Remark 4: The result in Proposition 5 can be proven without the considered *structural* assumptions on $\mathcal{A}_m$ and $\mathcal{B}_m$.

VI. CONCLUSION

We have constructed an analytical counterexample to the long-standing conjecture of whether MRAC with un-normalized estimators introduced by Monopoli yields bounded solutions—for all initial conditions and all tuning parameters [14]. A special mathematical machinery has been developed to solve the problem. In particular, a rather counter-intuitive, but powerful, freezing property of the un-normalized estimator has been established. Given the wide application of this kind of degenerate gradient flows, see e.g., [4], [20], and references therein, we believe this result would be of interest in other areas of systems theory. The counterexample has been constructed analyzing an homogeneous auxiliary system (15), which corresponds to the quasi-steady-state model of the original adaptive system when the adaptation gain tends to infinity.

In order to come to terms with the limits of adaptive control it is necessary to revisit the “fixes” that were introduced in an *ad-hoc* fashion during the effervescent 1980s [18]—normalization being one of them. As discussed in [15], normalization slows down the adaptation penalizing the parameter convergence rate and consequently the overall performance of the scheme. This is clearly manifested by the inability of normalized schemes to stabilize systems with nonglobally Lipschitz nonlinearities. Aware of this situation the authors recently proposed in [1] and [11], an adaptive scheme for nonlinear systems that does not rely on normalization. This framework allows to formulate MRAC as a problem of state and parameter estimation, yielding *new parameterizations* for which we dispose of alternative tuning knobs to shape the resulting error equations. Unfortunately, our first proposed scheme in the MRAC framework, reported in [19], suffers from the same drawbacks as M-MRAC. Currently, we are working on a new design that hopefully will overcome this problem.

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Output Feedback Stabilization and Approximate and Restricted Tracking for a Class of Cascaded Systems

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Abstract—In this note, we discuss the problems of output feedback stabilization for a class of cascaded systems and of (approximate and restricted) output regulation for general nonlinear systems. It is shown that (global) output feedback stabilization for a class of systems in feedforward form can be achieved with a dynamic feedback law, yielding bounded control, and relying on the introduction of a reduced-order observer. The above result, together with standard tools borrowed from the output regulator theory, is instrumental to construct dynamic control laws achieving (approximate) disturbance rejection and output tracking in the presence of (small) disturbance/reference signals generated by a known exosystem.

Index Terms—Cascaded systems, output feedback, output regulation.

I. INTRODUCTION

We discuss the problems of global output feedback stabilization and approximate/restricted output tracking for a class of cascaded

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systems in feedforward form. These results are based on recent developments on the global asymptotic stabilization of such systems by means of saturated partial state feedback control, see [17]. The problem of output feedback stabilization of feedforward systems has been widely addressed in the recent literature. Worth noting, to put the present contribution in the proper context, are the results in [22]. Therein, the output feedback stabilization problem for a class of cascaded systems is solved using a sort of separation principle, i.e., a state feedback control law is combined with a full-order observer, and it is shown that, under suitable technical assumptions, the resulting closed-loop system is globally asymptotically stable. Note that the state feedback stabilizing control law in [22] has to be arbitrarily small, so that its *action* on the observer can be neglected. Moreover, as the result requires the construction of a full order observer, its recursive application, which is in principle possible, requires the introduction of *redundant* observers. Note that more general results on the output feedback stabilization of feedforward systems have been proposed; see, e.g., [1]. However these results are in general semiglobal and rely on the fulfillment of some feasibility conditions, given in terms of parameterized Riccati equations. On the other hand, the problem of output regulation, and robust output regulation, of nonlinear systems has been extensively dealt with, within the last three decades [8], [14], [4], [18], [13], while extensions (to the case of unknown and nonlinear exosystems) and special cases (e.g., systems with input saturation or cascaded systems) can be found in the more recent work [26], [6], [20], and the references therein. On the basis of the seminal works [8] and [15] (see also [14]), it is nowadays apparent that, when a disturbance generated by an exosystem is driving the plant to be controlled, a controller that achieves stabilization and disturbance rejection (or output tracking) necessarily needs to *incorporate* a component constructed from a copy (or multiple copies) of the exosystem, known as internal model, together with a stabilizing component. The application of this principle in the presence of constant disturbances or constant references to be tracked is at the basis of the use of the classical *integral action*, which has to be introduced with an appropriate gain or with the addition of proportional/derivative action, to preserve stability of the closed-loop system. The extension of the above argument to nonconstant disturbances or references is fairly involved. To begin with it is in general assumed that the exosystem is Poisson stable and, whenever global problems are studied, this assumption is strengthened requiring (see, however, the recent developments in [5] and [12]) that the exosystem generates only sinusoidal signals. Then, it is noted that the key ingredient in the study of nonlinear regulation problems is the existence, for the system in closed loop with a (dynamic) controller and driven by the exosystem, of an output zeroing manifold. Goal of the controller is to render such a manifold attractive, invariant, and such that the (well-defined) dynamics on the manifold possess specific properties. Necessary conditions for the existence of such a manifold, and of a controller achieving the previously listed objectives, have been given by Isidori and Byrnes (see, e.g., [14]). Note that, unlike the case of linear systems, for general nonlinear systems the construction of the internal model is far from obvious. In the perspective of [9] (see also [7], [13], and [10]), the internal model is constructed from the parallel interconnection of a copy of the exosystem (which has to be linear) and a finite number of linear systems with eigenvalues multiple of the eigenvalues of the exosystem. This internal model, known as *k-fold* internal model, is able to guarantee approximate regulation, i.e., the regulated output will not contain frequency components corresponding to eigenvalues of the exosystem, but will not go asymptotically to zero. In the geometric perspective of Isidori and Byrnes, the exosystem is constructed using an immersion condition. This requires a realizability assumption, which

holds for linear exosystem and polynomial systems [26]. Notably, this perspective allows to cope with an unknown exosystem and to address in a simple way robustness issues [3]. Finally, in the perspective of repetitive control (see [29] and the references therein) the internal model is constructed using an infinite-dimensional component (in general a delay line) in positive feedback with a low-pass filter. This internal model is able to potentially generate all periodic signals of a given fixed period and, therefore, may be used to achieve asymptotic regulation. Note that the internal model employed in repetitive control can be interpreted as an *asymptotic* version of the *k-fold* internal model of [9]. Once the internal model is constructed, the main difficulty in the solution of global output regulation problems is the design of the stabilizing component. This design requires the construction of a (dynamic) output feedback stabilizing control law for an extended system, namely the system to be controlled in cascaded interconnection with the internal model. This problem can sometimes be resolved relaxing the global stability requirement and using semi-global stabilization tools [26], or restricting the class of systems under consideration; see, e.g., [18].

On the basis of the previous considerations, the contribution of the present note is twofold. First, the output feedback stabilization problem for a class of feedforward systems is solved by means of a dynamic output feedback control law, yielding bounded control action (as in [22]) and (unlike [22]) relying on a reduced-order observer. The stability analysis of the closed loop system is performed using the generalization of the small gain theorem given in [24] and the design tools developed in [17]. Second, the problem of *restricted and approximate* output regulation for general nonlinear systems, satisfying specific properties, is addressed and solved. The term *restricted* refers to the fact that the disturbances to be rejected and/or the references to be tracked have to be sufficiently small, so that the control law achieving exact regulation is smaller, in amplitude, than the stabilizing control. The term *approximate* refers to the fact that the regulated output will not vanish asymptotically, but only certain harmonic components will be cancelled. It is therefore obvious that, in the construction of the internal model we follow the perspective of the *k-fold* internal model, which allows to pose the output feedback regulation problem in a convenient form (see Section III).

The note is organized as follows. In Section II, the global output feedback stabilization problem for a class of cascaded systems is addressed and solved. In Section III, we address the problem of restricted and approximate output regulation for a class of nonlinear systems. Note that, for simplicity, we consider only a disturbance rejection problem, as the tracking problem can be posed and solved using similar considerations. In Section IV, we combine the results of Sections II and III to solve the problem of disturbance rejection by output feedback for a class of feedforward systems. Finally, Section V contains the discussion of a case study.

II. OUTPUT FEEDBACK STABILIZATION OF FEEDFORWARD SYSTEMS WITH BOUNDED CONTROL

To begin with, we recall some facts that are instrumental to develop the main results of this note [17]. Consider a nonlinear cascaded system described by equations of the form

$$\begin{aligned}\dot{z} &= Jz + h(\xi) + r(\xi)u \\ \dot{\xi} &= f(\xi) + g(\xi)u\end{aligned}\quad (1)$$

with state $(z, \xi) \in \mathbb{R}^p \times \mathbb{R}^n$, input $u \in \mathbb{R}$ and $f(0) = 0, h(0) = 0$. System (1) is in block feedforward form and, as such, can be asymptotically stabilized following one of the available constructive methodologies, see e.g., [17], [25], [21], [27], [19]. Here we recall one of

the main results of [17], which deals with the global asymptotic stabilization of the system (1) by means of bounded, partial state feedback. In particular, defining the saturation function $\sigma(s)$ as $\sigma(s) = \text{sign}(s) \min\{1, |s|\}$, which belongs to the sector $[0, 1]$, we state the following result.

Proposition 2.1: [17]: Consider the nonlinearity $\sigma(\cdot)$ and system (1). Suppose the following hold.

- H1) The system $\dot{\xi} = f(\xi) + g(\xi)u$ is ISS with respect to u , and $\dot{\xi} = f(\xi)$ is LES.
- H2) $J + J' = 0$!
- H3) The linear approximation of (1) is controllable.

Then, there exist $\epsilon^* > 0$ and a matrix² $K \in \mathbb{R}^{1 \times p}$ such that, for all $\epsilon \in (0, \epsilon^*)$, the static partial-state feedback control law

$$u = -\epsilon \sigma \left(\frac{1}{\epsilon} K z \right) \quad (2)$$

globally asymptotically stabilizes system (1). Furthermore, the system

$$\begin{aligned} \dot{z} &= Jz + h(\xi) - r(\xi)\epsilon \sigma \left(\frac{1}{\epsilon} K z \right) + r(\xi)w \\ \dot{\xi} &= f(\xi) - g(\xi)\epsilon \sigma \left(\frac{1}{\epsilon} K z \right) + g(\xi)w \end{aligned} \quad (3)$$

is ISS with respect to the new input w , with the restriction $|w| \leq \mu$, with $\mu < \epsilon$.

As detailed in [17], this stabilization result is based on a generalization of the nonlinear small gain theorem (see [24]) and relies on the properties of the approximation of system (1) for small $\|\xi\|$. Therefore, to state the main result on output feedback stabilization, we introduce the matrices

$$H = \left. \frac{\partial h(\xi)}{\partial \xi} \right|_{\xi=0} \quad F = \left. \frac{\partial f(\xi)}{\partial \xi} \right|_{\xi=0} \quad R = r(0) \quad G = g(0).$$

Proposition 2.2: Consider system (1) with the output

$$y = Dz. \quad (4)$$

Assume that H1)–H3) hold and, moreover, that

- H4) the pair $\{D, J\}$ is observable.

Then, there exist $\epsilon^* > 0$ and matrices L and K such that for all $\epsilon \in (0, \epsilon^*)$ the dynamic controller

$$\dot{\hat{z}} = J\hat{z} + L(D\hat{z} - y) + Ru \quad (5)$$

$$u = -\epsilon \sigma \left(\frac{1}{\epsilon} K \hat{z} \right) \quad (6)$$

globally asymptotically stabilizes system (1).

Proof: Consider the first equation of system (1) with the observer (5) and define the error variable $e = \hat{z} - z$. Note now that, selecting the matrix L such that $J + LD$ is Hurwitz, the system composed of (1) and (5) can be written in the form

$$\begin{aligned} \dot{\hat{z}} &= J\hat{z} + \mathcal{H}\zeta + Ru \\ \dot{\zeta} &= \mathcal{F}\zeta + \mathcal{G}u + \delta(\xi, u) \end{aligned} \quad (7)$$

where $\zeta = [e', \xi']'$ and the matrices $\mathcal{H}$, $\mathcal{F}$, and $\mathcal{G}$ are defined as

$$\mathcal{F} = \begin{bmatrix} J + LD & -H \\ 0 & F \end{bmatrix} \quad \mathcal{G} = \begin{bmatrix} 0 \\ G \end{bmatrix} \quad \mathcal{H} = [LD \quad 0].$$

¹H2) can be replaced by $J'Q + QJ = 0$ for some $Q = Q' > 0$.

²A constructive procedure to compute the matrix K is given in [17].

The vector $\delta(\xi, u)$ represents the difference between the nonlinear vector field $[(h(\xi) + Ru - r(\xi)u)', (f(\xi) + g(\xi)u)']'$ and its (linear) approximation for small $\|\xi\|$, namely $[(H\xi)', (F\xi + Gu)']'$. System (7) satisfies the assumptions of Proposition 2.1,³ therefore there exists a matrix K and a positive constant ϵ^* such that, for all $\epsilon \in (0, \epsilon^*)$, the controller given by (6) renders the equilibrium $[\hat{z}', \zeta']' = [0, 0]'$ globally asymptotically stable. As $z = \hat{z} - e$, the equilibrium $[z', \xi']' = [0, 0]'$ of (1) is globally asymptotically stable, as well. ■

The result of Proposition 2.2 can be improved, following the approach in [17], by showing that system (1) in closed loop with the controller (5)–(6), in which u is replaced by $u = -\epsilon \sigma((1/\epsilon)K\hat{z}) + v$, is ISS with respect to the new input v and with restriction $|v| < \epsilon$, see [17], as in Proposition 2.1. For brevity, we omit the detail. Thus, Proposition 2.2 can be applied iteratively to construct output feedback stabilizing control laws for a general class of systems in feedforward form, namely systems described by equations of the form

$$\begin{aligned} \dot{z}_1 &= J_1 z_1 + h_1(z_2, \dots, \xi) + r_1(z_2, \dots, \xi)u & y_1 &= D_1 z_1 \\ \dot{z}_2 &= J_2 z_2 + h_2(z_3, \dots, \xi) + r_2(z_3, \dots, \xi)u & y_2 &= D_2 z_2 \\ &\vdots & & \\ \dot{z}_m &= J_m z_m + h_m(\xi) + r_m(\xi)u & y_m &= D_m z_m \\ \dot{\xi} &= f(\xi) + g(\xi)u \end{aligned} \quad (8)$$

as detailed in the following statement, the proof of which is trivial hence omitted.

Corollary 2.1: Consider system (8) and assume the following hold.

- H1') The linear approximation of (8) is controllable.
- H2') The system $\dot{\xi} = f(\xi) + g(\xi)u$ is ISS and $\dot{\xi} = f(\xi)$ is locally exponentially stable.
- H3') $J_i + J_i^T = 0$, for $i = 1, \dots, m$.⁴
- H4') The pairs $\{D_i, J_i\}$, for $i = 1, \dots, m$, are observable.

Then, there exist $\epsilon^* > 0$ and matrices L_i and K_i , for $i = 1, \dots, m$, such that, for all $\epsilon \in (0, \epsilon^*)$, the dynamic controller

$$\begin{aligned} \dot{\hat{z}}_1 &= J_1 \hat{z}_1 + L_1(D_1 \hat{z}_1 - y_1) + R_1 u \\ \dot{\hat{z}}_2 &= J_2 \hat{z}_2 + L_2(D_2 \hat{z}_2 - y_2) + R_2 u \\ &\vdots \\ \dot{\hat{z}}_m &= J_m \hat{z}_m + L_m(D_m \hat{z}_m - y_m) + R_m u \\ u &= -\frac{\epsilon}{2} \sigma \left(\frac{2}{\epsilon} K_m \hat{z}_m \right) - \dots - \frac{\epsilon}{2^m} \sigma \left(\frac{2^m}{\epsilon} K_1 \hat{z}_1 \right) \end{aligned} \quad (9)$$

with $R_i = r_i(0)$, globally asymptotically stabilizes (8).

Remark 2.1: The results of Propositions 2.2 and 2.1 have to be compared with the results in [22], where the problem of output feedback stabilization for a class of cascaded systems is addressed and solved using bounded control laws and a full order observer. Following the procedure presented therein, the construction of an output feedback stabilizing control law for (8) requires at each step the construction of a full order observer, i.e., the state ξ has to be reconstructed m times, the state z_m $m - 1$ times, and so on. This redundancy does not arise in our design. However, similar to [22], the computation of the stabilizing gains K_i in the construction of the observer (9), needs to be performed in steps, i.e., the gains K_m to K_i need to be computed before K_{i-1} can be calculated. Also, the results in [22] apply to a slightly more

³The proof that the cascade (7) is controllable is based on a simple application of Hautus test. For brevity we omit the details.

⁴This assumption can be replaced by $J_i^T P_i + P_i J_i = 0$, for some $P_i = P_i' > 0$, and for $i = 1, \dots, m$.

general class of systems, i.e., the matrices J_i are allowed to have multiple eigenvalues on the imaginary axis. It is fair to say, though, that the design proposed therein leads to more complex formulas than the one proposed in this note.

III. DISTURBANCE REJECTION OF NONLINEAR SYSTEMS WITH BOUNDED CONTROL

In this section, we consider systems described by equations of the form

$$\begin{aligned}\dot{x} &= f(x) + g(x)u + \beta_1(x)w \\ y &= h(x)\end{aligned}\quad (10)$$

with state $x \in \mathbb{R}^n$, control $u \in \mathbb{R}$, output $y \in \mathbb{R}$, and disturbance $w \in \mathbb{R}^p$, which is generated by an exosystem of the form

$$\dot{w} = Sw \quad (11)$$

and we address the problem of restricted approximate output regulation by bounded feedback, i.e., the problem of determining a bounded control law such that the effect of w on y is *attenuated* (in a way that will be specified) provided that $\|w\|$ is sufficiently small.

Throughout this section, the following assumptions are made.

- A1) System (10) is ISS with respect to u and w and $\dot{x} = f(x)$ is locally exponentially stable.
- A2) System (11) is *neutrally stable*, i.e., without loss of generality, $S + S' = 0$.

Consider now the system (10). Assumption A1) implies that, for each $\delta > 0$, there exist constants $\epsilon_u^* > 0$ and $\epsilon_w^* > 0$ such that if $|u(t)| \leq \epsilon_u^*$ and $\|w(t)\| \leq \epsilon_w^*$ for all $t > 0$, then the state of the nonlinear system (10) enters in finite time in the region $\|x\| \leq \delta$, and remains therein for all t sufficiently large. As a result, following the approach in [17], it is enough to consider the approximation of the system for small $\|x\|$.

Proposition 3.1: Consider the system (10) and suppose that the disturbance w is generated by the exosystem (11). Suppose that Assumptions A1) and A2) hold. Suppose moreover that the linear approximation of the system around the origin has no transmission zeros on the imaginary axis.⁵

Then, there exist positive constants ϵ_w^* , ϵ_u^* , and δ_y , and matrices L and K such that for all $\epsilon \in (0, \epsilon_u^*)$ and for all disturbances such that $\|w(t)\| \leq \epsilon_w^*$ for all $t \geq 0$, the bounded dynamic output feedback control law

$$\dot{z} = Sz + Ly \quad u = -\epsilon\sigma\left(\frac{1}{\epsilon}Kz\right) \quad (12)$$

yields closed loop trajectories with the following properties.

- P0) The closed-loop system with $w = 0$ is globally asymptotically stable.
- P1) There exists $T > 0$ such that $\|y(t)\| \leq \delta_y$ for all $t \geq T$.
- P2) There exists a periodic function $y_{ss}(t)$ independent of the initial conditions such that, for any initial condition, one has $\lim_{t \rightarrow \infty} (y(t) - y_{ss}(t)) = 0$. Moreover, $y_{ss}(t)$ does not contain frequency components corresponding to eigenvalues of the matrix S .

Proof: According to the results in [17] there exists a positive constant ϵ_u^* and a matrix K such that, for $w = 0$ and all $\epsilon \in (0, \epsilon_u^*)$, the closed-loop system (10)–(12), which is given by the cascade

$$\dot{z} = Sz + Lh(x) \quad \dot{x} = f(x) - g(x)\epsilon\sigma\left(\frac{1}{\epsilon}Kz\right)$$

⁵This assumption may be relaxed requiring only that the transmission zeros are not eigenvalues of the exosystem.

is globally asymptotically and locally exponentially stable. Note now that for small enough disturbances w the well defined steady state trajectories [15] of the closed-loop system (10)–(12) will be given by $z_{ss}(t) = \pi_z(w(t))$, $x_{ss}(t) = \pi_x(w(t))$, where the functions $\pi_z(w)$ and $\pi_x(w)$ are such that $\pi_z(0) = 0$ and $\pi_x(0) = 0$ and solve (locally) the set of partial differential equations

$$\begin{aligned}\frac{\partial \pi_z(w)}{\partial w} Sw &= S\pi_z(w) + Lh(\pi_x(w)) \\ \frac{\partial \pi_x(w)}{\partial w} Sw &= -g(\pi_x(w))\epsilon\sigma\left(\frac{1}{\epsilon}K\pi_z(w)\right) \\ &\quad + f(\pi_x(w)) + b_1(\pi_x(w))w.\end{aligned}$$

Hence, the claim is a consequence of the general results in [9]. ■

Remark 3.1: The suggested design is based on a linear way of thinking, hence, for general nonlinear systems and as pointed out in [13], there is no guarantee that the *resonant* frequencies of the steady-state output will be unaffected by the disturbance. A way to partly overcome this problem is to use the k -fold internal model of [9]. For example, if the exosystem generating the disturbance is such that

$$S = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}$$

then a controller that uses the dynamic compensator $\dot{z} = S_h z + L_y y$ with $S_h = \text{blockdiag}\{S, 2S, \dots, kS\}$ with $k \geq 1$, is such that the steady-state output response is periodic and does not have components at angular velocities $\omega, 2\omega, \dots, k\omega$.

IV. ASYMPTOTIC STABILIZATION AND DISTURBANCE REJECTION FOR CASCADED SYSTEMS BY MEANS OF OUTPUT FEEDBACK

The combination of the results in Propositions 2.2 and 3.1 provide a solution to the problem of approximate and restricted output regulation by means of error feedback for a class of feedforward systems, namely systems of the form

$$\begin{aligned}\dot{z} &= Jz + h(\xi) + r(\xi)u + r_1(\xi)w \\ \dot{\xi} &= f(\xi) + g(\xi)u + g_1(\xi)w \\ y &= Cz\end{aligned}\quad (13)$$

where y represents the measured output, that needs to be regulated to zero, and $w \in \mathbb{R}^p$ a disturbance that is generated by an exosystem of the form $\dot{w} = Sw$.

Proposition 4.1: Consider the system (13), where w is a disturbance generated by an exosystem of the form (11), with $S + S' = 0$. Assume the following hold.

- The system $\dot{\xi} = f(\xi) + g(\xi)u + g_1(\xi)w$ is ISS with respect to both u and w , and $\dot{\xi} = f(\xi)$ is LES.
- $J + J' = 0$.
- The pair $\{C, J\}$ is observable.

Then, there exist positive constants ϵ_w^* , ϵ_u^* , and δ_y , column vectors L_1, L_2 , and R and row vectors K_1 and K_2 such that for all $\epsilon \in (0, \epsilon_u^*)$ and for all disturbances such that $\|w(t)\| \leq \epsilon_w^*$ for all $t \geq 0$, the dynamic control law

$$\begin{aligned}\dot{\zeta} &= S\zeta + L_2 y \\ \dot{z} &= Jz + L_1(C\zeta - y) + Ru \\ u &= -\epsilon\sigma\left(\frac{1}{\epsilon}K_1\zeta\right) - \frac{\epsilon}{2}\sigma\left(\frac{2}{\epsilon}K_2\zeta\right)\end{aligned}\quad (14)$$

achieves *global asymptotic stability* for $w = 0$, and for $w \neq 0$, yields closed-loop trajectories with the following properties.

- P1) There exists $T > 0$ such that $\|y(t)\| \leq \delta_y$ for all $t \geq T$.

- P2) There exists a periodic function $y_{ss}(t)$ independent of the initial conditions such that, for any initial condition, one has $\lim_{t \rightarrow \infty} (y(t) - y_{ss}(t)) = 0$. Moreover, $y_{ss}(t)$ does not contain frequency components corresponding to eigenvalues of the matrix S .

Proof: First, consider the system

$$\begin{aligned}\dot{z} &= Jz + h(\xi) + r(\xi)u \\ \dot{\xi} &= f(\xi) + g(\xi)u \\ y &= Cz\end{aligned}\quad (15)$$

which is (13) for $w = 0$, and note that, by the result of Proposition 2.2, an output feedback dynamic controller of the form

$$\dot{\hat{z}} = J\hat{z} + L_1(C\hat{z} - y) + Ru \quad u = -\epsilon\sigma\left(\frac{1}{\epsilon}K_1\hat{z}\right) + v$$

with $R = r(0)$, achieves global asymptotic, local exponential and *input to state* stability for system (15) with the restriction $|v| < \epsilon$. The matrix L_1 is such that $J + L_1C$ is Hurwitz, whereas the matrix K_1 is such that the gain of the linearized closed-loop system from v to the “output” $K_1\hat{z}$ is less than or equal to one (see [17]). With the previous controller, and using the coordinate transformation $e = \hat{z} - z$, (13) writes

$$\begin{aligned}\dot{\hat{z}} &= J\hat{z} + L_1Ce - R\epsilon\sigma\left(\frac{1}{\epsilon}K_1\hat{z}\right) + Rv \\ \dot{e} &= (J + L_1C)z - h(\xi) + \delta_1(\xi, \hat{z}) + (R - r(\xi))v - r_1(\xi)w \\ \dot{\xi} &= f(\xi) - g(\xi)\epsilon\sigma\left(\frac{1}{\epsilon}K_1\hat{z}\right) + g(\xi)v + g_1(\xi)w \\ y &= C(\hat{z} - e)\end{aligned}\quad (16)$$

where $\delta_1(\xi, \hat{z}) = -(R - r(\xi))\epsilon\sigma((1/\epsilon)K_1\hat{z})$ denotes higher order terms. System (16), which can be written in compact form as

$$\dot{x} = \alpha(x) + b(x)v + b_1(x)w \quad y = C_*x$$

with

$$\begin{aligned}x &= \begin{bmatrix} \hat{z} \\ e \\ \xi \end{bmatrix} & \alpha(x) &= \begin{bmatrix} J\hat{z} + L_1Ce - R\epsilon\sigma\left(\frac{1}{\epsilon}K_1\hat{z}\right) \\ (J + L_1C)z - h(\xi) + \delta_1(\xi, \hat{z}) \\ f(\xi) - g(\xi)\epsilon\sigma\left(\frac{1}{\epsilon}K_1\hat{z}\right) \end{bmatrix} \\ b(x) &= \begin{bmatrix} R \\ R - r(\xi) \\ g(\xi) \end{bmatrix} & b_1(x) &= \begin{bmatrix} 0 \\ -r_1(\xi)w \\ g_1(\xi)w \end{bmatrix} \\ C_* &= [C \quad -C \quad 0]\end{aligned}$$

represents a globally asymptotically stable system that is driven by the disturbance w , generated by the exosystem (11). Moreover, (16) is input to state stable with respect to both v and w . Therefore, the result of Proposition 3.1 can be applied to show that (P1) and (P2) hold, provided that $|w|$ is small enough, i.e., there exist matrices L_2 and K_2 such that the dynamic control law

$$\dot{\zeta} = S\zeta + L_2y \quad v = -\frac{\epsilon}{2}\sigma\left(\frac{2}{\epsilon}K_2\zeta\right)$$

yields closed-loop trajectories with the properties P1) and P2). ■

It must be noted that the problem considered in this section has been solved, by means of state feedback, but without restrictions on the amplitude of the disturbance in [20].

V. OUTPUT REGULATION OF THE TORA

In this section, we illustrate the contribution of the note considering the disturbance rejection problem for a nonlinear benchmark system: The *translational oscillator with a rotational actuator* (TORA) [2]. The (robust) disturbance rejection problem for such a system has also been studied, from an L_2 perspective, in [28] and, from the perspective of local (exact) regulation, in [11, Ch. 6], whereas various constructive nonlinear control methodologies have been tested on this system; see, for example, [16], [22], [25], and [23]. The system is four-dimensional, with states the translational and angular positions x_d and ϕ , respectively, and the corresponding velocities v_d and ω . In the coordinates

$$\begin{aligned}x_1 &= x_d + \gamma \sin \phi & x_2 &= v_d + \gamma \omega \cos \phi \\ x_3 &= \phi & x_4 &= \psi(x_3)\omega\end{aligned}\quad (17)$$

and for some appropriate function $\psi(x_3)$, the system can be written in *block feedforward* form. The available measurements for control are the positions x_d and ϕ or, equivalently, the states x_1 and x_3 . In [17], the (global) output feedback stabilization problem has been solved by means of a dynamic controller. According to the construction therein, and following [28] for the introduction of disturbances into the closed loop, the model of the “controlled” TORA, subject to a sinusoidal disturbance w acting on the *translational movement*, is described by

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \gamma \sin x_3 - p(x_3)\gamma \cos(x_3)w \\ \dot{x}_3 &= \alpha(x_3)x_4 \\ \dot{x}_4 &= -x_3\alpha(x_3) - x_4 - z - \epsilon\sigma\left(\frac{1}{\epsilon}kx_1\right) \\ &\quad + v - p(x_3)\psi(x_3)w \\ \dot{z} &= -\lambda(x_3)z - b\epsilon\sigma\left(\frac{1}{\epsilon}kx_1\right) \\ &\quad + bv + p(x_3)\psi(x_3)w\end{aligned}\quad (18)$$

where $\alpha(x_3)$, $\psi(x_3)$, $\lambda(x_3)$, and $p(x_3)$ are known functions, and γ, b, k, ϵ are known parameters of the system and of the stabilizing controller. The output to be regulated is the horizontal position of the system given by

$$y = x_d = x_1 - \gamma \sin(x_3). \quad (19)$$

System (18) is GAS (LES) for $w = 0$ and $v = 0$ and it ISS with respect to w and v . Moreover, due to ISS of the subsystem of $x_3 - x_4 - z$, $|x_3|$ will be small enough in finite time, and therefore the approximation $\sin(x_3) = x_3$ can be used.⁶ For the purpose of illustration consider the response of (18) driven by a nonzero disturbance w , generated by the exosystem (11), as depicted in Fig. 1(left graphs). To attenuate the effect of the disturbance on the output (19) we introduce, as discussed in Section IV, an “internal model”-based dynamic compensator with a bounded action, namely

$$\dot{\zeta} = S\zeta + L_2y \quad v = -\frac{\epsilon}{2}\sigma\left(\frac{2}{\epsilon}K_2\zeta\right). \quad (20)$$

The response of the closed-loop system is shown in Fig. 1(right graphs). Note the improved asymptotic behavior of the regulated output x_d .

In Fig. 2, we plot the spectrum decomposition of the signal $y(t)$ when only a stabilizing controller is applied (top graph) and when the controller (20) is used (middle graph). When the primary oscillation frequency ω is attenuated [by means of the dynamic feedback (20)] it is possible to detect a harmonic at 3ω . This third harmonic can also be

⁶Hence the output y can be regarded *linear*, i.e., $y = x_1 - \gamma x_3$.

Fig. 1. Response of the TORA system, driven by a sinusoidal disturbance, in closed loop with a stabilizing controller (left graphs) and in closed loop with the control law (20) (right graphs).

Fig. 2. Spectrum decomposition of the signal $y(t)$ when only a stabilizing control law is used (top graph), when the controller (20) is used (graph in the middle) and when a controller with a k -fold internal model is used (bottom graph). Note the different scale on the $|Y(j\omega)|$ axis.

attenuated if a k -fold internal model is used, as explained in Remark 3.1. The result is illustrated in the bottom graph of Fig. 2.

VI. CONCLUSION AND OPEN PROBLEMS

The problem of global asymptotic stabilization of a class of feed-forward systems has been addressed and solved using dynamic output feedback control laws delivering a bounded action. This result has been exploited to solve the problem of approximate and restricted output regulation for general nonlinear systems, provided that the exogenous signals are generated by a linear exosystem. Combination of these two results provide straightforward answers to the problem of stabilization and approximate and restricted output regulation for systems in feed-forward form, by means of error feedback. The theory has been illustrated via a simple simulation example. Extensions of the proposed re-

sults to solve the problem of exact (asymptotic) regulation, and to include nonlinear exosystems, exploiting the results in [20] and [12], are under investigation.

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Kalman Filtering in Extended Noise Environments

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Abstract—This note introduces an extended environment for Kalman filtering that considers also the presence of additive noise on input observations in order to solve the problem of optimal (minimal variance) estimation of noise-corrupted input and output sequences. This environment includes as subcases both errors-in-variables filtering (optimal estimate of inputs and outputs from noisy observations) and traditional Kalman filtering (optimal estimate of state and output in presence of state and output noise). A Monte Carlo simulation shows that the performance of this extended filtering technique leads to the expected minimal variance estimates.

Index Terms—Errors-in-variables filtering, Kalman filtering, optimal filtering, recursive filtering.

I. INTRODUCTION

Very few algorithms, if any, have seen so many applications in different areas as Kalman filtering that constitutes the *de facto* standard for information retrieval from noisy data generated by known processes and affected by noise with known statistical properties. This wide success could be considered somehow surprising in front of the relatively simple stochastic environment considered by this approach and of the requirement of information, usually unknown, on noise statistics. These drawbacks are, in fact, overcome by the robustness of the filter versus modeling errors and by the possibility of monitoring its performance and of deducing from the innovation of nonoptimal filters some information about the unknown noise properties [1]–[4].

A limit of the stochastic environment of Kalman filtering concerns its asymmetrical description of the uncertainties on the observations; in fact, while the output is considered as affected by additive noise, the input is assumed as exactly known. This condition is met in all control applications where the process input is generated by known laws but can be restrictive in other contexts.

Symmetrical environments are, instead, at the basis of errors-in-variables (EIV) models that consider all system attributes as affected by unknown additive disturbances. This symmetry allows, in many cases, to avoid system orientation, i.e., the necessity of partitioning observations into inputs and outputs [5].

Kalman filtering cannot be directly applied, as discussed in [6], to EIV filtering problems, i.e., to the optimal reconstruction of inputs and outputs of EIV models on the basis of their noise-corrupted observations. The problem of EIV filtering has been recently solved in [6] and [7] by making reference to both behavioral and state-space contexts, starting from the solution of optimal EIV interpolation. The numerical aspects of this problem have then been investigated in [8] where an high-efficiency algorithm, based on the properties of Cholesky factorization, has been described.

Optimal EIV filtering can be approached also in a deterministic context, as an optimization problem along the line followed by Roorda and Heij [9], as described in [6]. An approach of this kind has been recently used also in [10].

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