A Remark on the Stability of Interconnected Nonlinear Systems

A. Rapaport and A. Astolfi

Abstract—We use a notion of generalized L_2 -gain for nonlinear systems, where the gain is considered as a function of the state instead of a (global) constant, for characterizing stability properties of interconnected systems. The obtained results generalize the well-known small-gain theorem.

Index Terms—Interconnected systems, L_2 -gain, nonlinear systems, small-gain theorem, stability.

I. INTRODUCTION

The L_2 -gain analysis of affine nonlinear systems has been widely studied in the literature. For a state-of-the-art, see e.g., [5] and [17] for the smooth case, and [9] for the nondifferentiable one. All these studies deal with the existence of a solution V to a partial differential inequality $\mathcal{H}(V, \gamma) \leq 0$, which reflects the *dissipative* behavior of the system for a particular supply rate $r(u, y) = \gamma^2 |u|^2 - |y|^2$, where u and y are respectively the input and output of the system (see [18] for a general framework on dissipativity).

In [10], we have proposed a definition of a *generalized* L_2 -gain property, where the L_2 -gain constant γ is replaced by $\mu = \mu(\cdot)$, a positive function of the state. This new notion of gain stems from the observation that nonlinear systems have nonuniform properties, i.e., the L_2 -gain of a nonlinear system changes with the operating point (this remark has already been made by several authors; see, for instance, [4] and [6]). Our approach is different from the usual ones [18], in the sense that we consider the dissipativity of dynamical systems with respect to supply rates which are functions not only of the norms of the input and the output (as usual) but also of the state of the system: $r(x, u, y) = \mu(x)^2 |u|^2 - |y|^2$. This allows us to preserve the structure of the Hamiltonian $\mathcal{H}(V, \mu)$ which defines the partial differential inequality, where only γ is replaced by $\mu(\cdot)$. Typically, for nonlinear systems, μ cannot be bounded on the whole state space and a finite L_2 -gain cannot be guaranteed in the usual sense (i.e., there does not exist any nonnegative solution to the partial differential inequality with constant γ on the whole state space). Nevertheless, one may require that the function μ is bounded from above by a certain constant $\gamma > 0$ when the state of the system evolves outside a neighborhood of the origin (If this neighborhood can be made arbitrarily small, adjusting the function μ , the system is then said to have *practical* L_2 -gain γ , see [10]). These concepts are illustrated by several examples in [10], where the *practical* L_2 disturbance attenuation problem is also investigated (see also [2] for a complete application).

Other definitions of input-output gains for nonlinear systems have been also proposed in the literature, such as the input-output stability (IOS) [15] (see also integral IOS (iISS) [1], [11], output-to-state stability (OSS) [14] and input-to-state stability with respect to sets [13]), where asymptotic gains are sought as functions of the norm of the input, characterized then by slightly different differential inequalities.

In this note, we show that our concept of *generalized* L_2 -gain provides also some new results concerning the stability of interconnected nonlinear systems, which generalizes in some extent the well known small-gain theorem (see, for instance, [3]).

Several extensions of the small-gain theorem have already been proposed in the literature (see [7] and [16]), where the (constant) gains are replaced by functions of the norm of the input(s). Alternatively, we propose in this note an extension of the small-gain theorem in terms of generalized L_2 gains.

The note is organized as follows: In Section II, we recall our definition of generalized L_2 -gain property. In Sections III and IV, we present an extension of the small-gain theorem (without input) and then give input-output properties of interconnected systems. Finally, in Section V, we illustrate these new results on an example.

II. PRACTICAL L_2 -GAIN PROPERTY

Consider an input-output dynamical system:

$$\begin{cases} \dot{x} = f(x) + k(x)w \\ y = h(x) \end{cases} \quad (1)$$

in which $x \in \mathbb{R}^n$, $w \in \mathbb{R}^q$ and $y \in \mathbb{R}^s$ are, respectively, the state, the input, and the output. The functions f , k and h are assumed to be smooth and such that $f(0) = 0$ and $h(0) = 0$. The notation $x(t, x_0, w)$ stands for the absolutely continuous solution of (1) at time $t > 0$ with the initial condition $x(0) = x_0$ under the input $w(\cdot)$.

In the following, $\mathcal{O}$ is an open set of $\mathbb{R}^n$, μ a positive function defined on $\mathcal{O}$ and γ a positive number.

Definition 1: System (1) has **generalized L_2 -gain** μ on $\mathcal{O}$ if there exists a nonnegative *storage* function V , locally bounded on $\mathcal{O}$, such that for any initial condition $x_0 \in \mathcal{O}$ and input $w \in L_2$ we have

$$\begin{aligned} \forall T > 0 \text{ s.t. } x(t, x_0, w) \in \mathcal{O} \quad \forall t \in (0, T] \\ V(x_0) \geq V(x(T)) \\ - \frac{1}{2} \int_0^T \mu(x(\tau))^2 |w(\tau)|^2 - |y(\tau)|^2 d\tau. \end{aligned} \quad (2)$$

Unlike the linear case, the gain $\mu(\cdot)$ is here sought as a function of the state and not just a constant. The connection with the L_2 -gain in the usual sense is the following.

Definition 2: System (1) has L_2 -gain γ on $\mathcal{O}$ if it has generalized L_2 -gain μ on $\mathcal{O}$ for a certain μ bounded above by γ .

Remark 1: The usual L_2 -gain property is exactly recovered with $\mathcal{O} = \mathbb{R}^n$.

Although the notion of L_2 -gain is defined for nondifferentiable functions [9], only C^1 conditions shall be considered in the following. For a C^1 candidate *storage function* V , let $\mathcal{H}(V, \mu)$ be the Hamiltonian function, defined as

$$\mathcal{H}(V, \mu)(x) = L_f V(x) + \frac{[L_k V(x)]^2}{2\mu(x)^2} + \frac{|h(x)|^2}{2}.$$

We then recall the key result (see [10]).

Theorem 1: If there exist a C^1 nonnegative function V and a positive function μ , defined on $\mathcal{O}$, such that

$$\mathcal{H}(V, \mu)(x) \leq 0 \quad \forall x \in \mathcal{O} \quad (3)$$

then the system (1) has generalized L_2 -gain μ on $\mathcal{O}$.

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Fig. 1. Interconnection $\Sigma_{1,2}$.

III. STABILITY OF INTERCONNECTED SYSTEMS

Our motivation for studying interconnected systems comes from a natural extension of the small gain theorem (see [3]), which claims the stability of the interconnected system when the product of the L_2 -gains of the two subsystems is strictly less than one.

Consider two input-output systems

$$\Sigma_i : \begin{cases} \dot{x}_i = f_i(x_i) + k_i(x_i)w_i \\ y_i = h_i(x_i), \end{cases} \quad i = 1, 2$$

with $(x_1, w_1, y_1) \in \mathbb{R}^{n_1} \times \mathbb{R}^q \times \mathbb{R}^s$ and $(x_2, w_2, y_2) \in \mathbb{R}^{n_2} \times \mathbb{R}^s \times \mathbb{R}^q$.

Assume that Σ_1 and Σ_2 have generalized L_2 -gains, respectively μ_1 on $\mathcal{O}_1 \subset \mathbb{R}^{n_1}$ and μ_2 on $\mathcal{O}_2 \subset \mathbb{R}^{n_2}$, continuous and bounded on their domains, with radially unbounded C^1 storage functions V_1 and V_2 , that fulfill the condition (3) of Theorem 1. Denote $\mathcal{H}_1$ and $\mathcal{H}_2$ their associated Hamiltonian functions, and define the numbers

$$\gamma_i = \sup_{x_i \in \mathcal{O}_i} \mu_i(x_i) < +\infty, \quad i = 1, 2.$$

Definition 3: Let S_i be a subset of $\mathbb{R}^{n_i}$. The system Σ_i is S_i —observable if for any $t \geq 0$, one has the following property:

$$\begin{aligned} \{w_i(s) = 0, y_i(s) = 0 \quad \forall s \in [0, t]\} \\ \implies \{x_i(s) \in S_i \quad \forall s \in [0, t]\}. \end{aligned}$$

The system Σ_i is zero-state observable when $S_i = \{0\}$.

Lemma 2: If Σ_i is $(\mathbb{R}^{n_i} \setminus \mathcal{O}_i)$ —observable, then $V_i(x_i) > 0, \forall x_i \in \mathcal{O}_i$.

Proof: Take $w_i(t) = 0, \forall t \geq 0$ and $x_{i,0} \in \mathcal{O}_i$, then

$$V_i(x_i(t)) - V_i(x_{i,0}) \leq -\frac{1}{2} \int_0^t |y_i(\tau)|^2 d\tau$$

as long as $x_i(t) \in \mathcal{O}_i$. As $V_i(x_i(t)) \geq 0$, we have

$$V_i(x_{i,0}) \geq \frac{1}{2} \int_0^t |y_i(\tau)|^2 d\tau.$$

If $V_i(x_{i,0}) = 0$ then $y(s) = 0, \forall s \in [0, t]$ which implies that $x_i(s) \in \mathbb{R}^{n_i} \setminus \mathcal{O}_i, \forall s \in [0, t]$ and contradicts $x_{i,0} \in \mathcal{O}_i$. ■

The interconnected system $\Sigma_{1,2}$ is the closed-loop system described by the coupling equations (see Fig. 1)

$$w_1 = y_2 + u_1 \quad w_2 = y_1 + u_2$$

with state $x \in \mathbb{R}^{n_1+n_2}$, input $u \in \mathbb{R}^{s+q}$, and output $y \in \mathbb{R}^{s+q}$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Denote $\Gamma = \{(x_1, x_2) \in \mathcal{O}_1 \times \mathcal{O}_2 \mid \mu_1(x_1)\mu_2(x_2) \geq 1\}$.

In this section, we present a result for the stability with no input, in the spirit of the small-gain theorem.

Proposition 3: Assume that $\mathbb{R}^{n_i} \setminus \mathcal{O}_i$ are bounded. Let $u \equiv 0$, then we have the following properties.

- 1) If $\gamma_1\gamma_2 < 1$, the solutions of $\Sigma_{1,2}$ converge toward a bounded limit set. If furthermore $\mathcal{O}_i = \mathbb{R}^{n_i} \setminus \{0\}$ and Σ_i are zero-state observable ($i = 1, 2$), then $\Sigma_{1,2}$ is globally asymptotically stable.
- 2) If $\Gamma = \{0\}$, $\mathcal{O}_i = \mathbb{R}^{n_i}$ and Σ_i are zero-state observable ($i = 1, 2$), then $\Sigma_{1,2}$ is globally asymptotically stable.
- 3) If Γ is bounded and if at least one storage function V_i is such that

$$\gamma_i = \max_{x_i \in \mathcal{O}_i} \mu_i \text{ and } \mathcal{H}_i(V_i, \mu_i)(x_i) \leq -[\mathcal{K}_i(h_i(x_i))]^2 \quad \forall x_i \in \mathcal{O}_i$$

where $\mathcal{K}_i$ is a super-linear function (i.e., such that $\lim_{|\xi| \rightarrow \infty} |\mathcal{K}_i(\xi)|/|\xi| = +\infty$), then the solutions of $\Sigma_{1,2}$ converge toward a bounded limit set.

Proof: Consider the candidate Lyapunov function $V(x_1, x_2) = V_1(x_1) + \lambda V_2(x_2)$ where λ is a positive number (note that V is proper). Then, we have

$$\begin{aligned} \dot{V} &= L_{f_1}V_1(x_1) + L_{k_1}V_1(x_1)h_2(x_2) + \lambda(L_{f_2}V_2(x_2) \\ &\quad + L_{k_2}V_2(x_2)h_1(x_1)) \\ &\leq -\frac{[L_{k_1}V_1(x_1)]^2}{2\mu_1(x_1)^2} - \frac{|h_1(x_1)|^2}{2} - \lambda \frac{[L_{k_2}V_2(x_2)]^2}{2\mu_2(x_2)^2} \\ &\quad - \lambda \frac{|h_2(x_2)|^2}{2} + L_{k_1}V_1(x_1)h_2(x_2) \\ &\quad + \lambda L_{k_2}V_2(x_2)h_1(x_1) \\ &= -\frac{1}{2} \left(\frac{L_{k_1}V_1(x_1)}{\mu_1(x_1)} - \mu_1(x_1)h_2(x_2) \right)^2 \\ &\quad - \frac{\lambda}{2} \left(\frac{L_{k_2}V_2(x_2)}{\mu_2(x_2)} - \mu_2(x_2)h_1(x_1) \right)^2 \\ &\quad - \frac{|h_1(x_1)|^2}{2} (1 - \lambda\mu_2(x_2)^2) \\ &\quad - \frac{|h_2(x_2)|^2}{2} (\lambda - \mu_1(x_1)^2) \\ &\leq -\frac{1}{2} (1 - \lambda\mu_2(x_2)^2) |h_1(x_1)|^2 \\ &\quad - \frac{1}{2} (\lambda - \mu_1(x_1)^2) |h_2(x_2)|^2 \end{aligned} \quad (4)$$

for any $x \in \mathcal{O}_1 \times \mathcal{O}_2$.

- 1) When $\gamma_1\gamma_2 < 1$, we can choose λ such that $\lambda > \gamma_1^2$ and $1/\lambda > \gamma_2^2$. From the LaSalle invariance principle, the state x converges toward the smallest V -level set that contains $(\mathbb{R}^{n_1} \setminus \mathcal{O}_1) \times (\mathbb{R}^{n_2} \setminus \mathcal{O}_2)$, which is bounded by hypothesis. When $\mathcal{O}_i = \mathbb{R}^{n_i} \setminus \{0\}$ ($i = 1, 2$), this level set is exactly $V^{-1}(0)$. When the two systems Σ_i are zero-state observable, then $V^{-1}(0) = \{0\}$ (by Lemma 2) and the global asymptotic stability of $\Sigma_{1,2}$ is then guaranteed.
- 2) When Γ is reduced to $\{0\}$, by continuity of μ_i on $\mathbb{R}^{n_i}$, we have

$$\{(x_1, x_2) \in \mathcal{O}_1 \times \mathcal{O}_2 \mid \mu_1(x_1)\mu_2(x_2) = 1\} = \{(0, 0)\}.$$

Notice that in this case one has

$$\gamma_i = \max_{x_i \in \mathbb{R}^{n_i}} \mu_i(x_i), \quad i = 1, 2$$

and these two maxima are achieved exactly at 0. Take $\lambda = \gamma_1^2 = \gamma_2^{-2}$ and then the right term in the inequality (4) is negative for

any $x \neq 0$. When the systems Σ_i are zero-state observable, then by Lemma 2, $V^{-1}(0) = \{0\}$, and we conclude the global asymptotic stability of $\Sigma_{1,2}$.

- 3) Assume that the condition is fulfilled, for instance, for $i = 1$. Instead of (4), we have

$$\dot{V} \leq -\frac{1}{2} [(1 - \lambda\mu_2(x_2)^2)|h_1(x_1)|^2 + \mathcal{K}_1(h_1(x_1))^2] - (\lambda - \mu_1(x_1)^2) \frac{|h_2(x_2)|^2}{2}$$

for any $x \in \mathcal{O}_1 \times \mathcal{O}_2$. We then take $\lambda = \gamma_1^2$ and consider the set

$$\bar{\Phi}_1 = \left\{ (x_1, x_2) \in \mathcal{O}_1 \times \mathcal{O}_2 \mid \lambda\mu_2(x_2)^2 \geq 1 + \left[\frac{\mathcal{K}_1(h_1(x_1))}{h_1(x_1)} \right]^2 \right\}.$$

Let us show that $\bar{\Phi}_1$ is a bounded set.

The function μ_2 being bounded by γ_2 , any x_1 such that $(x_1, x_2) \in \bar{\Phi}_1$ fulfills $[\mathcal{K}_1(h_1(x_1))/h_1(x_1)]^2 \leq \gamma_1^2\gamma_2^2 - 1$. The super-linearity of $\mathcal{K}_1$ provides then the boundedness of x_1 on $\bar{\Phi}_1$. Take now $\bar{x}_1 \in \mathcal{O}_1$ such that $\mu_1(\bar{x}_1) = \gamma_1$. Then, any x_2 such that $(x_1, x_2) \in \bar{\Phi}_1$ fulfills $\mu_1(\bar{x}_1)\mu_2(x_2) \geq 1$, i.e., $(\bar{x}_1, x_2) \in \Gamma$. The boundedness of Γ implies then that x_2 is bounded on $\bar{\Phi}_1$.

Finally, by LaSalle invariance principle, $x(t)$ converges toward the smallest V -level set that contains $\bar{\Phi}_1 \cup (\mathbb{R}^{n_1} \setminus \mathcal{O}_1) \times (\mathbb{R}^{n_2} \setminus \mathcal{O}_2)$. ■

Remarks 2:

- i) When systems Σ_1, Σ_2 have L_2 -gains γ_1, γ_2 on $\mathbb{R}^{n_1}, \mathbb{R}^{n_2}$, point 1 is the natural extension of the well known *small-gain theorem*, that claims the internal stability of interconnected systems with no input when $\gamma_1\gamma_2 < 1$.
- ii) Points 2) and 3) show clearly the advantage of considering generalized L_2 -gains as we can guarantee the global asymptotic stability even when $\gamma_1\gamma_2 = 1$ (see the example in Section V).

IV. INPUT-OUTPUT PROPERTY OF INTERCONNECTED SYSTEMS

We study now the L_2 bounded input-bounded output property of the interconnected system $\Sigma_{1,2}$.

Proposition 4: When the input u belongs to L_2 , then the output y has the following properties.

- 1) If $\gamma_1\gamma_2 < 1$, then y belongs to $L_2([0, T])$ for any $T > 0$ such that $x(t)$ stays in $\mathcal{O}_1 \times \mathcal{O}_2$, for $t \leq T$.
- 2) If $\Gamma = \{0\}$, then y belongs to $L_2([0, T])$ for any $T > 0$ such that $x(t)$ stays in $\mathcal{O}_1 \times \mathcal{O}_2 \setminus \{0\}$, for $t \leq T$.
- 3) If Γ is bounded and if at least one storage function V_i is such that

$$\gamma_i = \max_{x_i \in \mathcal{O}_i} \mu_i \text{ and } \mathcal{H}_i(V_i, \mu_i)(x_i) \leq -[\mathcal{K}_i(h_i(x_i))]^2 \quad \forall x_i \in \mathcal{O}_i$$

where $\mathcal{K}_i$ is a super-linear function, then the output y belongs in $L_2([0, T])$ as long as the state evolves in $\mathcal{O}_1 \times \mathcal{O}_2 \setminus \bar{\Phi}_i$, where

$$\bar{\Phi}_i = \left\{ (x_1, x_2) \in \mathcal{O}_1 \times \mathcal{O}_2 \mid \gamma_i^2 \mu_j(x_j)^2 \geq 1 + \left[\frac{\mathcal{K}_i(h_i(x_i))}{h_i(x_i)} \right]^2 \right\} \quad (j \neq i)$$

is a compact set.

Proof: Consider the function $V(x) = V_1(x_1) + \lambda V_2(x_2)$ where λ is a positive number. By hypothesis, V is proper. From the property (2) fulfilled by V_1 and V_2 , we have

$$\begin{aligned} V(x(T)) - V(x(0)) &\leq \frac{1}{2} \int_0^T \mu_1(x_1(\tau))^2 |y_2(\tau) + u_1(\tau)|^2 \\ &\quad - |y_1(\tau)|^2 + \lambda [\mu_2(x_2(\tau))^2 |y_1(\tau) \\ &\quad + u_2(\tau)|^2 - |y_2(\tau)|^2] d\tau \\ &\leq \frac{1}{2} \int_0^T |y_1(\tau)|^2 [\lambda \mu_2(x_2(\tau))^2 - 1] \\ &\quad + 2|y_1(\tau)|\lambda \mu_2(x_2(\tau))^2 |u_2(\tau)| \\ &\quad + |y_2(\tau)|^2 [\mu_1(x_1(\tau))^2 - \lambda] \\ &\quad + 2|y_2(\tau)|\mu_1(x_1(\tau))^2 |u_1(\tau)| \\ &\quad + \mu_1(x_1(\tau))^2 |u_1(\tau)|^2 \\ &\quad + \lambda \mu_2(x_2(\tau))^2 |u_2(\tau)|^2 d\tau \end{aligned}$$

for T such that $x(t) \in \mathcal{O}_1 \times \mathcal{O}_2, \forall t \leq T$. V being nonnegative, we obtain from the previous inequality

$$\begin{aligned} &\frac{1}{2} \int_0^T |y_1(\tau)|^2 [1 - \lambda \mu_2(x_2(\tau))^2] \\ &\quad + |y_2(\tau)|^2 [\lambda - \mu_1(x_1(\tau))^2] d\tau \\ &\quad - \int_0^T |y_1(\tau)|\lambda \mu_2(x_2(\tau))^2 |u_2(\tau)| \\ &\quad + |y_2(\tau)|\mu_1(x_1(\tau))^2 |u_1(\tau)| d\tau \\ &\leq V(x(0)) + \frac{1}{2} \int_0^T \mu_1(x_1(\tau))^2 |u_1(\tau)|^2 \\ &\quad + \lambda \mu_2(x_2(\tau))^2 |u_2(\tau)|^2 d\tau. \end{aligned}$$

Finally, applying the Cauchy-Schwartz inequality on the second integral of the left term of the previous inequality, we obtain

$$\begin{aligned} &\frac{1}{2} \int_0^T |y_1(\tau)|^2 [1 - \lambda \mu_2(x_2(\tau))^2] + |y_2(\tau)|^2 \\ &\quad \times [\lambda - \mu_1(x_1(\tau))^2] d\tau \\ &\quad - \lambda \|u_2\|_{[0, T]} \int_0^T |y_1(\tau)|\mu_2(x_2(\tau))^2 d\tau \\ &\quad - \|u_1\|_{[0, T]} \int_0^T |y_2(\tau)|\mu_1(x_1(\tau))^2 d\tau \\ &\leq V(x(0)) + \frac{1}{2} \int_0^T \mu_1(x_1(\tau))^2 |u_1(\tau)|^2 \\ &\quad + \lambda \mu_2(x_2(\tau))^2 |u_2(\tau)|^2 d\tau. \end{aligned} \quad (5)$$

- 1) If $\gamma_1\gamma_2 < 1$, we choose $\lambda > 0$ such that $\lambda - \gamma_1^2 \geq \varepsilon_1 > 0$ and $1 - \lambda\gamma_2^2 \geq \varepsilon_2 > 0$. Then, the inequality (5) provides

$$\begin{aligned} &\frac{\varepsilon_2}{2} \|y_1\|_{[0, T]}^2 - \|y_1\|_{[0, T]} \|u_2\|_{[0, T]} + \frac{\varepsilon_1}{2} \|y_2\|_{[0, T]}^2 \\ &\quad - \lambda \|y_2\|_{[0, T]} \|u_1\|_{[0, T]} \leq V(x(0)) + \frac{\lambda}{2} \|u_1\|_{[0, T]}^2 + \frac{1}{2} \|u_2\|_{[0, T]}^2 \end{aligned}$$

and we conclude that $\|y\|_{[0, T]}$ is bounded as long as $x(t) \in \mathcal{O}_1 \times \mathcal{O}_2, \forall t \leq T$.

- 2) When $\Gamma = \{0\}$, the maxima of the functions μ_i are reached exactly at 0, and $\gamma_1\gamma_2$ is necessarily equal to one. We then take

$\lambda = \gamma_1^2 = \gamma_2^{-2}$. By continuity of the functions μ_i at 0, we have: $\forall \eta > 0, \exists \varepsilon_1 > 0, \exists \varepsilon_2 > 0$ such that

$$\lambda - \mu_1(x_1)^2 \geq \varepsilon_1 > 0 \text{ and } 1 - \lambda\mu_2(x_2)^2 \geq \varepsilon_2 > 0$$

for any $(x_1, x_2) \in \mathcal{O}_1 \times \mathcal{O}_2 \setminus B(0, \eta)$, and we obtain, as before, that $\|y\|_{[0, T]}$ is bounded as long as $x(t) \in \mathcal{O}_1 \times \mathcal{O}_2$ and $x(t) \notin B(0, \eta)$ for any $t \leq T$. This property is fulfilled for any arbitrary small $\eta > 0$.

- 3) Assume that the condition is fulfilled, for instance, for $i = 1$. Then, instead of (5), one can write

$$\begin{aligned} & \frac{1}{2} \int_0^T |y_1(\tau)|^2 [1 - \lambda\mu_2(x_2(\tau))^2] \\ & + \mathcal{K}_1(y_1(\tau))^2 d\tau - \lambda\gamma_2^2 \|y_1\|_{[0, T]} \|u_2\|_{[0, T]} \\ & + \frac{\lambda - \gamma_1^2}{2} \|y_2\|_{[0, T]}^2 - \gamma_1^2 \|y_2\|_{[0, T]} \|u_1\|_{[0, T]} \\ & \leq V(x(0)) + \frac{\gamma_1^2}{2} \|u_1\|_{[0, T]}^2 + \frac{\lambda\gamma_2^2}{2} \|u_2\|_{[0, T]}^2 \end{aligned} \quad (6)$$

as long as $x(t) \in \mathcal{O}_1 \times \mathcal{O}_2, \forall t \leq T$. Take $\lambda = \gamma_1^2 + \varepsilon$ with $\varepsilon > 0$, and consider the set

$$\Phi_1^\varepsilon = \left\{ (x_1, x_2) \in \mathcal{O}_1 \times \mathcal{O}_2 \mid 1 - (\gamma_1^2 + \varepsilon)\mu_2(x_2)^2 + \left[\frac{\mathcal{K}_1(h_1(x_1))}{h_1(x_1)} \right]^2 < \varepsilon \right\}.$$

Then, from (6), we conclude, as before, that $\|y\|_{[0, T]}$ stays bounded, as long as $x(t) \in \mathcal{O}_1 \times \mathcal{O}_2$ and $x(t) \notin \Phi_1^\varepsilon, \forall t \leq T$. This property is fulfilled for any arbitrary small number ε . Finally, μ_2 being bounded, we have $\bigcap_{\varepsilon > 0} \Phi_1^\varepsilon = \bar{\Phi}_1$, which is bounded, by compactness of Γ and super-linearity of $\mathcal{K}_1$.

V. ILLUSTRATIVE EXAMPLE

Consider the following interconnected system:

$$\begin{aligned} \Sigma_1 : \begin{cases} \dot{x}_1 = -x_1 - x_1^3 + \alpha_1 y_2 \\ y_1 = x_1 \end{cases} \quad \Sigma_2 : \begin{cases} \dot{x}_2 = -x_2 - x_2^3 + \alpha_2 y_1 \\ y_2 = x_2 \end{cases} \end{aligned} \quad (7)$$

Σ_1 and Σ_2 have generalized L_2 -gain on $\mathbb{R}$

$$\mu_i(x_i) = \frac{|\alpha_i|}{\sqrt{1 + x_i^2}}, \quad i = 1, 2$$

with storage functions $V_i(x_i) = x_i^2/2$ ($i = 1, 2$) as

$$\mathcal{H}(V_i, \mu_i)(x_i) = -\frac{x_i^4}{2}, \quad i = 1, 2$$

thus $\mathcal{K}_i(y_i) = (y_i^2/\sqrt{2})$. Furthermore, one has $|\alpha_i| = \max_{x_i \in \mathbb{R}} \mu_i(x_i) = \mu_i(0)$. Then, the set

$$\Gamma = \{(x_1, x_2) \in \mathbb{R}^2 \mid (1 + x_1^2)(1 + x_2^2) \leq \alpha_1^2 \alpha_2^2\}$$

is bounded, whatever are α_1, α_2 and we conclude that $\Sigma_{1,2}$ is stable for zero input.

- 1) If $|\alpha_1 \alpha_2| < 1$, by the usual small gain theorem, $\Sigma_{1,2}$ is globally asymptotically stable.
- 2) If $|\alpha_1 \alpha_2| = 1$, $\Sigma_{1,2}$ is also globally asymptotically stable (although the usual small gain theorem is not conclusive).

- 3) If $|\alpha_1 \alpha_2| > 1$, we have

$$\begin{aligned} \bar{\Phi}_1 &= \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid (1 + \frac{1}{2}x_1^2)(1 + x_2^2) \leq \alpha_1^2 \alpha_2^2 \right\} \\ \bar{\Phi}_2 &= \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid (1 + x_1^2)(1 + \frac{1}{2}x_2^2) \leq \alpha_1^2 \alpha_2^2 \right\} \end{aligned}$$

and we conclude that the state converges toward a bounded set S , which is the intersection of two level sets, which are the smallest $V_1 + \alpha_1^2 V_2$ (respectively, $\alpha_2^2 V_1 + V_2$) level set containing $\bar{\Phi}_1$ (respectively, $\bar{\Phi}_2$).

When $(x_1, x_2) \in \bar{\Phi}_1$, we have

$$\begin{aligned} x_1^2 &\leq 2(\alpha_1^2 \alpha_2^2 - 1) - x_2^2(2 + x_1^2) \quad \text{and} \\ x_2^2 &\leq (\alpha_1^2 \alpha_2^2 - 1) - \frac{1}{2}x_1^2(1 + x_2^2). \end{aligned}$$

Thus

$$\begin{aligned} V_1(x_1) + \alpha_1^2 V_2(x_2) &\leq \min \left[(\alpha_1^2 \alpha_2^2 - 1) \right. \\ &\quad \left. - \frac{1}{2}x_2^2(2 - \alpha_1^2 + x_1^2) \right. \\ &\quad \left. \frac{1}{2}\alpha_1^2(\alpha_1^2 \alpha_2^2 - 1) \right. \\ &\quad \left. - \frac{1}{4}x_1^2(\alpha_1^2 - 2 + \alpha_1^2 x_2^2) \right] \end{aligned}$$

which is equivalent to

$$\frac{1}{2}x_1^2 + \frac{1}{2}\alpha_1^2 x_2^2 \leq \max \left(1, \frac{1}{\alpha_1^2} \right) (\alpha_1^2 \alpha_2^2 - 1).$$

- In a similar way, when $(x_1, x_2) \in \bar{\Phi}_2$, we have

$$\frac{1}{2}\alpha_2^2 x_1^2 + \frac{1}{2}x_2^2 \leq \max \left(1, \frac{1}{\alpha_2^2} \right) (\alpha_1^2 \alpha_2^2 - 1).$$

So, the set S has the following representation

$$S = \left\{ (x_1, x_2) \mid \begin{aligned} & x_1^2 + \alpha_1^2 x_2^2 \leq \max(\alpha_1^2, 2)(\alpha_1^2 \alpha_2^2 - 1) \\ & \alpha_2^2 x_1^2 + x_2^2 \leq \max(\alpha_2^2, 2)(\alpha_1^2 \alpha_2^2 - 1) \end{aligned} \right\}.$$

Simulations has been run for $\alpha_1 = 2$ and $\alpha_2 = 3$. The interconnected system has then two equilibriums which belong to the set S , as illustrated on Fig. 2.

Remarks: The systems Σ_1, Σ_2 in (7) are also ISS; see [12]

$$\begin{aligned} \nabla V_i(x_i)(-x_i - x_i^3 + \alpha_i u_i) &\leq -x_i^4 \quad \text{when} \\ |x_i| &\geq |\alpha_i|(|u_i|), \quad i = 1, 2. \end{aligned}$$

So, V_1, V_2 are ISS-Lyapunov functions with ISS-gains χ_1, χ_2 , as functions of the norm of the inputs $u_1 = y_2$ and $u_2 = y_1$

$$\chi_i(|u_i|) = |\alpha_i| |u_i|, \quad i = 1, 2.$$

Then, according to [7], the global asymptotic stability of the interconnected system is guaranteed when $\chi_1 \circ \chi_2(r) < r, \forall r > 0$, i.e., when $|\alpha_1 \alpha_2| < 1$, which does not bring in this example more information than the usual small-gain theorem. On the contrary, the proposed generalization of the small gain theorem allows not only to conclude global asymptotic stability for $|\alpha_1 \alpha_2| = 1$, but also global boundedness for $|\alpha_1 \alpha_2| > 1$.

Fig. 2. Boundary of S in plain line, * the two equilibriums.

VI. CONCLUSION

This note presents an extension of the small-gain theorem, using generalized L_2 gain instead of usual (constant) L_2 gain. It is shown that global asymptotic stability with no input can be guaranteed even when the product of the two L_2 gains (in the usual sense) is equal to one. Furthermore, convergence toward a bounded set can be guaranteed even when this product is larger than one.

Further results providing a characterization of the smallest generalized L_2 gains should lead to sharper bounds.

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