V. CONCLUSION

In this note, the problem of finite time stabilization is considered for a class of uncertain chained systems. Using homogeneity, time-rescaling and Lyapunov function techniques, a finite-time stabilizing feedback law is designed in order to guarantee both Lyapunov stability and finite time convergence in any given settling time for the closed-loop system. The numerical results in a wheeled mobile robot demonstrate the effectiveness of the proposed control design.

REFERENCES

- [1] A. Astolfi and W. Schaefelberger, "State and output feedback stabilization of multiple chained systems with discontinuous control," presented at the IEEE Conf. Decision Control, Kobe, Japan, 1996.
- [2] S. Bhat and D. Bernstein, "Continuous finite-time stabilization of the translational and rotational double integrators," *IEEE Trans. Autom. Control*, vol. 43, no. 5, pp. 678–682, May 1998.
- [3] A. Bloch, M. Reyhanoglu, and N. McClamroch, "Control and stabilization of nonholonomic dynamic systems," *IEEE Trans. Autom. Control*, vol. 37, no. 11, pp. 1746–1757, Nov. 1992.
- [4] S. Ge, Z. Wang, and T. Lee, "Adaptive stabilization of uncertain nonholonomic systems by state and output feedback," *Automatica*, vol. 39, pp. 1451–1460, 2003.
- [5] J. Hespanha, S. Liberzon, and A. Morse, "Logic-based switching control of a nonholonomic system with parametric modeling uncertainty," *Syst. Control Lett.*, vol. 38, pp. 167–177, 1999.
- [6] Y. Hong, "Finite-time stabilization and stabilizability of a class of controllable systems," *Syst. Control Lett.*, vol. 46, pp. 231–236, 2002.
- [7] Y. Hong, J. Wang, and D. Cheng, "Adaptive finite time stabilization of a class of nonlinear systems," presented at the IEEE Conf. Decision Control, Paradise Island, Bahamas, 2004.
- [8] Z. P. Jiang, "Robust exponential regulation of nonholonomic systems with uncertainties," *Automatica*, vol. 36, pp. 189–209, 2000.
- [9] N. Marchand and M. Alamir, "Discontinuous exponential stabilization of chained form systems," *Automatica*, vol. 39, pp. 343–348, 2003.
- [10] R. M'Closkey and R. Murray, "Exponential stabilization of driftless nonlinear control systems using homogeneous feedback," *IEEE Trans. Autom. Control*, vol. 42, no. 5, pp. 614–628, May 1997.
- [11] O. J. Sordalen and O. Egeiland, "Exponential stabilization of nonholonomic chained systems," *IEEE Trans. Autom. Control*, vol. 40, no. 1, pp. 35–49, Jan. 1995.
- [12] Z. Xi, G. Feng, Z. Jiang, and D. Cheng, "A switching algorithm for global exponential stabilization of uncertain chained systems," *IEEE Trans. Autom. Control*, vol. 48, no. 10, pp. 1793–1798, Oct. 2003.

Is Normalization Necessary for Stable Model Reference Adaptive Control?

Nikita Barabanov, Romeo Ortega, and Alessandro Astolfi

Abstract—One of the longest standing open questions in adaptive control concerns the correctness of the stability claim of the un-normalized model reference scheme proposed by Monopoli in 1974. Although provably correct solutions to the problem now abound, in particular, it is well known that adding a normalization to Monopoli's original scheme ensures global convergence, it is interesting to know whether this technique-driven modification is really necessary or only required to complete the stability proof in the absence of more elaborate arguments. In this note, we construct a counterexample that provides a definite-unfortunately, negative-answer to the claim. Instrumental for the establishment of this result is a technical lemma that shows that, under some conditions on the regressor that may appear in Monopoli's scheme, the parameter error freezes as the adaptation gain goes to infinity. On the lighter side, we also prove that Monopoli's scheme is semiglobally stable, underscoring the relevance of this important contribution.

Index Terms—Adaptive control, model reference control, nonlinear control, stability of adaptive systems.

I. INTRODUCTION

Model reference adaptive control (MRAC) is unquestionably the most widely studied problem in the adaptive literature that has a very long history going back to the 1950s and extending to the present time.¹ The earliest attempts to solve the MRAC problem followed the classical path of designing an observer, that had to be made adaptive because of the unknown plant parameters, and then feeding back the observed state [10]. A first major breakthrough, essentially due to [3] and [13], was the introduction of the so-called direct control parameterization which revealed that the estimation of the plant state could be obviated and only a "good" estimation of the controller parameters was needed to achieve the asymptotic model matching objective.

A second fundamental development, also reported in [13], was the derivation of a suitable error signal, called the augmented error, that can be used to identify the controller parameters with a quality that, as time evolves, is nondegrading. (More precisely, the norm of the parameter estimation error is a nonincreasing function.) Motivated by this important property Monopoli also presented some arguments intended to establish global convergence of his scheme, that in the sequel we will call M-MRAC. A flaw in the proof of M-MRAC was indicated in [7]. As pointed out in that paper, the authors do not provide a counterexample to the claim of stability but only question the correctness of the proof, and the problem of deciding whether or not M-MRAC is globally convergent remained unsolved for 30 years (see [16].) The purpose of this note is to give a definite *negative* answer to the question. For, we construct a *bona-fide* analytical counterexample proving

Manuscript received December 7, 2004; revised April 25, 2005. Recommended by Associate Editor S. S. Ge.

N. Barabanov is with the Department of Mathematics, North Dakota State University, Fargo, ND 58102 USA (e-mail: nikita.barabanov@ndsu.nodak.edu).

R. Ortega is with the Laboratoire des Signaux et Systèmes, CNRS-SUP-ELEC, Gif-sur-Yvette 91192, France (e-mail: ortega@lss.supelec.fr).

A. Astolfi is with the Electrical Engineering Department, Imperial College, London SW7 2BT, U.K. (e-mail: a.astolfi@ic.ac.uk).

Digital Object Identifier 10.1109/TAC.2005.854625

¹The interested reader is referred to [14] for a vivid description of the history as well as to the existing textbooks [21], [17], [12], [9] for further information on MRAC.

that the equations describing M-MRAC may generate unbounded trajectories—the qualifier “analytical” is essential to distinguish it from a simple “numerical” counterexample.

It is now well known that M-MRAC may be rendered globally convergent adding a normalization to the estimator, thus one could question the interest of analyzing the un-normalized M-MRAC. Leaving aside the aesthetical beauty of closing a long-standing open question, there are practical reasons that justify a study of the necessity, and advantages or disadvantages, of normalization—in the spirit of [15]. Indeed, the original (and perhaps only) motivation to introduce normalization is to be able to complete the proof of global convergence of MRAC. However, these proofs rely on complicated derivations that obscure the main stability issues, and only for the case of plants with relative degree one some passivity arguments can be invoked to explain the stabilization mechanism (see [2]). For larger relative degrees the task of parameter estimation and control have to be decoupled and one has to go through noninsightful and painful calculations to infer some properties on the tracking error out of the ones we have for the augmented (or prediction) error used in the estimation. This is achieved, in essence, “slowing down” the adaptation via the introduction of signal normalization. In spite of the tons of publications on the field, this highly unsatisfactory situation prevails to date and, in the authors’ opinion, partially justifies the contempt with which the adaptive control field is perceived by other control groups and the lack of interest manifested by our graduate students to pursue this topic. One objective of this note is to contribute, if modestly, to correct this situation.

The remaining of the note is organized as follows. The classical MRAC problem and one of Monopoli’s proposed solutions are briefly reviewed in Section II. Establishing that M-MRAC is incorrect is complicated by the fact, mentioned previously, that the quality of the estimation (measured in terms of the norm of the parameter error) is non-degrading as time evolves. Clearly, if it continues improving, we will eventually enter the domain of attraction of the system in closed-loop with the parameterized controller and ensure, at least, boundedness of solutions. We give in Section III a key lemma which proves that under some conditions on the regressor, that we will prove appear in M-MRAC, the *parameter error freezes* as the adaptation gain goes to infinity. This rather surprising property is then used in Section IV to construct our counterexample. We wrap up the note with a positive result that establishes semiglobal stability of M-MRAC in Section V and some concluding remarks in Section VI.

Caveat: This is an abridged version of the full paper where all the proofs are omitted. The proofs may be found in [22].

II. MONOPOLI’S MRAC

The classical MRAC problem is to develop a dynamical controller capable of causing the output of an imprecisely known single-input–single-output (SISO) linear time-invariant (LTI) continuous-time plant to asymptotically track the output of a prespecified reference model with known input. We will consider in this note only the problem of regulation to zero of the plant output, therefore, assume that the input to the reference model is zero. We will show that M-MRAC, which is the standard MRAC without estimator normalization, may generate unbounded trajectories even for this simplest task.

In this section, we recall the general formulation of MRAC as presented, for instance, in [2]; see also [17] and [9]. The plant is assumed to be minimum phase and described by $y = H(p)u$, where $H(p) \in \mathbb{R}(p)$, $p := (d/dt)$, is strictly proper. The controller is set-up generating a vector signal, called the regressor, as

$$\phi = \begin{bmatrix} F_u(p)u \\ F_y(p)y \end{bmatrix}$$

Fig. 1. Block diagram representation of error (1).

where the vector filters $F_u(p) \in \mathbb{R}^{n_u \times 1}(p)$, $F_y(p) \in \mathbb{R}^{n_y \times 1}(p)$ are strictly proper and proper, respectively. Then, we close the loop by setting $u = \hat{\theta}^T \phi$, where $\hat{\theta} \in \mathbb{R}^q$, $q = n_u + n_y$ denotes the parameter estimate.

Defining the parameter estimation error $\tilde{\theta} \triangleq \hat{\theta} - \theta$, where $\theta \in \mathbb{R}^q$ is some *constant* parameterization of the controller, we can represent the closed-loop system with the block diagram given in Fig. 1, where we have introduced the obvious partition $\hat{\theta} = \text{col}(\hat{\theta}_u, \hat{\theta}_y)$. Some elementary block diagram manipulations lead to the, so-called, error equation

$$y = H_\theta(p)(\hat{\theta}^T \phi) + \epsilon_t \quad (1)$$

where the parameterized (called “tuned” in [2]) transfer function is defined as

$$H_\theta(p) = \frac{H(p)}{1 - \theta_u^T F_u(p) - H(p)\theta_y^T F_y(p)}$$

and ϵ_t is an exponentially decaying term due to initial conditions.²

We make now the following standard assumption of MRAC.

Matching Assumption: $\exists \theta \in \mathbb{R}^q$ such that $H_\theta(p) = H_m(p)$, where $H_m(p)$ is the reference model whose relative degree is equal to the one of the plant.

Under this assumption, and neglecting the exponentially decaying terms, (1) takes the form

$$y = H_m(p)(\hat{\theta}^T \phi). \quad (2)$$

The adaptation law of M-MRAC is given by

$$\begin{aligned} e_a &= y + \hat{\theta}^T \xi - H_m(p)(\hat{\theta}^T \phi) \\ \dot{\hat{\theta}} &= -\gamma \xi e_a \end{aligned}$$

where $\gamma > 0$ is the adaptation gain and

$$\xi = H_m(p)\phi. \quad (3)$$

The signal e_a is called the augmented error and was introduced by Monopoli to ensure some nice properties of the estimator; see Remark 2. The vector signal ξ is, for obvious reasons, referred as filtered regressor.

Adding and subtracting $H_m(p)(\hat{\theta}^T \phi)$ to e_a we get $e_a = \hat{\theta}^T \xi$, which replaced in the estimator yields

$$\dot{\hat{\theta}} = -\gamma \xi \xi^T \hat{\theta}. \quad (4)$$

The equations of M-MRAC consist then of (3), (4), and

$$\phi = W(p)(\hat{\theta}^T \phi) \quad (5)$$

with

$$W(p) = \begin{bmatrix} F_u(p)H^{-1}(p)H_m(p) \\ F_y(p)H_m(p) \end{bmatrix} \in \mathbb{R}^{q \times 1}(p)$$

which is obtained using (2) and the plant inverse—that we recall is assumed minimum phase and of the same relative degree as H_m . Con-

²These terms will be omitted (without loss of generality) in the sequel.

sequently, all the transfer functions of $W(p)$ are strictly proper and stable.

To formulate our problem we find convenient to give a state–space representation of (3)–(5) as

$$\begin{aligned}\dot{x}_m &= \mathcal{A}_m x_m + \mathcal{B}_m C_W x_W \\ \dot{\hat{\theta}} &= -\gamma x_m^\top C_m^\top C_m x_m \hat{\theta} \\ \dot{x}_W &= A_W x_W + b_W \hat{\theta}^\top C_W x_W\end{aligned}\quad (6)$$

where $\mathcal{A}_m = \text{block diag}\{A_m\}$, $\mathcal{B}_m = \text{block diag}\{b_m\}$, $C_m = \text{blockdiag}\{c_m^\top\}$, and $(A_m, b_m, c_m), (A_W, b_W, C_W)$ are minimal state–space realizations of $H_m(p)$ and $W(p)$, respectively; see, e.g., [17] for the details. (For ease of reference, we note that $\phi = C_W x_W$ while $\xi = C_m x_m$.)

We are in position to present the following.

M-MRAC Conjecture: Consider the nonlinear system (3)–(5), with $W(p)$ and $H_m(p)$ strictly proper and stable-equivalently, consider (6) with A_m and A_W Hurwitz. Then, for all initial conditions and all $\gamma > 0$, all solutions of the system are bounded.³

The main contribution of the note is the construction of a counterexample to this conjecture.

Remark 1: In the standard MRAC scenario, one takes

$$\begin{aligned}F_u(p) &= \frac{1}{\delta(p)} [1, p, \dots, p^{(n-2)}]^\top \\ F_y(p) &= \frac{1}{\delta(p)} [1, p, \dots, p^{(n-2)}, \delta(p)]^\top\end{aligned}$$

with $\delta(p) = p^{(n-1)} + \sum_{i=0}^{n-2} \delta_i p^i$, a designer chosen Hurwitz polynomial. If the plant is order $n \geq 2$ with relative degree d and unitary high frequency gain, the matching assumption is satisfied with $H_m(p) = (1)/(D_m(p))$, and $D_m(p) = p^d + \sum_{i=0}^{d-1} d_{mi} p^i$ an arbitrary Hurwitz polynomial.

Remark 2: Evaluating the derivative of $V(\hat{\theta}) = |\hat{\theta}|^2$, where $|\cdot|$ is the Euclidean norm, along the trajectories of (4) we get

$$\dot{V} = -2\gamma(\hat{\theta}^\top \xi)^2 \leq 0 \quad (7)$$

from which we immediately conclude that $\hat{\theta} \in \mathcal{L}_\infty$, $|\hat{\theta}|$ is monotone nonincreasing, and $\hat{\theta}^\top \xi \in \mathcal{L}_2$ —independently of the behavior of ξ . Unfortunately, it is not known whether these properties imply some suitable property for $\hat{\theta}^\top \phi$, for instance, $\hat{\theta}^\top \phi \in \mathcal{L}_2$, which as seen in (2) is enough to prove convergence of the output to zero. To relate $\hat{\theta}^\top \phi$ with $\hat{\theta}^\top \xi$ we invoke the swapping lemma (see, e.g., [21]) which establishes that

$$H_m(p)[\hat{\theta}^\top \phi] = \hat{\theta}^\top \xi - H_m^c(p) \left[\dot{\hat{\theta}}^\top H_m^b(p)(\phi) \right]$$

where $H_m^c(p) = c_m^\top (pI - A_m)^{-1}$ and $H_m^b(p) = (pI - A_m)^{-1} b_m$. From the expression above we see the key role played by $\dot{\hat{\theta}}$. It turns out that one way to complete the stability proof—after a lengthy and uninspiring signal chasing—is ensuring that $\dot{\hat{\theta}} \in \mathcal{L}_2$, a property which is enforced with the normalized estimator

$$\dot{\hat{\theta}} = -\frac{\gamma \xi}{1 + |\xi|^2} e_a.$$

It is rather significant that after 30 years it was still an open issue as to whether the simpler un-normalized law solves (or does not solve) the MRAC problem [16].

Remark 3: It has been proven in [6] that the conjecture is true in the scalar case, i.e., when $q = 1$. In [6], this is established proving conver-

gence of the parameter estimate and invoking detectability arguments, see also [16]. In our setting, we see that this follows from the fact that, in the scalar case, $\hat{\theta} \xi \in \mathcal{L}_2$, with $\hat{\theta} \in \mathcal{L}_\infty$ and $\hat{\theta} \not\rightarrow 0$, implies $\xi \in \mathcal{L}_2$. After some signal chasing, it can be shown that this ensures not just boundedness, but also global convergence. However, in the vector case the previous implication is not true, and ξ may increase exponentially.

III. KEY FREEZING PROPERTY OF THE ESTIMATOR

In Remark 2, we established that $|\hat{\theta}|$ is monotone nonincreasing independently of the properties of ξ . On the other hand, we can apply to the third equation of (6) some basic linear time-varying systems analysis to conclude boundedness of x_W in spite of the presence of sufficiently small variations of $|\hat{\theta}|$. If $x_W \in \mathcal{L}_\infty$ it follows immediately from the first equation of (6) that $x_m \in \mathcal{L}_\infty$, and the M-MRAC conjecture would be true. Therefore, to prove the possible existence of unbounded trajectories we have to rule out this stabilization mechanism, giving conditions on ξ (and γ) that freeze $|\hat{\theta}|$ —roughly speaking, by driving x_m and $\hat{\theta}$ to infinity along orthogonal directions. Of course, we also need later to prove that this class of signals are solutions of the M-MRAC equations.

Before presenting this—rather surprising—freezing property in full generality let us work out in detail an example that captures the essence of the property.

Proposition 1: Consider the estimator (4) with $q = 2$ and $\xi = \begin{bmatrix} e^t \\ 1 \end{bmatrix}$.

Then, all solutions starting in the line $\hat{\theta}^\top(0)\xi(0) = 0$ satisfy

$$\lim_{\gamma \rightarrow \infty} \lim_{t \rightarrow \infty} \{|\hat{\theta}(0)| - |\hat{\theta}(t)|\} = 0.$$

Lemma 1 generalizes the result given previously.

Lemma 1: Consider the estimator (4) with $\xi \in \mathcal{C}^0$. Assume

- i) $\hat{\theta}^\top(0)\xi(0) = 0$;
- ii) the filtered regressor ξ is such that for some positive numbers $\alpha_1 > \alpha_2, C_1, C_2, C_3, M$ we have

$$\left| \frac{d\xi}{dt} \right| \leq M|\xi| \quad \forall t > 0 \quad (8)$$

$$C_2 \leq |\xi| \leq C_3 e^{\alpha_2 t} \quad \forall t > 0 \quad (9)$$

$$\int_\tau^t |\xi(s)|^2 ds \geq C_1(e^{\alpha_1 t} - e^{\alpha_1 \tau}) \quad \forall t > \tau. \quad (10)$$

Then

$$\begin{aligned}\lim_{\gamma \rightarrow \infty} \max_{t \geq 0} \left\{ \frac{\hat{\theta}^\top \xi}{|\xi|} \right\} &= 0 \\ \lim_{\gamma \rightarrow \infty} \lim_{t \rightarrow \infty} \{|\hat{\theta}(0)| - |\hat{\theta}(t)|\} &= 0.\end{aligned}$$

Before closing this section, we make the following important observation. The freezing property stated in the lemma concerns the behavior of the trajectories as γ goes to infinity while we are, of course, interested in the case of finite adaptation gain. However, notice that applying to (6) the transformation

$$x_m \rightarrow \frac{1}{M} x_m \quad x_W \rightarrow \frac{1}{M} x_W \quad \gamma \rightarrow M^2 \gamma \quad (11)$$

with $M > 0$, yields the same system but with adaptation gain M^2 times bigger. As all trajectories of the system are preserved under this transformation we can assume that γ is as big as we need.

IV. COUNTEREXAMPLE

The counterexample to M-MRAC conjecture is a slightly damped oscillator with a fast regressor filter obtained selecting $b_W \in \mathbb{R}^2$ and

³Actually, [13] claims not just boundedness of trajectories but also global convergence of y to zero; see also [16].

$A_W = J - \alpha I$, $\mathcal{A}_m = -\lambda I$, $C_W = \mathcal{B}_m = C_m = I$, where $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and λ, α are some positive numbers. This yields the six-dimensional system

$$\dot{x}_W = (J - \alpha I)x_W + b_W \tilde{\theta}^\top x_W \quad (12)$$

$$\dot{x}_m = -\lambda x_m + x_W \quad (13)$$

$$\dot{\tilde{\theta}} = -\gamma x_m x_m^\top \tilde{\theta} \quad (14)$$

for which we are going to prove the existence of unbounded solutions. See [22] for the underlying MRAC problem and the correspondence with the notation of [16].

The construction of the counterexample proceeds in five steps.

Step 1) From Lemma 1, we know that, under some conditions on the regressor, adaptation is frozen which, roughly speaking, amounts to saying that the vectors x_m and $\tilde{\theta}$ are orthogonal. If this is the case, and assuming that they are nonzero and continuous, we have $\dot{\tilde{\theta}} = hJx_m$, where h is a scalar function that does not change sign. On the other hand, if x_m and $\tilde{\theta}$ are orthogonal then $|\tilde{\theta}|$ is constant, and if furthermore $|\tilde{\theta}|$ and $|x_m|$ stay bounded away from zero, then

$$\tilde{\theta} \rightarrow \frac{\beta}{|x_m|} Jx_m$$

where $\beta = |\tilde{\theta}|$.⁴ This discussion motivates us to consider the auxiliary system

$$\begin{aligned} \dot{x}_W &= (J - \alpha I)x_W - b_W \frac{1}{|x_m|} x_m^\top Jx_W \\ \dot{x}_m &= -\lambda x_m + x_W \end{aligned} \quad (15)$$

where we have used $J^\top = -J$. Notice that we can write (12) as a perturbation of the first equation of (15)

$$\begin{aligned} \dot{x}_W &= (J - \alpha I - b_W \mu^\top)x_W - b_W \frac{1}{|x_m|} x_m^\top Jx_W \\ \dot{x}_m &= -\lambda x_m + x_W \end{aligned} \quad (16)$$

where $\mu = \frac{1}{|x_m|} Jx_m - \tilde{\theta}$.

Step 2) In Proposition 2, we present a general result of instability for homogeneous four dimensional systems. Specifically, we show that if the trajectories have some specific *finite time* behavior then it is possible to prove the existence of unbounded solutions. Homogeneity states that given any solution $x(t)$ and any numbers k, T the function $kx(t+T)$ is also a solution of system, which allows us to patch pieces of solutions, defined in finite time intervals, to generate the unbounded trajectories. We note at this point that the auxiliary system (15) is homogeneous.

Step 3) We then verify that the unbounded signal x_m generated as explained previously satisfies the conditions of Lemma 1. As a consequence, the freezing property ensures that the trajectories of (12)–(14) and (15) are “close,” for sufficiently large adaptation gains.

Step 4) To make the last statement formal we study the perturbed system (16) as follows. Assume the conditions of Proposition 2 are valid for the auxiliary system (15) [this will be checked in Step 5)]. Note, that all the conditions concern the behavior of the solutions on *finite* intervals. Then, we invoke continuous dependence of the solutions on these intervals, to prove that, for *all* sufficiently small functions μ , we will have unbounded trajectories of (16) close to the

Fig. 2. Closed domains D_1, D_2 of Proposition 2. (Note that $\xi = x_m$ in the counterexample.)

unbounded trajectories of (15) created in Step 2). In this step, we use the property of invariance under the scaling (11) to increase the adaptation gain and make the norm of the perturbation, $|\mu|$, as small as required.

Step 5) It only remains to show that the trajectories of (15) satisfy, for some values of the parameters α, λ, b_W , the conditions of Proposition 2. This step is carried out numerically.

For ease of reference, we organize the various statements into three propositions.

Proposition 2: Consider an homogeneous four-dimensional system $\dot{x} = f(x)$ with $x \in \mathbb{R}^4$ partitioned (for notational convenience) as $x = \text{col}(x_W, x_m)$. Assume that there exist closed domains (see Fig. 2)

$$D_1 = \{(r \cos(\varphi), r \sin(\varphi)) : r \in [r_1^0, r_2^0], \varphi \in [\varphi_1^0, \varphi_2^0]\}$$

$$D_2 = \{(r \cos(\varphi), r \sin(\varphi)) : r \in [r_3^0, r_4^0], \varphi \in [\varphi_3^0, \varphi_4^0]\}$$

with

$$0 < r_1^0 < r_3^0 < r_4^0 < r_2^0$$

$$0 < \varphi_1^0 < \varphi_3^0 < \varphi_4^0 < \varphi_2^0 < \pi$$

such that

A) for some positive numbers $T_{\min}, T_{\max}$ for any $x_m^0 \in D_1$ there exists a number $T(x_m^0) \in [T_{\min}, T_{\max}]$ such that the solution $(x_W(t), x_m(t))$ of system (15) with initial data

$$x_W(0) = \begin{bmatrix} x_{W1}(0) \\ x_{W2}(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad x_m(0) = x_m^0;$$

A1) reaches the half plane $\{x_{W1} > 0, x_{W2} = 0\}$ for the first time at $t = T(x_m^0)$, and $x_m(t) \neq 0$ for all $t \in [0, T(x_m^0)]$;

A2) the point $x_m(T(x_m^0))/|x_W(T(x_m^0))|$ belongs to D_2 ;

B) the numbers $M = \max_{x_m^0 \in D_1} \{x_{W1}(T(x_m^0))\}$, $m = \min_{x_m^0 \in D_1} \{x_{W1}(T(x_m^0))\}$ satisfy the inequalities

$$m > 1 \quad 2mT_{\min} > MT_{\max}. \quad (17)$$

Then, the system $\dot{x} = f(x)$ has an exponentially increasing solution.

Proposition 3: The signal x_m constructed in Proposition 2 satisfies the bounds (8)–(10) of Lemma 1.

Proposition 4: If the trajectories of (15) satisfy the conditions A) and B) of Proposition 2, then for sufficiently large (but finite) γ , M-MRAC system (12)–(14) has exponentially increasing solutions.

Propositions 2–4 establish the existence of unbounded solutions of M-MRAC provided the trajectories of (15) verify conditions A) and B) of Proposition 2. As the M-MRAC conjecture pertains to arbitrary values of $\alpha, \lambda > 0$ and $b_W \in \mathbb{R}^2$, to disprove the claim it suffices to establish the existence of some set of parameters such that these conditions are fulfilled. We make at this point the observation that continuous dependence of solutions with respect to parameters allows us to consider the case $\alpha = 0$, then the conditions will hold also for sufficiently small $\alpha > 0$.

⁴Without loss of generality we take $\beta = 1$ in the sequel.

Fig. 3. Unbounded trajectories of (15).

We carried out this test numerically. For, we find more convenient to write (15), with $\alpha = 0$, in polar coordinates setting

$$x_W = r_1 \begin{bmatrix} \cos(\varphi_1) \\ \sin(\varphi_1) \end{bmatrix} \quad x_m = r_2 \begin{bmatrix} \cos(\varphi_2) \\ \sin(\varphi_2) \end{bmatrix}.$$

Then, (15) may be written in the form

$$\begin{aligned} \dot{r}_1 &= r_1 \sin(\varphi_1 - \varphi_2) [b_1 \cos(\varphi_1) - b_2 \sin(\varphi_1)] \\ \dot{\varphi}_1 &= 1 + \sin(\varphi_1 - \varphi_2) [-b_1 \sin(\varphi_1) + b_2 \cos(\varphi_1)] \\ \dot{r}_2 &= -\lambda r_2 + r_1 \cos(\varphi_1 - \varphi_2) \\ \dot{\varphi}_2 &= \frac{r_1}{r_2} \sin(\varphi_1 - \varphi_2). \end{aligned}$$

Introduce now the variables

$$z = \frac{r_1}{r_2} \quad \beta = \varphi_1 - \varphi_2 \quad r = \ln(r_1) \quad \varphi = \varphi_1.$$

In these variables, the equations become

$$\begin{aligned} \dot{z} &= \{ \lambda + \sin(\beta) [b_1 \cos(\varphi) + b_2 \sin(\varphi)] - z \cos(\beta) \} z \\ \dot{\beta} &= 1 + \sin(\beta) [-b_1 \sin(\varphi) + b_2 \cos(\varphi) - z] \\ \dot{\varphi} &= 1 + \sin(\beta) [-b_1 \sin(\varphi) + b_2 \cos(\varphi)] \\ \dot{r} &= \sin(\beta) [b_1 \cos(\varphi) + b_2 \sin(\varphi)]. \end{aligned} \quad (18)$$

There is a big domain in the space (λ, b_W) for which the required behavior is observed. For instance

$$\begin{aligned} \lambda &= 10 \quad b_W = \begin{bmatrix} 0 \\ 5 \end{bmatrix} \\ r_1^0 &= 10.0 \quad r_2^0 = 10.1 \quad r_3^0 = 10.03 \quad r_4^0 = 10.09 \\ \varphi_1^0 &= 0.1925 \quad \varphi_2^0 = 0.1935, \varphi_3^0 = 0.1928 \quad \varphi_4^0 = 0.1930. \end{aligned}$$

To check the conditions of Proposition 2 we used MATLAB simulations with the MATLAB function *ode45* with relative tolerance equal to 10^{-7} , absolute tolerance equal to 10^{-12} , and refine equal to 8. We integrate the system (18) with initial conditions $(z(0), \beta(0), \varphi(0), r(0))$

such that $\varphi(0) = 0, r(0) = 0$, and the pairs $(z(0), \beta(0))$ belong to a grid of the rectangle $R_0 = [10.0; 10.01] \times [0.1925; 0.1935]$ with step 10^{-4} for each coordinate independently. We integrate equations up to the first positive number T_1 such that $\varphi(T_1) = 2\pi$ (or, equivalently, $x_{W2}(T_1) = 0$). The simulation results are the following: the values T_1 belong to the interval $[6.2842; 6.2885]$; the values $z(T_1)$ range in the interval $[10.00626337; 10.00626343]$, the values $\beta(T_1)$ belong to the interval $[0.192864503; 0.192864505]$, the value $r(T_1)$ lie in the interval $[0.0103; 0.0117]$. Taking into account that the absolute tolerance is equal to 10^{-12} , we can conclude that the whole rectangle R_0 maps into rectangle $R_1 = \{(z, \beta)\} = [10.03; 10.09] \times [0.1928; 0.1930]$ (even into a smaller rectangle). Therefore, condition (ii) of Proposition 2 is fulfilled.

We have carried out simulations first of the auxiliary system (15) with the parameters above and $\alpha = 0.001$. Instability is clearly depicted in Fig. 3. Then, we simulated the M-MRAC (12)–(14). Figs. 4 and 5 show the behavior for the initial conditions $x_W(0) = (1, 0), x_m(0) = (0, -1), \hat{\theta}(0) = (1, 0)$, and $\gamma = 100, 1e3, 1e4, 1e5, 1e6$. As predicted by the theory, as γ increases, $|\hat{\theta}|$ freezes and the cosine of the angle between $\hat{\theta}$ and x_m decreases, showing that $\hat{\theta} \rightarrow Jx_m$.

V. SEMIGLOBAL STABILITY OF M-MRAC

In this section, we show that, despite the negative result established so far, system (6) possesses some strong local stability properties. These properties can be reinterpreted (and/or exploited) to derive a semiglobal-in-the-parameter-error stability result.

Proposition 5: Consider the system (6) with $\mathcal{A}_W$ Hurwitz, $\mathcal{A}_m = -\lambda I$, with $\lambda > 0, \gamma > 0, \mathcal{B}_m = I$, and $\mathcal{C}_m = I$, i.e.,

$$\begin{aligned} \dot{x}_m &= -\lambda x_m + C_W x_W \\ \dot{\hat{\theta}} &= -\gamma x_m^\top x_m \hat{\theta} \\ \dot{x}_W &= A_W x_W + b_W \hat{\theta}^\top C_W x_W. \end{aligned} \quad (19)$$

Fig. 4. $\|\hat{\theta}\|$ of (12)–(14) for various values of γ . Fig. 5. Cosine of the angle between $\hat{\theta}$ and x_m of (12)–(15) for various values of γ .

Let $F_W = A_W + b_W \hat{\theta}^\top C_W$. For any $K > 0$ there is a $\lambda^* = \lambda^*(K) > 0$ such that if $\lambda > \lambda^*$ then any trajectory of system (19) such that $\|\hat{\theta}(0)\| < K$ is bounded and it is such that

$$\lim_{t \rightarrow \infty} x_W(t) = 0 \quad \lim_{t \rightarrow \infty} x_m(t) = 0.$$

Finally, as a simple consequence of a trivial scaling property one has the following fact.

Proposition 6: Consider system (19). Suppose that for some Hurwitz matrix A_W , and some constants $\lambda > 0$ and $\gamma > 0$ all trajectories

of the system starting in the set

$$\Omega_K = \{(x_W, x_m, \tilde{\theta}) \in \mathbb{R}^q \mid \|\tilde{\theta}\| < K\}$$

are bounded and converge to zero. Then, for all $\tilde{\gamma} > 0$ all trajectories of the system

$$\begin{aligned} \dot{x}_m &= -\lambda x_m + C_W x_W \\ \dot{\tilde{\theta}} &= -\tilde{\gamma} x_m x_m^\top \tilde{\theta} \\ \dot{x}_W &= A_W x_W + b_W \tilde{\theta}^\top C_W x_w \end{aligned} \quad (20)$$

starting in Ω_K are bounded and converge to zero.

Remark 4: The result in Proposition 5 can be proven without the considered *structural* assumptions on $\mathcal{A}_m$ and $\mathcal{B}_m$.

VI. CONCLUSION

We have constructed an analytical counterexample to the long-standing conjecture of whether MRAC with un-normalized estimators introduced by Monopoli yields bounded solutions—for all initial conditions and all tuning parameters [14]. A special mathematical machinery has been developed to solve the problem. In particular, a rather counter-intuitive, but powerful, freezing property of the un-normalized estimator has been established. Given the wide application of this kind of degenerate gradient flows, see e.g., [4], [20], and references therein, we believe this result would be of interest in other areas of systems theory. The counterexample has been constructed analyzing an homogeneous auxiliary system (15), which corresponds to the quasi-steady-state model of the original adaptive system when the adaptation gain tends to infinity.

In order to come to terms with the limits of adaptive control it is necessary to revisit the “fixes” that were introduced in an *ad-hoc* fashion during the effervescent 1980s [18]—normalization being one of them. As discussed in [15], normalization slows down the adaptation penalizing the parameter convergence rate and consequently the overall performance of the scheme. This is clearly manifested by the inability of normalized schemes to stabilize systems with nonglobally Lipschitz nonlinearities. Aware of this situation the authors recently proposed in [1] and [11], an adaptive scheme for nonlinear systems that does not rely on normalization. This framework allows to formulate MRAC as a problem of state and parameter estimation, yielding *new parameterizations* for which we dispose of alternative tuning knobs to shape the resulting error equations. Unfortunately, our first proposed scheme in the MRAC framework, reported in [19], suffers from the same drawbacks as M-MRAC. Currently, we are working on a new design that hopefully will overcome this problem.

REFERENCES

- [1] A. Astolfi and R. Ortega, “Immersion and invariance for nonlinear control systems design,” *IEEE Trans. Autom. Control*, vol. 48, no. 4, pp. 590–606, Apr. 2003.
- [2] B. D. O. Anderson, R. Bitmead, C. Johnson, P. Kokotovic, R. Kosut, I. Mareels, L. Praly, and B. Riedle, *Stability of Adaptive Systems: Passivity and Averaging Analysis*. Cambridge, MA: MIT Press, 1986.
- [3] K. Åström and B. Wittenmark, “On self-tuning regulators,” *Automatica*, vol. 9, pp. 185–199, 1973.
- [4] R. Brockett, “The rate of descent for degenerate gradient flows,” presented at the MTNS 2000, Perpignan, France, Jun. 2000.
- [5] C. Desoer and M. Vidyasagar, *Feedback Systems: Input-Output Properties*. New York: Academic, 1975.
- [6] A. Feuer, “Adaptive control of single-input–single-output linear systems,” Ph.D. dissertation, Yale Univ., New Haven, CT, 1978.
- [7] A. Feuer, B. Barmish, and A. Morse, “An unstable dynamical system associated with model reference adaptive control,” *IEEE Trans. Autom. Control*, vol. AC-23, no. 3, pp. 499–500, Jun. 1978.
- [8] G. Goodwin and K. Sin, *Adaptive Filtering Prediction and Control*. Upper Saddle River, NJ: Prentice-Hall, 1984.
- [9] P. Ioannou and J. Sun, *Robust Adaptive Control*. Upper Saddle River, NJ: Prentice-Hall, 1996.
- [10] R. Kalman, “Design of a self optimizing control system,” *Trans. ASME*, pp. 468–478, 1958.
- [11] D. Karagiannis, A. Astolfi, and R. Ortega, “Two results for adaptive output feedback stabilization of nonlinear systems,” *Automatica*, vol. 39, pp. 857–866, 2003.
- [12] I. Mareels and J. W. Polderman, *Adaptive Systems: An Introduction*. Boston, MA: Birkhäuser, 1996.
- [13] R. V. Monopoli, “Model reference adaptive control with an augmented error,” *IEEE Trans. Autom. Control*, vol. AC-19, no. 4, pp. 474–484, Oct. 1974.
- [14] A. S. Morse, “Overcoming the obstacle of high-relative degree,” *Eur. J. Control*, vol. 2, no. 1, pp. 29–35, 1996.
- [15] A. S. Morse, “A comparative study of normalized and unnormalized tuning errors in parameter adaptive control,” *Int. J. Adapt. Control Signal Processing*, vol. 6, pp. 309–318, 1992.
- [16] —, “Is Monopoli’s model reference adaptive control correct?,” in *Open Problems in Mathematical System Theory and Control*. New York: Springer-Verlag, 2002.
- [17] K. Narendra and A. Annaswamy, *Stable Adaptive Systems*. Upper Saddle River, NJ: Prentice-Hall, 1989.
- [18] R. Ortega and T. Yu, “Robustness of adaptive controllers: A survey,” *Automatica*, vol. 25, no. 5, pp. 651–677, 1989.
- [19] R. Ortega, A. Astolfi, and L. Hsu, “Immersion and invariance model reference adaptive control: New parameterizations for the problem,” presented at the 2003 IEEE Conf. Decision and Control, Maui, HI, Dec. 10–13, 2003.
- [20] A. Rantzer, “To estimate the $\mathcal{L}_2$ -gain of two dynamic systems,” in *Open Problems in Mathematical Systems and Control Theory*, V. B. Blondel, E. D. Sontag, M. Vidyasagar, and J. C. Willems, Eds. New York: Springer-Verlag, 1999.
- [21] S. Sastry and M. Bodson, *Adaptive Control: Stability, Convergence and Robustness*. Upper Saddle River, NJ: Prentice-Hall, 1989.
- [22] R. Ortega, N. Barabanov, and A. Astolfi, “Monopoli’s model reference adaptive controller is incorrect,” presented at the IFAC Workshop on Adaptation and Learning in Control and Signal Processing (ALCOSP 04), Yokohama, Honshu, Japan, Aug. 30th–Sep. 1st 2004.

Output Feedback Stabilization and Approximate and Restricted Tracking for a Class of Cascaded Systems

Alessandro Astolfi, Georgia Kaliora, and Zhong-Ping Jiang

Abstract—In this note, we discuss the problems of output feedback stabilization for a class of cascaded systems and of (approximate and restricted) output regulation for general nonlinear systems. It is shown that (global) output feedback stabilization for a class of systems in feedforward form can be achieved with a dynamic feedback law, yielding bounded control, and relying on the introduction of a reduced-order observer. The above result, together with standard tools borrowed from the output regulator theory, is instrumental to construct dynamic control laws achieving (approximate) disturbance rejection and output tracking in the presence of (small) disturbance/reference signals generated by a known exosystem.

Index Terms—Cascaded systems, output feedback, output regulation.

I. INTRODUCTION

We discuss the problems of global output feedback stabilization and approximate/restricted output tracking for a class of cascaded

Manuscript received Aug 31, 2004; revised April 20, 2005. Recommended by Associate Editor W. Kang. The work of A. Astolfi and G. Kaliora was supported in part by the EPSRC Portfolio Award, under Grant GR/S61256/01. The work of A. Astolfi was supported in part by the Leverhulme Trust. The work of Z.P. Jiang was supported in part by the National Science Foundation under Grants INT-9987317, ECS-0093176, and OISE-0408925.

A. Astolfi and G. Kaliora are with the Department of Electrical and Electronic Engineering, Imperial College London, London SW7 2AZ, U.K. (e-mail: a.astolfi@ic.ac.uk; g.kaliora@tiscali.co.uk).

Z. P. Jiang is with the Department of Electrical and Computer Engineering, Polytechnic University, Brooklyn, NY 11201 USA (e-mail: zjiang@control.poly.edu).

Digital Object Identifier 10.1109/TAC.2005.854626