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‘Unforced’ Navier–Stokes solutions derived from convection in a curved channel

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Steady Boussinesq flow in a weakly curved channel driven by a horizontal temperature gradient is considered. Linear variation in the transverse direction is assumed so that the problem reduces to a system of ordinary differential equations. A series expansion in G , a parameter proportional to the Grashof number and the square root of the curvature, reveals a real singularity and anticipates hysteresis. Numerical solutions are found using path-continuation and the bifurcation diagrams for different parameter values are obtained. Multivalued solutions are observed as G and the Prandtl number vary. Often fields with the imposed structure that satisfy all the governing equations are insensitive to the boundary conditions and can be regarded as perturbations of the homogeneous (or ‘unforced’) problem. Four such unforced solutions are found. In two of these the velocity remains coupled with temperature which, formally, scales as $1/G$ as $G \rightarrow 0$. The other two are purely hydrodynamic. The existence of such solutions is due to the unbounded nature of the domain. It is shown that these occur not only for the Dean equations, but constitute previously unreported solutions of the full Navier–Stokes equations in an annulus of arbitrary curvature. Two additional unforced solutions are found for large curvature.

Key words:

1. Introduction

The study of internal flows in curved geometries subject to temperature gradients is important in a great range of industrial processes and natural phenomena. Applications include cooling systems, mould injection, biological flows, double glazed windows, chemical reactors, magma flows inside cracks of rocks and microelectronics. Laminar convection plays an important role in heat control in the latter, due to the low levels of noise, ease of maintenance and overall simplicity (Terekhov & Ekaid 2011). Although it is theoretically convenient to consider channels with straight walls, in practice they are often slightly curved.

Many authors have worked on this type of problem. For example, Prusa & Yao (1982) solve the flow in a horizontally heated curved tube numerically. Their model includes buoyancy and centrifugal effects, which we also consider in this paper, although instead of a pipe we have a weakly curved channel. A comprehensive review on flow through curved pipes can be found in Berger *et al.* (1983). Buoyancy driven flow in a straight channel inclined to the vertical is presented in Subramanian *et al.* (2016). For zero inclination this reduces to standard Rayleigh–Bénard convection. In the absence of curvature, it is

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straightforward to find an analytical solution for the base state, which is then perturbed to find the convection patterns.

In standard convection problems, the channel boundaries are considered to be at uniform temperature. However, in practice there may well be a temperature gradient in the transverse direction. This will drive fluid motion even before convectational instability sets in, just as in the inclined channel.

In our work, we combine two novel aspects. Firstly, the flow is driven by a horizontal temperature gradient that varies linearly in the transverse direction, (which results in a similarity reduction), and secondly, the motion takes place in a weakly curved channel. In such circumstances, there is an additional centrifugal effect, and the governing equations are the thermally modified Dean equations rather than the Navier–Stokes (NS) (Yao & Berger 1978). Despite its practical nature, this problem has not been previously considered, yet it can be solved numerically as well as investigated analytically. The resulting steady flow is non-trivial. It retains the key features of non-linearity and three-dimensionality even though it is governed by ODEs. We do not make an extensive analysis of stability here, but convection in this geometry is of undoubted theoretical and practical interest.

In addition to numerical methods, analytical representations, such as power series in a parameter, can reveal interesting features. The groundwork for this approach was laid down by Van Dyke (1974), and it has been employed in a number of situations. For instance, Yao & Berger (1978) investigate the flow in heated curved pipes by means of a double power series in the Dean number (K) and in the product of the Reynolds and Rayleigh numbers. Convergence considerations often limit series solutions to small or moderate parameter values. Nevertheless, there are methods to analytically extend the validity of series solutions, such as high-order Hermite–Padé approximants, which were introduced in Drazin & Tourigny (1996). This technique was employed in Boshier & Mestel (2014, 2017) to extend the study of flow in a slowly curving pipe to all values of K . Using analytical continuation we find here several interesting features, including the existence of non-trivial solutions of the unforced equations, when either the boundary conditions are made homogeneous, or the parameter G linking the hydrodynamic and thermal equations is set to zero. These unforced solutions often dominate the local flows and are the focus of the second part of this paper.

We find of particular interest that some of the unforced solutions completely decouple from the temperature field, which leads us to study their existence for the full steady NS equations in an annulus of arbitrary curvature. Many authors have explored similarity reductions in axisymmetric flows to reduce the problem to ordinary differential equations (Vyskrebtsov 2001). Additions to the axisymmetric flows of Squire (1951) are presented in Agrawal (1957). This work was itself extended by Weidman (2016) to study axisymmetric flow impinging on a radially stretching or shrinking sheet. In our problem, the flow is axisymmetric in an annulus, with a linear dependence on the transverse coordinate. Perhaps closest to our work are the self-similar flows through a circular pipe with porous wall conditions studied in Smirnov (1981). There too, linear dependence with the transverse coordinate is considered and multiple solutions are found, especially for strong wall suction. Interestingly, non-trivial solutions also exist for the case of impermeable walls, as the wall suction tends to zero. Nevertheless, our annular solutions are entirely distinct from Smirnov (1981) even as the inner radius tends to zero, because of our no-slip condition. Also closely related to this work are the similarity solutions to flow in a channel driven by suction at the walls described in Zaturka *et al.* (1988), and in a porous pipe in Banks & Zaturka (1992). **The latter considers flow in an annular pipe**

with no azimuthal velocity. In §6 below we find two additional solutions of this form with impermeable walls for the high curvature case.

The outline of this paper is as follows. We start by deriving the governing equations in §2, which under our assumptions reduce to ordinary differential equations (ODEs). In §3 we obtain an analytical representation of the solution as a power series in G . We find that a real singularity in the complex G -plane limits the convergence of the series and anticipates multiple solutions. In §4 we use high-order Hermite–Padé approximants to extend the series solution beyond the circle of convergence, which reveals multiple solution branches and predicts the existence of unforced solutions. We then proceed in §5 to solve the equations numerically, using path-continuation. This approach confirms four previously unreported unforced solutions. In §6, we generalise our problem and seek unforced solutions of the full NS equations in an annulus of arbitrary curvature. By performing continuation in the curvature, we extend the solutions already found for the Dean equations to larger curvatures. In addition, we find two extra unforced solutions in the large curvature limit.

2. Governing equations

We consider a buoyancy driven flow in an infinitely long and weakly curved vertical channel. A concise derivation of the equations and explanation of the key points of our model is presented in this section. Full details are given in appendix A. The derivation of the governing equations for this configuration is analogous to that for heated curved pipes, presented in Yao & Berger (1978).

The steady NS equations, coupled with the steady heat equation govern the velocity, $\mathbf{u}$, pressure, p , and temperature, T . These are given by

$$\nabla \cdot (\rho \mathbf{u}) = 0, \quad (2.1)$$

$$\rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \rho \mathbf{g} + \mu \nabla^2 \mathbf{u}, \quad (2.2)$$

$$\mathbf{u} \cdot \nabla T = \kappa \nabla^2 T, \quad (2.3)$$

where ρ is the density, $\mathbf{g}$ the acceleration of gravity, μ the dynamic viscosity and κ the thermal diffusivity.

We consider a cylindrical coordinate system (r, ϕ, y) , as depicted in figure 1, where the curvature is greatly exaggerated. We have that y is in the transverse direction, r is across the gap, and ϕ along the channel. The length scales are the width of the gap $2L$, and the radius of curvature R . The geometry is unbounded in the y direction. We assume that the channel is very weakly curved, so that to leading order gravity only acts in the negative ϕ direction. Furthermore, we assume fully developed flow in this direction, thus independent of ϕ . We employ the Boussinesq assumption (Boussinesq 1897), so that density variations are only important in the gravitational term. We also make the Dean approximation (Dean 1927), regarding the channel curvature, $\lambda = 2L/R$, as small, and neglect terms of $\mathcal{O}(\lambda)$. Once all assumptions are taken into account, the system of equations can be described in the local Cartesian reference frame (x, y, z) , depicted in figure 1. We also make the equations non-dimensional (details in appendix A). The following nomenclature is adopted,

$$\mathbf{u} = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}, w \right) \quad (2.4)$$

where ψ is a stream function, and w the velocity component in z . We adopt θ as the non-dimensional temperature. Taking all assumptions into account, we obtain the thermally

FIGURE 1. Depiction of curved channel. Gravity acts in the negative z direction and y is in the transverse direction. On the right we indicate the cylindrical and Cartesian reference frames. The width of the channel is $2L$, and the radius of curvature R , are such that $L \ll R$.

modified Dean equations

$$\mathcal{J}(w, \psi) = \nabla^2 w + G\theta, \quad (2.5)$$

$$\mathcal{J}(\nabla^2 \psi, \psi) = \nabla^4 \psi + 2w \frac{\partial w}{\partial y}, \quad (2.6)$$

$$\mathcal{J}(\theta, \psi) = Pr^{-1} \nabla^2 \theta, \quad (2.7)$$

where

$$\mathcal{J}(A, B) = \frac{\partial A}{\partial x} \frac{\partial B}{\partial y} - \frac{\partial A}{\partial y} \frac{\partial B}{\partial x}, \quad (2.8)$$

and

$$G = \frac{g\beta\Delta TL^3}{\nu^2} \left(\frac{L}{R}\right)^{\frac{1}{2}}, \quad Pr = \frac{\nu}{\kappa}, \quad (2.9)$$

i.e. G is proportional to the Grashof number and the square root of the curvature, Pr is the Prandtl number, ν the kinematic viscosity. The resulting system of equations is essentially the thermally coupled NS equations in two dimensions, with a centrifugal body force acting towards the inside wall.

These equations are valid for any buoyancy driven flow in a weakly curved channel. We now particularise to flow driven by a linearly varying temperature gradient, by imposing at the walls

$$\theta(\pm 1, y) = \pm y \quad (2.10)$$

where the inner and outer wall are located, respectively, at $x = -1$ and $x = 1$. We seek solutions linear in y , assuming $w(x, y) = \hat{w}(x)y$, $\theta(x, y) = \hat{\theta}(x)y$, $\psi(x, y) = \hat{\psi}(x)y$. Dropping the hats, the system of equations becomes

$$w'\psi - w\psi' = w'' + G\theta, \quad (2.11)$$

$$\psi'''\psi - \psi''\psi' = \psi'''' + 2w^2, \quad (2.12)$$

$$\theta'\psi - \theta\psi' = Pr^{-1}\theta'', \quad (2.13)$$

where the primes denote derivatives with respect to x . Our variables w , ψ and θ are functions of x only, as now our solutions are self-similar in y .

We also wish to impose no-slip condition on the walls, thus, our complete set of boundary conditions is

$$w = \psi = \psi' = 0, \quad \theta = \pm 1 \quad \text{at} \quad x = \pm 1. \quad (2.14)$$

We have assumed that to leading order gravity only acts in the negative z direction. A fairly simple extension of this work would be to consider an inclined channel. This requires gravity to have components in the x - and z - directions. We derive this system in Appendix B. However, as the dominant characteristics of the flow are unchanged, we do not consider the channel inclination further.

3. Extended Stokes series

We shall solve the system numerically in §5, but it is useful first to obtain an analytical representation of the solution by expanding the variables in powers of G ,

$$w = \sum_{n=0}^{\infty} w_n G^n, \quad \theta = \sum_{n=0}^{\infty} \theta_n G^n, \quad \psi = \sum_{n=0}^{\infty} \psi_n G^n \quad (3.1)$$

and truncate at N coefficients. The leading order coefficients are given by the straight channel solution, $w_0 = 0$, $\psi_0 = 0$, $\theta_0 = x$ and $w_1 = (x - x^3)/6$. We can derive recurrence relations to compute higher order coefficients of the series. For example, for θ we have

$$\theta_n'' = \sum_{i,j:i+j=n} Pr (\theta_i' \psi_j - \theta_i \psi_j') \quad (3.2)$$

which we can use to determine

$$\theta_2 = \frac{1}{59875200} Pr (x^2 - 1) (45x^{10} - 263x^8 + 562x^6 + 562x^4 - 703x^2 - 5323). \quad (3.3)$$

The series coefficients are polynomials in x and Pr . We find that the ψ and θ series are even in G , while the w series is odd.

By computing the series coefficients using recurrence relations, our non-linear system reduces to a sequence of linear problems. Naturally, it becomes increasingly computationally expensive to compute higher order coefficients. We have used the software MAPLE to compute up to $N = 236$ coefficients of the series.

The velocity and temperature fields for $Pr = 1.1$ and $G = 21$ are presented in figure 2. We recall that the fluid velocity is $\mathbf{u} = (\psi, -y\psi', yw)$ and the temperature is $y\theta$. We can see that for $y > 0$, the inner wall is cold and the fluid sinks ($\theta < 0, w < 0$), whereas it rises by the warm, outer wall. For this moderate value of G , the temperature profile is almost linear. As $\psi < 0$, we see that the x -velocity is always towards the inner wall. Fluid then departs the inner wall in the positive and negative y -directions. The outer wall behaviour is reversed.

3.1. Limits of validity

This solution is only valid inside the radius of convergence of the series, G_c , which is limited by the nearest singularity in the complex G -plane (Van Dyke 1974). We can estimate G_c , using a Domb Sykes plot, following Domb & Sykes (1957). In case of an algebraic singularity, the ratios of consecutive coefficients of a series $\sum c_n G^n$, should be

FIGURE 2. Velocity and temperature profiles for $Pr = 1.1$ and $G = 21$. We see $w < 0$ close to the inner wall which is cold for $y > 0$, whereas $w > 0$ near the outer (hot) wall. The temperature profile is approximately linear. The x -velocity is towards the inner wall, for all y .

linear in $1/n$, for large n

$$\frac{c_n}{c_{n-1}} = \frac{1}{G_c} \left(1 - \frac{1 + \alpha}{n} + \mathcal{O}\left(\frac{1}{n^2}\right) \right), \quad (3.4)$$

where α is the exponent of the singularity.

All measures of the series solution will have the same singularity structure. For this discussion we chose to compute the ratios of ψ at $x = 0.5$, which we refer to as,

$$f(G) = \sum_{n=2}^N c_n G^n. \quad (3.5)$$

Since the series is even, we started by computing the ratios c_n/c_{n-2} . However, this Domb-Sykes plot alternated between two lines, indicating that we should consider instead the ratios c_n/c_{n-4} . We find that the singularity is of the type $f(G) \sim (G_c^4 - G^4)^{1/2}$, which corresponds to four singularities $\{\pm G_c, \pm iG_c\}$. The existence of a real and positive singularity suggests that a solution branch will turn at $G = G_c$. We will observe later that the first fold in the bifurcation diagram indeed occurs at G_c .

For $Pr = 1$ the Domb-Sykes plot is presented in figure 3. Using polynomial extrapolation we find that $G_c \simeq 22.8$. The value of G_c for different values of Pr can be observed in figure 3. For $Pr = 0$ we find $G_c \simeq 44.7$. Furthermore, as $Pr \rightarrow \infty$ we observe that $G_c \sim Pr^{-\frac{1}{2}}$.

4. Analytic continuation by high-order Hermite–Padé approximants

In this section we explore a method to bypass the limitation of our series solution and extend our analysis to higher parameter values by means of generalised Hermite–Padé approximants. This method was implemented in Boshier & Mestel (2014, 2017), and is introduced and described in detail in Drazin & Tourigny (1996). The key points of the method are as follows. From a series expansion

$$U_N(G) = \sum_{n=1}^N c_n G^n + \mathcal{O}(G^{N+1}), \quad (4.1)$$

we construct a polynomial,

$$F_d(u, G) = \sum_{m=1}^d \sum_{k=0}^m f_{m-k,k} G^{m-k} u^k = 0, \quad (4.2)$$

FIGURE 3. **Left:** Domb-Sykes plot for $Pr = 1$. Since the ratios of the coefficients are positive, this indicates a real singularity in the complex G -plane which anticipates a turning point at $G = G_c$. As $n \rightarrow \infty$ the ratio approaches G_c^{-4} . **Right:** G_c for different values of Pr . We observe that $G_c \sim Pr^{-\frac{1}{2}}$ as $Pr \rightarrow \infty$.

where the coefficients $f_{i,j}$, are computed such that $u = U_N$ is a solution to equation (4.2). The maximum order of the approximant, d , is determined by how many coefficients of the series are available. In order to ensure that the number of unknown coefficients $f_{i,j}$ equals the number of equations, we truncate u at N where,

$$N = \frac{1}{2} (d^2 + 3d - 2). \quad (4.3)$$

The main advantage of this method is the possibility of finding multiple solution branches. For each order d of the approximant a polynomial is constructed and its roots for a certain G value yield d solutions. Some of these roots are spurious, but we can track actual flow solutions as they appear consistently as the order d is increased.

The highest order of approximant that we have obtained is $d = 14$. In figure 4, we observe a typical bifurcation diagram. The case considered is $Pr = 0.5$ and the measure selected is $\psi(0.5)$. We show the results obtained with approximant order $d = 13$ and $d = 14$. In the interest of clarity we omit spurious solutions. Many interesting aspects of the solution are revealed. First we note the symmetry of the solution branches about $G = 0$. This occurs as the system of equations (2.11)-(2.13) is invariant under the transformation $G \rightarrow -G$, $w \rightarrow -w$, $\psi \rightarrow \psi$ and $\theta \rightarrow \theta$. Secondly, we observe three turning points, the first at $G \simeq 29.1$ corresponds to the value predicted by Domb-Sykes analysis, and two more. One at $G \simeq 2.7$, and the last one apparently at $G \approx 50$ is not fully resolved as the approximants have yet to agree. We shall see that in fact the third turning point occurs at a much larger value of G . The final, and more surprising observation, is the existence of unforced solutions, i.e., non-trivial solutions at $G = 0$. There are three such solutions predicted with $\psi < 0$, where the solution branches cross the line $G = 0$. The two that are fully resolved at this approximant order are highlighted with circles. We also note two isolated points at $G = 0$ with $\psi > 0$, these originate from complex branches that cross the real (G, ψ) plane, and are highlighted with squares in the diagram. They correspond to complex branches with $\psi(-0.5)$, under the invariance $G \rightarrow iG$, $w \rightarrow iw$, $x \rightarrow -x$, $\psi \rightarrow -\psi$ and $\theta \rightarrow \theta$.

Up to the approximant order that we computed, the bifurcation diagram is not fully resolved after the first two turning points. In order to ascertain the complete bifurcation diagram, and shed some light on the nature of the unforced solutions, we solve the problem using a numerical approach. Using numerical continuation we were able to extend

FIGURE 4. **Left:** Bifurcation diagram for $Pr = 0.5$, the selected measure is $\psi(0.5)$. The approximants for $d = 13$ and $d = 14$ are shown. The key features observed in this diagram are: multivalued solution, symmetry in G and non-trivial solutions for $G = 0$, which are highlighted with squares for positive ψ , and circles for negative values. **Right:** Comparison between analytical continuation and numerical results. The latter allows us to extend the bifurcation diagram and reveals two additional unforced solutions.

the bifurcation diagram and find additional solution branches. We can observe in figure 4 a portion of the diagram for $Pr = 0.5$ obtained both with analytical and numerical continuation. There are two additional fully-resolved non-trivial solutions with $G = 0$. In order to perform a comprehensive analysis of this problem we consider different Pr values and further explore the nature of the unforced solutions.

5. Numerical path-continuation

We discretise our system of equations using second order centred finite differences with ghost nodes at the walls to impose Neumann boundary conditions. We use a minimum of 1000 mesh points in our results, however, higher resolution is used in certain computations to ensure accuracy throughout.

Since we are interested in the dependence of the solution on G , and in different solution branches, it is advantageous to write our system in the form,

$$F(U, G) = 0 \quad (5.1)$$

where U is the solution, which in our case corresponds to velocity and temperature fields. In this form we can use standard path-continuation techniques to obtain the bifurcation diagram for G at different Pr values. These techniques are variations of Newton's method applied to equation (5.1), that make use of the Jacobian of F . However, near turning points or bifurcation points, the Jacobian becomes singular making continuation less straightforward. In our work we overcome this by using a pseudo-arclength continuation method that is presented in depth in Chan (1984). A thorough discussion of continuation and path-following techniques is presented in Allgower & Georg (1993).

Henceforth, unless stated otherwise, we use the infinity norm of ψ and θ as measures, which we write as $\|\psi\|$ and $\|\theta\|$, respectively. In the interest of clarity, we do not draw w as its results are similar to ψ and we also omit the symmetry of the ψ diagram. We are using the infinity norm as this is a global quantity and arguably better to depict our solution, than the value at a single point. When we performed analytical continuation we did not use this measure, as it would require constructing a series and its extension at every point, which would be unnecessarily expensive computationally.

FIGURE 5. Bifurcation diagrams for $Pr = 0.5$. On the left we have the logarithm of $\|\psi\|$ as a function of G , and we observe four distinct solutions with $G = 0$. In the interest of clarity, the region near $G = 0$ has been magnified to display all relevant behaviour. We have marked with a circle the point for which we show the solution profiles in figure 6. On the right we have the logarithm of $\|\theta\|$, and observe that two of the solutions with $G = 0$ have finite θ , and in the other two we have $\theta \sim 1/G$ as $G \rightarrow 0$.

The solution is found to be multivalued in all cases studied, and exhibits four solutions at $G = 0$ for every Pr value considered, except for the two special cases $Pr = 1$ and $Pr = 0$ discussed below. The bifurcation diagrams are shown for $Pr = 0.5$, in figure 5, and for $Pr = 1.1$, in figure 7. As we consider different measures from the ones in the previous section the ψ diagram for $Pr = 0.5$ differs from figure 4, but the turning points occur at the same values of G .

For $Pr = 0.5$, in figure 6, we show the velocity and temperature profiles of the point marked in figure 5 with $G = 2.02 \times 10^3$. The velocity and temperature profiles exhibit the same key characteristics that we found for moderate G , before the first turning point (figure 2), except for the temperature profile that is no longer approximately linear. This implies that the nonlinear terms in equation (2.13) become important, which we expect since the magnitude of ψ has increased several orders. In addition, the temperature is not bounded by the boundary conditions, $|\theta| > 1$, but is still of the same order.

We identify two types of unforced solutions. In the first case the temperature fully decouples from the problem and we refer to these as hydrodynamic unforced solutions. We may think of these as zero gravity solutions, with non-zero flow and a passive temperature field. In the second case $\theta \sim 1/G$ as $G \rightarrow 0$, so that the buoyancy term remains in equation (2.11). We refer to these as thermally induced. It would of course be physically questionable were the actual temperature to become infinite, as say the Boussinesq approximation would break down. However, if the wall temperature difference $\Delta T \rightarrow 0$, it is quite consistent for the temperature, T and the buoyancy term to remain $O(1)$ even though θ is formally infinite. Following this solution branch through $G = 0$ leads to other physical solution branches. In this case, as $|\theta| \gg 1$, the solution with the boundary conditions $\theta(\pm 1) = \pm 1$ is a small perturbation of an unforced solution with $\theta(\pm 1) = 0$. The existence of both kinds of unforced solutions can be explained by the unbounded nature of the domain which permits advection from $y = \pm\infty$. Nevertheless, their existence has not been previously reported.

In the case $Pr = 1$ we observe only two unforced flows, shown in figure 8. These must be the unforced hydrodynamic flows, which exist independently of Pr . The scaled temperature field is formally infinite as $G \rightarrow 0$, but does not modify the flow. This can be explained as follows. Regarding ψ as given, we can rewrite equations (2.11) and (2.12)

FIGURE 6. Velocity and temperature profiles for $Pr = 0.5$ and $G = 2.02 \times 10^3$. We see $w < 0$ close to the inner wall which has imposed $\theta = -1$, whereas $w > 0$ near the outer ($\theta = 1$) wall. The x -velocity is towards the inner wall. Note that the temperature is not bounded by the boundary conditions, $|\theta| > 1$.

FIGURE 7. Bifurcation diagrams for $Pr = 1.1$. On the left we have the logarithm of $\|\psi\|$ as a function of G , and we observe four distinct solutions with $G = 0$. In the interest of clarity, the region near $G = 0$ has been magnified to display all relevant behaviour. On the right we have the logarithm of $\|\theta\|$, and again we can observe four unforced solutions, where in two of those cases $\theta \sim 1/G$ as $G \rightarrow 0$.

FIGURE 8. Bifurcation diagrams for $Pr = 1$. On the left we have the logarithm of $\|\psi\|$ as a function of G , and we observe two solutions with $G = 0$. On the right we have the logarithm of $\|\theta\|$, and we observe that both solutions at $G = 0$ are singular. for $Pr = 1$ each pair of thermal and hydrodynamic unforced solutions merges into one solution. Despite the temperature being singular we have $\theta \sim G^{-1/2}$ as $G \rightarrow 0$, thus the velocity field remains unaffected by temperature.

FIGURE 9. Bifurcation diagrams for $Pr = 0.5$ and 10 . Illustration of the change in order of appearance of the hydrodynamic solution (marked with squares) and the thermally induced one (marked with circles). This swap occurs at $Pr = 1$.

in the self-adjoint Sturm-Liouville form,

$$(pw')' + qw = -Gp\theta, \quad (5.2)$$

$$(p\theta')' + q\theta = 0, \quad (5.3)$$

where

$$p' = -p\psi, \quad q = p\psi'. \quad (5.4)$$

Multiplying equation (5.2) by θ and equation (5.3) by w , subtracting and integrating between $x = \pm 1$, we obtain

$$-\int_{-1}^1 Gp\theta^2 dx = [pw'\theta - pw\theta']_{-1}^1, \quad (5.5)$$

Imposing the boundary conditions we obtain

$$p(1)w'(1) + p(-1)w'(-1) = \int_{-1}^1 Gp\theta^2 dx. \quad (5.6)$$

There is no reason why the LHS should vanish (for example if w is odd it cannot). As $G \rightarrow 0$, we infer that $\theta \propto G^{-1/2}$. It follows that the $G\theta$ term becomes negligible, and the associated flow approaches the unforced hydrodynamic flow. This behaviour is confirmed numerically.

Another characteristic of the system is that as Pr passes through 1 , the hydrodynamic unforced solutions and the thermally induced ones swap their order of appearance. In order to illustrate this, we show in figure 9, the bifurcation diagram for $Pr = 0.5$ and 10 , for the first unforced hydrodynamic solution and the first thermally induced one.

The other special case is the limit $Pr \rightarrow 0$, when equation (2.13) reduces to $\theta'' = 0$, with the linear solution, $\theta(x) = x$. Therefore, in this case solutions with $\theta \sim 1/G$ are not possible. Indeed, the only unforced solutions we find decouple from θ . Figure 10, shows the bifurcation diagram for $Pr = 0$.

The fully hydrodynamic unforced solutions are naturally independent of the Pr number. The w and ψ profiles are presented in figure 11. In the top solution all variables are $\mathcal{O}(1)$ and there are no small lengthscales. In the lower solution however, the flow is large in magnitude, suggesting that it is inertially dominated. If the viscous terms in equations (2.11) and (2.12) are neglected, then the equations have the general solution for arbitrary

FIGURE 10. Bifurcation diagram for $Pr = 0$, showing $\log \|\psi\|$ as a function of G . There are two fully hydrodynamic solutions for $G = 0$.

constants A , C , x_0

$$w = A\psi, \quad \psi = CAi \left[(2A^2)^{1/3}(x - x_0) \right] \quad (5.7)$$

where Ai is an Airy function. The constant x_0 can be chosen to satisfy the solid wall condition $\psi(-1) = 0$. The no-slip condition at the inner wall requires a viscous layer of thickness $\mathcal{O}(C^{-1/2})$. Note that matching with the core requires that $C < 0$, so that the core flow is directed towards the viscous stagnation point. For $x \gg x_0$, the Airy function decays exponentially and the viscous terms become important before the outer wall is reached. It is worth observing that although asymptotic matching is useful here, strictly speaking C has a fixed finite value rather than being arbitrarily large. Nevertheless, as shown in the figure, there is good agreement between the numerical solution and the Airy approximation.

We have not yet remarked on the stability of our solutions. We can obtain some information in a fairly straightforward manner by computing the eigenvalues of the Jacobian of our discretised system, equation (5.1) (Sharma *et al.* 2015). This analysis indicates that the lower solution branch is stable, but after the first turning point the solution branches are unstable. We should stress that these conclusions only address temporal stability to modes of the same spatial structure as the solution.

Our main focus so far has been on the study of steady state solutions and we have not considered what happens for large G . For example, for $Pr = 0.5$ we have not found any solution for $G \gtrsim 2.7 \times 10^3$. Numerical solution of the the time dependent equations,

$$\frac{\partial w}{\partial t} + w'\psi - w\psi' = w'' + G\theta, \quad (5.8)$$

$$\frac{\partial \psi''}{\partial t} + \psi''' \psi - \psi'' \psi' = \psi'''' + 2w^2, \quad (5.9)$$

$$\frac{\partial \theta}{\partial t} + \theta' \psi - \theta \psi' = Pr^{-1} \theta'', \quad (5.10)$$

indicates that this system becomes singular in finite time. This is reminiscent of the work of Childress *et al.* (1989). We find that near some critical time t_0 , away from the walls our variables behave as

$$w(x, t) \sim \frac{f(x)}{t_0 - t}, \quad \psi(x, t) \sim \frac{g(x)}{t_0 - t}, \quad \theta(x, t) \sim \frac{h(x)}{t_0 - t}. \quad (5.11)$$

FIGURE 11. ψ and w profiles for fully hydrodynamic unforced solutions. **Top:** First hydrodynamic unforced solution. **Bottom:** Second hydrodynamic unforced solution, with inviscid core and boundary layer on inner wall. There is good agreement between the numerical solution and the Airy function, equation 5.7

The buoyancy and diffusive terms become negligible away from the walls as they behave as $(t_0 - t)^{-1}$ rather than $(t_0 - t)^{-2}$. There are boundary layers near the walls of thickness $(t_0 - t)^{1/2}$. As an example, we set $Pr = 0.5$, for which the stable steady state ceases for $G \geq 29.1$. If we follow the time evolution of a zero initial state for $G = 60$, we find that the solution becomes singular at $t_0 \simeq 2.55$. We illustrate this case in 12, where the $(t_0 - t)^{-1}$ behaviour is demonstrated by linear fitting.

For any value of Pr , as the solution branches are followed, eventually the velocity becomes large, resulting in a wall boundary layer. Adequate resolution requires a high number of mesh points which in turn leads to conditioning issues for the Jacobian matrix. Away from the walls the flow is inertially dominated, unaffected by the temperature similar to equation (5.7). For the rest of the paper we therefore focus attention on the unforced hydrodynamic solutions. In order to ascertain whether these are generic to the NS, rather than an artifice of the Dean approximation, in the next section we relax the assumption of small curvature so that our infinite channel transforms into the gap between two concentric cylinders. This annular geometry can then support any value of curvature.

6. Unforced flow in an infinite annulus

We consider steady and axisymmetric flow inside an annulus. The coordinate system is (r, ϕ, y) , and the inner and outer radii are given by r_a and r_b , respectively, as drawn in

FIGURE 12. $\log(|\psi|_{x=0.5})$ as a function of $\log(t_0 - t)$ near time singularity for $Pr = 0.5$ and $G = 60$. Evolution from a zero initial state. The solution becomes singular at $t_0 \simeq 2.55$. The behaviour $(t_0 - t)^{-1}$ is evident (dashed grey line).

FIGURE 13. Depiction of annulus geometry and coordinate system (r, ϕ, y) . The annulus is unbounded in the y direction.

figure 13. We introduce a Stokes stream function, $y\psi(r)$, and the velocity field is described by,

$$\mathbf{u} = \left(-\frac{\psi}{r}, yw, y\frac{\psi'}{r} \right). \quad (6.1)$$

In addition we scale w and ψ with the Reynolds number. Thus, the NS equations reduce to:

$$-\psi w' + \psi' w - \frac{\psi w}{r} = w''r + w' - \frac{w}{r}, \quad (6.2)$$

$$-\psi''' \psi + \psi'' \psi' + 3 \frac{\psi'' \psi}{r} - \frac{(\psi')^2}{r} - 3 \frac{\psi' \psi}{r^2} + 2w^2 r = \psi''''r - 2\psi''' + 3 \frac{\psi''}{r} - 3 \frac{\psi'}{r^2} + F, \quad (6.3)$$

where F is a fictitious forcing which we use as a continuation parameter. We seek non-trivial solutions with $F = 0$. We impose no-slip boundary conditions at the wall,

$$w(r_{a,b}) = \psi(r_{a,b}) = \psi'(r_{a,b}) = 0. \quad (6.4)$$

The curvature is given by,

$$\lambda = \frac{2(r_b - r_a)}{r_b + r_a}. \quad (6.5)$$

which has the maximum value of $\lambda = 2$, that occurs in the limit $r_b \gg r_a$. In order to reduce the number of parameters in our problem we have fixed the gap width to $r_b - r_a = 2$, same as the channel.

FIGURE 14. Dependence of $\|\psi\|$ with λ , for the two hydrodynamic unforced solutions. The dashed grey lines mark the solutions obtained with the Dean equations, which we retrieve when $\lambda \rightarrow 0$. We can also observe a different structure as $\lambda \rightarrow 2$. The results with w are similar.

FIGURE 15. Solutions of the unforced ψ equation near the maximum value of λ . For $\lambda = 1.998$ (highlighted by the dashed-grey line), we can observe the ψ profiles for the two solutions found, on the right. These solutions have $w = 0$.

We use second order centred finite differences to discretise our equations and the path-continuation techniques described in the previous section. We use the two fully hydrodynamic solutions obtained with the Dean equations as initial estimates for small λ . Once we obtain a converged solution, we perform continuation in λ to find the solutions for larger curvatures where we observe a more intricate structure as $\lambda \rightarrow 2$. In figure 14, we show $\|\psi\|$ as a function of λ . As expected, for small curvature values the solutions tend to the Dean solutions, indicated by a dashed line in figure 14.

We also look for unforced solutions for the ψ equation decoupled from w , i.e. $w = 0$. In this case, we find that for $\lambda \approx 2$, with $r_a/r_b \rightarrow 0$, two ψ -only solutions exist. In figure 15 we can observe $\|\psi\|$ as a function of λ , as well as the ψ profiles for $\lambda = 1.998$. These solutions only exist for $r_b \gg r_a$. We find that along the upper branch, the solution has $\psi \rightarrow \infty$ as $\lambda \rightarrow 2$.

We can rewrite equation (6.3), in a more compact form, if we consider the variable transformation $\xi = r^2$ (Yuan & Finkelstein 1956),

$$\psi' \psi'' - \psi''' \psi - 4\psi''' = 2\xi \psi'''' + \hat{F}, \quad (6.6)$$

where $\hat{F}$ is a continuation parameter and $'$ now denotes a ξ -derivative. The boundary

conditions in this case are

$$\psi(r_{a,b}^2) = \psi'(r_{a,b}^2) = 0. \quad (6.7)$$

We have constructed a power series in $\hat{F}$, and performed analytical continuation with high order Hermite–Padé approximants, which confirmed the existence of the two newly found solutions with $\hat{F} = 0$.

7. Summary and concluding remarks

Horizontal temperature gradients always drive motion. For the idealised case of two vertical walls at constant temperature, the flow is one-dimensional. With our two generalisations, including channel curvature and y -variation, the equilibrium flow is three-dimensional and is governed by nonlinear ODEs. The solutions have interesting structure, with multiple solutions even as $G \rightarrow 0$.

Some of the solutions may be considered “local”, in the sense that for given y the temperature lies between the boundary values $\theta = \pm y$. But in others, including those as $G \rightarrow 0$, the local temperature is relatively large, suggesting that advection of heat from large $|y|$ is important. This is physically plausible. As the temperature is unbounded as $y \rightarrow \pm\infty$, it is always possible that advection from infinity should dominate over the local boundary conditions. When this occurs, the solutions are perturbations of what would be obtained with the homogeneous conditions $\theta(\pm 1) = 0$. We call such solutions “unforced”. We have distinguished two types of unforced solutions for $G = 0$, hydrodynamic and thermal. In the first case the temperature fully decouples from the hydrodynamics, while in the latter, θ becomes singular, $\theta \sim 1/G$ as $G \rightarrow 0$ so that the buoyancy forcing remains significant. If we regard $G \rightarrow 0$ as $\Delta T \rightarrow 0$, then we note that the physical temperature T remains finite in this limit.

We have presented here results only for moderate Prandtl number. At high Pr the structure of the unforced solutions is similar, while the first critical G value is reduced. The behaviour of the subsequent turning points is, however, more involved.

Stability information can be obtained from the eigenvalues of the Jacobian of the discretised system. This analysis only considers modes of the same spatial structure as the solution, but nevertheless, it reveals that past the first turning point the solution branches are unstable. In order to ascertain the large G behaviour we study the time dependent system, equations (5.8)–(5.10). We then find that the system exhibits a finite-time singularity.

Relaxing the assumption of small curvature, we have shown that the unforced hydrodynamic solutions extend to previously unreported solutions of the full Navier–Stokes in an annulus. These solutions can also be understood as the local solution being dominated by the counterrotating shear flows at $y = \pm\infty$. However, the annular configuration also yields two extra unforced solutions with $w = 0$, which are harder to visualise. These only exist at high aspect ratio, when $r_b \gg r_a$.

The existence of solutions which do not depend on the local boundary conditions but are dominated by the behaviour at infinity may be important in many instances where we have a shear at large distances proportional to y . For example, in recent work, Vaz *et al.* (2018) we have demonstrated that when the fluid is electrically conducting, the unforced flows described in this paper are able to drive nonlinear dynamos, which may be significant in astrophysical contexts.

Appendix A. Derivation of governing equations

This appendix contains the details of the derivation of the governing equations of the flow in a weakly curving channel driven by buoyancy forces, presented in section 2.

The first step of this derivation is to incorporate the Boussinesq approximation into equation (2.2), which yields,

$$\rho_0 \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + (\rho_0 + \Delta\rho)\mathbf{g} + \mu \nabla^2 \mathbf{u} \quad (\text{A } 1)$$

where the subscript 0 designates a constant quantity and $\Delta\rho = \rho - \rho_0$. Assuming that density changes vary linearly with temperature, we have $\rho - \rho_0 = -\rho_0\beta(T - T_0)$, where β is the thermal expansion coefficient. Thus,

$$\rho_0 \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P - \rho_0 \mathbf{g} \beta (T - T_0) + \mu \nabla^2 \mathbf{u} \quad (\text{A } 2)$$

where P is a modified pressure gradient, $P = p + \rho_0 g z$, and z is the elevation.

Assuming fully developed flow in ϕ , the full system of equations can be written as,

$$\frac{1}{r} \frac{\partial(r u_r)}{\partial r} + \frac{\partial u_y}{\partial y} = 0 \quad (\text{A } 3)$$

$$\rho_0 \left(u_r \frac{\partial u_r}{\partial r} + u_y \frac{\partial u_r}{\partial y} - \frac{u_\phi^2}{r} \right) = -\frac{\partial P}{\partial r} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_r}{\partial r} \right) - \frac{u_r}{r^2} + \frac{\partial^2 u_r}{\partial y^2} \right] \quad (\text{A } 4)$$

$$\rho_0 \left(u_r \frac{\partial u_\phi}{\partial r} + \frac{u_r u_\phi}{r} + u_y \frac{\partial u_\phi}{\partial y} \right) = -\frac{1}{r} \frac{\partial P}{\partial \phi} + \rho_0 g \beta (T - T_0) \quad (\text{A } 5)$$

$$+ \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_\phi}{\partial r} \right) - \frac{u_\phi}{r^2} + \frac{\partial^2 u_\phi}{\partial y^2} \right] \quad (\text{A } 6)$$

$$\rho_0 \left(u_r \frac{\partial u_y}{\partial r} + u_y \frac{\partial u_y}{\partial y} \right) = -\frac{\partial P}{\partial y} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u_y}{\partial r} \right) + \frac{\partial^2 u_y}{\partial y^2} \right] \quad (\text{A } 7)$$

$$u_r \frac{\partial T}{\partial r} + u_y \frac{\partial T}{\partial y} = \kappa \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{\partial^2 T}{\partial y^2} \right] \quad (\text{A } 8)$$

where it has been considered that the components of gravity are $(0, -g, 0)$.

As the channel is almost straight, we expect that the velocity component in the ϕ direction will be much larger than that in the r and y directions. We also assume that $R \gg L$, such that,

$$r = R + Lx^* \simeq R \quad \text{and} \quad \frac{\partial}{\partial r} \sim \frac{1}{L} \gg \frac{1}{r}. \quad (\text{A } 9)$$

Defining $U_0 = \nu/L$ to be a typical scale of u_ϕ , it is expected that a suitable scale for the pressure is $P \sim \rho_0 U_0^2 (L/R)$. The variables are scaled in the following manner,

$$u_\phi = U_0 u_z^*, \quad u_{r,y} = U_0 \left(\frac{L}{R} \right)^{1/2} u_{x,y}^*, \quad P = \rho_0 U_0^2 \left(\frac{L}{R} \right) P^*, \quad (\text{A } 10)$$

where the asterisk denotes a dimensionless variable and (x, y, z) is the local Cartesian reference frame depicted in figure 1, where $z \simeq R\phi$. Furthermore, a dimensionless temperature can be defined by $\theta = (T - T_0)/\Delta T$, where ΔT is a scale of the temperature difference between the channel walls.

The second step, is to scale all variables according to equation (A 10), and including

the Dean approximation by neglecting terms of $\mathcal{O}(\lambda)$. Hence, we have,

$$\frac{\partial u_x^*}{\partial x^*} + \frac{\partial u_y^*}{\partial y^*} = 0 \quad (\text{A } 11)$$

$$K \left(u_x^* \frac{\partial u_x^*}{\partial x^*} + u_y^* \frac{\partial u_x^*}{\partial y^*} - u_z^{*2} \right) = -K \frac{\partial P^*}{\partial x^*} + \frac{\partial^2 u_x^*}{\partial x^{*2}} + \frac{\partial^2 u_x^*}{\partial y^{*2}} \quad (\text{A } 12)$$

$$K \left(u_x^* \frac{\partial u_z^*}{\partial x^*} + u_y^* \frac{\partial u_z^*}{\partial y^*} \right) = G_r \theta + \frac{\partial^2 u_z^*}{\partial x^{*2}} + \frac{\partial^2 u_z^*}{\partial y^{*2}} \quad (\text{A } 13)$$

$$K \left(u_x^* \frac{\partial u_y^*}{\partial x^*} + u_y^* \frac{\partial u_y^*}{\partial y^*} \right) = -K \frac{\partial P^*}{\partial y^*} + \frac{\partial^2 u_y^*}{\partial x^{*2}} + \frac{\partial^2 u_y^*}{\partial y^{*2}} \quad (\text{A } 14)$$

$$K \left(u_x^* \frac{\partial \theta}{\partial x^*} + u_y^* \frac{\partial \theta}{\partial y^*} \right) = Pr^{-1} \left(\frac{\partial^2 \theta}{\partial x^{*2}} + \frac{\partial^2 \theta}{\partial y^{*2}} \right) \quad (\text{A } 15)$$

where $K \equiv \frac{U_0 L}{\nu} \left(\frac{L}{R} \right)^{\frac{1}{2}}$ is the Dean number and $G_r \equiv \frac{g \beta \Delta T L^2}{U_0 \nu}$ is a similar dimensionless parameter to the Grashof number. From this point onwards all variables are non-dimensional and it is no longer necessary to use the asterisk notation. By introducing a stream function ψ , and writing $w = u_z$, our system of equations becomes

$$K \mathcal{J}(w, \psi) = \nabla^2 w + G_r \theta, \quad (\text{A } 16)$$

$$K \mathcal{J}(\nabla^2 \psi, \psi) = \nabla^4 \psi + 2Kw \frac{\partial w}{\partial y}, \quad (\text{A } 17)$$

$$K \mathcal{J}(\theta, \psi) = Pr^{-1} \nabla^2 \theta, \quad (\text{A } 18)$$

where the operator $\mathcal{J}$ is defined in equation (2.8). We have three parameters in our equations, in order to reduce it to two, Pr and $G = G_r K$, we perform the following scaling,

$$\psi \longrightarrow K^{-1} \psi, \quad w \longrightarrow K^{-1} w,$$

which brings us to our final system of equations (2.5)-(2.7).

Appendix B. Inclined channel

We have assumed that to leading order gravity only acts in the negative z direction. If the channel is inclined at a certain angle to the horizontal we must allow for gravity to act in the x - and z -directions. In this case the governing equations are,

$$\mathcal{J}(w, \psi) = \nabla^2 w + \hat{G} \theta, \quad (\text{B } 1)$$

$$\mathcal{J}(\nabla^2 \psi, \psi) = \nabla^4 \psi + 2w \frac{\partial w}{\partial y} + \gamma \frac{\partial \theta}{\partial y}, \quad (\text{B } 2)$$

$$\mathcal{J}(\theta, \psi) = Pr^{-1} \nabla^2 \theta, \quad (\text{B } 3)$$

where now we have

$$\hat{G} = G \cos \alpha, \quad \gamma = \frac{g \beta \Delta T L^3}{\nu^2} \sin \alpha, \quad (\text{B } 4)$$

where α is the angle of inclination of the channel to the horizontal. The boundary conditions are the same as before, but now we must allow for more general y - linear

behaviour in order to balance the $\gamma\partial\theta/\partial y$ term in equation (B 2). Thus, we write $w(x, y) = yw_a(x) + w_b(x)$, $\psi(x, y) = y\psi_a(x) + \psi_b(x)$, and $\theta(x, y) = y\theta_a(x) + \theta_b(x)$. The system of equations separates naturally into

$$w'_a\psi_a - w_a\psi'_a = w''_a + \hat{G}\theta_a, \quad (\text{B } 5)$$

$$\psi'''_a\psi_a - \psi''_a\psi'_a = \psi''''_a + 2w_a^2, \quad (\text{B } 6)$$

$$\theta'_a\psi_a - \theta_a\psi'_a = Pr^{-1}\theta''_a, \quad (\text{B } 7)$$

with

$$\psi_a = \psi'_a = w_a = 0, \quad \theta_a = \pm 1 \quad (\text{B } 8)$$

on $x = \pm 1$, and

$$w'_b\psi_a - w_a\psi'_b = w''_b + \hat{G}\theta_b, \quad (\text{B } 9)$$

$$\psi'''_b\psi_a - \psi''_a\psi'_b = \psi''''_b + 2w_aw_b + \gamma\theta_a, \quad (\text{B } 10)$$

$$\theta'_b\psi_a - \theta_a\psi'_b = Pr^{-1}\theta''_b, \quad (\text{B } 11)$$

with homogeneous boundary conditions. We can retrieve the non-inclined case by setting $\gamma = 0$. The equations for w_a , ψ_a and θ_a are the same as equations (2.11)-(2.13). These can be solved first, exactly as previously. The solution for w_b , ψ_b and θ_b can then be found independently. This latter solution is linear in γ , and accounts for the inclination of the channel.

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