamma_\ell(\vec{p}) \leq \gamma(\vec{p}) \quad \forall \vec{p}, \vec{q} \in \mathbb{R}^d \setminus \{\vec{0}\}.$$

This makes it possible to construct the Wulff shape $\mathcal{W}$ and find γ^* . Let S^{d-1} be the unit sphere in $\mathbb{R}^d$. Then, see e.g. [20],

$$\vec{z}_{\mathcal{F}}(\vec{z}) := [\gamma(\vec{z})]^{-1} \vec{z} \quad \text{and} \quad \vec{z}_{\mathcal{W}}(\vec{z}) := \gamma'(\vec{z}), \quad \vec{z} \in S^{d-1},$$

parameterize the boundaries of $\mathcal{F}$ and $\mathcal{W}$, respectively. Moreover, $\gamma^*(\vec{q}) = |\vec{z}_{\mathcal{W}}(\vec{z})|^{-1}$ if $\gamma'(\vec{z})$ and $\vec{q} \in S^{d-1}$ point in the same direction. This defines γ^* for $\vec{q} \in S^{d-1}$, and via the 1-homogeneity on all of $\mathbb{R}^d \setminus \{\vec{0}\}$. In fact one can show that

$$\gamma^*(\vec{q}) = \sqrt{\vec{q} \cdot G_q^* \vec{q}} \quad \text{with} \quad G_q^* = \left(\sum_{\ell=1}^L [\mu_\ell(\vec{q})]^{r-2} G_\ell \right)^{-1}, \quad (1.19)$$

where $\sum_{\ell=1}^L [\mu_\ell(\vec{q})]^r = 1$ and $\mu_\ell(\vec{q}) \in (0, 1]$, $\ell = 1 \rightarrow L$. In fact, $\mu_\ell(\vec{q}) = \frac{\gamma_\ell(\vec{z})}{\gamma(\vec{z})}$, where $\frac{\vec{z} \cdot \vec{q}}{\gamma(\vec{z})} = \sup_{\vec{p} \in \mathbb{R}^d \setminus \{\vec{0}\}} \frac{\vec{p} \cdot \vec{q}}{\gamma(\vec{p})}$.

In Figure 1.1 we give the Frank diagrams and Wulff shapes in $\mathbb{R}^3$, i.e. $d = 3$, for anisotropies of the form (1.12) with $r = 1$; in particular for

$$G_\ell := R(\theta_{\ell,12}, \theta_{\ell,13})^T D(\varepsilon_\ell) R(\theta_{\ell,12}, \theta_{\ell,13}), \quad \text{where } D(\varepsilon) := \text{diag}(1, \varepsilon^2, \varepsilon^2) \quad (1.20)$$

Fig. 1.1 Frank diagrams and Wulff shapes for different choices of (1.12) with $r = 1$ and $\varepsilon_\ell \equiv 10^{-1}$.

and $R(\theta_{12}, \theta_{13}) := R_{12}(\theta_{12})R_{13}(\theta_{13})$. Here $\text{diag}(a, b, c)$ denotes a diagonal matrix with diagonal entries a, b, c and $R_{12}(\theta) := \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $R_{13}(\theta) := \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix}$ are rotation matrices through the given angle θ . For the anisotropies in Figure 1.1 we used $\varepsilon_\ell \equiv 10^{-1}$ with $\{(\theta_{1,12}, \dots, \theta_{L,12}), (\theta_{1,13}, \dots, \theta_{L,13})\} = \{(0, \frac{\pi}{2}, 0), (0, 0, \frac{\pi}{2})\}$, $\{(0, 0, 0, \frac{\pi}{2}), (0, \frac{\pi}{3}, \frac{2\pi}{3}, 0)\}$ and $\{(0, \frac{\pi}{3}, \frac{2\pi}{3}, 0, \frac{\pi}{3}, \frac{2\pi}{3}), (0, 0, 0, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})\}$, respectively. The same anisotropies with $\varepsilon_\ell \equiv 10^{-2}$ can be seen in Figure 1.2, where we note that the Wulff shapes develop facets and sharp edges; an effect which becomes more pronounced for even smaller ε_ℓ , and e.g. in the first example leads to a smooth (regularized) approximation of the l^1 -norm's Wulff shape, i.e. a cube. In addition, we consider an example of (1.12) with $r = 1$, where $L = 2$ and $G_1 = \text{diag}(1, 1, \varepsilon^2)$, $G_2 = \text{diag}(\varepsilon^2, \varepsilon^2, 1)$, so that it approximates for small ε the anisotropy

$$\gamma(\vec{p}) = |p_3| + \sqrt{p_1^2 + p_2^2}; \quad (1.21)$$

as considered in e.g. [19]. See Figure 1.3, where we show its Frank diagram and Wulff shape for $\varepsilon = 10^{-k}$, $k = 1 \rightarrow 2$. For the case of curves, $d = 2$, the anisotropies (1.12) with the choice $r = 1$ adequately approximate most Wulff shapes. However, the choice $r = 1$ leads to a restricted class of Wulff shapes if $d \geq 3$; but by choosing $r > 1$ in (1.12) one can model a whole new class of anisotropies for $d \geq 3$. We now demonstrate this in the case $d = 3$. Using the same definition for the anisotropic functions γ_ℓ as in the first example in Figure 1.2, but now choosing $r = 9$ and $r = 30$,

Fig. 1.2 Frank diagrams and Wulff shapes for different choices of (1.12) with $r = 1$ and $\varepsilon_\ell \equiv 10^{-2}$.

Fig. 1.3 Frank diagrams and Wulff shapes for (1.12) with $L = 2$, $r = 1$ and $\varepsilon = 10^{-k}$, $k = 1 \rightarrow 2$.

leads to the Frank diagrams and Wulff shapes shown in Figure 1.4. Similarly, using the anisotropic functions γ_ℓ with $\varepsilon = 10^{-2}$ as in Figure 1.3 and setting $r = 9$ and $r = 30$ gives the results in Figure 1.5.

We remark that (1.12) with $r = \infty$, i.e.

$$\gamma(\vec{p}) = \max_{\ell=1 \rightarrow L} \gamma_\ell(\vec{p})$$

Fig. 1.4 Frank diagrams and Wulff shapes for different choices of (1.12) with $\varepsilon_\ell \equiv 10^{-2}$ and $r = 9, 30$.

Fig. 1.5 Frank diagrams and Wulff shapes for different choices of (1.12) with $\varepsilon = 10^{-2}$ and $r = 9, 30$.

with positive semidefinite G_ℓ can be used to model very general crystalline Wulff shapes, see [17]. Similarly to those authors, one can choose $\gamma_\ell(\vec{p}) = |\vec{p} \cdot \vec{\eta}_\ell|$ where $\pm \vec{\eta}_\ell$, $\ell = 1 \rightarrow L$, are the vertices of the Wulff shape. This choice leads to matrices G_ℓ , which are positive semidefinite rather than positive definite. If one now perturbs G_ℓ slightly such that the resulting matrix is positive definite, and then replaces the l^∞ -norm by an l^r -norm with r large, one obtains a Wulff shape which approximates a polyhedral Wulff shape with vertices $\pm \vec{\eta}_1, \dots, \pm \vec{\eta}_L$. It was this viewpoint, that lead us to the construction of the anisotropy illustrated in Figure 1.4.

This paper is organised as follows. In Section 2 we introduce our novel variational formulation for anisotropic versions of mean curvature. In Section 3 we formulate a finite element approximation of anisotropic mean curvature flow and anisotropic surface diffusion, as well as related anisotropic evolution laws, and derive stability bounds. We discuss how the discrete equations arising at each time level can be solved

in Section 4, and conclude with the presentation of numerous numerical computations in Section 5.

2 Variational Formulation

In this section we derive a variational formulation for the anisotropic mean curvature vector $\kappa_\gamma \vec{v}$, making use of ideas from Riemannian geometry. We refer also to the work [9] by Bellettini and Paolini, who derived a related variational approach in the context of Finsler geometry. Their approach is not applicable to our situation as no simple variational characterization of the anisotropic mean curvature vector can be derived. For the ensuing analysis it is convenient to introduce the following lemma which states how the surface area element of an inner product of the form (1.10) leads to an anisotropic energy density.

Lemma 2.1 *Let $\vec{p} \in \mathbb{R}^d$ with $|\vec{p}| = 1$ and let $\{\vec{p}, \vec{\tau}_1, \dots, \vec{\tau}_{d-1}\}$ be an orthonormal basis of $\mathbb{R}^d$. Then it holds that*

$$\gamma_\ell(\vec{p}) = \sqrt{\vec{p} \cdot G_\ell \vec{p}} = \sqrt{\det(\vec{\tau}_i \cdot \tilde{G}_\ell \vec{\tau}_j)_{i,j=1}^{d-1}}, \quad (2.1)$$

where $\tilde{G}_\ell := [\det G_\ell]^{\frac{1}{d-1}} G_\ell^{-1}$, $\ell = 1 \rightarrow L$, are symmetric positive definite matrices.

Proof For a fixed $\ell \in \{1, \dots, L\}$, let $G = G_\ell$ and set $\tilde{G} := \lambda^2 G^{-1}$, where $\lambda \in \mathbb{R}_{>0}$ is to be determined. As G is symmetric positive definite, so is $\tilde{G}$ and we can define $\tilde{\Lambda} := \tilde{G}^{\frac{1}{2}}$. It holds that

$$|\tilde{\Lambda} \vec{p} \wedge \tilde{\Lambda} \vec{\tau}_1 \wedge \dots \wedge \tilde{\Lambda} \vec{\tau}_{d-1}|^2 = (\det \tilde{\Lambda})^2 |\vec{p} \wedge \vec{\tau}_1 \wedge \dots \wedge \vec{\tau}_{d-1}|^2 = (\det \tilde{\Lambda})^2, \quad (2.2)$$

where $\wedge$ is the standard wedge product on $\mathbb{R}^d$. On noting that $(\tilde{\Lambda}^{-1} \vec{p}) \cdot \tilde{\Lambda} \vec{\tau}_i = 0$, $i = 1 \rightarrow d-1$, we have that

$$|\tilde{\Lambda} \vec{p} \wedge \tilde{\Lambda} \vec{\tau}_1 \wedge \dots \wedge \tilde{\Lambda} \vec{\tau}_{d-1}|^2 = |\tilde{\Lambda} \vec{\tau}_1 \wedge \dots \wedge \tilde{\Lambda} \vec{\tau}_{d-1}|^2 \frac{[(\tilde{\Lambda}^{-1} \vec{p}) \cdot \tilde{\Lambda} \vec{p}]^2}{|\tilde{\Lambda}^{-1} \vec{p}|^2}. \quad (2.3)$$

Combining (2.2) and (2.3) yields that $|\tilde{\Lambda} \vec{\tau}_1 \wedge \dots \wedge \tilde{\Lambda} \vec{\tau}_{d-1}|^2 = (\det \tilde{\Lambda})^2 |\tilde{\Lambda}^{-1} \vec{p}|^2$ and hence that

$$\det(\vec{\tau}_i \cdot \tilde{G} \vec{\tau}_j)_{i,j=1}^{d-1} = (\det \tilde{G}) \vec{p} \cdot \tilde{G}^{-1} \vec{p} = \lambda^{2d-2} (\det G)^{-1} \vec{p} \cdot G \vec{p}. \quad (2.4)$$

It immediately follows from (2.4) that (2.1) holds for $\lambda^2 = [\det G]^{\frac{1}{d-1}}$. $\square$

We note that (2.1) is independent of the choice of the orthonormal basis for $\mathcal{T}_{\vec{p}} := \text{span}\{\vec{\tau}_1, \dots, \vec{\tau}_{d-1}\}$. Hence it can be used as an alternative definition for the anisotropic surface energy γ_ℓ . For ease of exposition we now assume that $L = 1$ and write $\tilde{G} := \tilde{G}_1$. It will become clear later on, that the analysis which follows has an interpretation for arbitrary γ of the form (1.12).

We will adopt the following notation in this section. For a compact orientable $(d-1)$ -dimensional C^1 -hypersurface $\Gamma \subset \mathbb{R}^d$ without boundary let $\vec{x} : \mathcal{D} \rightarrow \mathbb{R}^d$,

$\mathcal{D} \subset \mathbb{R}^{d-1}$, be a (local) parameterization of Γ . The Jacobian of $\vec{x}$ is given by $D\vec{x} = (\partial_1 \vec{x}, \dots, \partial_{d-1} \vec{x}) \in \mathbb{R}^{d \times (d-1)}$. Let $\vec{v} : \Gamma \rightarrow \mathbb{R}^d$ be a continuous unit normal vector field, so that for a point $\vec{z} \in \Gamma$ the tangent space to Γ is given by $T_{\vec{z}}\Gamma = \mathcal{T}_{\vec{v}(\vec{z})}$. In addition, for any number $n \in \mathbb{N}$, let $Id_n \in \mathbb{R}^{n \times n}$ be the identity matrix, and similarly for $\tilde{Id}_n \in (\mathbb{R}^{d \times d})^{n \times n}$.

We now introduce an anisotropic surface gradient and an anisotropic surface divergence. Throughout this section let $\{\vec{t}_1, \dots, \vec{t}_{d-1}\}$ be an orthonormal basis of $T_{\vec{z}}\Gamma$ with respect to the inner product $(\cdot, \cdot)_{\tilde{G}}$ on $\mathbb{R}^d$ induced by $\tilde{G}$, recall (1.10). Then for a smooth scalar field $g : \Gamma \rightarrow \mathbb{R}$ we define the anisotropic surface gradient

$$(\nabla_s^{\tilde{G}} g)(\vec{z}) := \sum_{i=1}^{d-1} (\partial_{\vec{t}_i} g)(\vec{z}) \vec{t}_i. \quad (2.5)$$

In particular, when $\tilde{G} = Id_d$, then (2.5) reduces to the isotropic surface gradient $(\nabla_s g)(\vec{z})$. Similarly, for a smooth vector field $\vec{g} : \Gamma \rightarrow \mathbb{R}^d$ we define the anisotropic surface divergence

$$(\nabla_s^{\tilde{G}} \cdot \vec{g})(\vec{z}) := \sum_{i=1}^{d-1} (\partial_{\vec{t}_i} \vec{g})(\vec{z}) \cdot \tilde{G} \vec{t}_i. \quad (2.6)$$

Moreover, the anisotropic surface gradient of $\vec{g}$ is the tensor $\nabla_s^{\tilde{G}} \vec{g} : \Gamma \rightarrow L(T_{(\cdot)}\Gamma, \mathbb{R}^d)$, where for each $\vec{z} \in \Gamma$ the linear map $(\nabla_s^{\tilde{G}} \vec{g})(\vec{z}) : T_{\vec{z}}\Gamma \rightarrow \mathbb{R}^d$ is defined by

$$(\nabla_s^{\tilde{G}} \vec{g})(\vec{z}) := \sum_{i=1}^{d-1} (\partial_{\vec{t}_i} \vec{g})(\vec{z}) \otimes \tilde{G} \vec{t}_i, \quad (2.7)$$

where $\otimes$ is the standard tensor product of two vectors in $\mathbb{R}^d$. Hence we have that $[(\nabla_s^{\tilde{G}} \vec{g})(\vec{z})] \vec{t}_i = \partial_{\vec{t}_i} \vec{g}$. We define also the inner product

$$(\nabla_s^{\tilde{G}} \vec{u}, \nabla_s^{\tilde{G}} \vec{v})_{\tilde{G}} := \sum_{i=1}^{d-1} (\partial_{\vec{t}_i} \vec{u}, \partial_{\vec{t}_i} \vec{v})_{\tilde{G}} \quad (2.8)$$

for smooth $\vec{u}, \vec{v} : \Gamma \rightarrow \mathbb{R}^d$. In addition, given a tensor $U : \Gamma \rightarrow L(T_{(\cdot)}\Gamma, \mathbb{R}^d)$ we define its adjoint map $U^* : \Gamma \rightarrow L(\mathbb{R}^d, T_{(\cdot)}\Gamma)$, where for each $\vec{z} \in \Gamma$ the linear map $U^*(\vec{z}) : \mathbb{R}^d \rightarrow T_{\vec{z}}\Gamma$ is defined by

$$([U(\vec{z})] \vec{t}) \cdot \tilde{G} \vec{v} = ([U^*(\vec{z})] \vec{v}) \cdot \tilde{G} \vec{t} \quad \forall \vec{v} \in \mathbb{R}^d, \vec{t} \in T_{\vec{z}}\Gamma. \quad (2.9)$$

Hence $U^* = \tilde{G}^{-1} U^T \tilde{G}$, with U^T being the standard transpose. Then the anisotropic surface divergence of U is given by $\nabla_s^{\tilde{G}} \cdot U : \Gamma \rightarrow \mathbb{R}^d$, where $(\nabla_s^{\tilde{G}} \cdot U)(\vec{z})$ is such that

$$(\nabla_s^{\tilde{G}} \cdot U)(\vec{z}) \cdot \tilde{G} \vec{v} = \nabla_s^{\tilde{G}} \cdot ([U^*(\vec{z})] \vec{v}) \quad \forall \vec{v} \in \mathbb{R}^d, \quad (2.10)$$

with the right hand side operator defined as in (2.6). Moreover, for $g : \Gamma \rightarrow \mathbb{R}$ we define the anisotropic Laplace–Beltrami operator $\Delta_s^{\tilde{G}}$ by $\Delta_s^{\tilde{G}} g := \nabla_s^{\tilde{G}} \cdot (\nabla_s^{\tilde{G}} g)$ and similarly $\tilde{\Delta}_s^{\tilde{G}} \vec{g} := \nabla_s^{\tilde{G}} \cdot (\nabla_s^{\tilde{G}} \vec{g})$ for $\vec{g} : \Gamma \rightarrow \mathbb{R}^d$.

Finally, we set

$$H := [D\vec{x}]^T \tilde{G} D\vec{x} = (h_{ij})_{i,j=1}^{d-1} = (\partial_i \vec{x} \cdot \tilde{G} \partial_j \vec{x})_{i,j=1}^{d-1}, \quad H^{-1} = (h^{ij})_{i,j=1}^{d-1}. \quad (2.11)$$

Here we have noted, on recalling that $\text{rank } D\vec{x} = d - 1$ due to the assumptions on Γ , that H is symmetric and positive definite.

Remark 2.1 We note that the definitions (2.5) – (2.10) are independent of the choice of the parameterization $\vec{x}$ and the orthonormal basis $\{\vec{t}_i\}_{i=1}^{d-1}$ of $T_{\vec{x}}\Gamma$, see e.g. Lemma 2.4 below. In particular, the definitions as well as the subsequent results derived in this section only depend on Γ and the values of the functions $g : \Gamma \rightarrow \mathbb{R}$ and $\vec{g} : \Gamma \rightarrow \mathbb{R}^d$ on it.

The following lemma shows that the integration on Γ with respect to the metric given by $\tilde{G}$ corresponds to integrating $\gamma(\vec{v})$ on Γ with respect to the standard $(d - 1)$ -dimensional surface measure.

Lemma 2.2 *Let $\mathcal{M} := \vec{x}(\mathcal{D}) \subset \Gamma$. Then it holds for any $g \in L^1(\mathcal{M}, \mathbb{R})$ that*

$$\int_{\mathcal{M}} g \gamma(\vec{v}) \, d\mathcal{H}^{d-1} = \int_{\mathcal{D}} g \circ \vec{x} \sqrt{\det H} \, d\mathcal{L}^{d-1}, \quad (2.12)$$

where $\mathcal{L}^{d-1}$ is the Lebesgue measure in $\mathbb{R}^{d-1}$.

Proof Let $\{\vec{t}_1, \dots, \vec{t}_{d-1}\}$ be an orthonormal basis for $\mathcal{T}_{\vec{v}}$. As $\text{rank } D\vec{x} = d - 1$, one has that $\vec{t}_k = \sum_{i=1}^{d-1} a_{ki} \partial_i \vec{x}$, $k = 1 \rightarrow d - 1$, for some $A := (a_{ki})_{k,i=1}^{d-1}$. It follows that $Q = D\vec{x} A^T$, where $Q := [\vec{t}_1 \dots \vec{t}_{d-1}] \in \mathbb{R}^{d \times d-1}$. Therefore we have that $Id_{d-1} = Q^T Q = A [D\vec{x}]^T D\vec{x} A^T$, $A^T A = ([D\vec{x}]^T D\vec{x})^{-1}$; and hence that $\det(Q^T \tilde{G} Q) = [\det([D\vec{x}]^T D\vec{x})]^{-1} \det([D\vec{x}]^T \tilde{G} D\vec{x})$, i.e.

$$[\det(\vec{t}_i \cdot \tilde{G} \vec{t}_j)_{i,j=1}^{d-1} \circ \vec{x}] \det([D\vec{x}]^T D\vec{x}) = \det([D\vec{x}]^T \tilde{G} D\vec{x}). \quad (2.13)$$

It follows from (2.1) and (2.13) that

$$\begin{aligned} \int_{\mathcal{M}} g \gamma(\vec{v}) \, d\mathcal{H}^{d-1} &= \int_{\mathcal{M}} g \sqrt{\det(\vec{t}_i \cdot \tilde{G} \vec{t}_j)_{i,j=1}^{d-1}} \, d\mathcal{H}^{d-1} \\ &= \int_{\mathcal{D}} g \circ \vec{x} \sqrt{\det(\vec{t}_i \cdot \tilde{G} \vec{t}_j)_{i,j=1}^{d-1} \circ \vec{x}} \sqrt{\det(\partial_i \vec{x} \cdot \partial_j \vec{x})_{i,j=1}^{d-1}} \, d\mathcal{L}^{d-1} \\ &= \int_{\mathcal{D}} g \circ \vec{x} \sqrt{\det(\partial_i \vec{x} \cdot \tilde{G} \partial_j \vec{x})_{i,j=1}^{d-1}} \, d\mathcal{L}^{d-1} = \int_{\mathcal{D}} g \circ \vec{x} \sqrt{\det H} \, d\mathcal{L}^{d-1}. \end{aligned} \quad (2.14)$$

This completes the proof. $\square$

In the following lemma we prove identities needed for the analysis of the anisotropic differential operators.

Lemma 2.3 *Let $\vec{t}_i = \sum_{j=1}^{d-1} b_{ij} \partial_j \vec{x}$, $i = 1 \rightarrow d - 1$, be orthonormal with respect to the inner product given by $\tilde{G}$, and $B := (b_{ij})_{i,j=1}^{d-1}$. Then it holds that*

$$H^{-1} = B^T B, \quad (2.15)$$

$$\text{and } \partial_k(\sqrt{\det H}) = \sqrt{\det H} \sum_{i,j=1}^{d-1} h^{ij} \partial_j \vec{x} \cdot \tilde{G} \partial_{ik}^2 \vec{x}, \quad k = 1 \rightarrow d - 1. \quad (2.16)$$

Proof Similarly to the proof of Lemma 2.2, as $\tilde{t}_i \cdot \tilde{G} \tilde{t}_j = \delta_{ij}$, $i, j = 1 \rightarrow d-1$, it holds that $Id_{d-1} = BH B^T$, and hence (2.15). For any $\vec{y} \in \mathcal{D}$ we have that $[\partial_k(\det H)](\vec{y}) = \lim_{\delta \rightarrow 0} \frac{1}{\delta} [\det H(\vec{y} + \delta \vec{e}_k) - \det H(\vec{y})]$. On noting that $\partial_i \vec{x}(\vec{y} + \delta \vec{e}_k) = \partial_i \vec{x}(\vec{y}) + \delta \partial_{ik}^2 \vec{x}(\vec{y}) + \mathcal{O}(\delta^2)$, we have that

$$h_{ij}(\vec{y} + \delta \vec{e}_k) = h_{ij} + \delta (\partial_i \vec{x} \cdot \tilde{G} \partial_{jk}^2 \vec{x} + \partial_j \vec{x} \cdot \tilde{G} \partial_{ik}^2 \vec{x}) + \mathcal{O}(\delta^2), \quad (2.17)$$

where, here and from now on, all the terms on the right hand side are evaluated at $\vec{y}$. It follows from (2.17) and the well known trace approximation for the determinant, $\det(Id_{d-1} + \delta A) = 1 + \delta \operatorname{tr} A + \mathcal{O}(\delta^2)$ for $A \in \mathbb{R}^{(d-1) \times (d-1)}$, that

$$\begin{aligned} \det H(\vec{y} + \delta \vec{e}_k) &= \det H \det (Id_{d-1} + \delta H^{-1} [F + F^T] + \mathcal{O}(\delta^2)) \\ &= \det H [1 + 2\delta \operatorname{tr} (H^{-1} F)] + \mathcal{O}(\delta^2), \end{aligned} \quad (2.18)$$

where $F := (\partial_i \vec{x} \cdot \tilde{G} \partial_{jk}^2 \vec{x})_{i,j=1}^{d-1}$. It follows from (2.18), on noting (2.11), that

$$\frac{1}{\delta} [\det H(\vec{y} + \delta \vec{e}_k) - \det H(\vec{y})] = 2 \det H \sum_{i,j=1}^{d-1} h^{ij} \partial_j \vec{x} \cdot \tilde{G} \partial_{ik}^2 \vec{x} + \mathcal{O}(\delta).$$

Letting $\delta \rightarrow 0$ yields the desired result (2.16). $\square$

In the following lemma we express the relevant differential operators in local coordinates.

Lemma 2.4 *It holds that*

$$(\nabla_s^{\tilde{G}} g) \circ \vec{x} = \sum_{i,j=1}^{d-1} h^{ij} \partial_i (g \circ \vec{x}) \partial_j \vec{x}, \quad (2.19a)$$

$$(\nabla_s^{\tilde{G}} \cdot \vec{g}) \circ \vec{x} = \sum_{i,j=1}^{d-1} h^{ij} \partial_j \vec{x} \cdot \tilde{G} \partial_i [\vec{g} \circ \vec{x}], \quad (2.19b)$$

$$(\nabla_s^{\tilde{G}} \vec{g}) \circ \vec{x} = \sum_{i,j=1}^{d-1} h^{ij} \partial_i [\vec{g} \circ \vec{x}] \otimes \tilde{G} \partial_j \vec{x}, \quad (2.19c)$$

where $g : \Gamma \rightarrow \mathbb{R}$ and $\vec{g} : \Gamma \rightarrow \mathbb{R}^d$. Moreover, for $\vec{u}, \vec{v} : \Gamma \rightarrow \mathbb{R}^d$ we have that

$$(\nabla_s^{\tilde{G}} \vec{u}, \nabla_s^{\tilde{G}} \vec{v})_{\tilde{G}} = \sum_{i,j=1}^{d-1} h^{ij} \partial_j [\vec{u} \circ \vec{x}] \cdot \tilde{G} \partial_i [\vec{v} \circ \vec{x}]. \quad (2.20)$$

Finally, we have that

$$(\Delta_s^{\tilde{G}} g) \circ \vec{x} = [\nabla_s^{\tilde{G}} \cdot (\nabla_s^{\tilde{G}} g)] \circ \vec{x} = [\det H]^{-\frac{1}{2}} \sum_{i,j=1}^{d-1} \partial_i [\sqrt{\det H} h^{ij} \partial_j (g \circ \vec{x})] \quad (2.21a)$$

and, similarly,

$$(\Delta_s^{\tilde{G}} \vec{g}) \circ \vec{x} = [\nabla_s^{\tilde{G}} \cdot (\nabla_s^{\tilde{G}} \vec{g})] \circ \vec{x} = [\det H]^{-\frac{1}{2}} \sum_{i,j=1}^{d-1} \partial_i [\sqrt{\det H} h^{ij} \partial_j (\vec{g} \circ \vec{x})]. \quad (2.21b)$$

Proof Recalling the notation in Lemma 2.3, we have that

$$\sum_{i,j=1}^{d-1} h^{ij} \partial_i (g \circ \vec{x}) \partial_j \vec{x} = \sum_{i,j=1}^{d-1} \left(\sum_{k=1}^{d-1} b_{ki} b_{kj} \right) \partial_i (g \circ \vec{x}) \partial_j \vec{x} = \sum_{k=1}^{d-1} (\partial_{\vec{t}_k} g) \vec{t}_k = \nabla_s^{\tilde{G}} g,$$

and similarly

$$\sum_{i,j=1}^{d-1} h^{ij} \partial_j \vec{x} \cdot \tilde{G} \partial_i [\vec{g} \circ \vec{x}] = \sum_{k=1}^{d-1} (\partial_{\vec{t}_k} \vec{g}) \cdot \tilde{G} \vec{t}_k = \nabla_s^{\tilde{G}} \cdot \vec{g}.$$

This proves (2.19a,b); and (2.19c), as well as (2.20), can be shown analogously. Moreover, it follows from (2.19a,b) that

$$\begin{aligned} [\nabla_s^{\tilde{G}} \cdot (\nabla_s^{\tilde{G}} g)] \circ \vec{x} &= \sum_{i,j=1}^{d-1} h^{ij} \partial_j \vec{x} \cdot \tilde{G} \partial_i \left[\sum_{k,l=1}^{d-1} h^{kl} \partial_k (g \circ \vec{x}) \right] \partial_l \vec{x} \\ &\quad + \sum_{i,j=1}^{d-1} h^{ij} \partial_j \vec{x} \cdot \tilde{G} \sum_{k,l=1}^{d-1} h^{kl} \partial_k (g \circ \vec{x}) \partial_{il}^2 \vec{x} =: S_1 + S_2. \end{aligned} \quad (2.22)$$

On noting (2.11), we have that

$$S_1 = \sum_{i,j=1}^{d-1} h^{ij} h_{jl} \partial_i \left[\sum_{k,l=1}^{d-1} h^{kl} \partial_k (g \circ \vec{x}) \right] = \sum_{k,l=1}^{d-1} \partial_l [h^{kl} \partial_k (g \circ \vec{x})]. \quad (2.23)$$

In addition, it follows from (2.16) that

$$S_2 = \sum_{k,l=1}^{d-1} \left[h^{kl} \partial_k (g \circ \vec{x}) \sum_{i,j=1}^{d-1} h^{ij} \partial_j \vec{x} \cdot \tilde{G} \partial_{il}^2 \vec{x} \right] = [\det H]^{-\frac{1}{2}} \sum_{k,l=1}^{d-1} h^{kl} \partial_k (g \circ \vec{x}) \partial_l (\sqrt{\det H}). \quad (2.24)$$

Combining (2.22), (2.23) and (2.24) yields the desired result (2.21a).

Similarly, it follows from (2.10), (2.19b,c) and (2.9) that for all $\vec{v} \in \mathbb{R}^d$

$$\begin{aligned} [\nabla_s^{\tilde{G}} \cdot (\nabla_s^{\tilde{G}} \vec{g})] \circ \vec{x} \cdot \tilde{G} \vec{v} &= \nabla_s^{\tilde{G}} \cdot \left([(\nabla_s^{\tilde{G}} \vec{g})^* \circ \vec{x}] \vec{v} \right) = \sum_{i,j=1}^{d-1} h^{ij} \partial_j \vec{x} \cdot \tilde{G} \partial_i \left[[(\nabla_s^{\tilde{G}} \vec{g})^* \circ \vec{x}] \vec{v} \right] \\ &= \sum_{i,j=1}^{d-1} h^{ij} \partial_j \vec{x} \cdot \tilde{G} \partial_i \left[\sum_{k,l=1}^{d-1} h^{kl} ([\partial_k (\vec{g} \circ \vec{x})] \cdot \tilde{G} \vec{v}) \partial_l \vec{x} \right] \\ &= \sum_{i,j=1}^{d-1} h^{ij} \partial_j \vec{x} \cdot \tilde{G} \left(\partial_i \left[\sum_{k,l=1}^{d-1} h^{kl} [\partial_k (\vec{g} \circ \vec{x})] \right] \cdot \tilde{G} \vec{v} \right) \partial_l \vec{x} \\ &\quad + \sum_{i,j=1}^{d-1} h^{ij} \partial_j \vec{x} \cdot \tilde{G} \left(\sum_{k,l=1}^{d-1} h^{kl} [\partial_k (\vec{g} \circ \vec{x})] \cdot \tilde{G} \vec{v} \right) \partial_{il}^2 \vec{x} =: \tilde{S}_1 + \tilde{S}_2. \end{aligned} \quad (2.25)$$

Similarly to (2.23) and (2.24), we have that

$$\tilde{S}_1 = \sum_{k,l=1}^{d-1} \partial_l [h^{kl} \partial_k (\vec{g} \circ \vec{x})] \cdot \tilde{G} \vec{v} \quad (2.26)$$

and

$$\tilde{S}_2 = [\det H]^{-\frac{1}{2}} \sum_{k,l=1}^{d-1} h^{kl} \partial_k (\vec{g} \circ \vec{x}) \cdot \tilde{G} \vec{v} \partial_l (\sqrt{\det H}). \quad (2.27)$$

Combining (2.25), (2.26) and (2.27) yields the desired result (2.21b). $\square$

In contrast to Lemma 1.1, we now compute the first variation of the anisotropic surface energy (1.1) in a general direction.

Lemma 2.5 *Let*

$$\Gamma(\delta) := \{\vec{z} + \delta \vec{g}(\vec{z}) : \vec{z} \in \Gamma\}, \quad (2.28)$$

where $\vec{g} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a smooth vector field. Then it follows that

$$\frac{d}{d\delta} |\Gamma(\delta)|_\gamma|_{\delta=0} = \int_\Gamma (\nabla_s^{\tilde{G}} \cdot \vec{g}) \gamma(\vec{v}) \, d\mathcal{H}^{d-1}. \quad (2.29)$$

Proof We define

$$c_{ij}(\delta) := (\partial_i [\vec{x} + \delta \vec{g} \circ \vec{x}] \cdot \tilde{G} \partial_j [\vec{x} + \delta \vec{g} \circ \vec{x}])$$

and let $C(\delta) = (c_{ij}(\delta))_{i,j=1}^{d-1}$. Assuming w.l.o.g. that $\vec{g}$ has compact support in $\vec{x}(\mathcal{D}) \subset \Gamma$, on employing a partition of unity, it follows, similarly to (2.14), that

$$\frac{d}{d\delta} |\Gamma(\delta)|_\gamma = \frac{d}{d\delta} \int_{\mathcal{D}} \sqrt{\det C(\delta)} \, d\mathcal{L}^{d-1}. \quad (2.30)$$

Now, similarly to (2.17), we have that

$$c_{ij}(\delta) = \partial_i \vec{x} \cdot \tilde{G} \partial_j \vec{x} + \delta \left(\partial_i \vec{x} \cdot \tilde{G} \partial_j [\vec{g} \circ \vec{x}] + \partial_i [\vec{g} \circ \vec{x}] \cdot \tilde{G} \partial_j \vec{x} \right) + \mathcal{O}(\delta^2). \quad (2.31)$$

Similarly to (2.18), it follows from (2.31) that

$$\begin{aligned} \det C(\delta) &= \det H \left(1 + 2\delta \operatorname{tr} [H^{-1} (\partial_j \vec{x} \cdot \tilde{G} \partial_i [\vec{g} \circ \vec{x}])_{i,j=1}^{d-1}] + \mathcal{O}(\delta^2) \right) \\ &= \det H \left(1 + 2\delta \sum_{i,j=1}^{d-1} h^{ij} \partial_j \vec{x} \cdot \tilde{G} \partial_i [\vec{g} \circ \vec{x}] + \mathcal{O}(\delta^2) \right). \end{aligned}$$

Therefore we have that

$$\sqrt{\det C(\delta)} = \sqrt{\det H} \left(1 + \delta \sum_{i,j=1}^{d-1} h^{ij} \partial_j \vec{x} \cdot \tilde{G} \partial_i [\vec{g} \circ \vec{x}] + \mathcal{O}(\delta^2) \right).$$

This implies, on noting (2.30), (2.19b) and (2.12) that

$$\begin{aligned} \frac{d}{d\delta} |\Gamma(\delta)|_\gamma|_{\delta=0} &= \int_{\mathcal{D}} \sqrt{\det H} \sum_{i,j=1}^{d-1} h^{ij} \partial_j \vec{x} \cdot \tilde{G} \partial_i [\vec{g} \circ \vec{x}] \, d\mathcal{L}^{d-1} \\ &= \int_{\mathcal{D}} \sqrt{\det H} (\nabla_s^{\tilde{G}} \cdot \vec{g}) \circ \vec{x} \, d\mathcal{L}^{d-1} = \int_\Gamma \gamma(\vec{v}) \nabla_s^{\tilde{G}} \cdot \vec{g} \, d\mathcal{H}^{d-1}; \end{aligned}$$

and hence the desired result (2.29). $\square$

We now introduce an anisotropic mean curvature vector using the $\tilde{G}$ inner product.

Lemma 2.6 *For $\vec{g} : \Gamma \rightarrow \mathbb{R}^d$ it holds that*

$$\int_{\Gamma} (\nabla_s^{\tilde{G}} \cdot \vec{g}) \gamma(\vec{v}) \, d\mathcal{H}^{d-1} = - \int_{\Gamma} (\vec{g} \cdot \tilde{G} \vec{\mathcal{K}}^{\tilde{G}}) \gamma(\vec{v}) \, d\mathcal{H}^{d-1}, \quad (2.32)$$

where the anisotropic mean curvature vector $\vec{\mathcal{K}}^{\tilde{G}}$ is given by

$$\vec{\mathcal{K}}^{\tilde{G}} = \Delta_s^{\tilde{G}} \vec{\text{id}} = \nabla_s^{\tilde{G}} \cdot (\nabla_s^{\tilde{G}} \vec{\text{id}}). \quad (2.33)$$

Proof Assuming w.l.o.g. that $\vec{g}$ has compact support in $\vec{x}(\mathcal{D}) \subset \Gamma$, on employing a partition of unity, it follows from (2.19b), (2.12), the symmetry of H and (2.21b), on recalling (1.10), that

$$\begin{aligned} \int_{\Gamma} (\nabla_s^{\tilde{G}} \cdot \vec{g}) \gamma(\vec{v}) \, d\mathcal{H}^{d-1} &= \int_{\mathcal{D}} \sqrt{\det H} \sum_{i,j=1}^{d-1} h^{ij} \partial_i \vec{x} \cdot \tilde{G} \partial_j [\vec{g} \circ \vec{x}] \, d\mathcal{L}^{d-1} \\ &= \int_{\mathcal{D}} \sum_{j=1}^{d-1} \left(\partial_j [\vec{g} \circ \vec{x}] \cdot \tilde{G} \sum_{i=1}^{d-1} \sqrt{\det H} h^{ij} \partial_i \vec{x} \right) \, d\mathcal{L}^{d-1} \\ &= - \int_{\mathcal{D}} (\vec{g} \circ \vec{x}) \cdot \tilde{G} \sum_{i,j=1}^{d-1} \partial_j [\sqrt{\det H} h^{ij} \partial_i \vec{x}] \, d\mathcal{L}^{d-1} \\ &= - \int_{\Gamma} (\vec{g}, \nabla_s^{\tilde{G}} \cdot (\nabla_s^{\tilde{G}} \vec{\text{id}}))_{\tilde{G}} \gamma(\vec{v}) \, d\mathcal{H}^{d-1}. \end{aligned} \quad (2.34)$$

Using the identity (2.33) then yields the desired result (2.32). $\square$

We now demonstrate that this anisotropic mean curvature vector is perpendicular to Γ , with respect to the inner product defined by $\tilde{G}$.

Lemma 2.7 *It holds that $\tilde{G} \vec{\mathcal{K}}^{\tilde{G}}$ is a multiple of $\vec{v}$, i.e. for any $\vec{z} \in \Gamma$ it holds that*

$$\vec{t} \cdot \tilde{G} \vec{\mathcal{K}}^{\tilde{G}}(\vec{z}) = 0 \quad \forall \vec{t} \in T_{\vec{z}} \Gamma. \quad (2.35)$$

Proof As $T_{\vec{z}} \Gamma$ is spanned by $\{\partial_k \vec{x}\}_{k=1}^{d-1}$, it is sufficient to show that $\vec{\mathcal{K}}^{\tilde{G}}(\vec{z}) \cdot \tilde{G} \partial_k \vec{x} = 0$, $k = 1 \rightarrow d-1$. But this follows on noting (2.33), (2.21b) and that

$$\begin{aligned} &[\det H]^{-\frac{1}{2}} \sum_{i,j=1}^{d-1} \partial_i [\sqrt{\det H} h^{ij} \partial_j \vec{x}] \cdot \tilde{G} \partial_k \vec{x} \\ &= [\det H]^{-\frac{1}{2}} \sum_{i,j=1}^{d-1} \partial_i [\sqrt{\det H} h^{ij} \partial_j \vec{x} \cdot \tilde{G} \partial_k \vec{x}] - \sum_{i,j=1}^{d-1} h^{ij} \partial_j \vec{x} \cdot \tilde{G} \partial_{ik}^2 \vec{x} \\ &= [\det H]^{-\frac{1}{2}} \partial_k (\sqrt{\det H}) - \sum_{i,j=1}^{d-1} h^{ij} \partial_j \vec{x} \cdot \tilde{G} \partial_{ik}^2 \vec{x} = 0, \end{aligned} \quad (2.36)$$

on recalling (2.11) and (2.16). $\square$

Remark 2.2 We note that our definition of $\tilde{\kappa}^{\tilde{G}}$, for the anisotropy (1.12) with $L = 1$, differs from the usual weighted mean curvature vector $\tilde{\kappa}_\gamma := \kappa_\gamma \tilde{\nu}$, recall (1.3). Combining (1.4), (2.29) with $\tilde{g} = g \tilde{\nu}$, (2.32) and (2.35) yields that

$$\tilde{\kappa}^{\tilde{G}} = [\gamma(\tilde{\nu})]^{-1} \tilde{G}^{-1} \tilde{\kappa}_\gamma, \quad \text{and hence that} \quad \gamma(\tilde{\nu}) \tilde{\nu}^T \tilde{G} \tilde{\kappa}^{\tilde{G}} = \kappa_\gamma. \quad (2.37)$$

Remark 2.3 Similarly to (1.3), we can also introduce a scalar anisotropic mean curvature. In fact, there exists a $\kappa^{\tilde{G}} : \Gamma \rightarrow \mathbb{R}$ such that $\tilde{\kappa}^{\tilde{G}} = \kappa^{\tilde{G}} \tilde{\nu}^{\tilde{G}}$, where $\tilde{\nu}^{\tilde{G}} := [\tilde{\nu}^T \tilde{G}^{-1} \tilde{\nu}]^{-\frac{1}{2}} \tilde{G}^{-1} \tilde{\nu}$; i.e. $\tilde{\nu}^{\tilde{G}}$ is orthonormal to $\mathcal{T}_{\tilde{\nu}}$ with respect to the $(\cdot, \cdot)_{\tilde{G}}$ inner product. Moreover, it follows, as in (2.34), that

$$\kappa^{\tilde{G}} = -\nabla_s^{\tilde{G}} \cdot \tilde{\nu}^{\tilde{G}},$$

which is the direct analogue of (1.3).

In the lemma which follows we prove integration by parts formulas for the gradient and divergence operators induced by $\tilde{G}$.

Lemma 2.8 *Let $u, v : \Gamma \rightarrow \mathbb{R}$. Then it holds that*

$$\int_\Gamma u \nabla_s^{\tilde{G}} \cdot (\nabla_s^{\tilde{G}} v) \gamma(\tilde{\nu}) \, d\mathcal{H}^{d-1} = - \int_\Gamma (\nabla_s^{\tilde{G}} u, \nabla_s^{\tilde{G}} v)_{\tilde{G}} \gamma(\tilde{\nu}) \, d\mathcal{H}^{d-1}; \quad (2.38a)$$

where we recall (1.10). Similarly, for functions $\tilde{u}, \tilde{v} : \Gamma \rightarrow \mathbb{R}^d$ it holds that

$$\int_\Gamma (\tilde{u}, \nabla_s^{\tilde{G}} \cdot (\nabla_s^{\tilde{G}} \tilde{v}))_{\tilde{G}} \gamma(\tilde{\nu}) \, d\mathcal{H}^{d-1} = - \int_\Gamma (\nabla_s^{\tilde{G}} \tilde{u}, \nabla_s^{\tilde{G}} \tilde{v})_{\tilde{G}} \gamma(\tilde{\nu}) \, d\mathcal{H}^{d-1}; \quad (2.38b)$$

where we recall (2.8).

Proof Applying Lemma 2.6 to $\tilde{g} := u \nabla_s^{\tilde{G}} v$ and noting (2.35) yields that

$$\int_\Gamma (\nabla_s^{\tilde{G}} \cdot \tilde{g}) \gamma(\tilde{\nu}) \, d\mathcal{H}^{d-1} = \sum_{i=1}^{d-1} \int_\Gamma \nabla_s^{\tilde{G}} \cdot [u(\partial_{\tilde{t}_i} v) \tilde{t}_i] \gamma(\tilde{\nu}) \, d\mathcal{H}^{d-1} = 0. \quad (2.39)$$

Moreover, we have from (2.6), (2.5), and on recalling $\Delta_s^{\tilde{G}} \cdot \equiv \nabla_s^{\tilde{G}} \cdot (\nabla_s^{\tilde{G}} \cdot)$, that

$$\begin{aligned} \nabla_s^{\tilde{G}} \cdot \tilde{g} &:= \sum_{i=1}^{d-1} \partial_{\tilde{t}_i} [u \nabla_s^{\tilde{G}} v] \cdot \tilde{G} \tilde{t}_i = \nabla_s^{\tilde{G}} u \cdot \tilde{G} \nabla_s^{\tilde{G}} v + \sum_{i=1}^{d-1} u \partial_{\tilde{t}_i} [\nabla_s^{\tilde{G}} v] \cdot \tilde{G} \tilde{t}_i \\ &= (\nabla_s^{\tilde{G}} u, \nabla_s^{\tilde{G}} v)_{\tilde{G}} + u \nabla_s^{\tilde{G}} \cdot (\nabla_s^{\tilde{G}} v). \end{aligned} \quad (2.40)$$

Integrating (2.40) multiplied by $\gamma(\tilde{\nu})$ over Γ yields the desired result (2.38a), on noting (2.39).

For $U = \nabla_s^{\tilde{G}} \tilde{v}$, we note from (2.10), (2.32) and (2.35) that

$$\int_\Gamma [\nabla_s^{\tilde{G}} \cdot (U^* \tilde{u})] \gamma(\tilde{\nu}) \, d\mathcal{H}^{d-1} = \sum_{i=1}^{d-1} \int_\Gamma \nabla_s^{\tilde{G}} \cdot [(\partial_{\tilde{t}_i} \tilde{v} \cdot \tilde{G} \tilde{u}) \tilde{t}_i] \gamma(\tilde{\nu}) \, d\mathcal{H}^{d-1} = 0. \quad (2.41)$$

Then, similarly to (2.40) it holds on noting (2.9), (2.10) and (2.8) that for any $\vec{z} \in \Gamma$

$$\begin{aligned}
[\nabla_s^{\tilde{G}} \cdot (U^* \vec{u})](\vec{z}) &= \left[\nabla_s^{\tilde{G}} \cdot \left(\sum_{i=1}^{d-1} (\partial_{\vec{t}_i} \vec{v} \cdot \tilde{G} \vec{u}) \vec{t}_i \right) \right](\vec{z}) \\
&= \sum_{i,j=1}^{d-1} \left[(\partial_{\vec{t}_j} \vec{v} \cdot \tilde{G} \partial_{\vec{t}_i} \vec{u}) (\vec{t}_j \cdot \tilde{G} \vec{t}_i) \right](\vec{z}) + \sum_{i,j=1}^{d-1} \partial_{\vec{t}_i} \left[(\partial_{\vec{t}_j} \vec{v} \cdot \tilde{G} \vec{u}(\vec{z})) \vec{t}_j \right](\vec{z}) \cdot \tilde{G} \vec{t}_i \\
&= (\nabla_s^{\tilde{G}} \vec{u}, \nabla_s^{\tilde{G}} \vec{v})_{\tilde{G}}(\vec{z}) + \nabla_s^{\tilde{G}} \cdot (U^* \vec{u}(\vec{z}))(\vec{z}) \\
&= (\nabla_s^{\tilde{G}} \vec{u}, \nabla_s^{\tilde{G}} \vec{v})_{\tilde{G}}(\vec{z}) + (\nabla_s^{\tilde{G}} \cdot U)(\vec{z}) \cdot \tilde{G} \vec{u}(\vec{z}) \\
&= (\nabla_s^{\tilde{G}} \vec{u}, \nabla_s^{\tilde{G}} \vec{v})_{\tilde{G}}(\vec{z}) + (\vec{u}, \nabla_s^{\tilde{G}} \cdot (\nabla_s^{\tilde{G}} \vec{v}))_{\tilde{G}}(\vec{z}). \tag{2.42}
\end{aligned}$$

We remark that in the last term of the second and third rows the term $\vec{u}(\vec{z})$ is taken to be constant when differentiating. Integrating (2.42) multiplied by $\gamma(\vec{v})$ over Γ , and noting (2.41), yields the desired result (2.38b). $\square$

Combining the results obtained so far yields the following weak characterization of the weighted mean curvature vector $\kappa_\gamma \vec{v}$, as opposed to (1.18), which, together with its generalization in Theorem 2.1 below, will lead to unconditionally stable fully discrete approximations in Section 3.

Lemma 2.9 *For smooth test functions $\vec{\xi} : \Gamma \rightarrow \mathbb{R}^d$ it holds that*

$$\int_{\Gamma} \kappa_\gamma \vec{v} \cdot \vec{\xi} \, d\mathcal{H}^{d-1} + \int_{\Gamma} (\nabla_s^{\tilde{G}} \vec{\text{id}}, \nabla_s^{\tilde{G}} \vec{\xi})_{\tilde{G}} \gamma(\vec{v}) \, d\mathcal{H}^{d-1} = 0. \tag{2.43}$$

Proof Choosing $\vec{v} = \vec{\text{id}}$ in (2.38b) yields, on noting (2.33) and (2.37), that

$$\begin{aligned}
\int_{\Gamma} (\nabla_s^{\tilde{G}} \vec{\text{id}}, \nabla_s^{\tilde{G}} \vec{\xi})_{\tilde{G}} \gamma(\vec{v}) \, d\mathcal{H}^{d-1} &= - \int_{\Gamma} (\vec{\xi}, \nabla_s^{\tilde{G}} \cdot (\nabla_s^{\tilde{G}} \vec{\text{id}}))_{\tilde{G}} \gamma(\vec{v}) \, d\mathcal{H}^{d-1} \\
&= - \int_{\Gamma} (\vec{\xi}, \vec{\kappa}^{\tilde{G}})_{\tilde{G}} \gamma(\vec{v}) \, d\mathcal{H}^{d-1} = - \int_{\Gamma} \kappa_\gamma \vec{v} \cdot \vec{\xi} \, d\mathcal{H}^{d-1}. \tag{2.44}
\end{aligned}$$

This proves the desired result (2.43). $\square$

We will now derive the analogue of (2.43) for a general anisotropy γ given by (1.12).

Theorem 2.1 *For $\vec{g}$ and $\Gamma(\delta)$ as in Lemma 2.5 it holds that*

$$\frac{d}{d\delta} |\Gamma(\delta)|_\gamma|_{\delta=0} = \sum_{\ell=1}^L \int_{\Gamma} \left[\frac{\gamma_\ell(\vec{v})}{\gamma(\vec{v})} \right]^{r-1} (\nabla_s^{\tilde{G}_\ell} \cdot \vec{g}) \gamma_\ell(\vec{v}) \, d\mathcal{H}^{d-1}, \tag{2.45}$$

and for $\ell \in \{1, \dots, L\}$ we have that

$$\int_{\Gamma} (\nabla_s^{\tilde{G}_\ell} \cdot \vec{g}) \left[\frac{\gamma_\ell(\vec{v})}{\gamma(\vec{v})} \right]^{r-1} \gamma_\ell(\vec{v}) \, d\mathcal{H}^{d-1} = - \int_{\Gamma} (\vec{g} \cdot \tilde{G}_\ell \vec{\kappa}^{\tilde{G}_\ell}) \gamma_\ell(\vec{v}) \, d\mathcal{H}^{d-1}; \tag{2.46}$$

where $\tilde{\mathcal{K}}^{\tilde{G}_\ell} := \nabla_s^{\tilde{G}_\ell} \cdot \left[\left[\frac{\gamma_\ell(\vec{v})}{\gamma(\vec{v})} \right]^{r-1} \nabla_s^{\tilde{G}_\ell} \text{id} \right]$, $\ell = 1 \rightarrow L$, are such that

$$\sum_{\ell=1}^L \gamma_\ell(\vec{v}(\vec{z})) \tilde{\mathcal{K}}^{\tilde{G}_\ell}(\vec{z}) \cdot \tilde{G}_\ell \vec{t} = 0 \quad \forall \vec{t} \in T_{\vec{z}} \Gamma \quad (2.47)$$

for any $\vec{z} \in \Gamma$. In addition, it holds that

$$\sum_{\ell=1}^L \gamma_\ell(\vec{v}) \tilde{G}_\ell \tilde{\mathcal{K}}^{\tilde{G}_\ell} = \mathcal{K}_\gamma \vec{v}. \quad (2.48)$$

Moreover, for functions $\vec{u}, \vec{v} : \Gamma \rightarrow \mathbb{R}^d$ it holds that

$$\begin{aligned} \sum_{\ell=1}^L \int_\Gamma \left(\vec{u}, \nabla_s^{\tilde{G}_\ell} \cdot \left[\left[\frac{\gamma_\ell(\vec{v})}{\gamma(\vec{v})} \right]^{r-1} \nabla_s^{\tilde{G}_\ell} \vec{v} \right] \right)_{\tilde{G}_\ell} \gamma_\ell(\vec{v}) \, d\mathcal{H}^{d-1} \\ = - \sum_{\ell=1}^L \int_\Gamma \left[\frac{\gamma_\ell(\vec{v})}{\gamma(\vec{v})} \right]^{r-1} (\nabla_s^{\tilde{G}_\ell} \vec{u}, \nabla_s^{\tilde{G}_\ell} \vec{v})_{\tilde{G}_\ell} \gamma_\ell(\vec{v}) \, d\mathcal{H}^{d-1}, \end{aligned} \quad (2.49)$$

which is the analogue of (2.38b). Finally, the corresponding analogue of (2.43) also holds, i.e. for $\vec{\xi} : \Gamma \rightarrow \mathbb{R}^d$ we have that

$$\int_\Gamma \mathcal{K}_\gamma \vec{v} \cdot \vec{\xi} \, d\mathcal{H}^{d-1} + \sum_{\ell=1}^L \int_\Gamma \left[\frac{\gamma_\ell(\vec{v})}{\gamma(\vec{v})} \right]^{r-1} (\nabla_s^{\tilde{G}_\ell} \text{id}, \nabla_s^{\tilde{G}_\ell} \vec{\xi})_{\tilde{G}_\ell} \gamma_\ell(\vec{v}) \, d\mathcal{H}^{d-1} = 0. \quad (2.50)$$

Proof Let $H_\ell, C_\ell \in \mathbb{R}^{(d-1) \times (d-1)}$ be defined as in (2.11) and the proof of Lemma 2.5, respectively, with $\tilde{G}$ replaced by $\tilde{G}_\ell$. Then, similarly to (2.30), we have that

$$\frac{d}{d\delta} |\Gamma(\delta)|_\gamma = \frac{d}{d\delta} \int_{\mathcal{D}} \left(\sum_{\ell=1}^L [\det C_\ell(\delta)]^{\frac{r}{2}} \right)^{\frac{1}{r}} \, d\mathcal{L}^{d-1}.$$

Hence it holds that

$$\begin{aligned} \frac{d}{d\delta} |\Gamma(\delta)|_\gamma \big|_{\delta=0} \\ = \int_{\mathcal{D}} \left(\sum_{\ell=1}^L [\det C_\ell(0)]^{\frac{r}{2}} \right)^{\frac{1}{r}-1} \sum_{\ell=1}^L [\det C_\ell(0)]^{\frac{r-1}{2}} \sqrt{\det H_\ell} (\nabla_s^{\tilde{G}_\ell} \cdot \vec{g}) \circ \vec{x} \, d\mathcal{L}^{d-1} \\ = \sum_{\ell=1}^L \int_\Gamma (\nabla_s^{\tilde{G}_\ell} \cdot \vec{g}) \frac{[\gamma_\ell(\vec{v})]^r}{[\gamma(\vec{v})]^{r-1}} \, d\mathcal{H}^{d-1}. \end{aligned}$$

Similarly to (2.21b), we have that

$$[\nabla_s^{\tilde{G}_\ell} \cdot \left(\left[\frac{\gamma_\ell(\vec{v})}{\gamma(\vec{v})} \right]^{r-1} \nabla_s^{\tilde{G}_\ell} \vec{g} \right)] \circ \vec{x} = [\det H_\ell]^{-\frac{1}{2}} \sum_{i,j=1}^{d-1} \partial_i \left[\left[\frac{\gamma_\ell}{\gamma} \right]^{r-1} \sqrt{\det H_\ell} h_\ell^{ij} \partial_j (\vec{g} \circ \vec{x}) \right], \quad (2.51)$$

where here and throughout this proof we use the short hand notation $\left[\frac{\gamma_\ell}{\gamma}\right]^{r-1} := \left[\frac{\gamma_\ell(\vec{v})}{\gamma(\vec{v})}\right]^{r-1} \circ \vec{x}$. As in (2.34), on assuming w.l.o.g. that $\vec{g}$ has compact support in $\vec{x}(\mathcal{D}) \subset \Gamma$, it then follows from (2.19b), (2.12), the symmetry of H_ℓ and (2.51), on recalling (1.10), that

$$\begin{aligned}
& \int_\Gamma (\nabla_s^{\tilde{G}_\ell} \cdot \vec{g}) \left[\frac{\gamma_\ell(\vec{v})}{\gamma(\vec{v})} \right]^{r-1} \gamma_\ell(\vec{v}) \, d\mathcal{H}^{d-1} \\
&= \int_{\mathcal{D}} \sum_{j=1}^{d-1} \left(\partial_j [\vec{g} \circ \vec{x}] \cdot \tilde{G}_\ell \sum_{i=1}^{d-1} \sqrt{\det H_\ell} h_\ell^{ij} \partial_i \vec{x} \right) \left[\frac{\gamma_\ell}{\gamma} \right]^{r-1} d\mathcal{L}^{d-1} \\
&= - \int_{\mathcal{D}} (\vec{g} \circ \vec{x}) \cdot \tilde{G}_\ell \sum_{i,j=1}^{d-1} \partial_j \left[\left[\frac{\gamma_\ell}{\gamma} \right]^{r-1} \sqrt{\det H_\ell} h_\ell^{ij} \partial_i \vec{x} \right] d\mathcal{L}^{d-1} \\
&= - \int_\Gamma \left(\vec{g}, \nabla_s^{\tilde{G}_\ell} \cdot \left[\left[\frac{\gamma_\ell(\vec{v})}{\gamma(\vec{v})} \right]^{r-1} \nabla_s^{\tilde{G}_\ell} \text{id} \right] \right)_{\tilde{G}_\ell} \gamma_\ell(\vec{v}) \, d\mathcal{H}^{d-1} \\
&= - \int_\Gamma (\vec{g}, \vec{\mathcal{Z}}^{\tilde{G}_\ell})_{\tilde{G}_\ell} \gamma_\ell(\vec{v}) \, d\mathcal{H}^{d-1}.
\end{aligned}$$

We now prove (2.47). As in the proof of Lemma 2.7, it is sufficient to show that $\sum_{\ell=1}^L \gamma_\ell(\vec{v}) (\vec{\mathcal{Z}}^{\tilde{G}_\ell}, \partial_k \vec{x})_{\tilde{G}_\ell} = 0$, $k = 1 \rightarrow d-1$. Similarly to (2.36), it first holds for $\ell = 1 \rightarrow L$ that

$$\begin{aligned}
& [\det H_\ell]^{-\frac{1}{2}} \sum_{i,j=1}^{d-1} \partial_i \left[\left[\frac{\gamma_\ell}{\gamma} \right]^{r-1} \sqrt{\det H_\ell} h_\ell^{ij} \partial_j \vec{x} \right] \cdot \tilde{G}_\ell \partial_k \vec{x} \\
&= [\det H_\ell]^{-\frac{1}{2}} \sum_{i,j=1}^{d-1} \partial_i \left[\left[\frac{\gamma_\ell}{\gamma} \right]^{r-1} \sqrt{\det H_\ell} h_\ell^{ij} \partial_j \vec{x} \cdot \tilde{G}_\ell \partial_k \vec{x} \right] - \left[\frac{\gamma_\ell}{\gamma} \right]^{r-1} \sum_{i,j=1}^{d-1} h_\ell^{ij} \partial_j \vec{x} \cdot \tilde{G}_\ell \partial_{ik}^2 \vec{x} \\
&= [\det H_\ell]^{-\frac{1}{2}} \left[\frac{\gamma_\ell}{\gamma} \right]^{r-1} \sum_{i,j=1}^{d-1} \partial_i \left[\sqrt{\det H_\ell} h_\ell^{ij} \partial_j \vec{x} \cdot \tilde{G}_\ell \partial_k \vec{x} \right] - \left[\frac{\gamma_\ell}{\gamma} \right]^{r-1} \sum_{i,j=1}^{d-1} h_\ell^{ij} \partial_j \vec{x} \cdot \tilde{G}_\ell \partial_{ik}^2 \vec{x} \\
&\quad + [\det H_\ell]^{-\frac{1}{2}} \sum_{i,j=1}^{d-1} \partial_i \left[\left[\frac{\gamma_\ell}{\gamma} \right]^{r-1} \left[\sqrt{\det H_\ell} h_\ell^{ij} \partial_j \vec{x} \cdot \tilde{G}_\ell \partial_k \vec{x} \right] \right] \\
&= \sum_{i,j=1}^{d-1} \partial_i \left[\frac{\gamma_\ell}{\gamma} \right]^{r-1} \left[h_\ell^{ij} \partial_j \vec{x} \cdot \tilde{G}_\ell \partial_k \vec{x} \right] = \sum_{i=1}^{d-1} \partial_i \left[\frac{\gamma_\ell}{\gamma} \right]^{r-1} \delta_{ik} = \partial_k \left[\frac{\gamma_\ell}{\gamma} \right]^{r-1}. \tag{2.52}
\end{aligned}$$

Multiplying (2.52) with $\gamma_\ell \equiv \gamma_\ell(\vec{v}) \circ \vec{x}$ and summing for $\ell = 1 \rightarrow L$ yields that

$$\sum_{\ell=1}^L \gamma_\ell(\vec{v}) (\vec{\mathcal{Z}}^{\tilde{G}_\ell}, \partial_k \vec{x})_{\tilde{G}_\ell} = \sum_{\ell=1}^L \gamma_\ell \partial_k \left[\frac{\gamma_\ell}{\gamma} \right]^{r-1} = (r-1) \left(\gamma^{1-r} \sum_{\ell=1}^L \gamma_\ell^{r-1} \partial_k \gamma_\ell - \partial_k \gamma \right) = 0,$$

on recalling (1.13).

Combining (1.4), (2.45) with $\vec{g} = g \vec{v}$, (2.46) and (2.47) yields (2.48), which is the analogue of (2.37). Next we note that the proof of (2.49) is a simple extension of that of (2.38b). Finally, similarly to (2.44), the desired result (2.50) now follows from (2.49), (2.46) and (2.48). $\square$

Remark 2.4 We note that in line with Remark 2.1, (2.50) is clearly independent from the chosen parameterization $\vec{x} : \mathcal{D} \rightarrow \mathbb{R}^d$ of Γ . In particular, in practice it is often advantageous to consider a global parameterization $\vec{x} : \Omega \rightarrow \Gamma$, where $\Omega \subset \mathbb{R}^d$ is a suitable compact reference manifold without boundary, recall (1.9). This will be the chosen viewpoint from now on in this paper.

3 Finite Element Approximation

Similarly to [4], we introduce the following finite element approximation, based on the seminal paper [14]. Let $0 = t_0 < t_1 < \dots < t_{M-1} < t_M = T$ be a partitioning of $[0, T]$ into possibly variable time steps $\tau_m := t_{m+1} - t_m$, $m = 0 \rightarrow M-1$. We set $\tau := \max_{m=0 \rightarrow M-1} \tau_m$. Let $\Gamma^m \subset \mathbb{R}^d$ be a $(d-1)$ -dimensional *polyhedral surface*, i.e. a union of non-degenerate $(d-1)$ -simplices with no hanging vertices (see [13, p. 164] for $d = 3$), approximating the closed surface $\Gamma(t_m)$, $m = 0 \rightarrow M$. We now parameterize the new closed surface Γ^{m+1} over Γ^m . Hence, given $\vec{X}^m$, a parameterization of Γ^m , we introduce the following finite element spaces. Let $\Gamma^m = \bigcup_{j=1}^J \bar{\sigma}_j^m$, where $\{\sigma_j^m\}_{j=1}^J$ is a family of mutually disjoint open $(d-1)$ -simplices with vertices $\{\vec{q}_k^m\}_{k=1}^K$, and set $h := \max_{j=1 \rightarrow J} \text{diam}(\sigma_j^m)$. Then for $m = 0 \rightarrow M-1$, let

$$\begin{aligned} \underline{V}(\Gamma^m) &:= \{\vec{\chi} \in C(\Gamma^m, \mathbb{R}^d) : \vec{\chi}|_{\sigma_j^m} \text{ is linear } \forall j = 1 \rightarrow J\} \\ &=: [W(\Gamma^m)]^d \subset H^1(\Gamma^m, \mathbb{R}^d), \end{aligned}$$

where $W(\Gamma^m) \subset H^1(\Gamma^m, \mathbb{R})$ is the space of scalar continuous piecewise linear functions on Γ^m , with $\{\phi_k^m\}_{k=1}^K$ denoting the standard basis of $W(\Gamma^m)$.

For scalar and vector functions $u, v \in L^2(\Gamma^m, \mathbb{R}^{(d)})$ we introduce the L^2 inner product $\langle \cdot, \cdot \rangle_m$ over the current polyhedral surface Γ^m as follows

$$\langle u, v \rangle_m := \int_{\Gamma^m} u \cdot v \, d\mathcal{H}^{d-1}. \quad (3.1)$$

Here and throughout this paper, $\cdot^{(*)}$ denotes an expression with or without the superscript $*$, and similarly for subscripts. Moreover, for $m \geq 1$, $\vec{X}^m \in \underline{V}(\Gamma^{m-1})$ and for $m \geq 0$ we also note that $\vec{X}^m = \text{id}$ is the identity function on Γ^m .

If u, v are piecewise continuous, with possible jumps across the edges of $\{\sigma_j^m\}_{j=1}^J$, we introduce the mass lumped inner product $\langle \cdot, \cdot \rangle_m^h$ as

$$\langle u, v \rangle_m^h := \frac{1}{d} \sum_{j=1}^J |\sigma_j^m| \sum_{k=1}^d (u \cdot v)((\vec{q}_{j_k}^m)^-), \quad (3.2)$$

where $\{\vec{q}_{j_k}^m\}_{k=1}^d$ are the vertices of σ_j^m , and where we define $u((\vec{q}_{j_k}^m)^-) := \lim_{\sigma_j^m \ni \vec{p} \rightarrow \vec{q}_{j_k}^m} u(\vec{p})$. Here $|\sigma_j^m| = \frac{1}{(d-1)!} |(\vec{q}_{j_2}^m - \vec{q}_{j_1}^m) \wedge \dots \wedge (\vec{q}_{j_d}^m - \vec{q}_{j_1}^m)|$ is the measure of σ_j^m . In addition, we introduce the unit normal $\vec{v}^m$ to Γ^m ; that is,

$$\vec{v}_j^m := \vec{v}^m|_{\sigma_j^m} := \frac{(\vec{q}_{j_2}^m - \vec{q}_{j_1}^m) \wedge \dots \wedge (\vec{q}_{j_d}^m - \vec{q}_{j_1}^m)}{|(\vec{q}_{j_2}^m - \vec{q}_{j_1}^m) \wedge \dots \wedge (\vec{q}_{j_d}^m - \vec{q}_{j_1}^m)|},$$

where we have assumed that the vertices $\{\vec{q}_{j_k}^m\}_{k=1}^d$ of σ_j^m are ordered such that $\vec{v}^m : \Gamma^m \rightarrow \mathbb{R}^d$ induces an orientation on Γ^m . Finally, we set $|\cdot|_{m(h)}^2 := \langle \cdot, \cdot \rangle_m^{(h)}$.

Hence, on recalling (1.16a) and (2.50), we propose the following approximation to (1.5a) with γ defined by (1.12) for a given $r \in [1, \infty)$: Find $\{\vec{X}^{m+1}, \kappa_\gamma^{m+1}\} \in \underline{V}(\Gamma^m) \times W(\Gamma^m)$ such that

$$\left\langle \frac{\vec{X}^{m+1} - \vec{X}^m}{\tau_m}, \chi \vec{v}^m \right\rangle_m^h - \langle \beta(\vec{v}^m) \kappa_\gamma^{m+1}, \chi \rangle_m^h = 0 \quad \forall \chi \in W(\Gamma^m), \quad (3.3a)$$

$$\langle \kappa_\gamma^{m+1} \vec{v}^m, \vec{\eta} \rangle_m^h + \sum_{\ell=1}^L \int_{\Gamma^m} \left[\frac{\gamma_\ell(\vec{v}^{m+1})}{\gamma(\vec{v}^{m+1})} \right]^{r-1} (\nabla_s^{\tilde{G}_\ell} \vec{X}^{m+1}, \nabla_s^{\tilde{G}_\ell} \vec{\eta})_{\tilde{G}_\ell} \gamma_\ell(\vec{v}^m) d\mathcal{H}^{d-1} = 0$$

$$\forall \vec{\eta} \in \underline{V}(\Gamma^m); \quad (3.3b)$$

where, as noted above, the inner products $\langle \cdot, \cdot \rangle_m$, $\langle \cdot, \cdot \rangle_m^h$ as well as $\nabla_s^{\tilde{G}_\ell}$ depend on m .

The corresponding approximation for motion by anisotropic surface diffusion, (1.5b), is given by: Find $\{\vec{X}^{m+1}, \kappa_\gamma^{m+1}\} \in \underline{V}(\Gamma^m) \times W(\Gamma^m)$ such that

$$\left\langle \frac{\vec{X}^{m+1} - \vec{X}^m}{\tau_m}, \chi \vec{v}^m \right\rangle_m^h - \langle \beta(\vec{v}^m) \nabla_s \kappa_\gamma^{m+1}, \nabla_s \chi \rangle_m = 0 \quad \forall \chi \in W(\Gamma^m), \quad (3.4)$$

and (3.3b) hold. We note that (3.3a,b) and (3.4), (3.3b) for the choice $r = 1$ in (1.12) reduce to a linear system for the unknowns $\vec{X}^{m+1}$ and κ_γ^{m+1} . In particular, we note that (3.3a,b) and (3.4), (3.3b) for the isotropic surface energy $\gamma(\vec{p}) = |\vec{p}|$ and the mobility $\beta \equiv 1$ reduce in the case $d = 3$ to the schemes considered in [4]. Note furthermore that in the case $d = 2$, the above schemes, for $r = 1$, collapse to the schemes considered in [3]. This can be seen from the fact that in this case

$$\begin{aligned} \gamma_\ell(\vec{v}^m) (\nabla_s^{\tilde{G}_\ell} \vec{X}^{m+1}, \nabla_s^{\tilde{G}_\ell} \vec{\eta})_{\tilde{G}_\ell} &= \gamma_\ell(\vec{v}^m) (\tilde{G}_\ell \partial_{\vec{t}_1} \vec{X}^{m+1}, \partial_{\vec{t}_1} \vec{\eta}) \\ &= [\gamma_\ell(\vec{v}^m)]^{-1} (G_\ell [\vec{X}_s^{m+1}]^\perp, \vec{\eta}_s^\perp), \end{aligned} \quad (3.5)$$

as $\vec{u} \cdot \tilde{G}_\ell \vec{v} = \vec{u}^\perp \cdot G_\ell \vec{v}^\perp$ for all $\vec{u}, \vec{v} \in \mathbb{R}^2$. Here $\vec{t}_1 = [(\vec{v}^m)^\perp]^\perp$, $\tilde{G}_\ell(\vec{v}^m)^\perp = [\gamma_\ell(\vec{v}^m)]^{-1} (\vec{v}^m)^\perp$ and $\cdot_s$ denotes the partial derivative with respect to the arclength s . Observe that the schemes in [3] are based on a formulation involving the last term in (3.5), and so do not rely on the definition of $\tilde{G}_\ell$ and the novel variational formulation introduced in Section 2.

In order to approximate (1.7), we adapt (3.3a) to

$$\left\langle \frac{\vec{X}^{m+1} - \vec{X}^m}{\tau_m}, \chi \vec{v}^m \right\rangle_m^h - \langle \beta(\vec{v}^m) \kappa_\gamma^{m+1}, \chi \rangle_m^h = - \frac{\langle \beta(\vec{v}^m) \kappa_\gamma^m, 1 \rangle_m^h}{\langle 1, 1 \rangle_m} \langle 1, \chi \rangle_m$$

$$\forall \chi \in W(\Gamma^m), \quad (3.6)$$

where we use the obvious interpretation of κ_γ^m on Γ^m , and where $\kappa_\gamma^0 \in W(\Gamma^0)$ is suitably chosen, see [4] for details in the isotropic case. Curvature flows for which the normal velocity is a nonlinear function of the anisotropic mean curvature of the form

(1.8) can be approximated by the scheme: Find $\{\vec{X}^{m+1}, \kappa_\gamma^{m+1}\} \in \underline{V}(\Gamma^m) \times W(\Gamma^m)$ such that

$$\left\langle \frac{\vec{X}^{m+1} - \vec{X}^m}{\tau_m}, \chi \vec{v}^m \right\rangle_m^h - \langle \beta(\vec{v}^m) f(\kappa_\gamma^{m+1}), \chi \rangle_m^h = 0 \quad \forall \chi \in W(\Gamma^m) \quad (3.7)$$

and (3.3b) hold. Under certain assumptions on f existence, uniqueness and stability can be shown for (3.7) and (3.3b), see [4] for the details in the isotropic case.

Remark 3.1 A possible approximation for (1.14a) and (1.17), (recall its weak form (1.16a), (1.18)), that can be used for an arbitrary differentiable anisotropy function γ satisfying (1.2), is the following. Find $\{\vec{X}^{m+1}, \kappa_\gamma^{m+1}\} \in \underline{V}(\Gamma^m) \times W(\Gamma^m)$ such that

$$\left\langle \frac{\vec{X}^{m+1} - \vec{X}^m}{\tau_m}, \chi \vec{v}^m \right\rangle_m^h - \langle \beta(\vec{v}^m) \kappa_\gamma^{m+1}, \chi \rangle_m^h = 0 \quad \forall \chi \in W(\Gamma^m), \quad (3.8a)$$

$$\langle \kappa_\gamma^{m+1} \vec{v}^m, \vec{\eta} \rangle_m^h + \langle \gamma(\vec{v}^m) \nabla_s \vec{X}^{m+1}, \nabla_s \vec{\eta} \rangle_m = \langle [\vec{v}^m [\gamma'(\vec{v}^m)]^T], \nabla_s \vec{\eta} \rangle_m \quad \forall \vec{\eta} \in \underline{V}(\Gamma^m); \quad (3.8b)$$

where for the purposes of this remark, we need to extend the definitions for $\langle \cdot, \cdot \rangle_m$ to matrix valued functions, similarly to (3.1). To this end, we define $A \cdot B := \sum_{k,l=1}^d A_{kl} B_{kl}$ for $A, B \in \mathbb{R}^{d \times d}$. The system (3.8a,b) uses a similar time discretization to the scheme: Find $\vec{X}^{m+1} \in \underline{V}(\Gamma^m)$ such that for all $\vec{\eta} \in \underline{V}(\Gamma^m)$

$$\langle [\beta(\vec{v}^m)]^{-1} \frac{\vec{X}^{m+1} - \vec{X}^m}{\tau_m}, \vec{\eta} \rangle_m^h + \langle \gamma(\vec{v}^m) \nabla_s \vec{X}^{m+1}, \nabla_s \vec{\eta} \rangle_m = \langle [\vec{v}^m [\gamma'(\vec{v}^m)]^T], \nabla_s \vec{\eta} \rangle_m, \quad (3.9)$$

which was studied in [12]. The key difference between the schemes (3.8a,b) and (3.9) is that the former approximates (1.14) and (1.17), whereas the latter approximates $\vec{x}_t = \beta(\vec{v}) \kappa_\gamma \vec{v}$ and (1.17); that is, the tangential movement is not specified in the former, whereas it is (approximately) zero in the latter. We note that although in practice the approximation (3.8a,b) behaves well, it does not appear possible to prove a stability result similar to Theorem 3.2, below. Moreover, the corresponding scheme for anisotropic surface diffusion, (3.4) and (3.8b), is unstable in practice, unless very small time step sizes τ_m are used; and even then, it usually eventually fails due to the coalescence of mesh points.

An alternative approximation to (1.5) is given in [22] and can be stated as: Let $\vec{X}^{m+1} := \vec{X}^m + \tau_m \vec{V}^{m+1}$, where $\vec{V}^{m+1} \in \underline{V}(\Gamma^m)$ is part of the solution of: Find $\{\vec{V}^{m+1}, \kappa_\gamma^{m+1}, \vec{\kappa}_\gamma^{m+1}, V^{m+1}\} \in \underline{V}(\Gamma^m) \times W(\Gamma^m) \times \underline{V}(\Gamma^m) \times W(\Gamma^m)$ such that for all $\vec{\eta} \in \underline{V}(\Gamma^m)$ and $\chi \in W(\Gamma^m)$

$$\langle \vec{\kappa}_\gamma^{m+1}, \vec{\eta} \rangle_m + \tau_m \langle \gamma(\vec{v}^m) \nabla_s \vec{V}^{m+1}, \nabla_s \vec{\eta} \rangle_m = -\langle \gamma(\vec{v}^m) \nabla_s \vec{X}^m, \nabla_s \vec{\eta} \rangle_m + \langle [\vec{v}^m [\gamma'(\vec{v}^m)]^T], \nabla_s \vec{\eta} \rangle_m, \quad (3.10a)$$

$$\langle \kappa_\gamma^{m+1}, \chi \rangle_m - \langle \vec{\kappa}_\gamma^{m+1}, \chi \vec{v}^m \rangle_m = 0, \quad (3.10b)$$

$$\langle V^{m+1}, \chi \rangle_m - \langle \nabla_s \kappa_\gamma^{m+1}, \nabla_s \chi \rangle_m = 0, \quad (3.10c)$$

$$\langle \vec{V}^{m+1}, \vec{\eta} \rangle_m - \langle V^{m+1} \vec{v}^m, \vec{\eta} \rangle_m = 0. \quad (3.10d)$$

The scheme (3.10a–d) is the straightforward adaptation of the isotropic scheme in [2] and is based on the variational formulation of

$$\vec{x}_t = \vec{\mathcal{V}} = \mathcal{V} \vec{v}, \quad \mathcal{V} = -\Delta_s \kappa_\gamma, \quad \kappa_\gamma = \vec{x}_\gamma \cdot \vec{v} \quad \text{and} \quad (1.17);$$

as opposed to the formulation (1.16a) and (2.50) considered in this paper for the class of anisotropies (1.12). We note that, crucially, there is no stability results for either (3.9) or (3.10a–d). This, and the fact that both approximations only change the parameterization $\vec{X}^m$ in the normal direction, means that great care must be taken in practice in order to avoid pathological mesh distortions and unphysical evolutions; see e.g. [22] for details.

Before we can proceed to prove existence and uniqueness to our approximations, we have to make the following very mild assumption on the polyhedral surfaces at each time level.

($\mathcal{A}$) We assume for $m = 0 \rightarrow M$ that $|\sigma_j^m| > 0$ for all $j = 1 \rightarrow J$. For $k = 1 \rightarrow K$, let $\mathcal{T}_k^m := \{\sigma_j^m : \vec{q}_k^m \in \overline{\sigma_j^m}\}$ and set

$$\Lambda_k^m := \cup_{\sigma_j^m \in \mathcal{T}_k^m} \overline{\sigma_j^m} \quad \text{and} \quad \vec{\omega}_k^m := \frac{1}{|\Lambda_k^m|} \sum_{\sigma_j^m \in \mathcal{T}_k^m} |\sigma_j^m| \vec{v}_j^m.$$

Then we further assume that $\dim \text{span}\{\vec{\omega}_k^m\}_{k=1}^K = d$, $m = 0 \rightarrow M - 1$.

Remark 3.2 We note that one can interpret $\vec{\omega}_k^m$ as a weighted normal defined at the node $\vec{X}^m(\vec{q}_k^m) = \vec{q}_k^m$ of the surface Γ^m , where in general $|\vec{\omega}_k^m| < 1$. In addition, we note that ($\mathcal{A}$) is only violated in very rare occasions. For example, if $d \leq 3$ it always holds for surfaces without self intersections.

Theorem 3.1 *Let the assumption ($\mathcal{A}$) hold and let $r = 1$. Then there exists a unique solution $\{\vec{X}^{m+1}, \kappa^{m+1}\} \in \underline{V}(\Gamma^m) \times W(\Gamma^m)$ to the systems (3.3a,b) and (3.4), (3.3b).*

Proof As (3.3a,b) is linear, existence follows from uniqueness. To investigate the latter, we consider the system: Find $\{\vec{X}, \kappa_\gamma\} \in \underline{V}(\Gamma^m) \times W(\Gamma^m)$ such that

$$\langle \vec{X}, \chi \vec{v}^m \rangle_m^h - \tau_m \langle \beta(\vec{v}^m) \kappa_\gamma, \chi \rangle_m^h = 0 \quad \forall \chi \in W(\Gamma^m), \quad (3.11a)$$

$$\langle \kappa_\gamma \vec{v}^m, \vec{\eta} \rangle_m^h + \sum_{\ell=1}^L \int_{\Gamma^m} (\nabla_s^{\tilde{G}_\ell} \vec{X}, \nabla_s^{\tilde{G}_\ell} \vec{\eta})_{\tilde{G}_\ell} \gamma_\ell(\vec{v}^m) d\mathcal{H}^{d-1} = 0 \quad \forall \vec{\eta} \in \underline{V}(\Gamma^m). \quad (3.11b)$$

Choosing $\chi = \kappa_\gamma \in W(\Gamma^m)$ in (3.11a) and $\vec{\eta} = \vec{X} \in \underline{V}(\Gamma^m)$ in (3.11b) yields that

$$\sum_{\ell=1}^L \int_{\Gamma^m} (\nabla_s^{\tilde{G}_\ell} \vec{X}, \nabla_s^{\tilde{G}_\ell} \vec{X})_{\tilde{G}_\ell} \gamma_\ell(\vec{v}^m) d\mathcal{H}^{d-1} + \tau_m \langle \beta(\vec{v}^m) \kappa_\gamma, \kappa_\gamma \rangle_m^h = 0. \quad (3.12)$$

It follows from (3.12), (2.8), the positive definiteness of $\tilde{G}_\ell$, $\ell = 1 \rightarrow L$, and the positivity of β that $\kappa_\gamma \equiv 0$ and $\tilde{X} \equiv \tilde{X}^c \in \mathbb{R}^d$; and hence that

$$\langle \tilde{X}^c, \chi \tilde{v}^m \rangle_m^h = 0 \quad \forall \chi \in W(\Gamma^m). \quad (3.13)$$

Choosing $\chi = \phi_k^m$ in (3.13) yields that $\tilde{X}^c \cdot \tilde{\omega}_k^m = 0$ for $k = 1 \rightarrow K$. It follows from assumption (A) that $\tilde{X}^c = \vec{0}$. Hence we have shown that there exists a unique solution $\{\tilde{X}^{m+1}, \kappa_\gamma^{m+1}\} \in \underline{V}(\Gamma^m) \times W(\Gamma^m)$ to (3.3a,b).

The corresponding proof for the system (3.4), (3.3b) is similar with only a minor modification; see e.g. [4] for the isotropic case. $\square$

Remark 3.3 Let us remark that in the case $r > 1$ the nonlinear system (3.3a,b) for $\{\tilde{X}^{m+1}, \kappa_\gamma^{m+1}\}$ is not continuously dependent on the variable $\tilde{X}^{m+1}$. For certain choices of the unknown $\tilde{X}^{m+1}$, simplices might degenerate and in such a case the normal is discontinuous and therefore discontinuities appear in (3.3a,b). Due to this fact it is not possible to show existence of solutions to (3.3a,b) with the help of Brouwer's fixed point theorem. However, in practice we had no difficulties in finding solutions to the nonlinear systems (3.3a,b) and (3.4), (3.3b), and the employed iterative solvers always converged; see Section 4.

On the other hand, one could consider the obvious linearizations of (3.3a,b) and (3.4), (3.3b), obtained by replacing $\tilde{v}^{m+1}$ with $\tilde{v}^m$ in (3.3b). Then it is straightforward to show existence and uniqueness for the solutions of the arising linear systems for any $r \in [1, \infty)$. However, it then no longer appears possible to prove the corresponding stability result (3.19a,b), below, and numerical evidence suggests that instabilities occur unless very small time step sizes τ_m are used. Hence we prefer the (unconditionally stable) nonlinear schemes (3.3a,b) and (3.4), (3.3b).

We recall that for $d = 2$ and $r = 1$ unconditional stability of the schemes (3.3a,b) and (3.4), (3.3b) has been shown in [3]. We now prove the unconditional stability of these schemes for $d = 3$ and arbitrary $r \geq 1$.

Lemma 3.1 *Let $d = 3$ and $Y = \bigcup_{j=1}^J \sigma_j$ be a polyhedral surface. Then we have for $j = 1 \rightarrow J$ and $\ell = 1 \rightarrow L$ that*

$$\frac{1}{2} \int_{\sigma_j} \gamma_\ell(\tilde{v}^{\text{id}}) |\nabla_{\tilde{G}_\ell} \tilde{Y}|_{\tilde{G}_\ell}^2 \, d\mathcal{H}^2 \geq \int_{\tilde{Y}(\sigma_j)} \gamma_\ell(\tilde{v}^{\tilde{Y}}) \, d\mathcal{H}^2 = |\tilde{Y}(\sigma_j)|_{\gamma_\ell} \quad \forall \tilde{Y} \in \underline{V}(Y), \quad (3.14)$$

with equality for $\tilde{Y} = \vec{\text{id}} \in \underline{V}(Y)$. Here $\tilde{v}^{\tilde{Y}}$ is the unit normal on the polyhedral surface $\tilde{Y}(Y)$ and $|\cdot|_{\tilde{G}_\ell}^2 := (\cdot, \cdot)_{\tilde{G}_\ell}$.

Proof Let $\{\vec{t}_1, \vec{t}_2\}$ and $\{\vec{p}_1, \vec{p}_2\}$ be orthonormal bases of the plane containing σ_j , with respect to the $(\cdot, \cdot)_{\tilde{G}_\ell}$ inner product and the standard inner product, respectively. Then, on recalling (2.7) and (2.8), it holds on σ_j that

$$\gamma_\ell(\tilde{v}^{\text{id}}) |\nabla_{\tilde{G}_\ell} \tilde{Y}|_{\tilde{G}_\ell}^2 = \gamma_\ell(\tilde{v}^{\text{id}}) \sum_{i=1}^2 (\partial_{\vec{t}_i} \tilde{Y}, \partial_{\vec{t}_i} \tilde{Y})_{\tilde{G}_\ell} = \gamma_\ell(\tilde{v}^{\text{id}}) \sum_{i=1}^2 |\tilde{\Lambda}_\ell \partial_{\vec{t}_i} \tilde{Y}|^2, \quad (3.15)$$

where $\tilde{\Lambda}_\ell^2 = \tilde{G}_\ell$. Let $\vec{t}_i = \sum_{l=1}^2 a_{il} \vec{p}_l$ for some $A := (a_{il})_{i,l=1}^2$. Then, similarly to (2.13), one can show that

$$(\det A)^2 \det(\partial_{\vec{p}_i} \vec{Y} \cdot \partial_{\vec{p}_l} \vec{Y})_{i,l=1}^2 = \det(\partial_{\vec{t}_i} \vec{Y} \cdot \partial_{\vec{t}_l} \vec{Y})_{i,l=1}^2, \quad \text{with} \quad \det A = [\gamma_\ell(\vec{v}^{\text{id}})]^{-1}. \quad (3.16)$$

Similarly to (2.14), it follows from (3.16) that

$$\begin{aligned} |\vec{Y}(\sigma_j)|_{\gamma_\ell} &= \int_{\vec{Y}(\sigma_j)} \gamma_\ell(\vec{v}^{\vec{Y}}) \, d\mathcal{H}^2 \\ &= \int_{\sigma_j} \left[\frac{\det(\partial_{\vec{t}_i} \vec{Y} \cdot \tilde{G}_\ell \partial_{\vec{t}_l} \vec{Y})_{i,l=1}^2 \det(\partial_{\vec{p}_i} \vec{Y} \cdot \partial_{\vec{p}_l} \vec{Y})_{i,l=1}^2}{\det(\partial_{\vec{t}_i} \vec{Y} \cdot \partial_{\vec{t}_l} \vec{Y})_{i,l=1}^2} \right]^{\frac{1}{2}} d\mathcal{H}^2 \\ &= \int_{\sigma_j} \gamma_\ell(\vec{v}^{\text{id}}) \sqrt{\det(\partial_{\vec{t}_i} \vec{Y} \cdot \tilde{G}_\ell \partial_{\vec{t}_l} \vec{Y})_{i,l=1}^2} \, d\mathcal{H}^2 \\ &= \int_{\sigma_j} \gamma_\ell(\vec{v}^{\text{id}}) |\tilde{\Lambda}_\ell \partial_{\vec{t}_1} \vec{Y} \times \tilde{\Lambda}_\ell \partial_{\vec{t}_2} \vec{Y}| \, d\mathcal{H}^2. \end{aligned} \quad (3.17)$$

Next we note that

$$|\vec{a} \times \vec{b}| \leq |\vec{a}| |\vec{b}| \leq \frac{1}{2} (|\vec{a}|^2 + |\vec{b}|^2) \quad \forall \vec{a}, \vec{b} \in \mathbb{R}^3 \quad (3.18)$$

with equality if and only if $\vec{a} \cdot \vec{b} = 0$ and $|\vec{a}| = |\vec{b}|$. The desired results (3.14) then follow immediately on combining (3.17), (3.15) and (3.18); and on noting that if $\vec{Y}$ is the identity function on σ_j , then $\partial_{\vec{t}_i} \vec{Y} = \vec{t}_i$ and $\tilde{\Lambda}_\ell \partial_{\vec{t}_i} \vec{Y} \cdot \tilde{\Lambda}_\ell \partial_{\vec{t}_l} \vec{Y} = (\vec{t}_i, \vec{t}_l)_{\tilde{G}_\ell} = \delta_{il}$ on σ_j . $\square$

Theorem 3.2 *Let $d = 3$ and $\{\vec{X}^m, \kappa_\gamma^m\}_{m=1}^M$ be a solution of (3.3a,b). Then for $k = 1 \rightarrow M$ we have that*

$$|\Gamma^k|_\gamma + \sum_{m=0}^{k-1} \tau_m \langle \beta(\vec{v}^m) \kappa_\gamma^{m+1}, \kappa_\gamma^{m+1} \rangle_m^h \leq |\Gamma^0|_\gamma, \quad (3.19a)$$

where we recall (1.1). Similarly, the solution to (3.4), (3.3b) satisfies

$$|\Gamma^k|_\gamma + \sum_{m=0}^{k-1} \tau_m \langle \beta(\vec{v}^m) \nabla_s \kappa_\gamma^{m+1}, \nabla_s \kappa_\gamma^{m+1} \rangle_m \leq |\Gamma^0|_\gamma \quad (3.19b)$$

for $k = 1 \rightarrow M$.

Proof As the two proofs are almost identical, it is sufficient to show (3.19b). Choosing $\chi = \kappa_\gamma^{m+1} \in W(\Gamma^m)$ in (3.4) and $\vec{\eta} = \frac{\vec{X}^{m+1} - \vec{X}^m}{\tau_m} \in \underline{V}(\Gamma^m)$ in (3.3b) yields that

$$\begin{aligned} \sum_{\ell=1}^L \int_{\Gamma^m} \left[\frac{\gamma_\ell(\vec{v}^{m+1})}{\gamma(\vec{v}^{m+1})} \right]^{r-1} (\nabla_s^{\tilde{G}_\ell} \vec{X}^{m+1}, \nabla_s^{\tilde{G}_\ell} (\vec{X}^{m+1} - \vec{X}^m))_{\tilde{G}_\ell} \gamma_\ell(\vec{v}^m) \, d\mathcal{H}^2 \\ + \tau_m \langle \beta(\vec{v}^m) \nabla_s \kappa_\gamma^{m+1}, \nabla_s \kappa_\gamma^{m+1} \rangle_m = 0. \end{aligned} \quad (3.20)$$

Then it holds for any $\ell \in \{1, \dots, L\}$ and $j \in \{1, \dots, J\}$, on recalling (3.14), that

$$\begin{aligned}
& \int_{\sigma_j^m} \gamma_\ell(\vec{v}^m) (\nabla_s^{\tilde{G}_\ell} \vec{X}^{m+1}, \nabla_s^{\tilde{G}_\ell} (\vec{X}^{m+1} - \vec{X}^m))_{\tilde{G}_\ell} d\mathcal{H}^2 \\
&= \frac{1}{2} \int_{\sigma_j^m} \gamma_\ell(\vec{v}^m) |\nabla_s^{\tilde{G}_\ell} \vec{X}^{m+1}|_{\tilde{G}_\ell}^2 d\mathcal{H}^2 - \frac{1}{2} \int_{\sigma_j^m} \gamma_\ell(\vec{v}^m) |\nabla_s^{\tilde{G}_\ell} \vec{X}^m|_{\tilde{G}_\ell}^2 d\mathcal{H}^2 \\
&\quad + \frac{1}{2} \int_{\sigma_j^m} \gamma_\ell(\vec{v}^m) |\nabla_s^{\tilde{G}_\ell} (\vec{X}^{m+1} - \vec{X}^m)|_{\tilde{G}_\ell}^2 d\mathcal{H}^2 \\
&\geq \int_{\sigma_j^{m+1}} \gamma_\ell(\vec{v}^{m+1}) d\mathcal{H}^2 - \int_{\sigma_j^m} \gamma_\ell(\vec{v}^m) d\mathcal{H}^2.
\end{aligned} \tag{3.21}$$

Multiplying (3.21) with the constant $\left[\frac{\gamma_\ell(\vec{v}^{m+1})}{\gamma(\vec{v}^{m+1})} \right]^{r-1}$ and summing for $\ell = 1 \rightarrow L$ and $j = 1 \rightarrow J$ yields, on employing a Hölder inequality, that

$$\begin{aligned}
& \sum_{\ell=1}^L \int_{\Gamma^m} \left[\frac{\gamma_\ell(\vec{v}^{m+1})}{\gamma(\vec{v}^{m+1})} \right]^{r-1} (\nabla_s^{\tilde{G}_\ell} \vec{X}^{m+1}, \nabla_s^{\tilde{G}_\ell} (\vec{X}^{m+1} - \vec{X}^m))_{\tilde{G}_\ell} \gamma_\ell(\vec{v}^m) d\mathcal{H}^2 \\
&\geq \sum_{\ell=1}^L \int_{\Gamma^{m+1}} \frac{[\gamma_\ell(\vec{v}^{m+1})]^r}{[\gamma(\vec{v}^{m+1})]^{r-1}} d\mathcal{H}^2 - \sum_{\ell=1}^L \int_{\Gamma^m} \left[\frac{\gamma_\ell(\vec{v}^{m+1})}{\gamma(\vec{v}^{m+1})} \right]^{r-1} \gamma_\ell(\vec{v}^m) d\mathcal{H}^2 \\
&\geq \int_{\Gamma^{m+1}} \gamma(\vec{v}^{m+1}) d\mathcal{H}^2 - \int_{\Gamma^m} \gamma(\vec{v}^m) d\mathcal{H}^2.
\end{aligned} \tag{3.22}$$

Combining (3.22) and (3.20), yields that

$$|\Gamma^{m+1}|_\gamma - |\Gamma^m|_\gamma + \tau_m \langle \beta(\vec{v}^m) \nabla_s \kappa_\gamma^{m+1}, \nabla_s \kappa_\gamma^{m+1} \rangle_m \leq 0. \tag{3.23}$$

Summing (3.23) for $m = 0 \rightarrow k-1$ yields the desired result (3.19b). $\square$

We note that (3.19a,b) are the discrete analogues of (1.6). They are also the natural extensions of the stability results for curves, $d = 2$, that were derived in [3] for the case $r = 1$. Combining the approach for curves there and the ideas presented above allows one to extend the stability results for $d = 2$ to $r > 1$. To our knowledge, (3.19a,b) are the first stability results for a direct approximation of (1.5a,b) in higher space dimensions.

Remark 3.4 Similarly to [3, Remark 2.7], it is worthwhile to consider continuous in time semidiscrete versions of our schemes. For example, we replace (3.4), (3.3b) by

$$\langle \vec{X}_t, \chi \vec{v}^h \rangle^h - \langle \beta(\vec{v}^h) \nabla_s \kappa_\gamma, \nabla_s \chi \rangle = 0 \quad \forall \chi \in W(\Gamma^h(t)), \tag{3.24a}$$

$$\begin{aligned}
& \langle \kappa_\gamma \vec{v}^h, \vec{\eta} \rangle^h + \sum_{\ell=1}^L \int_{\Gamma^h} \left[\frac{\gamma_\ell(\vec{v}^h)}{\gamma(\vec{v}^h)} \right]^{r-1} (\nabla_s^{\tilde{G}_\ell} \vec{X}, \nabla_s^{\tilde{G}_\ell} \vec{\eta})_{\tilde{G}_\ell} \gamma_\ell(\vec{v}^h) d\mathcal{H}^{d-1} = 0 \\
& \quad \forall \vec{\eta} \in \underline{V}(\Gamma^h(t)); \tag{3.24b}
\end{aligned}$$

where we always integrate over the current surface $\Gamma^h(t)$ (with normal $\vec{v}^h(t)$) described by the identity function $\vec{X}(t) \in \underline{V}(\Gamma^h(t))$. In addition, $\langle \cdot, \cdot \rangle^{(h)}$ is the same

as $\langle \cdot, \cdot \rangle_m^{(h)}$ with Γ^m and $\vec{X}^m$ replaced by $\Gamma^h(t)$ and $\vec{X}(t)$, respectively. It is straightforward to show that (3.24a,b) conserves the enclosed volume exactly; since on choosing $\chi \equiv 1$ in (3.24a) and taking into account (3.2) yields that

$$0 = \langle \vec{X}_t, \vec{v}^h \rangle^h = \int_{\Gamma^h} \vec{X}_t \cdot \vec{v}^h \, d\mathcal{H}^{d-1} = \frac{d}{dt} [\text{Vol}(\Gamma^h(t))]. \quad (3.25)$$

Of course, (3.25) applies to the corresponding semidiscrete analogue of (3.6) with (3.3b), as it is based solely on (3.24a) with $\chi \equiv 1$.

In [3, Remark 2.7] we showed for a closed curve that the scheme (3.24a,b) for $d = 2$ and $r = 1$, and the corresponding semidiscrete analogues of (3.3a,b) and (3.6) with (3.3b) will always equidistribute the nodes along the polygonal approximation to the curve according to some nontrivial weighting function, if the corresponding intervals are not locally parallel. In particular, for $L = 1$ in (1.12) we obtained an equidistribution with respect to γ ; that is, $\gamma(\vec{X}(\vec{q}_{k+1}) - \vec{X}(\vec{q}_k))$ is constant for all k with $\{\vec{q}_k\}_{k=1}^K$ sequential around the polygon. Moreover, in [4] we derived for $d = 3$ in the isotropic case a similar, but weaker, criterion that is satisfied by the triangulations of the evolving polyhedral surfaces for our schemes.

It is possible to formulate criteria which are fulfilled by the schemes in this paper, and which lead to good mesh properties also in the anisotropic case. This discussion is a natural generalization of the isotropic case and involves γ -conformal mappings, i.e. roughly speaking mappings that preserve geometric information which are formulated with the help of the anisotropy function γ . We do not give the details here, but refer to [4] for the corresponding results in the isotropic case.

4 Solution of the discrete system

We introduce the matrices $\vec{N}_m \in (\mathbb{R}^d)^{K \times K}$ and $M_{\beta,m}, A_{(\beta,m)} \in \mathbb{R}^{K \times K}$ with entries

$$\begin{aligned} [M_{\beta,m}]_{kl} &:= \langle \beta(\vec{v}^m) \phi_k^m, \phi_l^m \rangle_m^h, & [\vec{N}_m]_{kl} &:= \int_{\Gamma^m} \pi^m[\phi_k^m \phi_l^m] \vec{v}^m \, d\mathcal{H}^{d-1}, \\ [A_m]_{kl} &:= \langle \nabla_s \phi_k^m, \nabla_s \phi_l^m \rangle_m, & [A_{\beta,m}]_{kl} &:= \langle \beta(\vec{v}^m) \nabla_s \phi_k^m, \nabla_s \phi_l^m \rangle_m, \end{aligned} \quad (4.1)$$

where $\pi^m : C(\Gamma^m, \mathbb{R}) \rightarrow W(\Gamma^m)$ is the standard interpolation operator at the nodes $\{\vec{q}_k^m\}_{k=1}^K$. In addition, given an approximation $\vec{v}^{m+1,i}$ to $\vec{v}^{m+1}$, where $\vec{v}^{m+1,i}$ is a constant vector on any σ_j^m , $j = 1 \rightarrow J$, we introduce $\vec{A}_m^i \in (\mathbb{R}^{d \times d})^{K \times K}$ with entries $[\vec{A}_m^i]_{kl} \in \mathbb{R}^{d \times d}$, $k, l = 1 \rightarrow K$, defined by

$$\begin{aligned} \vec{z} \cdot [\vec{A}_m^i]_{kl} \vec{y} &= \sum_{\ell=1}^L \int_{\Gamma^m} \left[\frac{\gamma_\ell(\vec{v}^{m+1,i})}{\gamma(\vec{v}^{m+1,i})} \right]^{r-1} (\nabla_s^{\tilde{G}_\ell}(\vec{z} \phi_k^m), \nabla_s^{\tilde{G}_\ell}(\vec{y} \phi_l^m))_{\tilde{G}_\ell} \gamma_\ell(\vec{v}^m) \, d\mathcal{H}^{d-1} \\ &= \sum_{\ell=1}^L \left\langle \left[\frac{\gamma_\ell(\vec{v}^{m+1,i})}{\gamma(\vec{v}^{m+1,i})} \right]^{r-1} \gamma_\ell(\vec{v}^m) \nabla_s^{\tilde{G}_\ell} \phi_k^m, \nabla_s^{\tilde{G}_\ell} \phi_l^m \right\rangle_m \vec{z} \cdot \tilde{G}_\ell \vec{y} \quad \forall \vec{z}, \vec{y} \in \mathbb{R}^d, \end{aligned} \quad (4.2)$$

where we have noted (2.8). Hence the computation of $\vec{A}_m^i$ reduces to assembling matrices of the form

$$\left(\left\langle \left[\frac{\gamma_\ell(\vec{V}^{m+1,i})}{\gamma(\vec{V}^{m+1,i})} \right]^{r-1} \gamma_\ell(\vec{V}^m) \nabla_s^{\tilde{G}_\ell} \phi_k^m, \nabla_s^{\tilde{G}_\ell} \phi_l^m \right\rangle_m \right)_{k,l=1}^K.$$

This is straightforward, since assembling e.g. $(\langle \nabla_s^{\tilde{G}_\ell} \phi_k^m, \nabla_s^{\tilde{G}_\ell} \phi_l^m \rangle_m)_{k,l=1}^K$ is very similar to assembling A_m in (4.1), on recalling (2.5).

4.1 The linear case

First, we will consider the linear case $r = 1$, when the definition (4.2) is independent of i , so that we can write $\vec{A}_m$. We can then formulate (3.3a,b), for $r = 1$, as: Find $\{\delta \vec{X}^{m+1}, \kappa_\gamma^{m+1}\} \in (\mathbb{R}^d)^K \times \mathbb{R}^K$, such that

$$\begin{pmatrix} \tau_m M_{\beta,m} & -\vec{N}_m^T \\ \vec{N}_m & \vec{A}_m \end{pmatrix} \begin{pmatrix} \kappa_\gamma^{m+1} \\ \delta \vec{X}^{m+1} \end{pmatrix} = \begin{pmatrix} 0 \\ -\vec{A}_m \vec{X}^m \end{pmatrix}, \quad (4.3)$$

where, with the obvious abuse of notation, $\delta \vec{X}^{m+1} = (\delta \vec{X}_1^{m+1}, \dots, \delta \vec{X}_K^{m+1})^T$ and $\kappa_\gamma^{m+1} = (\kappa_{\gamma,1}^{m+1}, \dots, \kappa_{\gamma,K}^{m+1})^T$ are the vectors of coefficients with respect to the standard basis of $\vec{X}^{m+1} - \vec{X}^m$ and κ_γ^{m+1} , respectively. We can transform (4.3) to

$$\kappa_\gamma^{m+1} = \frac{1}{\tau_m} M_{\beta,m}^{-1} \vec{N}_m^T \delta \vec{X}^{m+1}, \quad (4.4a)$$

$$(\vec{A}_m + \frac{1}{\tau_m} \vec{N}_m M_{\beta,m}^{-1} \vec{N}_m^T) \delta \vec{X}^{m+1} = -\vec{A}_m \vec{X}^m. \quad (4.4b)$$

As (4.4b) is clearly symmetric and positive definite, there exists a unique solution to (4.4b). Moreover, the solution to (4.4a,b) uniquely solves (3.3a,b).

In order to adapt (4.4a,b) to the approximation (3.4), (3.3b) of motion by anisotropic surface diffusion, i.e. (4.3) with $M_{\beta,m}$ replaced by $A_{\beta,m}$, we need the following definitions; see [6] for their equivalents in the isotropic case. Let S_m be the inverse of $A_{\beta,m}$ restricted on the subspace $(\ker A_{\beta,m})^\perp \equiv (\text{span}\{1\})^\perp$, where $1 := (1, \dots, 1)^T \in \mathbb{R}^K$. Let $\vec{\Pi}_m$ be the orthogonal projection onto $\mathcal{R}_m^\perp := \{\vec{X} \in (\mathbb{R}^d)^K : \vec{X}^T \vec{N}_m 1 = 0\}$; that is, $\vec{\Pi}_m := \vec{I}d_K - \frac{\vec{w}\vec{w}^T}{\vec{w}^T \vec{w}}$, where $\vec{w} := \vec{N}_m 1$. One can then employ a Schur complement approach to yield

$$\kappa_\gamma^{m+1} = \frac{1}{\tau_m} S_m \vec{N}_m^T \delta \vec{X}^{m+1}, \quad (4.5a)$$

$$\vec{\Pi}_m (\vec{A}_m + \frac{1}{\tau_m} \vec{N}_m S_m \vec{N}_m^T) \vec{\Pi}_m \delta \vec{X}^{m+1} = -\vec{\Pi}_m \vec{A}_m \vec{X}^m. \quad (4.5b)$$

As (3.4), (3.3b) has a unique solution, it is easily established that there exists a unique solution to (4.5b). Moreover, the system (4.5b) is symmetric and positive definite on $\mathcal{R}_m^\perp$, see [6] for the relevant details.

The Schur complement approaches (4.4b) and (4.5b) can be easily solved with a conjugate gradient solver. As the anisotropic evolution laws considered in this paper can in practice lead to a very non-uniform distribution of mesh points, it is essential

to employ a suitable preconditioner for the above mentioned systems. The following diagonal preconditioner $\tilde{G}_m \in [\mathbb{R}^{d \times d}]^{K \times K}$ with diagonal entries

$$[\tilde{G}_m]_{kk} = \left[[\tilde{A}_m]_{kk} + \frac{1}{\tau_m} [M_{\beta,m}]_{kk}^{-1} \text{diag}(\{([\tilde{N}_m]_{kk} \cdot \vec{e}_i)^2\}_{i=1}^d) \right]^{-1}, \quad (4.6)$$

where $\vec{e}_i, i = 1 \rightarrow d$, are the standard basis vectors in $\mathbb{R}^d$, worked very well in practice for the system (4.4b). Naturally, the definition (4.6) can easily be adapted to the Schur system (4.5b). Here we used the preconditioner $\tilde{I}_m \tilde{H}_m \tilde{I}_m$, where $\tilde{H}_m \in [\mathbb{R}^{d \times d}]^{K \times K}$ is obtained from (4.6) by replacing $[M_{\beta,m}]_{kk}$ with $[A_{\beta,m}]_{kk}$. Note that for the computations reported on in Section 5, the preconditioned conjugate gradient solver was up to 5 times faster than the standard conjugate gradient solver.

4.2 The nonlinear case

In order to find the solution to the schemes (3.3a,b) and (3.4), (3.3b) in the nonlinear setting, when γ is given by (1.12) with $r > 1$, we employ the following iterative solution method. For ease of exposition, we only describe it for the scheme (3.3a,b), as it naturally generalizes to all the other approximations.

At each time step, given $\{\vec{X}^{m+1,0}, \kappa^{m+1,0}\}$ we seek for $i \geq 0$ solutions $\{\vec{X}^{m+1,i+\frac{1}{2}}, \kappa^{m+1,i+\frac{1}{2}}\} \in \underline{V}(\Gamma^m) \times W(\Gamma^m)$ such that for all $\chi \in W(\Gamma^m)$ and $\vec{\eta} \in \underline{V}(\Gamma^m)$

$$\left\langle \frac{\vec{X}^{m+1,i+\frac{1}{2}} - \vec{X}^m}{\tau_m}, \chi \vec{v}_m^h \right\rangle - \langle \beta(\vec{v}^m) \kappa_\gamma^{m+1,i+\frac{1}{2}}, \chi \rangle_m^h = 0, \quad (4.7a)$$

$$\begin{aligned} & \langle \kappa_\gamma^{m+1,i+\frac{1}{2}} \vec{v}_m^h, \vec{\eta} \rangle_m^h \\ & + \sum_{\ell=1}^L \int_{\Gamma^m} \left[\frac{\gamma_\ell(\vec{v}^{m+1,i})}{\gamma(\vec{v}^{m+1,i})} \right]^{r-1} (\nabla_{\tilde{G}_\ell} \vec{X}^{m+1,i+\frac{1}{2}}, \nabla_{\tilde{G}_\ell} \vec{\eta})_{\tilde{G}_\ell} \gamma_\ell(\vec{v}^m) d\mathcal{H}^{d-1} = 0. \end{aligned} \quad (4.7b)$$

Clearly, on recalling the Schur complement approach (4.4a,b) for the linear system (4.3) and on noting the definition (4.2), the coefficient vector $\delta \vec{X}^{m+1,i+\frac{1}{2}} \in \mathbb{R}^K$ is the unique solution of the linear system

$$(\vec{A}_m^i + \frac{1}{\tau_m} \vec{N}_m M_{\beta,m}^{-1} \vec{N}_m^T) \delta \vec{X}^{m+1,i+\frac{1}{2}} = -\vec{A}_m^i \vec{X}^m. \quad (4.8a)$$

On obtaining $\delta \vec{X}^{m+1,i+\frac{1}{2}}$ from (4.8a), we set

$$\vec{X}^{m+1,i+1} = (1 - \mu) \vec{X}^{m+1,i} + \mu (\vec{X}^m + \delta \vec{X}^{m+1,i+\frac{1}{2}}), \quad (4.8b)$$

where $\mu \in (0, 2)$ is a fixed relaxation parameter. The iteration (4.8a,b) is repeated until $\|\vec{X}^{m+1,i+1} - \vec{X}^{m+1,i}\|_\infty < \text{tol}$, where $\text{tol} = 10^{-7}$ is a chosen tolerance. In practice, the iteration (4.8a,b) always converged, provided $\mu < 1$ was chosen sufficiently small.

Table 5.1 Absolute errors $\|\vec{X} - \vec{x}\|_{L^\infty}$ for the test problem, with $T = \frac{1}{2}\bar{T} = \frac{1}{8}$ and $T = \bar{T} - \tau$, respectively.

K	$\varepsilon = 0.5$			$\varepsilon = 0.1$		
	$h_{\vec{X}^0}$	$T = \frac{1}{2}\bar{T}$	$T = \bar{T} - \tau$	$h_{\vec{X}^0}$	$T = \frac{1}{2}\bar{T}$	$T = \bar{T} - \tau$
50	7.1104e-01	4.2822e-02	8.6025e-02	9.0598e-01	1.5049e-02	1.6111e-02
194	4.5309e-01	2.7105e-02	1.1153e-01	6.4277e-01	4.1042e-02	1.0648e-01
770	2.5646e-01	1.0536e-02	1.0260e-01	3.7441e-01	2.1118e-02	1.1484e-01
3074	1.3437e-01	3.1780e-03	7.6203e-02	1.9555e-01	6.6810e-03	9.2615e-02
12290	6.8515e-02	8.7318e-04	4.9815e-02	9.9150e-02	1.8467e-03	6.3176e-02
49154	3.4561e-02	2.3086e-04	3.0128e-02	4.9826e-02	4.9084e-04	4.9324e-02

5 Numerical Results

All the result reported on in this section are for the evolution of hypersurfaces in $\mathbb{R}^3$, so $d = 3$ throughout. Unless otherwise stated, we use a constant mobility $\beta \equiv 1$; and for the anisotropy (1.12), we take G_ℓ of the form (1.20) with $\varepsilon_\ell = \varepsilon$, $\ell = 1 \rightarrow L$.

Throughout this section we use essentially uniform time steps, i.e. $\tau_m = \tau$, $m = 0 \rightarrow M-2$, and $\tau_{M-1} = T - t_{m-1}$. For later purposes, we define

$$\vec{X}(t) := \frac{t-t_{m-1}}{\tau_{m-1}} \vec{X}^m + \frac{t_m-t}{\tau_{m-1}} \vec{X}^{m-1} \quad t \in [t_{m-1}, t_m] \quad m \geq 1.$$

Finally, we note that we implemented the approximations within the finite element toolbox ALBERTA, see [24].

5.1 Anisotropic mean curvature flow

It was shown in [25] that for the anisotropic mobility $\beta = \gamma$, an exact solution to (1.5a) is given by

$$\Gamma(t) = \{\vec{q} \in \mathbb{R}^d : \gamma^*(\vec{q}) = \sqrt{1 - 2(d-1)t}\}, \quad (5.1)$$

i.e. shrinking boundaries of Wulff shapes. Using (5.1) we now perform a convergence test for our approximation (3.3a,b) with $\beta = \gamma$. An exact solution to (1.5a) with $\beta = \gamma$ defined by (1.12) with $L = 1$ and $G_1 = \text{diag}(1, \varepsilon^2, \varepsilon^2)$, on noting (1.19) and (5.1), is given by

$$\vec{x}(\cdot, t) = (1 - 4t)^{\frac{1}{2}} G_1^{\frac{1}{2}} \vec{\text{id}}_{S^2}, \quad t \in [0, \bar{T}), \quad \bar{T} = 0.25; \quad (5.2)$$

where $\vec{\text{id}}_{S^2}$ is the identity function on the unit sphere $\Omega \equiv S^2 \subset \mathbb{R}^3$. For $\varepsilon = 0.5$ and $\varepsilon = 0.1$ we report on the error $\|\vec{X} - \vec{x}\|_{L^\infty}$ in Table 5.1. Here we always compute the error $\|\vec{X} - \vec{x}\|_{L^\infty} := \max_{m=1 \rightarrow M} \|\vec{X}(t_m) - \vec{x}(\cdot, t_m)\|_{L^\infty}$, where $\|\vec{X}(t_m) - \vec{x}(\cdot, t_m)\|_{L^\infty} := \max_{k=1 \rightarrow K} \left\{ \min_{\vec{y} \in \Omega} |\vec{X}^m(\vec{q}_k^m) - \vec{x}(\vec{y}, t_m)| \right\}$ between $\vec{X}$ and the true solution on the interval $[0, T]$ by employing a Newton method. We used $\tau = 0.125 h_{\vec{X}^0}^2$ and either $T = \frac{1}{2}\bar{T}$ or $T = \bar{T} - \tau$. We note that the experiments indicate that the convergence rate for the error away from the singularity is $O(h^2)$, and up to the singularity at time $\bar{T}$ is of order less than $O(h)$, which corresponds to the results obtained for the

Fig. 5.1 Plots of $\bar{X}(t)$ at times $t = 0, \frac{1}{2}\bar{T}, \bar{T} - \tau$.

Fig. 5.2 Plots of $\bar{X}(t)$ at times $t = 0, 0.01, 0.03, 0.06, T = 0.08$ (all on the same scale), and $\bar{X}(T)$ (rescaled).

isotropic case in [4]. In Figure 5.1, we present the evolution for the case $K = 3074$ and $\varepsilon = 0.5$. Here and throughout the paper, we use the same scaling for all the plots in a figure, unless otherwise stated.

An experiment for anisotropic mean curvature flow for the anisotropy function γ as in the second row in Figure 1.2 and $\beta = \gamma$ can be seen in Figure 5.2. The initial surface is given by a unit sphere, and the discretization parameters are $K = 770$, $J = 1536$, $\tau = 10^{-3}$ and $T = 0.08$. It is interesting to note that the initially spherical surface shrinks to a point by adopting an elongated shape with rounded facets that are aligned with the Wulff shape. In comparison, the same evolution for $\beta = 1$ is shown in Figure 5.3, where we integrated until time $T = 0.12$. Here the surface attains a much flatter shape. These different shapes can be explained by the fact that the chosen γ is smaller on the top and bottom facets of the Wulff shape, and hence for $\beta = \gamma$ the surface shrinks faster on the vertical sides.

Fig. 5.3 Plots of $\vec{X}(t)$ at times $t = 0, 0.01, 0.03, 0.06, T = 0.12$ (all on the same scale), and $\vec{X}(T)$ (rescaled).

Fig. 5.4 Plots of $\vec{X}(t)$ at times $t = 0, 0.1, 0.2, 0.25$.

5.2 Regularized crystalline mean curvature flow

In this subsection, we report on numerical experiments that approximate motion by crystalline mean curvature flow, see e.g. [27, 1]. To this end, we used the anisotropy as depicted in the first row of Figure 1.2, but here chose a smaller $\varepsilon = 10^{-5}$. The evolution of an initial letter “L”, that is made up of forty unit cubes and has total dimension $8 \times 2 \times 4$, can be seen in Figure 5.4, where we used the discretization parameters $K = 2818, J = 5632, \tau = 10^{-3}$ and $T = 0.25$. We note that, as predicted in [7, p. 193]; see also [8], for the true crystalline mean curvature flow, we observe facet breaking. We note that the triangulations in this simulation deteriorate slightly, with a small band of vertices forming on the surface. Although this is an undesirable effect of the tangential motion induced by our scheme, the meshes are still well behaved and the linear systems at each time step are still easy to solve. We repeated this experiment for the mobility $\beta = \gamma$ and obtained very similar results, and so omitted them here.

In a second experiment, we investigated the evolution of a dumbbell-like initial surface under motion by (regularized) crystalline mean curvature flow. Here we observe that, as in the isotropic case, a pinch-off occurs in finite time. In particular, the middle neck connecting the two cubes is thinning under the flow, until it shrinks to a line. See Figure 5.5 for details of the evolution. The initial surface has total dimen-

Fig. 5.5 Plots of $\vec{X}(t)$ at times $t = 0, 0.1, 0.18$.

Fig. 5.6 A plot of $\vec{X}(50)$ for $\varepsilon = 10^{-k}$, $k = 0 \rightarrow 3$.

sions $8 \times 3 \times 3$, with the thin middle neck having dimensions $2 \times 1 \times 1$. The chosen discretization parameters were $K = 1826$, $J = 3648$, $\tau = 10^{-4}$ and $T = 0.18$.

5.3 Anisotropic surface diffusion

In this subsection we present numerical results for our approximation (3.4), (3.3b) for a variety of anisotropies. We begin with the evolution of the unit sphere towards the Wulff shape for the anisotropies as depicted in the first row of Figures 1.1 and 1.2. In particular, we take $r = 1$, $L = 3$ and $\varepsilon = 10^{-k}$, $k = 0 \rightarrow 3$. The discretization parameters are $K = 770$ and $J = 1536$, $\tau = 10^{-3}$ and $T = 50$. See Figure 5.6 for the results, where the solutions have reached a numerically steady state. We note that our scheme manages to approximate these strongly anisotropic evolutions in a stable fashion, with the vertices of the triangulation distributed such that more mesh points are represented near the edges, and less vertices on the nearly flat facets of the Wulff shapes. We chose $T = 50$, in order to highlight the tangential movement of vertices induced by our scheme.

In addition, we present corresponding computations for the remaining anisotropies introduced in Section 1. In all the cases, we start from the unit sphere and use the discretization parameters as in Figure 5.6, with $T = 0.5$, and only varied the parameters r , L and $\{G_\ell\}_{\ell=1}^L$ associated with (1.12). The results for $r = 1$, $L = 4$ and $\varepsilon = 10^{-k}$, $k = 1 \rightarrow 3$, can be seen in Figure 5.7. Moreover, experiments for $r = 1$, $L = 6$ and $\varepsilon = 10^{-k}$, $k = 1 \rightarrow 3$, can be seen in Figure 5.8, while an approximation for (1.21), i.e. $r = 1$, $L = 2$ and $\varepsilon = 10^{-k}$, $k = 1 \rightarrow 2$, is shown in Figure 5.9.

Experiments for the anisotropies in Figure 1.4, i.e. $r = 9$ and $r = 30$, with $L = 3$, $\varepsilon = 10^{-k}$, $k = 1 \rightarrow 2$, and for the discretization parameters as before except $T = 0.02$, are reported on in Figures 5.10 and 5.11, respectively. Similarly, computations for the anisotropies at the bottom of Figure 1.5 are shown in Figure 5.12. Once again, we note

Fig. 5.7 A plot of $\vec{X}(T)$ for $\varepsilon = 10^{-k}$, $k = 1 \rightarrow 3$.

Fig. 5.8 A plot of $\vec{X}(T)$ for $\varepsilon = 10^{-k}$, $k = 1 \rightarrow 3$.

Fig. 5.9 A plot of $\vec{X}(0)$ and $\vec{X}(T)$ for $\varepsilon = 10^{-k}$, $k = 1 \rightarrow 2$.

Fig. 5.10 A plot of $\vec{X}(0)$ and $\vec{X}(T)$ for $\varepsilon = 10^{-k}$, $k = 1 \rightarrow 2$.

Fig. 5.11 A plot of $\vec{X}(0)$ and $\vec{X}(T)$ for $\varepsilon = 10^{-k}$, $k = 1 \rightarrow 2$.

Fig. 5.12 A plot of $\vec{X}(0)$ and $\vec{X}(T)$ for $\varepsilon = 10^{-k}$, $k = 1 \rightarrow 2$.

Table 5.2 Relative volume loss and some errors with respect to the true asymptotic solution $\vec{x}^* := \lim_{t \rightarrow \infty} \vec{x}(\cdot, t)$.

K	$h_{\vec{x}^0}$	$\frac{ \mathbf{V}^0 - \mathbf{V}^M }{ \mathbf{V}^0 }$	$ \mathbf{V}^M - \mathbf{V}(0) $	$ \Gamma^M _\gamma - \Gamma^* _\gamma $	$\ \vec{X}(T) - \vec{x}^*\ _{L^\infty}$
26	9.1940e-01	6.34%	1.1300e-00	1.3065e-00	6.5865e-02
98	5.3327e-01	2.87%	3.8157e-01	4.3235e-01	3.0443e-02
386	2.7688e-01	0.89%	1.0673e-01	1.2047e-01	9.7475e-03
1538	1.3975e-01	0.24%	2.7836e-02	3.1428e-02	2.6504e-03
6146	7.0041e-02	0.06%	7.0769e-03	7.9960e-03	6.8220e-04
24578	3.5041e-02	0.02%	1.7805e-03	2.0125e-03	1.7235e-04

that our approximations are able to maintain very good meshes throughout, with no mesh distortions occurring in practice.

Furthermore, we perform the following convergence test for the scheme (3.4), (3.3b). For the anisotropy (1.12) with $L = r = 1$ and $G_1 = \text{diag}(1, \varepsilon^2, \varepsilon^2)$, with $\varepsilon = 0.5$, we choose as initial shape the unit sphere, and let $\tau = 0.125 h_{\vec{x}^0}^2$ with $T = 10$, by which time the numerical solutions have reached an ellipsoidal “steady state”. In fact, the true asymptotic solution $\vec{x}^*(\cdot) := \lim_{t \rightarrow \infty} \vec{x}(\cdot, t)$ is given by

$$\vec{x}^* = \rho G_1^{\frac{1}{2}} \vec{\text{id}}_{S^2}, \quad \text{where } \rho := \varepsilon^{-\frac{2}{3}}; \quad (5.3)$$

similarly to (5.2). In Table 5.2 we report on the relative volume loss compared to the volume $\mathbf{V}^0 = \text{Vol}(\Gamma^0)$ of the initial polyhedral surface Γ^0 , as well as the error $|\mathbf{V}^M - \mathbf{V}(0)| \equiv |\text{Vol}(\Gamma^M) - \text{Vol}(\Gamma(0))|$ and the indicative error $||\Gamma^M|_\gamma - |\Gamma^*|_\gamma| \equiv ||\Gamma^M|_\gamma - \lim_{t \rightarrow \infty} |\Gamma(t)|_\gamma|$, i.e. the differences in volume and in surface energy to the true asymptotic solution, which is given by (5.3). Here we note that $|\Gamma^*|_\gamma = 4\pi\rho^2\varepsilon^2$, with ρ defined as in (5.3). We report also on the error $\|\vec{X}(T) - \vec{x}^*\|_{L^\infty}$ between $\vec{X}(T)$ and the true asymptotic solution $\vec{x}^*$. We present the triangulations $\vec{X}(T)$ at the final time $T = 10$ for two of the experiments in Figure 5.13.

5.4 Regularized crystalline surface diffusion

In this subsection, we report on numerical experiments that approximate motion by crystalline surface diffusion, see e.g. [11, 28]. To this end, we used the anisotropy as depicted in the first row of Figure 1.2. The evolution of an initial letter “L”, that is made up of four unit cubes and has total dimension $2 \times 1 \times 3$, can be seen in

Fig. 5.13 Plots of $\vec{X}(T)$ for $K = 386$ and $K = 1538$.

Fig. 5.14 Plots of $\vec{X}(t)$ at times $t = 0, 0.01, 0.05, 0.1, 0.25$.

Fig. 5.15 Plots of $\vec{X}(t)$ at times $t = 0.25, 0.5, 1, 1.5, 2, 3$.

Figure 5.14, where we used the discretization parameters $K = 2306$, $J = 4608$, $\tau = 10^{-4}$ and $T = 0.25$. As can be seen from the meshes in Figure 5.14, the triangulations become very non-uniform due to the lack of tangential motion on the almost flat faces that are aligned with the Wulff shape. We also note that no facet breaking occurs in this evolution. We now show an example of facet breaking. Starting with the same initial surface as in Figure 5.4, we compute the evolution under surface diffusion until time $T = 3$, with the remaining parameters chosen to be $K = 1410$, $J = 2816$ and $\tau = 10^{-3}$. The evolution can be seen in Figure 5.15. We observe that new facets appear, and disappear again, as the evolution progresses.

An experiment for a cuboid of dimension $4 \times 1 \times 1$ is shown in Figure 5.16. For the computation we used a triangulation with $K = 1154$ vertices and $J = 2304$

Fig. 5.16 Plots of $\vec{X}(t)$ at times $t = 0, 0.05, 0.1, 0.2, 0.5, 1$.

Fig. 5.17 Plots of $\vec{X}(t)$ at times $t = 0, 0.05, 0.1, 0.2, 0.5, 1$.

triangles. The time step size was chosen to be $\tau = 10^{-4}$ with $T = 1$. We note that at time T the cuboid has reached a numerically steady state in the form of the Wulff shape, which is close to a cube. The observed dimensions are $1.59 \times 1.59 \times 1.59$. The relative volume loss for this experiment was 0.02%. For the interested reader we note that the preconditioned CG solver for this experiment took 117 minutes, while the standard CG solver needed 571 minutes.

An experiment for a much longer cuboid of dimension $12 \times 1 \times 1$ is shown in Figure 5.17. It is interesting to note that while for isotropic surface diffusion pinch-off occurs already for the evolution of an initial $8 \times 1 \times 1$ cuboid, see e.g. [2,4], this is not the case for the crystalline case approximated here. For the computation in Figure 5.17 we used a triangulation with $K = 3202$ vertices and $J = 6400$ triangles. The time step size was chosen to be $\tau = 10^{-4}$ with $T = 1$. We note that at time T , the cuboid has reached dimensions of about $6.07 \times 1.41 \times 1.41$. The relative volume loss for this experiment was 0.005%. Finally, we investigated the evolution of the dumbbell-like initial surface from Figure 5.5 under motion by (regularized)

Fig. 5.18 Plots of $\vec{X}(t)$ at times $t = 0, 0.5, 0.7, 1, 1.2, 3$.

crystalline surface diffusion. In contrast to the corresponding mean curvature flow calculation, and in line with the previous experiments, we observe that no pinch-off occurs. Instead, the thin middle neck widens so that the surface becomes a cuboid, which then evolves towards the (almost) cubic Wulff shape as in the simulations before. See Figure 5.18 for details of the evolution, where we used the same discretization parameters as in Figure 5.5. We note that at time $T = 3$, the cuboid has reached dimensions of about $6.9 \times 2.9 \times 2.9$.

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