Nonlinear Normal Modes in an Intrinsic Theory of Anisotropic Beams

Rafael Palacios*

Department of Aeronautics, Imperial College, London SW7 2AZ, United Kingdom

Abstract

Nonlinear normal modes in nonlinear oscillations of beams are derived from intrinsic equations, that is, using velocities and strains as primary degrees of freedom. Displacements and rotations are thus not system states but are instead obtained using the propagation of the local beam material reference frames, as in rigid-body dynamics. It is shown that the intrinsic variables suffice to describe the free vibrations of the beam. The approach does not need assumptions in the material properties, i.e., it is valid for general anisotropic behavior, or the beam kinematics, i.e., it is based on Cosserat's exact geometrical description of the deformable curve. Furthermore, the nonlinear modal equations in intrinsic coordinates are obtained from integrals involving only products of the mode shapes and known coefficients. Using this description, the nonlinear normal modes are sought through an asymptotic approximation to the invariant manifolds that define them in the space of intrinsic modal coordinates. Particular cases of homogeneous isotropic and composite cantilever beams are finally used to exemplify the approach.

*Tel.: +44 20 7594 5075; Fax: +44 20 7594 1974.

Email address: rpalacio@imperial.ac.uk (Rafael Palacios)

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Nomenclature

A	cross-sectional area
A_2, A_3	equivalent cross-sectional areas for shear stiffness
$\mathbf{c}$	cross-sectional flexibility matrix
$\mathbf{e}$	shear/bending coupling coefficients in equilibrium equations
$\mathbf{e}_1$	unit vector in the beam axial direction
E	Young modulus
$\mathbf{f}_1$	force vector per unit length in equilibrium equations
$\mathbf{f}_a$	applied forces per unit length
$\mathbf{F}$	beam internal forces
G	shear modulus
I_1, I_2, I_3	cross-sectional moments of inertia around all 3 axis
J	polar moment of inertia
$\mathcal{L}_1, \mathcal{L}_2$	matrix operators in nonlinear equilibrium equations
L	beam length
$\mathbf{m}$	cross-sectional mass matrix
$\mathbf{m}_a$	applied moments per unit length
$\mathbf{M}$	beam internal moments
N_a	number of axial modes
N_b	number of bending modes
q_{1j}	modal coordinates (velocities) of mode j
q_{2j}	modal coordinates (internal forces/moments) of mode j
Q_{1j}	generalized force corresponding to mode j
s	curvilinear coordinate (arc length)

t	time
$\mathbf{V}$	beam translational velocity vector
$\mathbf{x}_1$	velocity states in the intrinsic model
$\mathbf{x}_2$	internal-force/moment states in the intrinsic model
α, β	coefficients in nonlinear modal equations
δ_{ij}	Kronecker delta
η_1, η_2	modal coordinates of master mode
Φ_j	mode shapes of linear normal modes j
$\xi_0, \boldsymbol{\xi}$	components of the quaternion vector
ρ	material density
θ	composite ply orientation
ω_j	angular frequencies of linear normal mode j
$\boldsymbol{\Omega}$	beam angular velocity vector

1. Introduction

Nonlinear Normal Modes (NNMs) appear as a natural extension of linear vibration analysis for large-amplitude oscillations of autonomous mechanical systems, and this has brought considerable attention to them as a means to generate reduced-order models in nonlinear structural dynamics applications [1]. Two alternative definitions have been proposed, corresponding to the NNMs being either synchronous oscillations [2] or invariant manifolds [3] in the configuration space of the free structure. The first definition was later extended to general conservative systems by investigating periodic (but not necessarily synchronous) solutions of the system [4], while the second corresponds, if restricted to conservative systems, to the application of normal form theory [5, 6] to the limit cycles defined from each linear mode. For weakly nonlinear systems, the NNMs are typically obtained from an asymptotic approximation to the nonlinear dynamics projected on space of the linear normal modes (LNMs) [7, 8, 9, 10]. In more general situations, Galerkin-based methods have been also proposed to describe the NNM in a global basis [11, 12].

NNMs are a useful tool to obtain minimal descriptions in structural dynamics (i.e., a *proper model*, as defined, for example, in Ref. [13]) and also to identify the underlying structure of the system nonlinear response [1]. Moreover, although the principle of superposition does not hold for NNMs, component-mode synthesis techniques can be extended to situations with weakly coupled substructures that can be independently represented by NNMs [14]. This opens the door to the application of the methodology to investigate nonlinear oscillations of large-scale structural problems and po-

tentially to the efficient coupling of, for example, a substructure represented by NNMs with a second one given by a finite-element discretization. The literature on NNMs is quite extensive and the reader is referred for further background information on applications in structural dynamics to the excellent overviews in Refs. [15] and [16].

There is therefore a huge potential in the routine use of NNMs for non-linear structural analysis, yet a major obstacle that may be preventing their widespread adoption is that the methodology relies on quite involved numerical procedures. Aiming to overcome this, this work presents an alternative framework to study NNMs in geometrically-nonlinear beam dynamics. Instead of a description of the beam dynamics using displacements and rotations, the starting point will be an intrinsic beam theory such as those developed by Hegemier and Nair [17] and, more recently, by Hodges [18]. This approach draws from Kirchhoff's analogy between the spatial and time derivatives (see, for instance, Ref. [19]), and uses a two-field description of the beam dynamics on first-derivatives, i.e., strains and velocities. This results in a formulation that closely resembles that of rigid-body dynamics, with first order equations of motion in both beam strains and velocities. The main drawback of intrinsic models (also shared with rigid-body dynamics based on velocities) is that constraints in displacements cannot be imposed directly and need instead to be enforced using Lagrange multipliers. For structures with multiple closed kinematic chains this solution can be rather ineffective, but many situations of geometrically-nonlinear structural response correspond to mostly open kinematic chains (see, for example, Ref. [20]). In those cases, intrinsic formulations provide a simple and powerful

description of the nonlinear structural dynamics.

In this paper, the intrinsic beam equations are first introduced as in Hodges [18], together with the corresponding general form of the boundary conditions and the propagation equations. This form of the beam equations is then used to obtain their LNMs, which is done in some detail as no references of this procedure for an intrinsic model are available in the literature. The nonlinear equations are then projected on the intrinsic modal coordinates, and this form is used to obtain the NNMs (defined as invariant manifolds in the system dynamics, as in Ref. [3]), in the absence of internal resonances, using a truncated polynomial expansion. The formulation is general and accounts for material anisotropy and varying properties along the beam. To exemplify the methodology, numerical results are finally presented for isotropic and composite cantilever beams with constant cross section.

2. Intrinsic beam equations

Following Cosserat's model, a beam will be defined as a solid determined by the rigid motion of cross-sections linked to a deformable reference line. There will be no assumptions in terms of material or geometric characteristics of the cross section other than its area being small compared to the square of the typical scale in the beam deformations. The intrinsic equations that describe the beam dynamics were developed by Hodges [18], and will be written here as

$$\begin{aligned} \mathbf{m}\dot{\mathbf{x}}_1 - \mathbf{x}'_2 - \mathbf{e}\mathbf{x}_2 + \mathcal{L}_1(\mathbf{x}_1)\mathbf{m}\mathbf{x}_1 + \mathcal{L}_2(\mathbf{x}_2)\mathbf{c}\mathbf{x}_2 &= \mathbf{f}_1, \\ \mathbf{c}\dot{\mathbf{x}}_2 - \mathbf{x}'_1 + \mathbf{e}^T\mathbf{x}_1 - \mathcal{L}_1^T(\mathbf{x}_1)\mathbf{c}\mathbf{x}_2 &= \mathbf{0}. \end{aligned} \quad (1)$$

Dots ($\dot{\bullet}$) denote derivatives with time, t , while primes ($\bullet'$) are used for derivatives with the arc length, s . The first equation is the actual equation of motion, while the second one is the compatibility condition between strains and velocities that is added to ensure uniqueness of the solution (cross derivatives of the displacement/rotation field with respect to both space and time have to be consistent). The state vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ and the force vector $\mathbf{f}_1$ are given by

$$\mathbf{x}_1 = \begin{Bmatrix} \mathbf{V} \\ \boldsymbol{\Omega} \end{Bmatrix}, \quad \mathbf{x}_2 = \begin{Bmatrix} \mathbf{F} \\ \mathbf{M} \end{Bmatrix}, \quad \mathbf{f}_1 = \begin{Bmatrix} \mathbf{f}_a \\ \mathbf{m}_a \end{Bmatrix}, \quad (2)$$

where $\mathbf{V}(s, t)$ and $\boldsymbol{\Omega}(s, t)$ are the translational and angular inertial velocities; $\mathbf{F}(s, t)$ and $\mathbf{M}(s, t)$ are the sectional internal forces and moments; and $\mathbf{f}_a(s, t)$ and $\mathbf{m}_a(s, t)$ are the applied forces and moments per unit length. All vectors are expressed in their components in the local (deformed) material frame. Note that, as also done in Ref. [18], the equations of the elastic beam are not written in terms of strains, but rather in terms of internal forces and moments (their energy conjugates). The arc length along the beam is s and it is assumed that the beam is straight in its reference (undeformed) configuration with tangent vector $\mathbf{e}_1 = \{1, 0, 0\}^T$. Being based on vectors in the current configuration, the extension of Eq. (1) to initially curved and twisted beams is straightforward, but it is omitted here for simplicity in the presentation. The coefficients in the equation are the 6×6 (symmetric) cross-sectional mass ($\mathbf{m}$) and flexibility ($\mathbf{c}$) matrices, which can be obtained from an appropriate dimensional-reduction method [21]. They are, in general, full matrices that vary with the arc length, s . The constant matrix $\mathbf{e}$ and the

matrix operators $\mathcal{L}_1$ and $\mathcal{L}_2$ are finally defined as

$$\mathbf{e} = \begin{bmatrix} 0 & 0 \\ \tilde{\mathbf{e}}_1 & 0 \end{bmatrix}, \quad \mathcal{L}_1(\mathbf{x}_1) = \begin{bmatrix} \tilde{\mathbf{\Omega}} & 0 \\ \tilde{\mathbf{V}} & \tilde{\mathbf{\Omega}} \end{bmatrix}, \quad \text{and} \quad \mathcal{L}_2(\mathbf{x}_2) = \begin{bmatrix} 0 & \tilde{\mathbf{F}} \\ \tilde{\mathbf{F}} & \tilde{\mathbf{M}} \end{bmatrix}. \quad (3)$$

where $\tilde{\bullet}$ is the skew-symmetric (or cross-product) operator. The intrinsic formulation is therefore defined in terms of the spatial (force) and time (velocity) derivatives of the displacements and rotations of the beam.

Eq. (1) needs to be solved with the appropriate boundary and initial conditions, which are introduced in the next section. Ref. [18] presented a finite-difference solution to the problem, while Refs. [22] and [20] more recently introduced finite-element solutions. Once the force/velocity field is calculated, one can integrate in either the spatial or the time variable to obtain the beam displacements and rotations. This is discussed in Section 2.2.

2.1. Boundary Conditions

The problem is, in general, a 2-point boundary value problem in space and time, and needs to be solved with end conditions at $s = 0$ and $s = L$, for all t , as well as, $t = 0$ and $t = T$, for all s . Note that the spatial boundary conditions will satisfy

$$\begin{aligned} \mathbf{x}_1(0, t)\mathbf{x}_2(0, t) &= 0, \\ \mathbf{x}_1(L, t)\mathbf{x}_2(L, t) &= 0. \end{aligned} \quad (4)$$

This simply means that at the boundary the components of either the velocity or the internal forces have to be zero. In particular, for a cantilever beam of

length L , the spatial boundary conditions are

$$\begin{aligned}\mathbf{x}_1(0, t) &= 0, \\ \mathbf{x}_2(L, t) &= 0.\end{aligned}\tag{5}$$

2.2. Recovering displacements and rotations

Intrinsic beam equations, as defined by Hogdes [18] are analogous to the Euler's equations of motion for an unconstrained rigid body. Displacements and rotations are dependent variables, which only appear explicitly in the equations if the applied forces and moments in Eq. (2) depend on them (and, crucially for this work, they are not needed to describe the beam dynamics in the particular case of free vibrations). Eqs. (1) are complemented by output relations that give the local displacement and orientation as a function of either the internal forces and moments or the local inertial velocities. In the first case, displacements and rotations are obtained from the instantaneous spatial integration of their space derivatives (strains) [23], as with Frenet-Serret formulas in differential geometry. For the application in this paper, a direct time integration of the velocities, as done in rigid body dynamics, will be sufficient. The local orientation with respect to some inertial reference frame will be represented by means of quaternions, which need to satisfy the propagation equation [24],

$$\begin{aligned}\dot{\zeta}_0 &= -\frac{1}{2}\boldsymbol{\Omega}^T \boldsymbol{\zeta}, \\ \dot{\boldsymbol{\zeta}} &= \frac{1}{2}\left(\zeta_0 \boldsymbol{\Omega} - \tilde{\boldsymbol{\Omega}} \boldsymbol{\zeta},\right)\end{aligned}\tag{6}$$

The local rotation matrix $\mathbf{R}$ is then obtained from

$$\mathbf{R} = \mathbf{H}(\zeta_0, \boldsymbol{\zeta}) \mathbf{G}^T(\zeta_0, \boldsymbol{\zeta})\tag{7}$$

with $\mathbf{H} = \begin{bmatrix} -\boldsymbol{\zeta} & \zeta_0 \mathbf{I} + \tilde{\boldsymbol{\zeta}} \end{bmatrix}$ and $\mathbf{G} = \begin{bmatrix} -\boldsymbol{\zeta} & \zeta_0 \mathbf{I} - \tilde{\boldsymbol{\zeta}} \end{bmatrix}$. The position vector $\mathbf{p}$ along the beam with respect to the reference frame (and given in its components in that frame) is finally obtained from direct integration of the inertial velocities

$$\dot{\mathbf{p}} = \mathbf{R}\mathbf{V} \quad (8)$$

3. Linear normal modes

Eq. (1) is now linearized around $\mathbf{x} = 0$, to give

$$\begin{aligned} \mathbf{m}\dot{\mathbf{x}}_1 - \mathbf{x}'_2 - \mathbf{e}\mathbf{x}_2 &= \mathbf{f}_1, \\ \mathbf{c}\dot{\mathbf{x}}_2 - \mathbf{x}'_1 + \mathbf{e}^T \mathbf{x}_1 &= \mathbf{0}. \end{aligned} \quad (9)$$

The linear normal modes are obtained by solving the homogeneous equation obtained from imposing $\mathbf{f}_1 = 0$ in Eq. (9) together with the problem spatial boundary conditions. The nontrivial solutions of the resulting system will be then sought as

$$\begin{aligned} \mathbf{x}_1 &= \boldsymbol{\Phi}_{1j}(s) \sin(\omega_j t), \\ \mathbf{x}_2 &= \boldsymbol{\Phi}_{2j}(s) \cos(\omega_j t), \end{aligned} \quad (10)$$

with $j = 0, 1, 2, \dots$. Substituting Eq. (10) into Eq. (9) with $\mathbf{f}_1 = \mathbf{0}$ gives

$$\begin{aligned} \boldsymbol{\Phi}'_{1j} - \mathbf{e}^T \boldsymbol{\Phi}_{1j} &= -\omega_j \mathbf{c} \boldsymbol{\Phi}_{2j}, \\ \boldsymbol{\Phi}'_{2j} + \mathbf{e} \boldsymbol{\Phi}_{2j} &= \omega_j \mathbf{m} \boldsymbol{\Phi}_{1j}, \end{aligned} \quad (11)$$

which have to be solved with the appropriate set of boundary conditions.

Noting that they have to satisfy Eqs. (4), it is easy to prove that

$$\begin{aligned}\omega_i \int_0^L \Phi_{1i}^T \mathbf{m} \Phi_{1j} ds - \omega_j \int_0^L \Phi_{2i}^T \mathbf{c} \Phi_{2j} ds &= 0, \\ \omega_j \int_0^L \Phi_{1i}^T \mathbf{m} \Phi_{1j} ds - \omega_i \int_0^L \Phi_{2i}^T \mathbf{c} \Phi_{2j} ds &= 0.\end{aligned}\tag{12}$$

If $i \neq j$ then $\omega_i \neq \omega_j$ and as a result all the integrals in Eq. (12) are zero. This defines orthogonality conditions between the mode shapes, which, including a normalization, are

$$\begin{aligned}\int_0^L \Phi_{1i}^T \mathbf{m} \Phi_{1j} ds &= \delta_{ij}, \\ \int_0^L \Phi_{2i}^T \mathbf{c} \Phi_{2j} ds &= \delta_{ij}.\end{aligned}\tag{13}$$

The modal expansions will be then defined as

$$\begin{aligned}\mathbf{x}_1(s, t) &= \sum_{j=0}^{\infty} \Phi_{1j}(s) q_{1j}(t), \\ \mathbf{x}_2(s, t) &= \sum_{j=0}^{\infty} \Phi_{2j}(s) q_{2j}(t),\end{aligned}\tag{14}$$

where (q_{1j}, q_{2j}) are pairs of *intrinsic modal coordinates*. Note that the first mode corresponds to $j = 0$. Since this is a first-order theory, each natural frequency will be associated to two generalized coordinates¹. The linearized equations of motion in intrinsic modal coordinates are finally written as

$$\begin{aligned}\dot{q}_{1j} - \omega_j q_{2j} &= Q_{1j}, \\ \dot{q}_{2j} + \omega_j q_{1j} &= 0,\end{aligned}\tag{15}$$

¹Alternatively, the previous derivation can be carried out in the complex plane, starting from Eq. (10), in which case a single (complex) generalized coordinate would be used. The current form of the equations was found however to be more easily manageable.

with the generalized forces given by $Q_{1j} = \int_0^L \Phi_{1j}^T \mathbf{f}_1 ds$. This system of equations can be easily transformed to second-order differential form, by substituting the second of Eqs. (15) into the first one. That gives

$$\ddot{q}_{2j} + \omega_j^2 q_{2j} = \omega_j Q_{1j}. \quad (16)$$

3.1. Beams of constant cross section

The modal Eqs. (11) can be, in general, solved using a finite-element discretization. For a uniform beam of length L and constant properties, matrices $\mathbf{m}$ and $\mathbf{c}$ are constant and then Eqs. (11) have the analytical solution

$$\Phi_j(s) = e^{\mathbf{A}(\omega_j)s} \Phi_j(0), \quad (17)$$

with

$$\Phi_j = \begin{Bmatrix} \Phi_{1j} \\ \Phi_{2j} \end{Bmatrix} \quad \text{and} \quad \mathbf{A}(\omega_j) = \begin{bmatrix} \mathbf{e}^T & -\omega_j \mathbf{c} \\ \omega_j \mathbf{m} & -\mathbf{e} \end{bmatrix}. \quad (18)$$

In general the mass and compliance matrices, $\mathbf{m}$ and $\mathbf{c}$, respectively, are full but symmetric matrices and the solution to Eq. (18) can be sought using similarity transformations, as

$$\Phi_j(s) = \mathbf{P} e^{\mathbf{J}(\omega_j)s} \mathbf{P}^{-1} \Phi_j(0), \quad (19)$$

with $\mathbf{J}(\omega_j)$ the diagonal matrix with the eigenvalues of $\mathbf{A}(\omega_j)$ and $\mathbf{P}$, the matrix obtained with the corresponding eigenvectors as columns. The natural angular frequencies of the problem, ω_j , are finally obtained after enforcing non-trivial solutions of Eq. (19) that satisfy the boundary conditions. Examples of this will be presented for isotropic and cantilever beams in sections 5 and 6, respectively.

4. Nonlinear normal modes

The LNMs obtained in the previous section will be used now to project Eqs. (1). By substituting Eq. (14) into that equation, and after using the orthogonality conditions in Eq. (13), one obtains the equations of motion in intrinsic modal coordinates, which will be written as

$$\begin{aligned} \dot{q}_{1j} - \omega_j q_{2j} + \frac{1}{2} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \alpha_{1j}^{kl} q_{1k} q_{1l} + \frac{1}{2} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \alpha_{2j}^{kl} q_{2k} q_{2l} &= Q_{1j}, \\ \dot{q}_{2j} + \omega_j q_{1j} - \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \beta_j^{kl} q_{1k} q_{2l} &= 0, \end{aligned} \quad (20)$$

with

$$\begin{aligned} \alpha_{1j}^{kl} &= \int_0^L 2\mathbf{\Phi}_{1j}^T \mathcal{L}_1(\mathbf{\Phi}_{1k}) \mathbf{m} \mathbf{\Phi}_{1l} ds, \\ \alpha_{2j}^{kl} &= \int_0^L 2\mathbf{\Phi}_{1j}^T \mathcal{L}_2(\mathbf{\Phi}_{2k}) \mathbf{c} \mathbf{\Phi}_{2l} ds, \\ \beta_j^{kl} &= \int_0^L \mathbf{\Phi}_{2j}^T \mathcal{L}_1^T(\mathbf{\Phi}_{1k}) \mathbf{c} \mathbf{\Phi}_{2l} ds. \end{aligned} \quad (21)$$

This form of the equations is similar to that proposed by Nayfeh [7, p 607], but with only quadratic nonlinear terms due to the use of the intrinsic formulation. Thomas and Bilbao [25] also found a similar form for the modal projection of the intrinsic equations (using transverse displacements and the Airy function) of plate dynamics. Note that Eqs. (20) consider general anisotropic material properties that can vary along the beam length, that is, $\mathbf{m}$ and $\mathbf{c}$ are, in general, full matrices and a function of the arc length, s . To compare the results of the intrinsic modal coordinates with those of the corresponding displacement-based equations, e.g., Refs. [26, 10], a two-step post-process is needed: First, the inertial velocities ($\mathbf{x}_1$) and internal

forces ($\mathbf{x}_2$) are obtained from the intrinsic modal coordinates, using Eq. (14); second, the displacements and rotations are finally obtained from a recovery process from either the force or velocities as described in section 2.2. Note again that there is no need to introduce approximations in the propagation equations, Eqs. (6)-(8), even if the equations of motion are linearized.

NNMs are defined next as in Refs. [3, 8], that is, as the invariant manifolds of the free-vibration equations (20) that, for infinitesimal amplitudes, are tangent to the LNMs defined in Eq. (15). A master-slave approach is then used to describe the NNMs in the space of intrinsic modal coordinates, with master degrees of freedom corresponding to the LNMs that coincide with the NNMs at the zero-amplitude limit. A crucial difference however is that the present definition of the LNMs only corresponds to the conventional free vibrations modes (displacement-based beam dynamics) for infinitesimal values of the intrinsic modal coordinates, in which case the strain-displacement relations and velocity-displacement relations can also be linearized. This however will not be the case in the description of the free vibrations of the nonlinear system and, as a result, the independent coordinates used to describe the NNMs (the intrinsic modal coordinates of the master LNMs) define a very different parameterization of the manifold dynamics.

Assuming that there are no internal resonances, a 2-dimensional invariant manifold can be obtained with the previous procedure starting from each LNM. This section will describe the dynamics of individual NNMs and, without loss of generality, it will be assumed that the modes can be rearranged

such that the master linear mode is the 0-th mode, that is,

$$\begin{aligned} q_{10} &= \eta_1(t), \\ q_{20} &= \eta_2(t). \end{aligned} \tag{22}$$

The rest of the modal coordinates will be constrained by algebraic relations that enforce that motions remain on the invariant manifold. An asymptotic approximation to the NNMs can be now obtained. Up to third order, the non-zero terms in the constraint equations on the slave modal coordinates are

$$\begin{aligned} q_{1j} &= \mu_{1j}\eta_1\eta_2 + \eta_1 (\nu_{1j}\eta_1^2 + \nu_{3j}\eta_2^2) + H.O.T., \\ q_{2j} &= \mu_{2j}\eta_1^2 + \mu_{3j}\eta_2^2 + \eta_2 (\nu_{2j}\eta_2^2 + \nu_{4j}\eta_1^2) + H.O.T., \end{aligned} \tag{23}$$

for $j = 1, 2, \dots, \infty$. After some manipulation, and after defining $\tau_{2j}^{-1} = \omega_0\bar{\omega}_j(\bar{\omega}_j^2 - 4)$, with $\bar{\omega}_j = \frac{\omega_j}{\omega_0}$, the coefficients in the quadratic terms in Eq. (23) are obtained as

$$\begin{Bmatrix} \mu_{1j} \\ \mu_{2j} \\ \mu_{3j} \end{Bmatrix} = \tau_{2j} \begin{bmatrix} \bar{\omega}_j^2 & -\bar{\omega}_j & \bar{\omega}_j \\ -\bar{\omega}_j & \frac{\bar{\omega}_j^2-2}{2} & -1 \\ \bar{\omega}_j & -1 & \frac{\bar{\omega}_j^2-2}{2} \end{bmatrix} \begin{Bmatrix} \beta_j^{00} \\ \alpha_{1j}^{00} \\ \alpha_{2j}^{00} \end{Bmatrix}, \tag{24}$$

where the coefficients in the right-hand side are the constants of the nonlinear equations of motion in modal coordinates, Eq. (21). The coefficients in the cubic terms in Eq. (23) are obtained, after defining $\tau_{3j}^{-1} = \omega_0(\bar{\omega}_j^2 - 9)(\bar{\omega}_j^2 - 1)$,

as

$$\begin{pmatrix} \nu_{1j} \\ \nu_{2j} \\ \nu_{3j} \\ \nu_{4j} \end{pmatrix} = \tau_{3j} \times \begin{bmatrix} \frac{\bar{\omega}_j}{6}(\bar{\omega}_j^2 - 7) & -1 & -\bar{\omega}_j & \frac{1}{2}(\bar{\omega}_j^2 - 3) \\ -1 & \frac{\bar{\omega}_j}{6}(\bar{\omega}_j^2 - 7) & \frac{1}{2}(\bar{\omega}_j^2 - 3) & -\bar{\omega}_j \\ -\bar{\omega}_j & \frac{1}{2}(\bar{\omega}_j^2 - 3) & \frac{\bar{\omega}_j}{2}(\bar{\omega}_j^2 - 3) & -\bar{\omega}_j^2 \\ \frac{1}{2}(\bar{\omega}_j^2 - 3) & -\bar{\omega}_j & -\bar{\omega}_j^2 & \frac{\bar{\omega}_j}{2}(\bar{\omega}_j^2 - 3) \end{bmatrix} \begin{pmatrix} \Gamma_{1j} \\ \Gamma_{2j} \\ \Gamma_{3j} \\ \Gamma_{4j} \end{pmatrix}, \quad (25)$$

with

$$\begin{aligned} \Gamma_{1j} &= 6\alpha_{10}^{00}\mu_{2j} + 6\sum_{k=1}^{\infty}\beta_j^{0k}\mu_{2k}, \\ \Gamma_{2j} &= -3\alpha_{20}^{00}\mu_{1j} + 3\sum_{k=1}^{\infty}(\alpha_{2j}^{0k} + \alpha_{2j}^{k0})\mu_{3k}, \\ \Gamma_{3j} &= 2\alpha_{20}^{00}\mu_{2j} - 4\beta_0^{00}\mu_{3j} + 2\sum_{k=1}^{\infty}(\beta_j^{k0}\mu_{1k} + \beta_j^{0k}\mu_{3k}), \\ \Gamma_{4j} &= (2\beta_0^{00} - \alpha_{10}^{00})\mu_{1j} + \sum_{k=1}^{\infty}[(\alpha_{1j}^{0k} + \alpha_{1j}^{k0})\mu_{1k} + (\alpha_{2j}^{0k} + \alpha_{2j}^{k0})\mu_{2k}]. \end{aligned} \quad (26)$$

Substituting back the coordinate transformation defined by Eq. (23) into the nonlinear equations of motion, Eq. (20), yields the dynamic equations of the NNM. Retaining only terms up to third order, they are

$$\begin{aligned} \dot{\eta}_1 &= \omega_0\eta_2 - \frac{1}{2}(\alpha_{10}^{00}\eta_1^2 + \alpha_{20}^{00}\eta_2^2) - \eta_2(\Xi_1\eta_1^2 + \Xi_2\eta_2^2) + H.O.T., \\ \dot{\eta}_2 &= -\omega_0\eta_1 + \beta_0^{00}\eta_1\eta_2 + \eta_1(\Xi_3\eta_1^2 + \Xi_4\eta_2^2) + H.O.T., \end{aligned} \quad (27)$$

with

$$\begin{aligned}
\Xi_1 &= \sum_{k=1}^{\infty} \left(\frac{\alpha_{10}^{0k} + \alpha_{10}^{k0}}{2} \mu_{1k} + \frac{\alpha_{20}^{0k} + \alpha_{20}^{k0}}{2} \mu_{2k} \right), \\
\Xi_2 &= \sum_{k=1}^{\infty} \frac{\alpha_{20}^{0k} + \alpha_{20}^{k0}}{2} \mu_{3k}, \\
\Xi_3 &= \sum_{k=1}^{\infty} \beta_0^{0k} \mu_{2k}, \\
\Xi_4 &= \sum_{k=1}^{\infty} \left(\beta_0^{k0} \mu_{1k} + \beta_0^{0k} \mu_{3k} \right). \tag{28}
\end{aligned}$$

Thus only the quadratic terms in Eq. (23) need to be retained to compute the third-order dynamics of the NNM, while cubic terms are needed to represent the invariant manifold in the intrinsic modal coordinates. The coefficients in Eq. (27) are all explicitly known as a function of the linear mode shapes and natural frequencies through Eqs. (21) and (24). It should be noted that the natural symmetry of the intrinsic formulation yields a very manageable description of the dynamics of the NNMs and that these results are valid for arbitrary beam properties. The instantaneous values of the beam displacements and rotations can finally be obtained by using the kinematic relations (strain-displacement or velocity-displacement), as described in Section 2.2.

5. Application to cantilever isotropic beams

For the particular case of beams with neither elastic nor geometric couplings, both mass and flexibility matrices are diagonal. Their components

can be identified in the usual way as

$$\begin{aligned}\mathbf{m} &= \text{diag} \{ \rho A, \rho A, \rho A, \rho I_1, \rho I_2, \rho I_3 \}, \\ \mathbf{c}^{-1} &= \text{diag} \{ EA, GA_2, GA_3, GJ, EI_2, EI_3 \}.\end{aligned}\quad (29)$$

This simplification gives analytical expressions of Eq. (17). Although the process is basically equivalent to the classical modal analysis based on displacements, it was found convenient to provide some details of the procedure to obtain the linear vibration modes in an intrinsic beam formulation.

5.1. Linear normal modes

The solutions of Eq. (17) with the constant property matrices given by Eq. (29) will correspond, to the conventional axial, torsional and coupled bending/transverse shear vibration modes.

Axial modes.

Eqs. (17) reduce in this case to

$$\begin{Bmatrix} \Phi_{V_1} \\ \Phi_{F_1} \end{Bmatrix} = \exp \left(\begin{bmatrix} 0 & -\frac{\omega}{EA} \\ \rho A \omega & 0 \end{bmatrix} s \right) \begin{Bmatrix} \Phi_{V_1}(0) \\ \Phi_{F_1}(0) \end{Bmatrix}, \quad (30)$$

with $\Phi_{V_1}(0) = 0$ and $\Phi_{F_1}(L) = 0$. The eigenvalues are then $\omega_j^a = \sqrt{\frac{E}{\rho}} \lambda_j^a$, with

$$\lambda_j^a = \frac{2j-1}{2} \frac{\pi}{L}. \quad (31)$$

for $j = 1, 2, \dots, \infty$. The corresponding eigenvectors are, after normalization,

$$\begin{aligned}\Phi_{V_1j} &= \sqrt{\frac{2}{\rho AL}} \sin(\lambda_j^a s), \\ \Phi_{F_1j} &= -\sqrt{\frac{2EA}{L}} \cos(\lambda_j^a s).\end{aligned}\quad (32)$$

Torsional modes.

Eqs. (17) reduce in this case to

$$\begin{Bmatrix} \Phi_{\Omega_1} \\ \Phi_{M_1} \end{Bmatrix} = \exp \left(\begin{bmatrix} 0 & -\frac{\omega}{GJ} \\ \rho I_1 \omega & 0 \end{bmatrix} s \right) \begin{Bmatrix} \Phi_{\Omega_1}(0) \\ \Phi_{M_1}(0) \end{Bmatrix}, \quad (33)$$

with $\Phi_{\Omega_1}(0) = 0$ and $\Phi_{M_1}(L) = 0$. These equations are analogous to Eqs. (30) and their solution has the same form as Eqs. (31) and (32).

Bending/shear modes.

Bending modes in the $x - y$ plane are given by

$$\begin{Bmatrix} \Phi_{V_2} \\ \Phi_{\Omega_3} \\ \Phi_{F_2} \\ \Phi_{M_3} \end{Bmatrix} = \exp \left(\begin{bmatrix} 0 & 1 & -\frac{\omega}{GA_2} & 0 \\ 0 & 0 & 0 & -\frac{\omega}{EI_3} \\ \rho A \omega & 0 & 0 & 0 \\ 0 & \rho I_3 \omega & -1 & 0 \end{bmatrix} s \right) \begin{Bmatrix} \bar{\Phi}_{V_2} \\ \bar{\Phi}_{\Omega_3} \\ \bar{\Phi}_{F_2} \\ \bar{\Phi}_{M_3} \end{Bmatrix}, \quad (34)$$

with $\Phi_{V_2}(0) = 0$, $\Phi_{\Omega_3}(0) = 0$, $\Phi_{F_2}(L) = 0$ and $\Phi_{M_3}(L) = 0$. An analytical solution to this problem is quite involved [27], but it can be easily found if it is further assumed that $I_3 = 0$ and $G \rightarrow \infty$ (an Euler-Bernoulli beam). In such case, the eigenvalues are the solutions of the well-known formula

$$\cos(\lambda_j^b L) \cosh(\lambda_j^b L) + 1 = 0, \quad (35)$$

with $\omega_j^b = (\lambda_j^b)^2 \sqrt{\frac{EI_3}{\rho A}}$. The corresponding eigenvectors are

$$\begin{aligned} \Phi_{V_{2j}} &= \frac{1}{\sqrt{\rho A L}} [\cos(\lambda_j^b s) - \cosh(\lambda_j^b s) - \Lambda_j^b (\sin(\lambda_j^b s) - \sinh(\lambda_j^b s))], \\ \Phi_{\Omega_{3j}} &= \frac{\lambda_j^b}{\sqrt{\rho A L}} [-\sin(\lambda_j^b s) - \sinh(\lambda_j^b s) - \Lambda_j^b (\cos(\lambda_j^b s) - \cosh(\lambda_j^b s))], \\ \Phi_{F_{2j}} &= \lambda_j^b \sqrt{\frac{EI_3}{L}} [\sin(\lambda_j^b s) - \sinh(\lambda_j^b s) + \Lambda_j^b (\cos(\lambda_j^b s) + \cosh(\lambda_j^b s))], \\ \Phi_{M_{3j}} &= \sqrt{\frac{EI_3}{L}} [\cos(\lambda_j^b s) + \cosh(\lambda_j^b s) - \Lambda_j^b (\sin(\lambda_j^b s) + \sinh(\lambda_j^b s))], \end{aligned} \quad (36)$$

with

$$\Lambda_j^b = \frac{\cos(\lambda_j^b L) + \cosh(\lambda_j^b L)}{\sin(\lambda_j^b L) + \sinh(\lambda_j^b L)}. \quad (37)$$

Similar results are obtained for the LNM for bending motions in the $x - z$ plane.

5.2. Modal representation of nonlinear motions in the $x - y$ plane

Consider the particular case of a cantilever beam in the $x - y$ plane, that is, $V_3 = 0$, $\Omega_1 = 0$, and $\Omega_2 = 0$ and there will be a geometric coupling between axial and bending modes. The modal expansion of the non-zero states is given by

$$\begin{aligned} V_1 &= \sum_{j=1}^{\infty} \Phi_{V_1 j} q_{1j}^a, \\ F_1 &= \sum_{j=1}^{\infty} \Phi_{F_1 j} q_{2j}^a, \\ \begin{Bmatrix} V_2 \\ \Omega_3 \end{Bmatrix} &= \sum_{j=1}^{\infty} \begin{Bmatrix} \Phi_{V_2 j} \\ \Phi_{\Omega_3 j} \end{Bmatrix} q_{1j}^b, \\ \begin{Bmatrix} F_2 \\ M_3 \end{Bmatrix} &= \sum_{j=1}^{\infty} \begin{Bmatrix} \Phi_{F_2 j} \\ \Phi_{M_3 j} \end{Bmatrix} q_{2j}^b. \end{aligned} \quad (38)$$

The superscript a is introduced to identify the modal coordinates corresponding to the axial modes, which were given by Eq. (32). Superscript b refers to bending/shear modes, obtained from Eq. (34), with the normalization of Eq. (13) (for Euler-Bernoulli assumptions, they reduce to Eqs. (36)). The resulting equations of motion in intrinsic modal coordinates are then written

as

$$\begin{aligned}
\dot{q}_{1j}^a &= Q_j^a + \omega_j^a q_{2j}^a + \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} [A_{jkl} q_{1k}^b q_{1l}^b - B_{jkl} q_{2k}^b q_{2l}^b], \\
\dot{q}_{1j}^b &= Q_j^b + \omega_j^b q_{2j}^b + \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} [-A_{kjl} q_{1k}^a q_{1l}^b + ((1 - \sigma)C_{klj} + D_{kjl}) q_{2k}^a q_{2l}^b], \\
\dot{q}_{2j}^a &= -\omega_j^a q_{1j}^a + \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} (\sigma C_{jlk} - D_{jkl}) q_{1k}^b q_{2l}^b, \\
\dot{q}_{2j}^b &= -\omega_j^b q_{1j}^b + \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} (B_{kjl} - C_{kjl}) q_{1k}^a q_{2l}^b,
\end{aligned} \tag{39}$$

with $\sigma = \frac{EA}{GA_2}$ and with coefficients obtained from the integrals

$$\begin{aligned}
A_{jkl} &= \rho A \int_0^L \Phi_{V_{1j}} \Phi_{V_{2k}} \Phi_{\Omega_{3l}} ds, & C_{jkl} &= \frac{1}{EA} \int_0^L \Phi_{F_{1j}} \Phi_{F_{2k}} \Phi_{\Omega_{3l}} ds, \\
B_{jkl} &= \frac{1}{EI_3} \int_0^L \Phi_{V_{1j}} \Phi_{F_{2k}} \Phi_{M_{3l}} ds, & D_{jkl} &= \frac{1}{EI_3} \int_0^L \Phi_{F_{1j}} \Phi_{V_{2k}} \Phi_{M_{3l}} ds.
\end{aligned} \tag{40}$$

Eqs. (39) are a particular case of Eqs. (20) corresponding to isotropic cantilever beams. Note that the nonlinear terms in the bending/shear equations are products of the axial and bending/shear modal coordinates and therefore the axial modes are required to capture the nonlinear beam dynamics. In the axial equations, the nonlinear terms are coming exclusively from the bending degrees of freedom. When the natural frequencies of the axial modes are very high compared to the frequency range of interest, a quasi-static approximation can be introduced to the first and third of equations (39), i.e., $\dot{q}_{1j}^a = 0$ and $\dot{q}_{2j}^a = 0$, and then those equations can be used to obtain the instantaneous amplitude of the axial modes as a function of the modal coordinates of the bending/shear modes. This simplification however was not deemed necessary in this work.

5.3. Nonlinear normal modes for free vibrations in the $x - y$ plane

We will finally evaluate the NNM that coincides with the n -th linear bending mode of an isotropic cantilever beam at the zero-energy limit. The master modal coordinates will be then

$$\begin{aligned} q_{1n}^b &= \eta_1(t), \\ q_{2n}^b &= \eta_2(t), \end{aligned} \quad (41)$$

and the corresponding linear natural frequency will be $\omega_0 = \omega_n^b$. Using the coefficients defined in Eq. (40), the third-order approximation to the NNM dynamics of Eq. (27), becomes in this case

$$\begin{aligned} \dot{\eta}_1 &= \omega_0 \eta_2 - (\Xi_1 \eta_1^2 + \Xi_2 \eta_2^2) \eta_2, \\ \dot{\eta}_2 &= -\omega_0 \eta_1 + \Xi_4 \eta_1 \eta_2^2, \end{aligned} \quad (42)$$

with

$$\begin{aligned} \Xi_1 &= \sum_{j=1}^{\infty} [\mu_{1j}^a A_{jnn} - \mu_{2j}^a ((1 - \sigma)C_{jnn} + D_{jnn})], \\ \Xi_2 &= \sum_{j=1}^{\infty} -\mu_{3j}^a ((1 - \sigma)C_{jnn} + D_{jnn}), \\ \Xi_4 &= \sum_{j=1}^{\infty} \mu_{1j}^a (B_{jnn} - C_{jnn}), \end{aligned} \quad (43)$$

and

$$\begin{Bmatrix} \mu_{1j}^a \\ \mu_{2j}^a \\ \mu_{3j}^a \end{Bmatrix} = \tau_{2j}^a \begin{bmatrix} (\bar{\omega}_j^a)^2 & -\bar{\omega}_j^a & \bar{\omega}_j^a \\ -\bar{\omega}_j^a & \frac{(\bar{\omega}_j^a)^2 - 2}{2} & -1 \\ \bar{\omega}_j^a & -1 & \frac{(\bar{\omega}_j^a)^2 - 2}{2} \end{bmatrix} \begin{Bmatrix} \sigma C_{jnn} - D_{jnn} \\ -2A_{jnn} \\ 2B_{jnn} \end{Bmatrix}. \quad (44)$$

Therefore, the dynamics of the NNM (up to third order) for an isotropic cantilever beam are uniquely determined by the projection of the invariant manifold onto the space defined by the master n -th bending mode, and the axial modes. After some manipulation, and using the mode shapes of section 5.1, it can be also be proven² that $\Xi_4 = \Xi_1 + \Xi_2$.

The frequency for the nonlinear free vibrations is obtained after a change of variables to an amplitude/phase description, as

$$\begin{aligned}\eta_1 &= a \sin \phi, \\ \eta_2 &= a \cos \phi.\end{aligned}\tag{45}$$

Substituting Eq. (45) into Eq. (42), and after integration over a cycle, one can obtain the period of the NNM as

$$T = \int_0^{2\pi} \frac{1}{\omega_0 - a^2 f(\phi)} d\phi,\tag{46}$$

with $f(\phi) = \Xi_2 \cos^4 \phi + (\Xi_1 + \Xi_4) \sin^2 \phi \cos^2 \phi$. The amplitude a can be solved as a function of the phase ϕ and the initial amplitude a_0 as

$$a^2 = a_0^2 - \frac{a_0^4}{\omega_0} \left(\Xi_4 \sin^2 \phi + \frac{1}{2} \Xi_2 \sin^2 2\phi \right) + \mathcal{O}(a_0^5).\tag{47}$$

The integral in Eq. (46) can then be approximated as

$$T = \frac{2\pi}{\omega_0} + \frac{\pi a_0^2}{2\omega_0^2} (\Xi_2 + \Xi_4) + \frac{\pi a_0^4}{16\omega_0^3} (6\Xi_2^2 - \Xi_2\Xi_4 + 3\Xi_4^2) + \mathcal{O}(a_0^5).\tag{48}$$

While the third-order dynamics of the NNM can be described with only a partial knowledge of the system, the invariant manifold itself spans all axial

²For the general problem, defined by Eq. (27), it was also observed numerically that the third-order coefficients satisfy $\Xi_1 + \Xi_2 = \Xi_3 + \Xi_4$, but a formal proof is not yet available.

and bending modes and is defined by the constraint equations of Eq. (23). For the case of an isotropic cantilever beam, it can be seen that the constraint equations in the axial modal coordinates have zero third-order terms, and are therefore simplified to

$$\begin{aligned} q_{1j}^a &= \mu_{1j}^a \eta_1 \eta_2, \\ q_{2j}^a &= \mu_{2j}^a \eta_1^2 + \mu_{3j}^a \eta_2^2, \end{aligned} \quad (49)$$

while for the slave bending modal coordinates the quadratic terms in the constraint equations, Eq. (23), are set equal to zero, to give

$$\begin{aligned} q_{1j}^b &= \eta_1 (\nu_{1j}^b \eta_1^2 + \nu_{3j}^b \eta_2^2), \\ q_{2j}^b &= \eta_2 (\nu_{2j}^b \eta_2^2 + \nu_{4j}^b \eta_1^2). \end{aligned} \quad \text{if } j \neq n. \quad (50)$$

The coefficients in Eq. (50) are given by Eq. (25), with the right-hand side given by

$$\begin{aligned} \Gamma_{1j}^b &= 0, \\ \Gamma_{2j}^b &= \sum_{k=1}^{\infty} -6\mu_{3k}^a ((1-\sigma)C_{knj} + D_{kjn}), \\ \Gamma_{3j}^b &= \sum_{k=1}^{\infty} 2\mu_{1k}^a (B_{kjn} - C_{kjn}), \\ \Gamma_{4j}^b &= \sum_{k=1}^{\infty} 2[\mu_{1k}^a A_{kjn} - \mu_{2k}^a ((1-\sigma)C_{knj} + D_{kjn})]. \end{aligned} \quad (51)$$

Equations (41), (42), (49), and (50) thus describe the third-order dynamics of the NNM of an isotropic cantilever beam in intrinsic modal coordinates.

5.4. Numerical example

As an example of a typical application of the method, numerical results will be presented for very flexible cantilever beams with properties as in the

experiments of Ref. [28, p 379], that is, mass density 4430 kg/m^3 , Young modulus 127 GPa and dimensions $479.0 \times 50.8 \times 0.45 \text{ mm}$. The beam is modelled using the Euler-Bernoulli assumptions, for which the normalized linear axial and bending mode shapes are given by Eqs. (32) and (36), respectively. They are shown in Figs. 1 and 2.

To verify the implementation and investigate convergence with the number of modes, the transient response in Eqs. (39) is computed for free vibrations ($Q_j^a = 0$ and $Q_j^b = 0$) under large-amplitude non-zero initial conditions. A fourth-order explicit Runge-Kutta solver is used with a time step defined as $\frac{2\pi}{\omega_n} \times 10^{-2}$, where n is the index of the master bending mode. Convergence studies found that this timestep was sufficient for all results in this section. They were compared with results obtained using two Finite-Element Method (FEM) solutions introduced in Ref. [20]. The first approach is a FEM discretization of the intrinsic equations, using linear spatial interpolation and an implicit mid-point rule for time integration, while the second is a displacement-based FEM solver that follows the description of Ref. [29], and uses 2-noded linear elements, a parameterization of rotations using the cartesian rotation vector and an implicit Newmark-beta integration scheme. In all cases, the solutions are obtained with the same time step as the modal solutions.

Fig. 3 shows the nonzero components (in the local material coordinate system) of the beam tip velocities for the free vibration (unforced) problem with initial velocity in the second linear bending mode, i.e., $n = 2$. The initial amplitude is ($q_{12}^b(0) = 0.5$). Results are obtained with $N_b = 5$ and an increasing number of axial modes, N_a . FEM results obtained with 80

intrinsic elements define the reference for comparison. As it can be seen in the figure, at least four axial modes ($N_a = 4$) are needed to obtain a good approximation to the reference results. It was also checked that a converged solution required a minimum of four bending modes ($N_b > 4$). When $N_a = 0$ the nonlinear terms in the bending equations, 2nd and 4th of Eqs. (39), are zero and the solution in Fig. 3 corresponds to a linear problem on the bending modes given by Eq. (15).

The velocity distribution on the beam can now be integrated in time using the propagation equations of section 2.2 to obtain the displacements and rotations along the beam. This is shown in Fig. 4 for the non-zero components of the beam tip displacements and rotations in the inertial reference frame. The recovery of displacements can also be done from spatial integration (at a given time) of the distribution of internal forces and moments, as in Ref. [20]. As it was pointed out above, the evaluation of displacements and rotations from velocities (or internal forces and moments) does not require any simplification (beyond the algorithms for numerical integration). From this results, at least 4 axial modes ($N_a = 4$) are needed to capture the reduction in frequency in the nonlinear oscillations. Those results are compared in Fig. 4 with the solution of the displacement-based FEM discretization using 40 2-noded elements.

Once the modal equations and its convergence properties have been verified, we investigate the ability of the NNMs to describe the dynamics of the autonomous system. For that purpose, the unforced response in Eq. (39) are solved with initial conditions on the invariant manifold that defines a certain NNM. By definition, the system response will stay in that manifold

(however, since each NNM is only known by its cubic approximation, the initial conditions are also only approximated). Those results are compared with the direct computation of the NNM dynamics given by Eq. (42).

This comparison is shown in Figs. 5, 7, and 6 for the first NNM, that is, the NNM that coincides with the first bending mode at zero amplitude, defined by $n = 1$ in Eq. (41). The initial conditions are chosen as $\eta_1(0) = 0$ and $\eta_2(0) = 0.5$. Fig. 5 shows the evolution of the modal coordinates of the first bending mode (the master mode in the NNM), as obtained by the solution of either Eqs. (42) (NNM) or Eqs. (39). Convergence of the results is obtained with five axial ($N_a = 5$) and bending ($N_b = 5$) modes. The linear modal solution given by Eqs. (15) is also included for comparison. The rest of the modal coordinates in the NNM description is then obtained using the constraint equations and are shown in Figs. 6 and 7. Note that the NNM is able to capture very well the increase in frequency of the nonlinear oscillations of the master mode, while it gives a good approximation to the low-frequency dynamics of the slave modes (q_{11}^b). Fig. 8 shows snapshots of the beam at its maximum deformation point for different amplitudes of the NNM (defined by $\eta_1(0)$ from 0.05 to 0.65 in increments of 0.05). The corresponding reduction in the period oscillations is shown in Fig. 9. As it can be seen, for this highly-flexible beam the frequency shifting is small. The third-order approximation captures very well the amplitude and period of the NNM with tip displacements up to 50% of the beam length.

Similar results are obtained for the second NNM, that is, the NNM that coincides with the second bending mode at zero amplitude. Fig. 10 shows the velocities at the free end corresponding to initial NNM amplitudes equal

to $\eta_1(0) = 0.25$ and $\eta_2(0) = 0$ (non-zero velocities and zero internal forces). Results are compared with the 2-noded displacement-based FEM of Ref. [20]. Results in Fig. 10 compare the linear and nonlinear modal equations with $N_a = 5$ and $N_b = 5$, the linear and nonlinear FEM results, and the third-order approximation to the NNM given by Eqs. (42). A convergence study indicated that the number of nodes must be at least $N_a = 4$ and $N_b = 4$. As expected, an excellent agreement is found between the modal and the FEM solutions, and, as before, the linear solutions fail to capture the shift in frequency as well as the tangent component of the velocities. The third-order description approximate very well the dynamics of the NNM, although for this amplitude of oscillations there is already a small underprediction of the frequency of oscillations. This is further studied in Fig. 11, which shows the period of the oscillations of the second NNM (normalized by the period of the corresponding LNM) as a function of the NNM amplitude. Results compare the numerical results and also the second- and fourth-order approximations given by Eq. (48). For low-to-moderate amplitudes of the intrinsic modal coordinates the agreement is excellent between the different solutions. For values of $\eta_2(0)$ above around 0.2, the frequency obtained from the asymptotic NNM approximation diverges from that of the full modal equations. Note finally that the initial conditions in the modal equations are the third-order approximation of the NNM and thus when both the modal solution and the NNM approximation diverge, the modal equations are not solving the NNM dynamics either.

6. Application to cantilever composite beams with extension/twist coupling

For composite (or, more generally, anisotropic) beams, the cross-sectional mass and flexibility matrices that determine the beam dynamics in Eqs. (1) will be, in general, full matrices. Numerical methods based on homogeneization processes are available to compute the sectional properties for general geometries and material distributions. Here, we will use our implementation of the Variational-Asymptotic Beam Sectional (VABS) analysis, described in Ref. [21]. Once the sectional properties have been obtained, computing the NNMs requires first the evaluation of the LNMs from Eq. (11) and the problem boundary conditions. In general, a finite-element discretization of the intrinsic beam equations would be needed [20], although one could also obtain the LNMs in intrinsic coordinates by the differentiation (in both time and space) of the mode shapes in displacements given by standard finite-element solvers. For the case of prismatic bars with constant cross sections, the LNMs can be computed using the exponential map introduced in section 3.1.

To illustrate the application of the methodology to composite beams, this work will focus on prismatic cantilever beams with extension/twist elastic coupling. Geometric coupling, shear effects and rotational inertia are further assumed to be negligible. With those assumptions the mass and flexibility

matrices are

$$\mathbf{m} = \text{diag} \{ \rho A, \rho A, \rho A, \rho I_1, 0, 0 \},$$

$$\mathbf{c} = \begin{bmatrix} c_{11} & 0 & 0 & c_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ c_{14} & 0 & 0 & 0 & c_{45} & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix}, \quad (52)$$

where c_{14} are the elastic coupling coefficients. Both matrices are assumed to be constant along the length of the beam. Despite the relative simplicity of the problem, the elastic coupling will make the free vibrations to occur in the 3-D space. The beam with the properties given by Eq. (52) will therefore have pure bending LNMs in two directions (obtained as in section 5.1, with $EI_2 = c_{55}^{-1}$ and $EI_3 = c_{66}^{-1}$) and coupled extensional/torsional LNMs. The later ones can be easily obtained after defining the matrices $\bar{\mathbf{m}} = \text{diag} \{ \rho A, \rho I_1 \}$ and

$$\bar{\mathbf{c}} = \begin{bmatrix} c_{11} & c_{14} \\ c_{14} & c_{44} \end{bmatrix}.$$

The vector $\bar{\Phi} = \{ \Phi_{V_1}, \Phi_{\Omega_1}, \Phi_{F_1}, \Phi_{M_1} \}^T$, which defines the non-zero components of the coupled mode shapes, can be now obtained. Eq. (19) reduces in this case to $\bar{\Phi}_j(s) = \bar{\mathbf{P}} e^{\bar{\mathbf{J}} \omega_j s} \bar{\mathbf{P}}^{-1} \bar{\Phi}_j(0)$, with

$$\bar{\mathbf{P}} = \begin{bmatrix} i\Lambda_1 \bar{\mathbf{m}}^{-1} \mathbf{v}_1 & -i\Lambda_1 \bar{\mathbf{m}}^{-1} \mathbf{v}_1 & i\Lambda_2 \bar{\mathbf{m}}^{-1} \mathbf{v}_2 & -i\Lambda_2 \bar{\mathbf{m}}^{-1} \mathbf{v}_2 \\ \mathbf{v}_1 & \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_2 \end{bmatrix},$$

$$\bar{\mathbf{J}} = \text{diag} \{ i\Lambda_1, -i\Lambda_1, i\Lambda_2, -i\Lambda_2 \}, \quad (53)$$

where $\{\Lambda_1^2, \Lambda_2^2\}$ and $\{\mathbf{v}_1, \mathbf{v}_2\}$ are the eigenvalues and eigenvectors, respectively, of the matrix $\bar{\mathbf{m}}\bar{\mathbf{c}}$. The modes are uniquely defined then by $\bar{\Phi}_j(0)$, which is computed from the boundary and the normalization conditions.

Consider now the hollow composite shaft with elliptical section and constant ply orientation, θ of Fig. 12. Its geometry will be defined by its length, L , minor and minor sectional semi-axis, a and b , measured with respect to the outer wall, and the total laminate thickness, t (see Fig. 12). For the numerical results it will be $L = 25$ m, $t = 0.01$ m, and varying a and b (of the order of $10 - 25t$). Material properties correspond to a AS4/3506-1 unidirectional graphite-epoxy laminate ($E_{11} = 142$ GPa, $E_{22} = 9.8$ GPa, $G_{12} = 6$ GPa, $G_{23} = 4.8$ GPa, $\nu_{12} = 0.3$, $\nu_{23} = 0.42$, $\rho = 800$ kg/m). These properties are typical of the wing spar of a large solar-powered aircraft and present the desired elastic extension/twist coupling. Fig. 13 shows the normalized of the compliance matrix obtained from VABS analysis for fixed $b = 0.1$ m and varying section major semiaxis, a , and ply angle, θ (the bending compliance in the $x - y$ plane is obtained by exchanging a and b in the results). The mass coefficients are independent of the ply orientation and are easily evaluated for this geometry. The first natural frequencies of the cantilever shaft are shown in Fig. 14 for fixed $b = 0.1$ m and varying ply angle, θ , and major section semiaxis, a . For $\theta \neq 0$ and $\theta \neq 90^\circ$, axial and torsion modes are coupled ones, although their evolution with the ply angle depends on their uncoupled characteristics (they have been identified in the figures as *Coupled*[×], torsion at $\theta = 0$, and *Coupled*^{||}, axial at $\theta = 0$). The metric for coupling (shown in Fig. 15) is the root value of the force and moment component of the mode shapes, which after normalization is constant for all modes of a particular

kind. The coupled mode shapes are simply obtained from the corresponding values of Fig. 15 and the matrices introduced in Eq. (53).

The NNMs for this configuration can now be sought. Note that the condition $a/b = 1$ (a circular shaft) has 1:1 internal resonances, as the bending modes on both directions will have the same frequencies, and cannot be directly studied with the expressions presented in Section 4. Fig. 16 shows the time-history of several modal coordinates on the first NNM for $a/b = 1.5$ and $\theta = 30^\circ$. For small amplitudes, this NNM reduces to the first bending LNM in the $x - z$ plane, which is used as master mode in Eq. (22). Fig. 16 shows the shaft response to oscillations on the first NNM for initial conditions $(\eta_1, \eta_2)(0) = (0, 150)$, as obtained by the manifold equations and by the full modal solution. Note that the first *Coupled*^{||} mode (mode 3, axial-dominated) has larger amplitudes than the lower-frequency *Coupled*[×] ones. These results were obtained with 5 coupled and 5 bending modes on each direction (a total of 15 LNMs). Including additional modes does not change these results.

While the linear free vibrations in bending are restricted to the $x - y$ or $x - z$ planes, at large amplitudes the oscillations become three-dimensional. This is shown in Fig. 17, which includes the three components of the beam tip displacements for increasing values of the initial velocity on the first NNM (defined by the value of the master mode, $\eta_1(0)$). As before, results are obtained for $a/b = 1.5$, $\theta = 30^\circ$, and with an approximation with 15 LNMs. For small amplitudes, the beam remains in the $x - z$ plane, while the torsion-axial elastic coupling creates out-of-plane displacements, whose amplitude grows with the amplitude of the oscillations. These 3-D couplings

are one of the fundamental characteristics that need to be considered in the study of the nonlinear oscillations of composite structures.

7. Conclusions

This paper has presented a new procedure to evaluate the Nonlinear Normal Modes (NNMs) of beam structures of arbitrary material properties, based on a first-order description that uses internal forces/moments and translational/angular velocities as primary degrees of freedom. This intrinsic description of the beam dynamics is similar to Euler’s equations for the unconstrained dynamics of rigid bodies. In particular, for the case of free vibration, it is possible to describe the beam dynamics without ever computing the actual displacement and rotations along the beam. This formalism results in a compact closed-form solution of the nonlinear equations for straight anisotropic beams with varying properties and arbitrary spatial boundary conditions. A single NNM has been evaluated using a truncated polynomial expansion, but other procedures found in the literature would also benefit from this approach. The approach can also be extended to curved and 3-D multi-beam configurations, but this was not addressed in this work.

The decoupling between the dynamic equations in their intrinsic form and the strain-displacement and velocity-displacement kinematic relations, allows an *exact* (i.e. without geometric approximations) description of the beam displacements and rotations once the internal forces and the velocities have been computed. In particular, the propagation equations of rigid-body dynamics were used here to track the location of specific points on the beam structure from the local translational and angular velocities. One way of

interpreting this approach is by considering the intrinsic description of the beam dynamics as a co-rotational description in the limit to infinitesimally-small finite elements.

To illustrate the approach, NNMs have been obtained for the in-plane motions of a cantilever isotropic and composite beams under a number of approximations (negligible transverse shear, symmetric sections, etc.). For those cases, analytical expressions exist for the Linear Normal Modes (LNMs), which has been used to simplify the presentation. However, the evaluation of the coefficients of the nonlinear equations in intrinsic modal coordinates is obtained by simple product operations of the mode shapes and a set of known coefficients (the sectional properties). Thus, the only additional complexity in the solution of arbitrary configurations, is in the evaluation of their LNMs. In those cases, a finite-element discretization of the structure will be necessary. Note also that, in their linear form, the LNMs in intrinsic variables can be obtained by the appropriate differentiation of the LNMs given in their conventional description on displacements/rotations, which can be obtained from most commercial finite-element solvers. A detailed description of this procedure has not been presented here, but it can be easily identified by comparing the analytical modes obtained for a cantilever isotropic beam with the solution in displacements that can be found in many textbooks.

The analytical expressions to describe the NNM of isotropic cantilever beams have also shown that the third-order dynamics of the invariant manifold are uniquely determined by the axial modes and the master linear bending mode (the mode that coincides with the NNM at the zero energy). The constrained linear bending modes in the invariant manifold have cubic re-

lations with the master mode in an intrinsic theory, and therefore do not affect the manifold dynamics (in contrast to the coupling that occurs in displacement-based formulations). The situation is more complex for composite cases. In general, the beam response for nonlinear free vibrations becomes fundamentally three-dimensional, even though it might be contained into a plane of motion for very small amplitudes. These spatial couplings are a key characteristic in the free vibration analysis of general composite structures.

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Figure 1: First six axial modes for the isotropic cantilever beam.

Figure 2: First six bending modes for the isotropic cantilever beam.

Figure 3: Non-zero velocities at the free end (components in material frame) of the isotropic beam for initial velocity at second bending mode ($q_{12}^b(0) = 0.5$). Five bending modes, $N_b = 5$, and increasing number of axial modes, N_a . Results compared to FEM solution of intrinsic equations with 80 elements.

Figure 4: Non-zero displacements (components in spatial frame) and rotations at the free end of the isotropic beam for initial velocity at second bending mode ($q_{12}^b(0) = 0.5$). $N_b = 5$, and increasing number of axial modes, N_a . Results compared to displacement-based FEM solution with 40 spanwise elements.

Figure 5: Modal coordinates of the first bending mode of the isotropic beam for initial conditions on the first NNM $((\eta_1, \eta_2)(0) = (0, 0.5))$. Results from linear and nonlinear equations in intrinsic modal coordinates, and the NNM approximation ($N_a = 5$, $N_b = 5$).

Figure 6: Modal coordinates of the first two axial modes for initial conditions of the isotropic beam on the first NNM $((\eta_1, \eta_2)(0) = (0, 0.5))$. Results from nonlinear equations in intrinsic modal coordinates, and NNM equations ($N_a = 5$, $N_b = 5$).

Figure 7: Modal coordinates of the 2nd and 3rd bending modes of the isotropic beam for initial conditions on the first NNM $((\eta_1, \eta_2)(0) = (0, 0.5); N_a = 5, N_b = 5)$.

Figure 8: Shape at time of maximum deformation for the first NNM of the isotropic beam with $\eta_1(0)$ increasing from 0.05 to 0.65, in increments of 0.05 ($\eta_2(0) = 0$, $N_a = 5$, $N_b = 5$).

Figure 9: Period of the first NNM of the isotropic beam, normalized by the period of the first LNM, for varying initial conditions. Results from the solution of the NNM and modal equations, and from Eq. (48) ($\eta_2(0) = 0$; $N_a = 5$, $N_b = 5$).

Figure 10: Velocities (components in material frame) at the free end of the isotropic beam for initial conditions on the second NNM $((\eta_1, \eta_2)(0) = (0.25, 0))$; $N_a = 5$, $N_b = 5$ in modal solutions; 60 elements in displacement-based FEM solutions).

Figure 11: Period of the free vibration of the isotropic beam as a function of the initial conditions. Results from the solution of the NNM and modal equations, and from Eq. (48) ($\eta_1(0) = 0$; $N_a = 4$, $N_b = 4$).

Figure 12: Composite hollow elliptic shaft with constant ply angle, θ .

Figure 13: Compliance coefficients of the composite shaft with $b = 0.1$ m, $t = 0.01$ m, and varying section major semi-axis, a , and ply angle, θ .

Figure 14: Natural frequencies of the composite shaft with $L = 25$ m, $a = b = 0.1$ m, $t = 0.01$ m, and varying ply angle, θ .

Figure 15: Root values of the coupled LNMs of the composite shaft with $a = b = 0.1$ m, and varying ply angle, θ , in degrees.

Figure 16: Comparison between third-order approximation of the NNM dynamics of the composite shaft and the full modal solution, on selected (velocity) modal coordinates and for oscillations in the first NNM. Initial conditions on the master mode are $(\eta_1, \eta_2)(0) = (0, 150)$ ($\theta = 30^\circ$, $a/b = 1.5$, 15 LNMs).

Figure 17: Beam tip displacements (coordinates in spatial frame) in the third-order approximation of the NNM of the composite shaft with varying initial amplitude $\eta_1(0)$ in increments of 20 until 200 ($\eta_2(0) = 0$, $\theta = 30^\circ$, $a/b = 1.5$, 15 LNMs).