gets further coded. The latter scheme entails implementing a dynamic encoding scheme, much like that introduced in [10].

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A Remark on an Example By Teel–Hespanha With Applications to Cascaded Systems

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Abstract—The properties of a system, proposed by Teel and Hespanha, which is globally exponentially stable but with state that can be driven to infinity by an arbitrarily small exponentially decaying disturbance, are discussed in detail. These are used to propose a family of systems with a similar property and to argue that unstable behavior may be nongeneric and not detected by means of simulations. Finally, sufficient conditions for the existence of unbounded trajectories in cascaded systems are given.

Index Terms—Cascaded systems, nonlinear systems, stability.

I. INTRODUCTION

In [5], which builds upon the results in [4], the system (system (13) of [5])

$$\begin{aligned}\dot{x}_1 &= g(x_1 x_2) x_1 \\ \dot{x}_2 &= -2x_2 + z \\ \dot{z} &= -z\end{aligned}\quad (1)$$

where $g(\cdot)$ satisfies Assumption 1 below, has been considered.

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Assumption 1: [5, Ass. 1] The function $g(s)$ is such that

- i) it is continuous;
- ii) $|g(s)| \leq 1$ for all $s \in \mathbb{R}$;
- iii) $g(s) = -1$ for all $s \in (-\infty, 1/2] \cup [3/2, \infty)$;
- iv) $g(1) = 1$.

The authors have shown that if the initial condition $(x_1(0), x_2(0), z(0))$ is such that $x_1(0) \neq 0$, $x_2(0) = 1/x_1(0)$ and $z(0) = x_2(0)$ then the resulting trajectory is such that $x_1(t) = e^t x_1(0)$, i.e., it is exponentially diverging. This property, together with the fact that the (x_1, x_2) -subsystem with $z = 0$ is globally exponentially stable, is exploited to conclude that globally exponentially stable systems may be *destabilized* by arbitrarily small and exponentially decaying disturbances. This may not be a surprising fact, but the illuminating example in [5] shows that it may arise in very simple systems.

However, the aforementioned class of systems reserves further surprises. To begin with, one may be tempted to see the phenomenon highlighted in [5] by means of simulations. This yields the first surprise.

We run Matlab simulations using the ode45 command, setting the initial condition to (1, 1, 1) and selecting the function¹

$$g(s) = \begin{cases} -1 & \text{if } s \leq \frac{1}{2} \\ 4s - 3 & \text{if } \frac{1}{2} \leq s \leq 1 \\ -4s + 5 & \text{if } 1 \leq s \leq \frac{3}{2} \\ -1 & \text{if } s \geq \frac{3}{2} \end{cases}$$

which is such that Assumption 1 holds. This yields the results in Fig. 1, displaying the time histories of the state $x_1(t)$ for different values of the "Relative Tolerance" Matlab internal variable (the "Absolute Tolerance" has been set to 1/100 the "Relative Tolerance"), together with the function e^t . Surprisingly, the state x_1 does not behave as theoretically forecast, but *follows* the signal e^t for an initial period of time (which depends upon the simulation parameters), and then converges exponentially to zero. At this point, after carefully rechecking all the calculations in [5], and after convincing ourselves that they are correct (as one would expect), we are left wondering why the forecast instability is not exposed by the simulation.

We devote the rest of this note to address this question and some related issues. In particular, we show that system (1) can be simply modified to display a family of unbounded trajectories, which are also *captured* by simulations, and we provide some sufficient conditions for the existence of unbounded trajectories in a class of cascaded systems. Note that the problem of existence of unbounded trajectories has also been studied, for general nonlinear systems, in [2].

II. A CLASS OF SYSTEMS

Consider a system described by equations of the form

$$\begin{aligned}\dot{x}_1 &= g(x_1 x_2) x_1 \\ \dot{x}_2 &= -2x_2 + z \\ \dot{z} &= -\lambda z\end{aligned}\quad (2)$$

where $g(\cdot)$ satisfies Assumption 1, and the following assumption.

Assumption 2: The function $g(s)$ is such that

- i) $g(s)$ is C^1 for all $s \in (1/2, 1) \cup (1, 3/2)$;
- ii) $g(s) = 1 \Rightarrow s = 1$.

For system (2), the properties outlined in the following statement hold.

Proposition 1: Consider system (2) and assume the function $g(\cdot)$ satisfies Assumptions 1 and 2.

- i) If $\lambda > 1$, then all trajectories are bounded and converge exponentially to zero.

¹Similar conclusions can be drawn with other selections.

Fig. 1. Time histories of the state x_1 for different values of the “Relative tolerance” (Rt) (solid lines) together with the function e^t (dashed line).

- ii) If $\lambda = 1$ then almost² all trajectories are bounded and converge exponentially to zero.
- iii) For almost all λ such that $0 < \lambda < 1$ the set of initial conditions yielding unbounded trajectories has nonzero measure.

Proof: To begin with consider the variables (the variable ξ is the same as the one used in [5])

$$\xi = x_1 x_2 \quad \eta = x_1 z$$

and note that

$$\begin{aligned} \dot{\xi} &= (g(\xi) - 2)\xi + \eta \\ \dot{\eta} &= (g(\xi) - \lambda)\eta. \end{aligned} \quad (3)$$

Note that system (3) is forward complete and that the study of the asymptotic behavior of system (3) allows to draw conclusions on the asymptotic behavior of system (2). In fact, as x_2 and z are exponentially converging to zero for any $\lambda > 0$, we can conclude the following. If $\xi(t)$ converges to zero then, because $g(0) = -1$ by Assumption 1 and noting that $\dot{x}_1 = g(\xi)x_1$, $x_1(t)$ converges exponentially to zero, whereas if $\xi(t)$ converges to a nonzero constant then $x_1(t) = \xi(t)/x_2(t)$ diverges. We consider the previous three cases individually.

- If $\lambda > 1$, by Assumption 1, system (3) has only one equilibrium, namely $(\xi, \eta) = (0, 0)$. Consider the candidate Lyapunov function

$$V(\xi, \eta) = \frac{1}{2}\xi^2 + \frac{1}{2(\lambda - 1)}\eta^2$$

and note that

$$\dot{V} = \xi^2(g(\xi) - 2) + \xi\eta + \frac{g(\xi) - \lambda}{\lambda - 1}\eta^2$$

yielding

$$\dot{V} \leq -\xi^2 + \xi\eta - \eta^2 \leq 0.$$

²This means that the set of initial conditions yielding unbounded trajectories has zero Lebesgue measure.

As a result, the origin is a globally exponentially stable equilibrium, and this prove the first claim.

- If $\lambda = 1$, by Assumptions 1 and 2, system (3) has only two equilibria, namely $(\xi, \eta) = (0, 0)$ and $(\xi, \eta) = (1, 1)$. We show that almost all trajectories of system (3) are converging to the zero equilibrium. For, observe that $\eta(t)$ is bounded for all t and this implies that $\xi(t)$ is also bounded. Consider the variable $e = \eta - \xi$ and note that

$$\dot{e} = (g(\xi) - 2)e.$$

This shows that the set $e = 0$, which contains all the equilibria of the system, is invariant and globally attractive. This property, together with boundedness of trajectories, implies that all trajectories converge to the set $e = 0$. The (well-defined) restriction of system (3) to this invariant manifold is described by the equation

$$\dot{\chi} = (g(\chi) - 1)\chi. \quad (4)$$

The scalar system (4) has an equilibrium for $\chi = 1$, and one for $\chi = 0$. The equilibrium $\chi = 1$ is unstable, since $\dot{\chi} < 0$ for $\chi < 1$. As a result, the equilibrium $(\xi, \eta) = (1, 1)$ is unstable. Moreover, the set of initial conditions yielding trajectories converging to this unstable equilibrium is

$$\Omega = \{(\xi, \eta) \mid \xi = \eta = s \text{ and } s \geq 1\}.$$

The equilibrium $\chi = 0$ is locally asymptotically stable, hence the equilibrium $(\xi, \eta) = (0, 0)$ is also locally asymptotically stable. As a result, almost all trajectories of the system converge to the stable equilibrium. Finally, the set of initial conditions yielding unbounded trajectories for system (2) with $\lambda = 1$ is the set (already characterized in [5])

$$\tilde{\Omega} = \left\{ (x_1, x_2, z) \mid x_2 = z = \frac{1}{x_1} = s \text{ and } s \neq 0 \right\}$$

which is a set of zero Lebesgue measure. This completes the proof of the second claim.

- If $0 < \lambda < 1$ the system (3) has several equilibria. In fact, it is not difficult to see that the system has at least three equilibria. Note

that the nonzero equilibria $(\bar{\xi}, \bar{\eta})$ are such that $\bar{\xi} > 0$, $g(\bar{\xi}) = \lambda$ and $\bar{\eta} = (2 - \lambda)\bar{\xi}$. Moreover, for almost all $\lambda \in (0, 1)$ there is at least one nonzero equilibrium $(\bar{\xi}_*, \bar{\eta}_*)$ such that

$$\frac{dg}{d\xi}(\bar{\xi}_*) < 0.$$

Consider the linear approximation of system (3) around such an equilibrium $(\bar{\xi}_*, \bar{\eta}_*)$. Simple computations show that the eigenvalues of such a linearized system are

$$s_1 = -2 + \lambda \quad s_2 = \frac{dg}{d\xi}(\bar{\xi}_*)\bar{\xi}_*$$

hence they are both real and negative. This implies that the equilibrium $(\bar{\xi}_*, \bar{\eta}_*)$ is locally asymptotically stable, i.e., there is a set of initial condition of nonzero measure yielding trajectories converging to $(\bar{\xi}_*, \bar{\eta}_*)$. As a result, there is a set of initial conditions for system (2) with nonzero measure yielding unbounded trajectories. ■

The result of Proposition 1 can be interpreted as follows. If the signal $z(t)$ decays sufficiently fast then all trajectories of system (2) are bounded and exponentially converging. If $z(t) = z_0 e^{-t}$, for some non-zero z_0 , then system (2) has unbounded trajectories, but these are unstable, hence the presence of any small perturbation or integration error (such as the one introduced by the Matlab ode command) yields trajectories which converge exponentially to zero. (It would be interesting to investigate if system (2) with $\lambda = 1$ can be studied with the tools introduced in [3], which naturally yield *almost* stability results.) Finally, if the signal $z(t)$ decays slower than e^{-t} system (2) has a family of unbounded trajectories (for almost all λ) and these are (locally) asymptotically stable. As a result, a Matlab simulation for system (2) with $0 < \lambda < 1$ exposes the unbounded trajectories. For brevity, and because it is of no conceptual interest we do not display the result of such simulations.

III. SUFFICIENT CONDITIONS FOR UNBOUNDED TRAJECTORIES

Motivated by the previous discussion in this section we consider cascaded systems described by equations of the form

$$\begin{aligned} \dot{\zeta} &= f(\zeta, y, \mu) \\ \dot{x} &= A(\mu)x \\ y &= Cx \end{aligned} \quad (5)$$

with $\zeta \in \mathbb{R}^m$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}$, and $\mu \in \mathcal{D}$ for some open set $\mathcal{D} \subset \mathbb{R}^d$, and such that

$$\dot{\zeta} = f(\zeta, 0, \mu)$$

is globally asymptotically stable for all $\mu \in \mathcal{D}$, and³ $\sigma(A(\mu)) \in \mathcal{U}^-$ for all $\mu \in \mathcal{D}$. System (5) can be obtained applying a linearizing state feedback to a general nonlinear single-input single-output system with a well-defined relative degree and with asymptotically stable zero dynamics [1].

In addition we assume the following.

C1) The mapping f is such that

$$f(\zeta, y, \mu) = F(\zeta y, \mu)\zeta$$

for some mapping $F : \mathbb{R}^m \times \mathcal{D} \rightarrow \mathbb{R}^{m \times m}$.

C2) The output y is such that⁴

$$y^{(n)} = -k_0 y - k_1 \dot{y} - \dots - k_{n-1} y^{(n-1)}$$

³ $\sigma(A)$ denotes the spectrum of the matrix A .

⁴We use the notation $y^{(i)} = d^i y / dt^i$.

for some functions $k_i = k_i(\mu)$ such that the polynomial

$$p(s) = s^n + k_{n-1}s^{n-1} + \dots + k_1 s + k_0 \quad (6)$$

has all roots in the left half of the complex plane for all $\mu \in \mathcal{D}$.

Consider the variables

$$\xi_0 = \zeta y \quad \xi_1 = \zeta \dot{y} \quad \dots \quad \xi_{n-1} = \zeta y^{(n-1)} \quad (7)$$

and note that⁵

$$\begin{aligned} \dot{\xi}_0 &= F(\xi_0)\xi_0 + \xi_1 \\ \dot{\xi}_1 &= F(\xi_0)\xi_1 + \xi_2 \\ &\vdots \\ \dot{\xi}_{n-1} &= F(\xi_0)\xi_{n-1} - k_0 \xi_0 - k_1 \xi_1 - \dots - k_{n-1} \xi_{n-1}. \end{aligned} \quad (8)$$

We are now ready to state a simple preliminary result.

Lemma 1: Consider system (5) with Assumptions C1) and C2). Any nonzero equilibrium $(\bar{\xi}_0, \bar{\xi}_1, \dots, \bar{\xi}_{n-1})$ of system (8) such that

$$\text{rank} [\bar{\xi}_0, \bar{\xi}_1, \dots, \bar{\xi}_{n-1}] = 1 \quad (9)$$

corresponds to an unbounded trajectory of (5).

Proof: From the equilibrium $(\bar{\xi}_0, \bar{\xi}_1, \dots, \bar{\xi}_{n-1})$ and the rank condition (9) we can construct an initial condition

$$(\zeta(0), y(0), \dot{y}(0), \dots, y^{(n-1)}(0))$$

for system (5). The resulting trajectory is then such that

$$\zeta(t)y(t) = \bar{\xi}_0 \quad \zeta(t)\dot{y}(t) = \bar{\xi}_1, \dots, \zeta(t)y^{(n-1)}(t) = \bar{\xi}_{n-1}.$$

Hence the claim follows noting that, by Assumption C2), $y(t)$ converges to zero as t goes to infinity. ■

Remark 1: Condition (9) holds trivially if $n = 1$, i.e., the x subsystem is scalar, or if $m = 1$, i.e., the ζ subsystem is scalar.

The converse statement is not necessarily true. In fact there may be unbounded trajectories of system (5) which are *mapped* into the zero equilibrium of system (8) (this is the case if the trajectories of system (5) are diverging sufficiently slow, e.g., polynomially), or into unbounded trajectories (this is the case if the trajectories of system (5) are diverging sufficiently fast).

Note finally that any trajectory of system (5) escaping in finite time is *mapped* into a trajectory of system (8) escaping in finite time. The converse is not (in general) true, because not all trajectories of system (8) are trajectories of system (5).

The transformation of (a class of) unbounded trajectories into equilibria of an auxiliary system has the distinct advantage that we can study their existence using bifurcation diagrams (the variable μ is the bifurcation variable), and we can assess stability or instability of such unbounded trajectories using simple tools, e.g., Lyapunov first method and the center manifold theory.

The equilibria $(\bar{\xi}_0, \bar{\xi}_1, \dots, \bar{\xi}_{n-1})$ of (8) are obtained solving the equation

$$\begin{aligned} &\left((-1)^{n+1} F(\bar{\xi}_0)^n + (-1)^n k_{n-1} F(\bar{\xi}_0)^{n-1} + \dots \right. \\ &\quad \left. \dots + k_1 F(\bar{\xi}_0) - k_0 I \right) \bar{\xi}_0 = 0 \end{aligned} \quad (10)$$

in the unknown $\bar{\xi}_0$ and setting

$$\bar{\xi}_i = (-1)^i F^i(\bar{\xi}_0) \bar{\xi}_0. \quad (11)$$

⁵To simplify notation, we omit the argument μ in $F(\cdot)$.

Remark 2: Equation (11) implies that

$$\text{rank} [\bar{\xi}_0, \bar{\xi}_1, \dots, \bar{\xi}_{n-1}] = 1 \Leftrightarrow \text{rank} [\bar{\xi}_0, \bar{\xi}_{n-1}] = 1.$$

We are now ready to state sufficient conditions for the existence of unbounded trajectories for system (5) with Assumptions C1) and C2).

Proposition 2: Consider system (5) with Assumptions C1) and C2). Suppose (10) has a nonzero solution $\bar{\xi}_0$ such that

$$\text{rank} [\bar{\xi}_0, (-1)^{n-1} F^{n-1}(\bar{\xi}_0) \bar{\xi}_0] = 1.$$

Then system (5) has at least one unbounded trajectory.

Moreover such a trajectory is asymptotically stable (resp., unstable) if $\sigma(\Gamma) \in \mathcal{D}^-$ (resp., $\sigma(\Gamma) \cap \mathcal{D}^+ \neq \emptyset$), where

$$\Gamma = \begin{bmatrix} \Psi_0(\xi_0) & I & 0 & \cdots & 0 \\ \Psi_1(\xi_0) & F(\xi_0) & I & \cdots & 0 \\ \Psi_2(\xi_0) & 0 & F(\xi_0) & \ddots & 0 \\ \vdots & 0 & \vdots & \ddots & I \\ \Psi_{n-1}(\xi_0) - k_0 I & -k_1 I & -k_2 I & \cdots & F(\xi_0) - k_{n-1} I \end{bmatrix}$$

and

$$\Psi_i(\bar{\xi}_0) = \left[\frac{\partial}{\partial \xi_0} (F(\xi_0) \xi_i) \right]_{\xi_0 = \bar{\xi}_0, \xi_i = \bar{\xi}_i}$$

for $i = 0, \dots, n-1$, with $\bar{\xi}_i$ as in (11).

Proof: The existence of the unbounded solution is a consequence of Lemma 1, whereas its asymptotic stability (respectively, instability) results from the asymptotic stability (respectively, instability) of the system linearized along the unbounded trajectory. This system is described by equations of the form $\dot{\gamma} = \Gamma \gamma$. ■

The conditions in Proposition 2 take a very simple form for systems described by (5) with $m = 1$. In this case the matrix $F(\xi)$ is a scalar function and nonzero solutions of (10) can be obtained solving the (scalar) equation

$$(-1)^{n+1} F(\xi_0)^n + (-1)^n k_{n-1} F(\xi_0)^{n-1} + \cdots + k_1 F(\xi_0) - k_0 I = 0. \quad (12)$$

Note that [recall (6)]

$$-p(-s) = (-1)^{n+1} s^n + (-1)^n k_{n-1} s^{n-1} + \cdots + k_1 s - k_0$$

therefore (12) can be rewritten as

$$p(-F(\xi_0)) = 0.$$

This provides a very simple sufficient condition for the existence of unbounded trajectories, as summarized in the following statement.

Proposition 3: Consider system (5) with Assumptions C1) and C2) and $m = 1$. Let $I \subset \mathbb{R}$ be the image of the function $F(\xi)$.

System (5) has at least one unbounded trajectory if there is a root of $p(-s)$ in I .

Proposition 3 highlights a link between the image of $F(\xi_0)$ and the roots of the polynomial $p(-s)$, i.e., between the image of $F(\xi_0)$ and the mirror image in the right half complex plane of the roots of the asymptotically stable polynomial $p(s)$. In particular, if $I \supset \mathbb{R}^+$ and $p(s)$ has a real root then system (5) (under the stated assumptions) has unbounded trajectories.

Remark 3: The above link exists also in the general (i.e., $m > 1$) case. In fact, a necessary condition for the existence of a nonzero solution for (10) is

$$\det \left((-1)^{n+1} F(\xi_0)^n + (-1)^n k_{n-1} F(\xi_0)^{n-1} + \cdots + k_1 F(\xi_0) - k_0 I \right) = 0.$$

This shows a relation between $\sigma(F(\xi_0))$ and the roots of the polynomial $p(-s)$, i.e., between unstable eigenvalues of $F(\xi_0)$ and the mirror image in the right half complex plane of the roots of the asymptotically stable polynomial $p(s)$.

Remark 4: Applying the result in Proposition 3 to system (2), and with obvious redefinition of variables, (12) is

$$-(g(\xi_0) - 2)(g(\xi_0) - \lambda) = 0$$

and this clearly shows, recalling Assumption 1, that if $\lambda > 1$ the system (8) does not have nonzero equilibria, while nonzero equilibria exist if $\lambda \in (0, 1]$. Moreover, by Proposition 2, the nonzero equilibrium arising for $\lambda \in (0, 1)$ is stable, hence, the corresponding unbounded trajectory is stable.

IV. CONCLUSION

We conclude this note highlighting that the results in Proposition 3 provide two indications to avoid unbounded trajectories, provided the coefficients of $p(s)$ can be assigned.

- If I is bounded from above, the considered class of unbounded trajectories can be ruled out assigning a fast convergence rate to $y, \dot{y}, \dots$.
- The roots of $p(s)$ should be complex.

Interestingly, the second option does not require fast convergence rate, which is in general associated with high-gain control laws. For example, if we consider the system [we use the notation introduced in (5)]

$$\begin{aligned} \dot{\zeta} &= g(\zeta x_1) \zeta \\ \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - \gamma x_2 \end{aligned} \quad (13)$$

where $g(\cdot)$ satisfies Assumption 1, hence $I = [-1, 1]$, and $\gamma > 0$, we can draw the following conclusions.

- If $\gamma \geq 2$ the eigenvalues of the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -\gamma \end{bmatrix}$$

are real and at least the mirror image of one of them belongs to I . As a result the system has unbounded trajectories.

- If $0 < \gamma < 2$ the matrix A has complex conjugate eigenvalues, in modulo equal to one. In particular, as γ approaches zero these eigenvalues become *slow* and badly damped. However, despite the possibly slow convergence, in this case all trajectories of system (13) converge to zero.

The first claim is a consequence of the results in Section III. The second claim can be proven considering that any nontrivial trajectory of the system obtained from system (13) applying the partial coordinates transformation (7), which is described by equations of the form (8), intersects the positive ξ_0 -axis infinitely many times. Moreover, computing the time derivative of the function $\Xi(\zeta, x_1, x_2) = \xi_0^2 + \xi_1^2 = \zeta^2 x_1^2 + \zeta^2 x_2^2$ and exploiting the properties of the function $g(\cdot)$, we conclude that the intersection points approach

the origin. As a result, all trajectories of such a system converge to the origin. From this, from the fact that, sufficiently close to zero,

$$\Xi(t) \leq \Xi(0) \exp^{-2t}$$

and from the fact that x_1 and x_2 converge exponentially to zero but slower than $\exp^{-t}$, we conclude the claim.

The previous example shows that global boundedness and asymptotic convergence in a cascaded system may be obtained by judicious assignment of the eigenvalues of the *driving* system, and this may not necessarily mean the assignments of *fast* eigenvalues.

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Further Improvement of Free-Weighting Matrices Technique for Systems With Time-Varying Delay

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Abstract—A novel method is proposed in this note for stability analysis of systems with a time-varying delay. Appropriate Lyapunov functional and augmented Lyapunov functional are introduced to establish some improved delay-dependent stability criteria. Less conservative results are obtained by considering the additional useful terms (which are ignored in previous methods) when estimating the upper bound of the derivative of Lyapunov functionals and introducing the new free-weighting matrices. The

resulting criteria are extended to the stability analysis for uncertain systems with time-varying structured uncertainties and polytopic-type uncertainties. Numerical examples are given to demonstrate the effectiveness and the benefits of the proposed method.

Index Terms—Augmented Lyapunov functional, delay-dependent, linear matrix inequality (LMI), stability, time-varying delay.

I. INTRODUCTION

The problem of deriving delay-dependent stability criteria for linear delay systems has attracted the attention of many researchers in the last decade [1]–[21]. For systems with time-varying delay, fixed model transformations are the main methods to deal with delay-dependent stability problems [6], in which some inequalities such as Park and Moon *et al.*'s inequalities [3], [4] were used to estimate the upper bound of cross product terms. Recently, in order to reduce the conservatism, a free-weighting matrix method was proposed in [13]–[16] to investigate the delay-dependent stability, in which the bounding techniques on some cross product terms are not involved. However, when estimating the upper bound of the derivative of Lyapunov functional for systems with time-varying delay, there is room for investigation. For example, in [6], [10], [12]–[14], the derivative of $\int_{-h}^0 \int_{t+\theta}^t \dot{x}^T(s) Z \dot{x}(s) ds d\theta$ is often estimated as $h \dot{x}^T(t) Z \dot{x}(t) - \int_{t-d(t)}^t \dot{x}^T(s) Z \dot{x}(s) ds$ and the term $-\int_{t-h}^{t-d(t)} \dot{x}^T(s) Z \dot{x}(s) ds$ is ignored, which may lead to considerable conservativeness.

On the other hand, some other efforts on improving the delay-dependent conditions were made through introducing new Lyapunov functionals [11], [12], [16]. Indeed, the results in [11] and [12], which used modified Lyapunov functionals, are included or equivalent to those in [13] and [14]. The augmented Lyapunov functional introduced in [16] and [21] is only applicable for neutral systems with time-invariant delay, but is not easy to extend to time-varying delay case.

In this note, a new method that introduces the new free-weighting matrices is proposed to estimate the upper bound of the derivative of Lyapunov functional without ignoring some useful terms. Some less conservative delay-dependent stability criteria for systems with time-varying delay are presented by introducing new types of Lyapunov functionals. The resulting criteria are extended to the stability analysis for systems with time-varying structured uncertainties and polytopic-type uncertainties. Numerical examples are given to demonstrate the effectiveness and the merits of the proposed method.

II. PROBLEM FORMULATION

Consider the following nominal linear system with a time-varying delay:

$$\begin{cases} \dot{x}(t) = Ax(t) + A_d x(t - d(t)), t > 0 \\ x(t) = \phi(t), t \in [-h, 0] \end{cases} \quad (1)$$

where $x(t) \in \mathcal{R}^n$ is the state vector, A and A_d are constant matrices with appropriate dimensions, the time delay, $d(t)$, is a time-varying differentiable function that satisfies

$$0 \leq d(t) \leq h \quad (2)$$

and

$$|\dot{d}(t)| \leq \mu \quad (3)$$

where $h > 0$ and μ are constants. The initial condition, $\phi(t)$, is a continuous and differentiable vector-valued function of $t \in [-h, 0]$.

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