A STATE-SPACE ALGORITHM FOR THE SOLUTION OF THE 2-BLOCK SUPEROPTIMAL DISTANCE PROBLEM*

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Abstract. A state-space algorithm for computing the solution of the 2-block superoptimal distance problem (SODP) is presented. Given a rational and antistable matrix function $R(s) = [R_{11}(s) \ R_{12}(s)]$, find all stable approximations $Q(s)$ that lexicographically minimize the singular values of the error function $E(s) = [R_{11}(s) \ R_{12}(s) + Q(s)]$. Conditions are given for which the superoptimal approximation is unique. In addition, an a priori upper bound on the MacMillan degree of the approximation is given. The algorithm may be stopped after minimizing a given number of singular values. This premature termination of the algorithm carries with it an expected saving in the computational effort and a predictable reduction in the MacMillan degree of the approximation. The algorithm only requires standard linear algebraic computations and is, therefore, easily implemented.

Key words. superoptimal general distance problem, 2-block general distance problem, H_∞ -optimal control

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1. Notation and definitions.

$\mathbb{R}, \mathbb{R}_+, \mathbb{C}$	real, nonnegative and complex numbers
$\mathbb{C}_+(\bar{\mathbb{C}}_+), \mathbb{C}_-(\bar{\mathbb{C}}_-)$	open (respectively, closed) right half plane, open (respectively, closed) left half-plane
$\lambda(A), \lambda_{\max}(A)$	eigenvalue of square matrix A , largest eigenvalue of A
A^*	complex conjugate transpose of $A \in \mathbb{C}^{m \times l}$
$A \geq 0, A > 0$	A is positive semidefinite (respectively, positive definite)
$\mathcal{L}_{\infty}^{p \times m}$	space of $p \times m$ matrix functions with entries bounded on the $j\omega$ axis
$\ (\cdot)\ _{\infty}$	$\mathcal{L}_{\infty}$ -norm of matrices in $\mathcal{L}_{\infty}$
$\mathcal{H}_{\infty}^{+p \times m}, \mathcal{H}_{\infty}^{-p \times m}$	subspaces of $\mathcal{L}_{\infty}^{p \times m}$; matrices that are analytic in $\bar{\mathbb{C}}_+$ (respectively, $\bar{\mathbb{C}}_-$)
$[G(s)]^*, [G(s)]^{-*}$	$G(-\bar{s})^*$, the para-Hermitian conjugate of $G(s)$, $[G^*(s)]^{-1}$
$\deg [G(s)]$	MacMillan degree of the rational function $G(s)$
$[G(s)]_+, [G(s)]_-$	stable (respectively, antistable) projection of $G(s)$
Prefix $\mathbb{R}$	denotes real rational.

Associated with every rational transfer function matrix $G(s)$ is a *state-space realization* $G(s) = D + C(sI - A)^{-1}B$, where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times l}$, $C \in \mathbb{R}^{m \times n}$ and $D \in \mathbb{R}^{m \times l}$, and where $n \geq \deg [G(s)]$. The alternative notation $G(s) = (A, B, C, D)$ or

$$G(s) \stackrel{s}{=} \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

is also used. Occasionally we write

$$\begin{bmatrix} R_{11}(s) & R_{12}(s) \\ R_{21}(s) & R_{22}(s) \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right],$$

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in which $R_{ij}(s) \stackrel{s}{=} (A, B_j, C_i, D_{ij})$ for $i, j = 1, 2$. If D is nonsingular, then

$$(1) \quad [G(s)]^{-1} \stackrel{s}{=} (A - BD^{-1}C, -BD^{-1}, D^{-1}C, D^{-1}).$$

If $[G(s)]^{-1} = [G(s)]^*$, then $G(s)$ is said to be an *all-pass* system and satisfies $[G(s)] \cdot [G(s)]^* = [G(s)]^* [G(s)] = I$. Occasionally, we say that $G(s)$ is all-pass for some $\gamma \in \mathbb{R}_+$. This is taken to mean that $\gamma^{-1}G(s)$ is all-pass. The rational function $G(s)$ is called *stable* if it has no poles in $\bar{\mathbb{C}}_+$. If $G(s)$ has no poles in $\bar{\mathbb{C}}_-$ it is called *antistable*. If $G(s) \in \mathcal{H}_\infty^+$ has $\|G\|_\infty \leq 1$, it is called a *stable contraction*. The set of all stable contractions is denoted $\mathcal{B}\mathcal{H}_\infty^+$. We say that $G(s) \in \gamma^{-1}\mathcal{B}\mathcal{H}_\infty^+$ if $\gamma G(s) \in \mathcal{B}\mathcal{H}_\infty^+$. If a *basis change* T is introduced into a state-space realization of $G(s)$, this is taken to mean the similarity transformation

$$(2) \quad G(s) \stackrel{s}{=} (A, B, C, D) \xrightarrow{T} G(s) \stackrel{s}{=} (T^{-1}AT, T^{-1}B, CT, D).$$

In most cases, $G(s)$ will be abbreviated to G .

If $U \in \mathcal{L}_\infty^{l \times q}$ and

$$H = \begin{matrix} & m & l \\ p & \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} & \end{matrix} \in \mathcal{L}_\infty^{(p+q) \times (m+l)}$$

we define the *lower linear fractional map* (LLFM) $\mathcal{F}_l(H, U) := H_{11} + H_{12}U(I - H_{22}U)^{-1}H_{21}$ provided $[I - H_{22}(\infty)U(\infty)]$ is invertible. If $U \in \mathcal{L}_\infty^{m \times p}$ we define the *upper linear fractional map* (ULFM) $\mathcal{F}_u(H, U) := H_{22} + H_{21}U(I - H_{11}U)^{-1}H_{12}$ provided $[I - H_{11}(\infty)U(\infty)]$ is invertible. If $\mathcal{U}$ is a set, then $\mathcal{F}_l(H, \mathcal{U})$ is taken to denote the set $\{\mathcal{F}_l(H, U) : U \in \mathcal{U}\}$. If $G_1, G_2, G_3 \in \mathcal{L}_\infty$ have appropriate dimensions, and $\mathcal{U}$ is a set, then $G_1 + G_2\mathcal{U}G_3$ is taken to denote the set $\{G_1 + G_2UG_3 : U \in \mathcal{U}\}$.

A problem of presenting the superoptimal algorithm is the elaborate notation. To simplify the presentation, we adopt the following special notation and definitions. If $A \in \mathbb{C}^{p \times m}$, then $\sigma_i(A)$ denotes the i th largest singular value of A . If $G \in \mathcal{L}_\infty^{p \times m}$, then

$$s_i^\infty(G) := \sup_{\omega \in \mathbb{R}} \sigma_i[G(j\omega)]$$

denotes the supremum of the i th largest singular value of G over the imaginary axis (including ∞). Clearly, $s_1^\infty(G) = \|G\|_\infty$. If $R \in \mathbb{R}\mathcal{H}_\infty^{-p \times (l+m)}$ is partitioned as

$$(3) \quad R = \begin{matrix} & l & m \\ p & \begin{bmatrix} R_{11} & R_{12} \end{bmatrix} & \end{matrix} \in \mathbb{R}\mathcal{H}_\infty^{-p \times (l+m)},$$

we define the optimal level of R as

$$s_1(R) := \inf_{Q \in \mathcal{H}_\infty^{+p \times m}} s_1^\infty([R_{11} \quad R_{12} + Q]),$$

and the set of all optimal approximations of R as

$$\mathcal{S}_1(R) := \{Q \in \mathcal{H}_\infty^{+p \times m} : s_1^\infty([R_{11} \quad R_{12} + Q]) = s_1\}.$$

If p and m are both greater than 1, then we define the first and subsequent superoptimal levels of R as

$$s_i(R) := \inf_{Q \in \mathcal{S}_{i-1}(R)} s_i^\infty([R_{11} \quad R_{12} + Q]) \quad i = 1, 2, \dots,$$

and the set of all i th level superoptimal approximations of R as

$$\mathcal{S}_i(R) := \{Q \in \mathcal{S}_{i-1}(R) : s_i^\infty([R_{11} \quad R_{12} + Q]) = s_i(R)\} \quad i = 1, 2, \dots,$$

in which $s_0 = \infty$ and $S_0(R) = \mathcal{H}_\infty^{+p \times m}$. Clearly, the optimal level is equal to the first superoptimal level, and the set of all optimal approximations is equal to the set of all first-level superoptimal approximations. If $\gamma > s_1(R)$, then γ is said to be a suboptimal level of R . The set of all suboptimal approximations of R at level γ is defined as

$$\mathcal{S}(R, \gamma) := \{Q \in \mathcal{H}_\infty^{+p \times m} : s_1^\infty([R_{11} \quad R_{12} + Q]) \leq \gamma\}.$$

2. Introduction. It is well known [4], [14] that a large class of $\mathcal{H}_\infty$ -control problems may be reduced to the 2-block optimal distance problem via the Youla parametrization of all stabilizing controllers [18]. In the 2-block optimal distance problem, we are given $R \in \mathbb{R}\mathcal{H}_\infty^{-p \times (l+m)}$ partitioned as (3), and we seek to find the optimal level $s_1(R)$ and the set $S_1(R)$ of all optimal approximations of R . In control problems, R is a function of a plant model and various weighting functions [4], [14]. In general, the solution of the 2-block optimal distance problem is hardly ever unique [4], [14]. The question then arises as to whether any of these optimal solutions is best in some sense. One way of recovering uniqueness is to strengthen the optimality requirement. Specifically, we request that the second and subsequent singular values are minimized with respect to lexicographic ordering [19].

Problem 1. Suppose we are given a rational antistable matrix R partitioned as in (3). Then for $i = 1, 2, \dots$ we are required to find the superoptimal levels $s_1(R), \dots, s_i(R)$ and the set $S_i(R)$ of all i th level superoptimal approximations of R , where we set $s_0 = \infty$ and $S_0(R) = \mathcal{H}_\infty^{+p \times m}$.

When $R_{11} = 0$, Problem 1 reduces to the 1-block SODP first proposed by Young [19], who showed, using operator theoretic techniques, that the superoptimal approximation is unique [19]. State-space algorithms for calculating the solution are given in [9], [15], and [17], where it is shown that the superoptimal approximation has a surprisingly low MacMillan degree. Gu, Tsai, and Postlethwaite [8] give a solution to the 2-block SODP using a technique that is different from the one used in this paper. However, their algorithm is unnecessarily complicated and cannot be readily generalized to the solution of the 4-block SODP. In contrast, we give a new and simple algorithm for the solution of both the 1-block and 2-block SODPs, using essentially the same technique. More precisely, we prove that, under certain conditions, it is possible to construct a sequence of 2-block systems $R_1, \dots, R_k$ of the same form as R such that $s_i(R) = s_1(R_i)$ and $S_i(R) = S_{i-1}[S_1(R_i)]$, where $S_1, \dots, S_{k-1}$ is a sequence of operators to be defined later. Thus, the solution of the i th level SODP is reduced to the solution of i optimal problems. We also prove that there exist i th level superoptimal approximations of R of MacMillan degree $\leq (n-1) + (n-2) + \dots + (n-i)$, where $n = \deg(R)$. Furthermore, sufficient conditions are given for which the superoptimal approximation of R is unique. Finally, we demonstrate that the algorithm can be readily extended to the solution of the 4-block SODP.

The paper is laid out as follows. Section 3 gives a review of the 2-block optimal distance problem. Section 4 gives the solution of the second-level superoptimal problem and § 5 gives a cancellation analysis of this solution. Section 6 extends the solution to the full superoptimal problem and § 7 outlines the solution to the 4-block superoptimal problem. A few examples are given in § 8, and finally, the conclusions are given in § 9.

3. The 2-block optimal distance problem. This section outlines the solution of the 2-block optimal distance problem: for $R \in \mathbb{R}\mathcal{H}_\infty^{-p \times (l+m)}$ given by the state-space realization

$$(4) \quad R = {}_p[R_{11} \quad R_{12}] \stackrel{s}{=} \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & 0 \end{array} \right] \in \mathbb{R}\mathcal{H}_\infty^{-p \times (l+m)},$$

where

$$(5) \quad A \in \mathbb{R}^{n \times n}, \quad \operatorname{Re} [\lambda_i(A)] > 0 \quad \forall i,$$

find the optimal level $s_1(R)$ and the set $\mathbb{S}_1(R)$ of all optimal approximations of R . The solution to this problem has been developed in [2], [7], and [13]. We assume that the optimal level $s_1(R)$ can be calculated to any desired degree of accuracy using the γ -iteration [4], [7], and furthermore that

$$(6) \quad s_1(R) > \|R_{11}\|_\infty.$$

The following theorem gives the set $\mathbb{S}_1(R)$ in the form of a linear fractional map.

THEOREM 1 (see [2], [7], [13]). *Let $s_1 := s_1(R)$ and assume that (5) and (6) are satisfied. Then*

(i) *There exists an embedding of R of the form*

$$H := \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} = \begin{array}{c} p \\ l \\ m-1 \end{array} \left[\begin{array}{cc|c} R_{11} & R_{12} + Q_{12} & Q_{13} \\ R_{21} & R_{22} + Q_{22} & Q_{23} \\ 0 & Q_{32} & Q_{33} \end{array} \right] \begin{array}{c} l \\ m \\ p-1 \end{array}$$

such that $HH^* = H^*H = s_1^2 I$; $R_{ij}, (R_{21})^{-1} \in \mathbb{RH}_\infty^-, i, j = 1, 2$; $Q_{ij} \in \mathbb{RH}_\infty^+, i = 1, 2, 3$; $j = 2, 3$ and

$$(7) \quad \|H_{22}\|_\infty < s_1.$$

Furthermore,

$$\mathbb{S}_1(R) = \mathcal{F}_1(Q_a, s_1^{-1} \mathcal{BH}_\infty^{(p-1) \times (m-1)})$$

and

$$[R_{11} \quad R_{12} + \mathbb{S}_1(R)] = \mathcal{F}_l(H, \mathcal{U}_1)$$

where

$$\mathcal{U}_1 := \{[0_{(p-1) \times l} \quad u] : u \in s_1^{-1} \mathcal{BH}_\infty^{(p-1) \times (m-1)}\}, \quad Q_a = \begin{bmatrix} Q_{12} & Q_{13} \\ Q_{32} & Q_{33} \end{bmatrix}.$$

(ii) *There exist real matrices*

$$\begin{aligned} A_H &= \begin{array}{c} n \\ n-1 \end{array} \begin{bmatrix} A & 0 \\ 0 & A_Q \end{bmatrix}, & B_H &= \begin{array}{c} l \\ n-1 \end{array} \begin{bmatrix} B_1 & B_2 & 0 \\ 0 & B_{Q_2} & B_{Q_3} \end{bmatrix}, \\ C_H &= \begin{array}{c} p \\ l \\ m-1 \end{array} \begin{bmatrix} C_1 & C_{Q_1} \\ C_2 & C_{Q_2} \\ 0 & C_{Q_3} \end{bmatrix}, & D_H &= \begin{array}{c} l \\ p \\ l \\ m-1 \end{array} \begin{bmatrix} 0 & D_{12} & D_{13} \\ s_1 I & 0 & 0 \\ 0 & D_{32} & 0 \end{bmatrix}, \\ P_H &= \begin{array}{c} n \\ n-1 \end{array} \begin{bmatrix} P_1 & P_3 \\ P_3^* & P_2 \end{bmatrix}, & Q_H &= \begin{array}{c} n \\ n-1 \end{array} \begin{bmatrix} Q_1 & Q_3 \\ Q_3^* & Q_2 \end{bmatrix} \end{aligned}$$

such that

$$(8) \quad A_Q \in \mathbb{R}^{(n-1) \times (n-1)}, \quad \operatorname{Re} [\lambda_i(A_Q)] < 0 \quad \forall i$$

$$P_H Q_H = Q_H P_H = s_1^2 I, \quad D_H D_H^* = D_H^* D_H = s_1^2 I, \quad A_H^* Q_H + Q_H A_H + C_H^* C_H = 0,$$

$$(9) \quad D_H^* C_H + B_H^* Q_H = 0, \quad D_H B_H^* + C_H P_H = 0, \quad A_H P_H + P_H A_H^* + B_H B_H^* = 0.$$

Furthermore, H and Q_a have state-space realizations given by

$$(10) \quad H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \stackrel{s}{=} \begin{bmatrix} A_H & B_H \\ C_H & D_H \end{bmatrix}, \quad Q_a := \begin{bmatrix} Q_{12} & Q_{13} \\ Q_{32} & Q_{33} \end{bmatrix} \stackrel{s}{=} \begin{bmatrix} A_Q & B_{Q_2} & B_{Q_3} \\ C_{Q_1} & D_{12} & D_{13} \\ C_{Q_3} & D_{32} & 0 \end{bmatrix}.$$

Remark 1. In the interest of a clear presentation the optimal level s_1 is taken to have multiplicity one. Relaxing this assumption will lead to a messy indexing system without introducing any serious technical difficulties. See [7] for a full treatment of the general case for the optimal distance problem. The assumption in (4) that $R(\infty) = 0$ is used to simplify the notation and can be removed with minor modifications to the algorithm [7].

Remark 2. If $\min(p, m) = 1$, then the optimal, and hence, the superoptimal approximation of R is unique and given by Q_{12} . If $n = 1$, then Q_{12} is a constant equal to D_{12} . Using the results in [7] it can be shown that the error $[R_{11} \quad R_{12} + D_{12}]$ has rank one so that D_{12} is the superoptimal solution.

Remark 3. In this paper, we will not give an explicit construction of the optimal generator H . However, we make extensive use of its properties as summarized in Theorem 1. For a full construction of the optimal generator H , see [7, Thm. 4.4].

4. The second-level SODP. This section considers the solution of the second-level SODP.

Problem 2. Suppose we are given a rational antistable matrix R partitioned as in (4). Then we are required to find the second superoptimal level $s_2(R)$ and the set $\mathbb{S}_2(R)$ of all second-level superoptimal approximations of R .

In the following theorem we derive the solution of the second level SODP associated with R using H only. Since $HH^* = H^*H = s_1^2 I$, we can regard H as an all-pass embedding of H_{22} . The next theorem uses standard spectral factorization results to give a construction of another embedding $\bar{H}$ of H_{22} , which also satisfies $\bar{H}\bar{H}^* = \bar{H}^*\bar{H} = s_1^2 I$.

THEOREM 2. Let H be the generator of all optimal error functions of R given by Theorem 1. Then (i) There exists an embedding of H_{22} of the form

$$(11) \quad \bar{H} = \begin{bmatrix} \bar{H}_{11} & \bar{H}_{12} \\ \bar{H}_{21} & \bar{H}_{22} \end{bmatrix} = \begin{matrix} l & m-1 & p-1 \\ p-1 & & \\ & l & \\ m-1 & & \end{matrix} \begin{bmatrix} \hat{R}_{11} & \hat{R}_{12} + \bar{Q}_{12} & \bar{Q}_{13} \\ \bar{R}_{21} & \bar{R}_{22} + \bar{Q}_{22} & \bar{Q}_{23} \\ 0 & \bar{Q}_{32} & \bar{Q}_{33} \end{bmatrix}$$

such that

$$(12) \quad \bar{H}^* \bar{H} = \bar{H} \bar{H}^* = s_1^2 I,$$

$$(13) \quad \bar{R}_{21}^{-1}, \bar{R}_{21}, \bar{R}_{22}, \hat{R} := [\hat{R}_{11} \quad \hat{R}_{12}] \in \mathbb{RH}_{\infty}^{-}, \quad \bar{Q}_{13}^{-1}, \bar{Q}_{32}^{-1}, \bar{Q}_{12}, \bar{Q}_{13}, \bar{Q}_{22}, \bar{Q}_{32} \in \mathbb{RH}_{\infty}^{+}.$$

(ii) The set of all suboptimal approximations of $\hat{R}$ is given by

$$(14) \quad \mathbb{S}(\hat{R}, s_1) = \mathcal{F}_l(\bar{Q}_a, s_1^{-1} \mathcal{B} \mathcal{H}_{\infty}^{+(p-1) \times (m-1)})$$

and the corresponding set of all suboptimal error functions of $\hat{R}$ is given by

$$(15) \quad [\hat{R}_{11} \quad \hat{R}_{12} + \mathbb{S}(\hat{R}, s_1)] = \mathcal{F}_l(\bar{H}, \mathcal{U}_1),$$

where $\mathcal{U}_1$ is defined in Theorem 1 and

$$\bar{Q}_a := \begin{bmatrix} \bar{Q}_{12} & \bar{Q}_{13} \\ \bar{Q}_{32} & \bar{Q}_{33} \end{bmatrix}.$$

Proof. (i) Using (7), it is a simple exercise to give a construction of $\bar{H}$ using standard spectral factorization theory [1] using either transfer function or state-space techniques. In this paper, we give a state-space construction of $\bar{H}$. It follows from Theorem 1 that H_{22} has the state-space realization

$$H_{22} \stackrel{s}{=} \left[\begin{array}{c|c} A_Q & B_{Q_3} \\ \hline C_{Q_2} & 0 \\ C_{Q_3} & 0 \end{array} \right].$$

Hence, it follows from (7) that there exist stabilizing solutions $\bar{P}_2$ and $\bar{Q}_2$ to the algebraic Riccati equations

$$(16) \quad \begin{aligned} A_Q \bar{P}_2 + \bar{P}_2 A_Q^* + B_{Q_3} B_{Q_3}^* + s_1^{-2} \bar{P}_2 C_{Q_3}^* C_{Q_3} \bar{P}_2 &= 0, \\ A_Q^* \bar{Q}_2 + \bar{Q}_2 A_Q + C_{Q_2}^* C_{Q_2} + C_{Q_3}^* C_{Q_3} + s_1^{-2} \bar{Q}_2 B_{Q_3} B_{Q_3}^* \bar{Q}_2 &= 0, \end{aligned}$$

respectively [2]. Since $\bar{P}_2$ and $\bar{Q}_2$ are stabilizing,

$$(17) \quad \operatorname{Re} [\lambda_i (A_Q + s_1^{-2} \bar{P}_2 C_{Q_3}^* C_{Q_3})] < 0 \quad \forall i, \quad \operatorname{Re} [\lambda_i (A_Q + s_1^{-2} B_{Q_3} B_{Q_3}^* \bar{Q}_2)] < 0 \quad \forall i.$$

Define

$$\bar{R} := \bar{Q}_2 \bar{P}_2 - s_1^2 I.$$

Then using the fact that $\bar{P}_2$ and $\bar{Q}_2$ are continuous and decreasing functions of s_1 [10], it is easy to show that (7) implies that $\lambda_{\max}(\bar{P}_2 \bar{Q}_2) < s_1^2$ [11], from which it follows that $\bar{R}$ is invertible. Defining

$$(18) \quad \bar{H} = \begin{bmatrix} \hat{R}_{11} & \hat{R}_{12} + \bar{Q}_{12} & \bar{Q}_{13} \\ \bar{R}_{21} & \bar{R}_{22} + \bar{Q}_{22} & \bar{Q}_{23} \\ 0 & \bar{Q}_{32} & \bar{Q}_{33} \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{cc|cc} \hat{A} & 0 & \hat{B}_1 & \hat{B}_2 & 0 \\ 0 & A_Q & 0 & \bar{B}_{Q_2} & B_{Q_3} \\ \hline \hat{C}_1 & \bar{C}_{Q_1} & 0 & 0 & s_1 I \\ \bar{C}_2 & C_{Q_2} & s_1 I & 0 & 0 \\ 0 & C_{Q_3} & 0 & s_1 I & 0 \end{array} \right] \stackrel{s}{=} \left[\begin{array}{c|c} A_{\bar{H}} & B_{\bar{H}} \\ \hline C_{\bar{H}} & D_{\bar{H}} \end{array} \right]$$

in which

$$(19) \quad \bar{Q}_a := \begin{bmatrix} \bar{Q}_{12} & \bar{Q}_{13} \\ \bar{Q}_{32} & \bar{Q}_{33} \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A_Q & \bar{B}_{Q_2} & B_{Q_3} \\ \hline \bar{C}_{Q_1} & 0 & s_1 I \\ C_{Q_3} & s_1 I & 0 \end{array} \right],$$

and

$$(20) \quad \hat{R} = {}_{p-1} \begin{bmatrix} \hat{R}_{11} & \hat{R}_{12} \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & 0 & 0 \end{array} \right] \in \mathbb{R} \mathcal{H}_{\infty}^{-(p-1) \times (l+m-1)},$$

$$(21) \quad \begin{aligned} \hat{A} &:= -(A_Q + s_1^{-2} \bar{P}_2 C_{Q_3}^* C_{Q_3})^*, \\ \bar{C}_{Q_1} &:= -s_1^{-1} B_{Q_3}^* \bar{Q}_2, \quad \hat{B}_1 := s_1 \bar{R}^{-1} C_{Q_2}^*, \quad \hat{B}_2 := -s_1^{-1} C_{Q_3}^*, \\ \hat{C}_1 &:= s_1^{-1} B_{Q_3}^* \bar{R}, \quad \bar{C}_2 := -s_1^{-1} \hat{B}_1^* \bar{Q}_1, \quad \bar{B}_{Q_2} := -s_1^{-1} \bar{P}_2 C_{Q_3}^*. \end{aligned}$$

Then

$$(22) \quad \hat{A} \in \mathbb{R}^{(n-1) \times (n-1)}$$

since $A_Q \in \mathbb{R}^{(n-1) \times (n-1)}$. Furthermore, (17) implies that

$$(23) \quad \operatorname{Re} [\lambda_i(\hat{A})] > 0 \quad \forall i.$$

Equations (16) and definitions (21) are used to verify that the realization of $\bar{H}$ given in (18) satisfies the all-pass equations

$$\begin{aligned} A_{\bar{H}}^* Q_{\bar{H}} + Q_{\bar{H}} A_{\bar{H}} + C_{\bar{H}}^* C_{\bar{H}} &= 0, \quad D_{\bar{H}}^* C_{\bar{H}} + B_{\bar{H}}^* Q_{\bar{H}} = 0, \quad D_{\bar{H}} D_{\bar{H}}^* = D_{\bar{H}}^* D_{\bar{H}} = s_1^2 I, \\ A_{\bar{H}} P_{\bar{H}} + P_{\bar{H}} A_{\bar{H}}^* + B_{\bar{H}} B_{\bar{H}}^* &= 0, \quad D_{\bar{H}} B_{\bar{H}}^* + C_{\bar{H}} P_{\bar{H}} = 0, \quad P_{\bar{H}} Q_{\bar{H}} = Q_{\bar{H}} P_{\bar{H}} = s_1^2 I \end{aligned}$$

with

$$P_{\bar{H}} = \begin{bmatrix} \hat{P}_1 & I \\ I & \bar{P}_2 \end{bmatrix} := \begin{bmatrix} \bar{Q}_2 \bar{R}^{-*} & I \\ I & \bar{P}_2 \end{bmatrix}, \quad Q_{\bar{H}} = \begin{bmatrix} \bar{Q}_1 & -\bar{R}^* \\ -\bar{R} & \bar{Q}_2 \end{bmatrix} := \begin{bmatrix} \bar{P}_2 \bar{R} & -\bar{R}^* \\ -\bar{R} & \bar{Q}_2 \end{bmatrix}.$$

Hence, $\bar{H}$ is all-pass at s_1 from [5, Thm. 5.1] and this proves (12). The state-space realizations (18)–(20), definitions (21), and (16) yield the following identities

$$\begin{aligned} \bar{Q}_{13}^{-1} &\stackrel{s}{=} (A_Q - s_1^{-1} B_{Q_3} C_{Q_1}, -s_1^{-1} B_{Q_3}, s_1^{-1} C_{Q_1}, s_1^{-1} I) \\ &\stackrel{s}{=} (A_Q + s_1^{-2} B_{Q_3} B_{Q_3}^* \bar{Q}_2, -s_1^{-1} B_{Q_3}, s_1^{-1} \bar{C}_{Q_1}, s_1^{-1} I), \\ \bar{Q}_{32}^{-1} &\stackrel{s}{=} (A_Q - s_1^{-1} B_{Q_2} C_{Q_3}, -s_1^{-1} B_{Q_2}, s_1^{-1} C_{Q_3}, s_1^{-1} I) \\ &\stackrel{s}{=} (A_Q + s_1^{-2} \bar{P}_2 C_{Q_3}^* C_{Q_3}, -s_1^{-1} \bar{B}_{Q_2}, s_1^{-1} C_{Q_3}, s_1^{-1} I), \\ \bar{R}_{21}^{-1} &\stackrel{s}{=} (A - s_1^{-1} \hat{B}_1 \bar{C}_2, -s_1^{-1} \hat{B}_1, s_1^{-1} \bar{C}_2, s_1^{-1} I) \\ &\stackrel{s}{=} [-\bar{R}^{-1} (A_Q + s_1^{-2} B_{Q_3} B_{Q_3}^* \bar{Q}_2)^* \bar{R}, -s_1^{-1} \hat{B}_1, s_1^{-1} \bar{C}_2, s_1^{-1} I]. \end{aligned}$$

Hence, (17) and (23) prove (13).

(ii) The fact that $s_1(R)$ is a suboptimal level of $\hat{R}$ follows from part (i) since $\bar{Q}_{12} \in \mathbb{R} \mathcal{H}_{\infty}^+$ and

$$\|\bar{H}_{11}\|_{\infty} = \|\begin{bmatrix} \hat{R}_{11} & \hat{R}_{12} + \bar{Q}_{12} \end{bmatrix}\|_{\infty} \leq \|\bar{H}\|_{\infty} = s_1(R).$$

Finally, (14) and (15) follow from (i) and [7, Thm. 3.6]. $\square$

The next theorem, which is the main result, shows that the solution of the second-level SODP associated with R is equivalent to the solution of the optimal problem associated with $\hat{R}$.

THEOREM 3. *Let H and $\bar{H}$ be as given in Theorems 1 and 2, respectively. Then*

$$(24) \quad s_2(R) = s_1(\hat{R})$$

and

$$(25) \quad \mathbb{S}_2(R) = S[\mathbb{S}_1(\hat{R})],$$

where

$$S(\cdot) = \mathcal{F}_l\{Q_a, \mathcal{F}_u[\bar{Q}_a^{-1}, (\cdot)]\},$$

and where Q_a and $\bar{Q}_a$ are defined in Theorems 1 and 2, respectively.

Proof. Since $HH^* = H^*H = s_1^2 I$ and $\bar{H}\bar{H}^* = \bar{H}^*\bar{H} = s_1^2 I$, it follows that

$$(26) \quad H_{11}H_{21}^* = -H_{21}H_{22}^*, \quad \bar{H}_{11} = -\bar{H}_{12}H_{22}^*\bar{H}_{21}^*,$$

$$(27) \quad \bar{H}_{21}\bar{H}_{21}^* = s_1^2 I - H_{22}H_{22}^* = H_{21}H_{21}^*, \quad \bar{H}_{12}^*\bar{H}_{12} = s_1^2 I - H_{22}^*H_{22} = H_{12}^*H_{12}.$$

Define

$$(28) \quad V_\perp := H_{12}\bar{H}_{12}^{-1}, \quad W_\perp := H_{21}^*\bar{H}_{21}^*.$$

Then (27) implies that

$$V_\perp^*V_\perp = I_{p-1}, \quad W_\perp^*W_\perp = I_{l+m-1}.$$

Hence, there exist

$$V = [v \quad V_\perp] \in \mathbb{R}\mathcal{L}_\infty^{p \times p}, \quad W = [w \quad W_\perp] \in \mathbb{R}\mathcal{L}_\infty^{(l+m) \times (l+m)}$$

such that V and W are all-pass [5]. Using (26) and definition (28) we obtain

$$V_\perp^*H_{12} = \bar{H}_{12}^*H_{12}^*H_{12} = \bar{H}_{12}^*\bar{H}_{12}^*\bar{H}_{12} = \bar{H}_{12},$$

$$H_{21}W_\perp = H_{21}H_{21}^*\bar{H}_{21}^* = \bar{H}_{21}\bar{H}_{21}^*\bar{H}_{21}^* = \bar{H}_{21},$$

$$V_\perp^*H_{11}W_\perp = V_\perp^*H_{11}H_{21}^*\bar{H}_{21}^* = -V_\perp^*H_{12}H_{22}^*\bar{H}_{21}^* = -\bar{H}_{12}H_{22}^*\bar{H}_{21}^* = \bar{H}_{11}.$$

It follows that

$$(29) \quad \begin{bmatrix} V^* & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} W & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} v^*H_{11}w & v^*H_{11}W_\perp & v^*H_{12} \\ V_\perp^*H_{11}w & V_\perp^*H_{11}W_\perp & V_\perp^*H_{12} \\ H_{21}w & H_{21}W_\perp & H_{22} \end{bmatrix} \\ = \begin{bmatrix} v^*H_{11}w & v^*H_{11}W_\perp & v^*H_{12} \\ V_\perp^*H_{11}w & \bar{H}_{11} & \bar{H}_{12} \\ H_{21}w & \bar{H}_{21} & H_{22} \end{bmatrix}.$$

One can verify that

$$(30) \quad G_{22}, G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \text{ are all-pass at } \gamma \Rightarrow G_{12} = 0,$$

$$G_{21} = 0 \text{ and } G_{11} \text{ is all-pass at } \gamma.$$

Since V , W , $s_1^{-1}H$, and $s_1^{-1}\bar{H}$ are all-pass, we can use (30) and (29) to show that

$$(31) \quad \begin{bmatrix} V^* & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} W & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} v^*H_{11}w & v^*H_{11}W_\perp & v^*H_{12} \\ V_\perp^*H_{11}w & V_\perp^*H_{11}W_\perp & V_\perp^*H_{12} \\ H_{21}w & H_{21}W_\perp & H_{22} \end{bmatrix} \\ = \begin{bmatrix} s_1 a & 0 & 0 \\ 0 & \bar{H}_{11} & \bar{H}_{12} \\ 0 & \bar{H}_{21} & H_{22} \end{bmatrix},$$

where

$$(32) \quad a := s_1^{-1} v^* H_{11} w \in \mathbb{R} \mathcal{L}_\infty^{1 \times 1} \text{ is all-pass.}$$

Using the characterization of the set $[R_{11} \ R_{12} + \mathbb{S}_1(R)]$ given in Theorem 1, then with a slight abuse of notation, we have

$$(33) \quad \begin{bmatrix} v^* \\ V_\perp^* \end{bmatrix} [R_{11} \ R_{12} + \mathbb{S}_1(R)] \begin{bmatrix} w & W_\perp \end{bmatrix} \\ = \begin{bmatrix} v^* \\ V_\perp^* \end{bmatrix} \{ H_{11} + H_{12} \mathcal{U}_1 (I - H_{22} \mathcal{U}_1)^{-1} H_{21} \} \begin{bmatrix} w & W_\perp \end{bmatrix}.$$

Expanding the right-hand side of (33) and using (31) gives

$$(34) \quad V^* [R_{11} \ R_{12} + \mathbb{S}_1(R)] W = \begin{bmatrix} s_1 a & 0 & 0 \\ 0 & \mathcal{F}_1(\bar{H}, \mathcal{U}_1) \end{bmatrix} = \begin{bmatrix} s_1 a & 0 & 0 \\ 0 & \hat{R}_{11} & \hat{R}_{12} + \mathbb{S}(\hat{R}, s_1) \end{bmatrix},$$

where the second equality follows from part (ii) of Theorem 2. Suppose we write

$$(35) \quad V^* [R_{11} \ R_{12} + Q] W = \begin{bmatrix} s_1 a & 0 & 0 \\ 0 & \hat{R}_{11} & \hat{R}_{12} + \bar{Q} \end{bmatrix}.$$

Then (34) is equivalent to the following. For each $Q \in \mathbb{S}_1(R)$ there exists a $\bar{Q} \in \mathbb{S}(\hat{R}, s_1)$ such that (35) is satisfied. Conversely, for each $\bar{Q} \in \mathbb{S}(\hat{R}, s_1)$, there exists a $Q \in \mathbb{S}_1(R)$ such that (35) is satisfied. Furthermore, as Q ranges over the whole of $\mathbb{S}_1(R)$, $\bar{Q}$ ranges over the whole of $\mathbb{S}(\hat{R}, s_1)$. Since V and W are all-pass, we have

$$s_2(R) := \inf_{Q \in \mathbb{S}_1(R)} s_2^\infty([R_{11} \ R_{12} + Q]) = \inf_{Q \in \mathbb{S}_1(R)} s_2^\infty(V^*[R_{11} \ R_{12} + Q]W).$$

Using (34) and the fact that a is all-pass and fixed,

$$(36) \quad s_2(R) = \inf_{\bar{Q} \in \mathbb{S}(\hat{R}, s_1)} s_1^\infty([\hat{R}_{11} \ \hat{R}_{12} + \bar{Q}]).$$

Since the set $\mathbb{S}(\hat{R}, s_1)$ consists of all suboptimal approximations of $\hat{R}$, then it includes all the optimal ones. Therefore, the set $\mathbb{S}(\hat{R}, s_1)$ in (36) can be replaced by the set $\mathcal{H}_\infty^{+(p-1) \times (m-1)}$, and (24) follows. Since V and W are all-pass we can write (34) as

$$(37) \quad [R_{11} \ R_{12} + \mathbb{S}_1(R)] = V \begin{bmatrix} s_1 a & 0 & 0 \\ 0 & \hat{R}_{11} & \hat{R}_{12} + \mathbb{S}(\hat{R}, s_1) \end{bmatrix} W^*.$$

Hence, it follows that the subset of $\mathbb{S}_1(R)$ whose elements minimize the second singular value of the left-hand side of (37) can be obtained by replacing the set $\mathbb{S}(\hat{R}, s_1)$ by $\mathbb{S}_1(\hat{R})$ in the right-hand side of (37), which proves that

$$(38) \quad [R_{11} \ R_{12} + \mathbb{S}_2(R)] = V \begin{bmatrix} s_1 a & 0 & 0 \\ 0 & \hat{R}_{11} & \hat{R}_{12} + \mathbb{S}_1(\hat{R}) \end{bmatrix} W^*.$$

To prove (25) we expand the right-hand side of (38) as

$$(39) \quad [R_{11} \ R_{12} + \mathbb{S}_2(R)] = [v \ V_\perp] \begin{bmatrix} s_1 a & 0 & 0 \\ 0 & \hat{R}_{11} & \hat{R}_{12} + \bar{Q}_{12} \end{bmatrix} \begin{bmatrix} w^* \\ W_\perp^* \end{bmatrix} \\ + [v \ V_\perp] \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathbb{S}_1(\hat{R}) - \bar{Q}_{12} \end{bmatrix} \begin{bmatrix} w^* \\ W_\perp^* \end{bmatrix}.$$

Using (31), the first term in (39) is equal to $[R_{11} \ R_{12} + Q_{12}]$. Using definition (28) and the partitions of H and $\bar{H}$, it follows from (39) that

$$[R_{11} \ R_{12} + S_2(R)] = [R_{11} \ R_{12} + Q_{12}] + Q_{13}\bar{Q}_{13}^{-1}[0 \ S_1(\hat{R}) - \bar{Q}_{12}]\begin{bmatrix} \bar{R}_{21}^{-1}R_{21} & * \\ 0 & \bar{Q}_{32}^{-1}Q_{32} \end{bmatrix},$$

where $*$ denotes an expression that is irrelevant for the present purpose. Thus,

$$S_2(R) = Q_{12} + Q_{13}\bar{Q}_{13}^{-1}[S_1(\hat{R}) - \bar{Q}_{12}]\bar{Q}_{32}^{-1}Q_{32} = \mathcal{F}_l[K, S_1(\hat{R})],$$

where

$$K = \begin{bmatrix} Q_{12} - Q_{13}\bar{Q}_{13}^{-1}\bar{Q}_{12}\bar{Q}_{32}^{-1}Q_{32} & Q_{13}\bar{Q}_{13}^{-1} \\ \bar{Q}_{32}^{-1}Q_{32} & 0 \end{bmatrix}.$$

A simple calculation will verify that $\mathcal{F}_l[K, (\cdot)] = \mathcal{F}_l\{Q_a, \mathcal{F}_u[\bar{Q}_a^{-1}, (\cdot)]\}$, which proves (25).

Remark 4. Compared with R , $\hat{R}$ has a lower state dimension $\deg(\hat{R}) \leq \deg(R) - 1$ and

$$R = {}_p[R_{11} \ R_{12}] \in \mathbb{R}\mathcal{H}_{\infty}^{-} \Rightarrow \hat{R} = {}_{p-1}[\hat{R}_{11} \ \hat{R}_{12}] \in \mathbb{R}\mathcal{H}_{\infty}^{-}.$$

The solution of the second-level SODP associated with R is equivalent to the solution of an optimal problem of reduced dimensions and MacMillan degree.

To summarize the results of this section, the following procedure may be used to find the second-level solution of the 2-block SODP. We start with R partitioned as in (4).

(I) Find the optimal level $s_1 = s_1(R)$. If $s_1 = \|R_{11}\|_{\infty}$, stop. If $s_1 > \|R_{11}\|_{\infty}$, continue.

(II) Find H , the generator of all optimal error functions of R using Theorem 1.

(III) Find $\bar{H}$, the all-pass embedding of H_{22} using Theorem 2 and define $\hat{R} := (\bar{H}_{11})_-$.

The second superoptimal level of R is equal to the optimal level of $\hat{R}$ and the set of all second-level superoptimal approximations of R is given in terms of the optimal approximations of $\hat{R}$ as (25).

Remark 5. The calculation of the optimal levels $s_1(R)$ and $s_1(\hat{R})$ involves a binary search algorithm (the so-called γ -iteration), where each step involves the solution of two algebraic Riccati equations of order $n = \deg(R)$ and $n - 1 = \deg(\hat{R})$, respectively. The calculation of $\hat{R}$ involves the solution of the algebraic Riccati equations (16), which are of order $n - 1$. Hence, the calculation of the second-level superoptimal solution is essentially equivalent to the calculation of the optimal solutions for R and $\hat{R}$.

Remark 6. If assumption (6) is not satisfied, then the set of all optimal error functions of R can, in general, be generated by an LLFM of the form

$$[R_{11} \ R_{12} + S_1(R)] = \mathcal{F}_l(H, \mathcal{U}), \quad \mathcal{U} := \{[0_{p \times l} \ u]: u \in s_1^{-1} \mathcal{BH}_{\infty}^{p \times m}\},$$

where H is all-pass at s_1 and satisfies [7],

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} = {}_p \left[\begin{array}{cc|c} R_{11} & R_{12} + Q_{12} & Q_{13} \\ R_{21} & R_{22} + Q_{22} & Q_{23} \\ \hline 0 & Q_{32} & Q_{33} \end{array} \right] \quad \|H_{22}\|_{\infty} = s_1; H_{12}, H_{21} \text{ have full rank.}$$

It follows that (31) cannot be satisfied for any all-pass transformations V and W since H_{12} and H_{21} have full normal rank. This is because there do not exist vector transfer functions w and v such that the (1, 3) and (3, 1) blocks of (31) are satisfied. Consequently, it is not possible to carry out the diagonalization and the algorithm stops. If $\min(p, m) = 1$, then the optimal and superoptimal approximations are the same and the set of all optimal error functions has a continuum of solutions. This shows that if $s_1(R) = \|R_{11}\|_\infty$, then, in general, there does not exist a unique superoptimal approximation of R . See [11] for more details.

Remark 7. Previous solutions to the SODP are based on finding a maximal Schmidt pair, which are a pair of left and right singular vectors associated with the largest singular value $s_1(R)$ [3], [8], [9], [17]. This pair corresponds to the vectors v and w in (31). Here, in contrast, the solution of the SODP is obtained directly via the optimal generator H ; this is possible since all information about the optimal approximations is contained in this generator (Theorem 1). The advantage of this approach is that the same method can be used for the solution of the one-, two-, and four-block problems, since in each case, the optimal generators have the same form [7].

The solution of the second-level SODP associated with R is given in terms of the solution of the optimal problem associated with $\hat{R}$. Since $\hat{R}$ has the same form as R , Theorem 1 can be used to give the solution of the optimal problem associated with $\hat{R}$, and Problem 2 is solved. In § 6, the solution of Problem 2 is used to give the solution of the full SODP (Problem 1).

5. MacMillan degree bounds. In this section, we carry out a cancellation analysis of the second-level superoptimal algorithm developed in the previous section. We will prove the existence of second-level superoptimal approximations of R of MacMillan degree $\leq (n-1) + (n-2)$, where $n = \text{MacMillan degree of } R$. The following lemma gives some properties about linear fractional maps in a state-space setting.

LEMMA 1. Let $P, \hat{P}, \tilde{U} \in \mathbb{R}\mathcal{L}_\infty$ have state-space realizations $\tilde{U} \stackrel{s}{=} (\hat{A}, \hat{B}, \hat{C}, \hat{D})$

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right], \quad \hat{P} = \begin{bmatrix} \hat{P}_{11} & \hat{P}_{12} \\ \hat{P}_{21} & \hat{P}_{22} \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & \hat{D}_{11} & \hat{D}_{12} \\ \hat{C}_2 & \hat{D}_{21} & \hat{D}_{22} \end{array} \right],$$

respectively, and assume that the inverses

$\tilde{L} = (I - D_{22}\tilde{D})^{-1}$, $\tilde{K} = (I - \tilde{D}D_{22})^{-1}$, $\hat{K} = (I - \hat{D}_{11}D_{11})^{-1}$, $\hat{L} = (I - D_{11}\hat{D}_{11})^{-1}$ exist. Then

- (i) $\deg(\mathcal{F}_l(P, \tilde{U})) \leq \deg(P) + \deg(\tilde{U})$ and $\deg[\mathcal{F}_u(\hat{P}, \tilde{U})] \leq \deg(\hat{P}) + \deg(\tilde{U})$.

$$(ii) \mathcal{F}_l(P, \tilde{U}) \stackrel{s}{=} \left[\begin{array}{cc|c} A + B_2\tilde{D}\tilde{L}C_2 & B_2\tilde{K}\tilde{C} & B_1 + B_2\tilde{D}\tilde{L}D_{21} \\ \hline \tilde{B}\tilde{L}C_2 & \tilde{A} + \tilde{B}D_{22}\tilde{K}\tilde{C} & \tilde{B}\tilde{L}D_{21} \\ \hline C_1 + D_{12}\tilde{D}\tilde{L}C_2 & D_{12}\tilde{K}\tilde{C} & D_{11} + D_{12}\tilde{D}\tilde{L}D_{21} \end{array} \right].$$

- (iii) $\mathcal{F}_u\{P, \mathcal{F}_l[\hat{P}, (\cdot)]\} = \mathcal{F}_l[J, (\cdot)]$ where

$$J \stackrel{s}{=} \left[\begin{array}{cc|cc} A + B_1\hat{D}_{11}\hat{L}C_1 & B_1\hat{K}\hat{C}_1 & B_2 + B_1\hat{D}_{11}\hat{L}D_{12} & B_1\hat{K}\hat{D}_{12} \\ \hline \hat{B}_1\hat{L}C_1 & \hat{A} + \hat{B}_1D_{11}\hat{K}\hat{C}_1 & \hat{B}_1\hat{L}D_{12} & \hat{B}_2 + \hat{B}_1D_{11}\hat{K}\hat{D}_{12} \\ \hline C_2 + D_{21}\hat{D}_{11}\hat{L}C_1 & D_{21}\hat{K}\hat{C}_1 & D_{22} + D_{21}\hat{D}_{11}\hat{L}D_{12} & D_{21}\hat{K}\hat{D}_{12} \\ \hline \hat{D}_{21}\hat{L}C_1 & \hat{C}_2 + \hat{D}_{21}D_{11}\hat{K}\hat{C}_1 & \hat{D}_{21}\hat{L}D_{12} & \hat{D}_{22} + \hat{D}_{21}D_{11}\hat{K}\hat{D}_{12} \end{array} \right].$$

Proof. The proof follows by routine computation. $\square$

To give an upper bound on the degree of the second-level superoptimal approximations of R , we need to characterize the set $\mathbb{S}_1(\hat{R})$ as required by Theorem 3. Assume that the optimal level $s_1(\hat{R})$ can be calculated to any desired degree of accuracy using the γ -iteration, and furthermore, that

$$s_2(R) = s_1(\hat{R}) > \|\hat{R}_{11}\|_\infty.$$

Since $\hat{R}$ has the same form as R , Theorem 1 is used to give a characterization of the set $\mathbb{S}_1(\hat{R})$.

COROLLARY 1. *Theorem 1 is satisfied with all variables “hatted”; p , m , and n are replaced by $p - 1$, $m - 1$, and $n - 1$, respectively, and $\mathcal{U}_1$ and s_1 are replaced by $\mathcal{U}_2$ and s_2 , respectively.*

The next theorem proves a general cancellation phenomenon between the optimal and suboptimal approximations of a given 2-block system. This result is used to give an upper bound on the MacMillan degree of the second-level superoptimal approximations of R .

THEOREM 4. *Let $\bar{Q}_a$ be the generator of all suboptimal approximations of $\hat{R}$ at level s_1 . Then*

$$(40) \quad \hat{Q} \in \mathbb{S}_1(\hat{R}) \Rightarrow \deg[\mathcal{F}_u(\bar{Q}_a^{-1}, \hat{Q})] \leq \deg(\hat{Q}).$$

Hence,

$$(41) \quad \hat{Q} \in \mathbb{S}_1(\hat{R}) \Rightarrow \deg\{\mathcal{F}_l[Q_a, \mathcal{F}_u(\bar{Q}_a^{-1}, \hat{Q})]\} \leq \deg(\hat{Q}) + n - 1.$$

Finally, if $\hat{Q}_{12}$ is the central optimal approximation of $\hat{R}$, then

$$(42) \quad \deg\{\mathcal{F}_l[Q_a, \mathcal{F}_u(\bar{Q}_a^{-1}, \hat{Q}_{12})]\} \leq (n - 1) + (n - 2).$$

Proof. Corollary 1 implies that

$$(43) \quad \deg(\hat{Q}_a) \leq n - 2.$$

Since $\hat{Q} \in \mathbb{S}_1(\hat{R})$, then $\hat{Q}$ is generated by the LLFM

$$(44) \quad \hat{Q} = \mathcal{F}_l(\hat{Q}_a, \hat{U}) \text{ for some } \hat{U} \in \mathcal{U}_2.$$

Hence, $\deg(\hat{Q}) \leq \deg(\hat{U}) + n - 2$ from part (i) of Lemma 1. To simplify the proof, assume that, in fact,

$$(45) \quad \deg(\hat{Q}) = \deg(\hat{U}) + n - 2.$$

Using (19), (21), and (1) gives

$$\bar{Q}_a = \begin{bmatrix} \bar{Q}_{12} & \bar{Q}_{13} \\ \bar{Q}_{32} & \bar{Q}_{33} \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A_Q & \bar{B}_{Q_2} & B_{Q_3} \\ \hline \bar{C}_{Q_1} & 0 & s_1 I \\ C_{Q_3} & s_1 I & 0 \end{array} \right] \stackrel{s}{=} \left[\begin{array}{c|cc} -\hat{A}^* - \bar{B}_{Q_2} \hat{B}_2^* & \bar{B}_{Q_2} & B_{Q_3} \\ \hline -\hat{C}_1 \hat{P}_1 & 0 & s_1 I \\ -s_1 \hat{B}_2^* & s_1 I & 0 \end{array} \right]$$

and

$$\bar{Q}_a^{-1} \stackrel{s}{=} \left[\begin{array}{c|cc} -\hat{A}^* + s_1^{-1} B_{Q_3} \hat{C}_1 \hat{P}_1 & s_1^{-1} B_{Q_3} & s_1^{-1} \bar{B}_{Q_2} \\ \hline \hat{B}_2^* & 0 & s_1^{-1} I \\ s_1^{-1} \hat{C}_1 \hat{P}_1 & s_1^{-1} I & 0 \end{array} \right].$$

Part (iii) of Lemma 1, (44), Corollary 1, and (10) yield

$$(46) \quad \mathcal{F}_u(\bar{Q}_a^{-1}, \hat{Q}) = \mathcal{F}_u[\bar{Q}_a^{-1}, \mathcal{F}_l(\hat{Q}_a, \hat{U})] = \mathcal{F}_l(J, \hat{U}),$$

where

$$(47) \quad J \stackrel{s}{=} \left[\begin{array}{cc|cc} s_1^{-1}B_{Q_3}(\hat{C}_1\hat{P}_1 + \hat{D}_{12}\hat{B}_2^*) - \hat{A}^* & s_1^{-1}B_{Q_3}\hat{C}_{Q_1} & s_1^{-1}\bar{B}_{Q_2} + s_1^{-2}B_{Q_3}\hat{D}_{12} & s_1^{-1}B_{Q_3}\hat{D}_{13} \\ \hat{B}_{Q_2}\hat{B}_2^* & \hat{A}_Q & s_1^{-1}\hat{B}_{Q_2} & \hat{B}_{Q_3} \\ \hline s_1^{-1}(\hat{C}_1\hat{P}_1 + \hat{D}_{12}\hat{B}_2^*) & s_1^{-1}\hat{C}_{Q_1} & s_1^{-2}\hat{D}_{12} & s_1^{-1}\hat{D}_{13} \\ \hat{D}_{32}\hat{B}_2^* & \hat{C}_{Q_3} & s_1^{-1}\hat{D}_{32} & 0 \end{array} \right]$$

$$\stackrel{s}{=} \left[\begin{array}{cc|cc} A_{11} & A_{12} & B_{11} & B_{12} \\ A_{21} & A_{22} & B_{21} & B_{22} \\ \hline C_{11} & C_{12} & s_1^{-2}\hat{D}_{12} & s_1^{-1}\hat{D}_{13} \\ C_{21} & C_{22} & s_1^{-1}\hat{D}_{32} & 0 \end{array} \right].$$

Applying the change of basis $T = \begin{bmatrix} I & 0 \\ \hat{P}_3^* & I \end{bmatrix}$ to (47) and using (2),

$$(48) \quad J \stackrel{s}{=} \left[\begin{array}{cc|cc} \hat{A}_{11} & \hat{A}_{12} & \hat{B}_{11} & \hat{B}_{12} \\ \hat{A}_{21} & \hat{A}_{22} & \hat{B}_{21} & \hat{B}_{22} \\ \hline \hat{C}_{11} & \hat{C}_{12} & s_1^{-2}\hat{D}_{12} & s_1^{-1}\hat{D}_{13} \\ \hat{C}_{21} & \hat{C}_{22} & s_1^{-1}\hat{D}_{32} & 0 \end{array} \right]$$

in which

$$(49) \quad \begin{aligned} \hat{A}_{21} &= \hat{A}_Q\hat{P}_3^* + \hat{P}_3^*\hat{A}^* + \hat{B}_{Q_2}\hat{B}_2^* - s_1^{-1}\hat{P}_3^*B_{Q_3}(\hat{C}_1\hat{P}_1 + \hat{D}_{12}\hat{B}_2^* + \hat{C}_{Q_1}\hat{P}_3^*), \\ \hat{C}_{11} &= s_1^{-1}(\hat{C}_1\hat{P}_1 + \hat{D}_{12}\hat{B}_2^* + \hat{C}_{Q_1}\hat{P}_3^*), \quad \hat{C}_{21} = \hat{D}_{32}\hat{B}_2^* + \hat{C}_{Q_3}\hat{P}_3^* \\ \hat{A}_{22} &= \hat{A}_Q - s_1^{-1}\hat{P}_3^*B_{Q_3}\hat{C}_{Q_1}, \quad \hat{C}_{12} = s_1^{-1}\hat{C}_{Q_1}, \end{aligned}$$

$$(50) \quad \begin{aligned} \hat{B}_{21} &= s_1^{-1}[\hat{B}_{Q_2} - \hat{P}_3^*(\bar{B}_{Q_2} + s_1^{-1}B_{Q_3}\hat{D}_{12})], \quad \hat{C}_{22} = \hat{C}_{Q_3}, \\ \hat{B}_{22} &= \hat{B}_{Q_3} - s_1^{-1}\hat{P}_3^*B_{Q_3}\hat{D}_{13}. \end{aligned}$$

It follows from Corollary 1 that the all-pass equations (9) are satisfied for the state-space realization of the generator $\hat{H}$ for some real matrices

$$P_{\hat{H}} = \begin{matrix} n-1 \\ n-2 \end{matrix} \begin{bmatrix} \hat{P}_1 & \hat{P}_3 \\ \hat{P}_3^* & \hat{P}_2 \end{bmatrix}, \quad Q_{\hat{H}} = \begin{matrix} n-1 \\ n-2 \end{matrix} \begin{bmatrix} \hat{Q}_1 & \hat{Q}_3 \\ \hat{Q}_3^* & \hat{Q}_2 \end{bmatrix}.$$

This implies that

$$(51) \quad \begin{aligned} &A_{\hat{H}}P_{\hat{H}} + P_{\hat{H}}A_{\hat{H}}^* + B_{\hat{H}}B_{\hat{H}}^* \\ &= \begin{bmatrix} \hat{A}\hat{P}_1 + \hat{P}_1\hat{A}^* + \hat{B}_1\hat{B}_1^* + \hat{B}_2\hat{B}_2^* & \hat{A}\hat{P}_3 + \hat{P}_3\hat{A}^* + \hat{B}_2\hat{B}_{Q_2} \\ \hat{A}_Q\hat{P}_3^* + \hat{P}_3^*\hat{A}^* + \hat{B}_{Q_2}\hat{B}_2^* & \hat{A}_Q\hat{P}_2 + \hat{P}_2\hat{A}_Q^* + \hat{B}_{Q_2}\hat{B}_{Q_2}^* + \hat{B}_{Q_3}\hat{B}_{Q_3}^* \end{bmatrix} = 0 \end{aligned}$$

and

$$(52) \quad D_{\hat{H}} B_{\hat{H}}^* + C_{\hat{H}} P_{\hat{H}} \\ = \begin{bmatrix} \hat{C}_1 \hat{P}_1 + \hat{C}_{Q_1} \hat{P}_3^* + \hat{D}_{12} \hat{B}_2^* & \hat{C}_1 \hat{P}_3 + \hat{C}_{Q_1} \hat{P}_2 + \hat{D}_{12} \hat{B}_{Q_2}^* + \hat{D}_{13} \hat{B}_{Q_3}^* \\ -s_1^{-1} \hat{B}_1^* \hat{Q}_1 \hat{P}_1 + \hat{C}_{Q_2} \hat{P}_3^* + s_1 \hat{B}_1^* & -s_1^{-1} \hat{B}_1^* \hat{Q}_1 \hat{P}_3 + \hat{C}_{Q_2} \hat{P}_2 \\ \hat{C}_{Q_3} \hat{P}_3^* + \hat{D}_{32} \hat{B}_2^* & \hat{C}_{Q_3} \hat{P}_2 + \hat{D}_{32} \hat{B}_{Q_2}^* \end{bmatrix} = 0.$$

Hence, from (49), the (2, 1) block of (51), and the (1, 1) and (3, 1) blocks of (52), we have $\hat{A}_{21} = 0$, $\hat{C}_{11} = 0$, and $\hat{C}_{21} = 0$, so that (48) and (50) imply that

$$J \stackrel{\Delta}{=} \left[\begin{array}{c|cc} \hat{A}_Q - s_1^{-1} \hat{P}_3^* B_{Q_3} \hat{C}_{Q_1} & s_1^{-1} [\hat{B}_{Q_2} - \hat{P}_3^* (\bar{B}_{Q_2} + s_1^{-1} B_{Q_3} \hat{D}_{12})] & \hat{B}_{Q_3} - s_1^{-1} \hat{P}_3^* B_{Q_3} \hat{D}_{13} \\ \hline s_1^{-1} \hat{C}_{Q_1} & s_1^{-1} \hat{D}_{12} & s_1^{-1} \hat{D}_{13} \\ \hat{C}_{Q_3} & s_1^{-1} \hat{D}_{32} & 0 \end{array} \right].$$

Thus, $\deg(J) \leq n - 2$ since $\hat{A}_Q \in \mathbb{R}^{(n-2) \times (n-2)}$ and this, together with (46), implies that

$$\deg[\mathcal{F}_u(\bar{Q}_a^{-1}, \hat{Q})] \leq \deg(\hat{U}) + n - 2.$$

Hence, (45) proves (40), which, in turn, proves (41) from part (i) of Lemma 1. Finally, (42) follows from (41) since the central approximation $\hat{Q}_{12} \stackrel{\Delta}{=} (\hat{A}_Q, \hat{B}_{Q_2}, \hat{C}_{Q_1}, \hat{D}_{12})$ satisfies $\deg(\hat{Q}_{12}) \leq n - 2$. $\square$

6. The superoptimal algorithm. In this section, the full 2-block superoptimal algorithm is presented. Let $R_1 = R$, $\hat{R}_1 = \hat{R}$ and $S_1(\cdot) = S(\cdot)$, where $S(\cdot)$ is defined in Theorem 3. The results of §§ 3 and 4 indicate that the solution of the second level SODP associated with R_1 is given in terms of the solution of the ODP associated with $\hat{R}_1$ as $s_2(R) = s_1(\hat{R})$, $S_2(R) = S_1[S_1(\hat{R})]$, where $\hat{R}_1$ satisfies

$$\hat{R}_1 = \begin{bmatrix} l & m-1 \\ p-1 & \end{bmatrix} [\hat{R}_{1,11} \quad \hat{R}_{1,12}] \in \mathbb{R} \mathcal{H}_{\infty}^{-}, \quad \deg(\hat{R}_1) \leq n - 1.$$

If we set $R_2 = \hat{R}_1$, we can repeat this process to obtain a sequence of 2-block systems

$$R_i = \begin{bmatrix} l & p-i+1 \\ p-i+1 & \end{bmatrix} [R_{i,11} \quad R_{i,12}] \in \mathbb{R} \mathcal{H}_{\infty}^{-}, \quad i = 1, \dots, k$$

and a sequence of operators $S_0, S_1, \dots, S_{k-1}$, such that for $j = 1, \dots, k - 1$,

$$(53) \quad s_2(R_j) = s_1(R_{j+1}),$$

$$(54) \quad S_2(R_j) = S_{j-1}[S_1(R_{j+1})],$$

$$(55) \quad \deg(R_j) \leq n - j + 1$$

from Theorems 3 and 4. Hence, a simple induction argument will establish that for $i = 1, \dots, k$,

$$(56) \quad s_i(R_1) = s_1(R_i),$$

$$(57) \quad S_i(R_1) = S_0 \circ S_1 \circ \dots \circ S_{i-1}[S_1(R_i)],$$

where, for operators S_i and S_{i+1} , the operator $S_i \circ S_{i+1}(\cdot)$ is defined by $S_i[S_{i+1}(\cdot)]$. Equation (56) says that the i th superoptimal level for R_1 is equal to the optimal level for R_i , and (57) says that the set of all i th level superoptimal approximations of R_1 can be obtained from the set of all optimal approximations of R_i through the composite operator $S_0 \circ S_1 \circ \dots \circ S_{i-1}$. Thus, we obtain the following algorithm for the solution of the SODP.

ALGORITHM 1. Given $R \in \mathbb{R}\mathcal{H}_\infty^-$ partitioned as in (4):

(0) Set $i = 1$, $R_1 = R$ and $S_0 = I$;

(1) Find $s_i = s_1(R_i)$;

(2) If $i = n$ then set $k = \min(p, m)$; $R_k = R_i$; $s_j = 0$ for $j = n + 1, \dots, k$ and stop;

(3) If $s_i = \|R_{i,11}\|_\infty$ then set $k = i$ and stop;

If

$$(58) \quad s_i := s_1(R_i) > \|R_{i,11}\|_\infty$$

then continue;

(4) If $i = \min(p, m)$ then set $k = \min(p, m)$ and stop;

(5) Find $\hat{R}_i$ and $S_{i-1}(\cdot)$;

(6) If it is required to minimize more singular values, then set $R_{i+1} = \hat{R}_i$, $i = i + 1$ and go to step (1); else stop.

The superoptimal levels of R are given by $\{s_1, \dots, s_k\}$ and the set of all k th level superoptimal approximations of R is given by $\mathbb{S}_k(R) = S_0 \circ \dots \circ S_{k-1}[\mathbb{S}_1(R_k)]$.

Next, we generalize the cancellation analysis given in § 5. Theorems 3 and 4, when applied to R_{k-1} , imply that

$$Q \in \mathbb{S}_1(R_k) \Rightarrow S_{k-1}(Q) \in \mathbb{S}_2(R_{k-1}) \text{ and } \deg[S_{k-1}(Q)] \leq \deg(Q) + n - k + 1.$$

Since $\mathbb{S}_2(R_{k-1}) \subseteq \mathbb{S}_1(R_{k-1})$ (a second superoptimal approximation is also an optimal approximation), then by repeated application of Theorems 3 and 4 we obtain

$$(59) \quad Q \in \mathbb{S}_1(R_k) \Rightarrow \deg\{S_0 \circ \dots \circ S_{k-1}[\mathbb{S}_1(R_k)]\} \leq \deg(Q) + \sum_{i=1}^{k-1} n - i.$$

Since $\deg(R_k) \leq n - k + 1$ from (55), then it follows from Theorem 1 that the central optimal approximation $Q_{k,12}$ satisfies $\deg(Q_{k,12}) \leq n - k$. Hence, by taking $Q = Q_{k,12}$ in (59), it follows from (57) that there exist k th level superoptimal approximations of R whose MacMillan degree satisfies

$$(60) \quad \deg(Q_{so}) \leq \sum_{i=1}^k n - i.$$

The number of singular values that can be minimized is given by k , and the superoptimal approximation is unique only if the set $\mathbb{S}_1(R_k)$ has a unique element. If the algorithm stops at the end of step (3), then it follows from Remark 6 that the set $\mathbb{S}_1(R_k)$, in general, has a continuum of solutions, and the superoptimal approximation may not be unique. If the algorithm stops at the end of either steps (2) or (4), then the number of singular values that can be minimized is given by $k = \min(p, m)$, and it follows from Remark 2 that the set $\mathbb{S}_1(R_k)$ has a unique element. Hence, the superoptimal approximation is unique. It also follows from Remark 2 that if the algorithm stops at the end of step (2) then the last $[\min(p, m) - n]$ superoptimal levels are equal to zero.

Remark 8. The algorithm may be stopped prematurely at step (6). In this case, (56), (57), and (60) are still satisfied, which means that there is a saving in the complexity of the algorithm as well as a reduction in the MacMillan degree of the superoptimal approximations. This shows that the algorithm may be used to solve the related problem of minimizing a given number of singular values.

Condition (58) is sufficient for the existence and uniqueness of the superoptimal approximation. To see that this condition is not necessary for existence, consider the system $R = [R_{11} \ R_{12}] = [1/(s-1) \ 0]$. Then it is clear that for this example the

superoptimal approximations are the same as the optimal ones. It can be shown using the results in [7] that $s_1(R) = \|R_{11}\|_\infty = 1$ and that the set of all superoptimal approximations of R is given by $\mathcal{S}_1(R) = \{s/(s+1)u; u \in \mathcal{BH}_\infty^+\}$. Thus, the superoptimal approximation of R exists, although it is not unique. In this case, the superoptimal approach is not appropriate for restoring uniqueness to the optimal approximations, and other criteria must be used. To see that condition (58) is not necessary for uniqueness, consider the system $R = [R_{11} \ R_{12}] = [1/(s-1) \ 1/(s-1)]$. It can be shown using the results in [7] that $s_1(R) = \|R_{11}\|_\infty = 1$ and that the set of all superoptimal approximations of R has a unique element given by $Q_{12} = 1$. Although condition (58) is not satisfied for this example, we have a unique superoptimal approximation. A full discussion is given in [11].

7. The 4-block SODP. This section gives a brief outline of the solution of the 4-block SODP. Given $R \in \mathbb{S}\mathcal{H}_\infty^-$ partitioned as

$$R = \begin{array}{c} l \quad m \\ \begin{array}{cc} q & \\ p & \end{array} \left[\begin{array}{cc} R_{11} & R_{12} \\ R_{21} & R_{22} \end{array} \right],$$

we are required to find all stable Q that lexicographically minimize the singular values of the error function

$$E_Q = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} + Q \end{bmatrix}.$$

The optimal level $s_1(R) := \inf_{Q \in \mathcal{H}_\infty^+} s_1^\infty(E_Q)$ is obtained using the γ -iteration [7]. If we assume that

$$s_1 := s_1(R) > \max \{ \| [R_{11} \ R_{12}] \|_\infty, \| [R_{11}^* \ R_{21}^*] \|_\infty \},$$

then [7] gives the set of all optimal error functions of R as the LLFM

$$\mathcal{E}_1(R) := \left\{ E_Q = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} + Q \end{bmatrix} : Q \in \mathcal{H}_\infty^+, \|E_Q\|_\infty = s_1(R) \right\} = \mathcal{F}_l(H, \mathcal{U}_1),$$

where

$$\mathcal{U}_1 := \left\{ \begin{bmatrix} 0_{q \times l} & 0_{q \times (m-1)} \\ 0_{(p-1) \times l} & u \end{bmatrix} : u \in s_1^{-1} \mathcal{BH}_\infty^{(p-1) \times (m-1)} \right\},$$

and H satisfies $HH^* = H^*H = s_1^2 I$ and has the form

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} = \begin{array}{c} l \quad m \\ \begin{array}{cc} q & \\ p & \end{array} \left[\begin{array}{cc|cc} R_{11} & R_{12} & R_{13} & 0 \\ R_{21} & R_{22} + Q_{22} & R_{23} + Q_{23} & Q_{24} \\ \hline R_{31} & R_{32} + Q_{32} & R_{33} + Q_{33} & Q_{34} \\ 0 & Q_{42} & Q_{43} & Q_{44} \end{array} \right] \end{array}.$$

In addition, $Q_{ij} \in \mathbb{R}\mathcal{H}_\infty^+$, $i, j = 2, 3, 4$; R_{31}^{-1} , R_{13}^{-1} , $R_{ij} \in \mathbb{R}\mathcal{H}_\infty^-$, $i, j = 1, 2, 3$ and $\|H_{22}\|_\infty < s_1$. By using arguments essentially similar to those used in the proof of Theorem 3 we obtain the following result, which gives the solution of the second-level 4-block SODP.

THEOREM 5 [11]. Let H be the generator of all optimal error functions of R . Then

(i) There exists an embedding of H_{22} that is all-pass at s_1 and satisfies

$$\bar{H} = \begin{bmatrix} \bar{H}_{11} & \bar{H}_{12} \\ \bar{H}_{21} & H_{22} \end{bmatrix} = \begin{matrix} & \begin{matrix} l & m-1 & q & p-2 \end{matrix} \\ \begin{matrix} q \\ p-1 \\ l \\ m-2 \end{matrix} & \left[\begin{array}{cc|cc} \hat{R}_{11} & \hat{R}_{12} & \bar{R}_{13} & 0 \\ \hat{R}_{21} & \hat{R}_{22} + \bar{Q}_{22} & \bar{R}_{23} + \bar{Q}_{23} & \bar{Q}_{24} \\ \hline \bar{R}_{31} & \bar{R}_{32} + \bar{Q}_{32} & R_{33} + Q_{33} & Q_{34} \\ 0 & \bar{Q}_{42} & Q_{43} & Q_{44} \end{array} \right] \end{matrix}$$

$\hat{R}_{12}, \hat{R}_{21}, \hat{R}_{22}, \bar{R}_{31}^{-1}, \bar{R}_{13}^{-1}, \bar{R}_{ij} \in \mathbb{RH}_{\infty}^{-}$; $i, j = 1, 2, 3$, $\bar{Q}_{24}^{-1}, \bar{Q}_{42}^{-1}, \bar{Q}_{ij} \in \mathbb{RH}_{\infty}^{+}$; $i, j = 2, 3, 4$, $\hat{R}_{11} \in \mathbb{RL}_{\infty}$.

Hence, the set of all suboptimal error functions of $\hat{R}$ is generated by the LLFM

$$\mathcal{E}(\hat{R}, s_1) := \left\{ \hat{E}_Q = \begin{bmatrix} \hat{R}_{11} & \hat{R}_{12} \\ \hat{R}_{21} & \hat{R}_{22} + Q \end{bmatrix} : Q \in \mathcal{H}_{\infty}^{+}, \|\hat{E}_Q\|_{\infty} \leq s_1 \right\} = \mathcal{F}_l(\bar{H}, \mathcal{U}_1).$$

(ii) The solution of the second-level SODP associated with R is given by (24) and (25), where

$$Q_a = \begin{bmatrix} Q_{22} & Q_{24} \\ Q_{42} & Q_{44} \end{bmatrix}, \quad \bar{Q}_a = \begin{bmatrix} \bar{Q}_{22} & \bar{Q}_{24} \\ \bar{Q}_{42} & \bar{Q}_{44} \end{bmatrix}.$$

Remark 9. A minor complication arises in the solution of the 4-block SODP since $\hat{R}_{11}$ is not necessarily antistable. This can be resolved by noting that

1. There exists an all-pass matrix $X \in \mathbb{RH}_{\infty}^{-}$ such that $\tilde{R}_{11} := \hat{R}_{11}X$, $\tilde{R}_{21} := \hat{R}_{21}X \in \mathbb{RH}_{\infty}^{-}$.

2. The ODP associated with $\hat{R}$ is equivalent to the ODP associated with

$$\tilde{R} := \begin{bmatrix} \tilde{R}_{11} & \tilde{R}_{12} \\ \tilde{R}_{21} & \tilde{R}_{22} \end{bmatrix} = \begin{bmatrix} \hat{R}_{11} & \hat{R}_{12} \\ \hat{R}_{21} & \hat{R}_{22} \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \in \mathbb{RH}_{\infty}^{-},$$

since X is all-pass [11].

8. Examples. This section gives a few examples to illustrate the superoptimal algorithm.

Example 1. We give a trivial example as a check, and to illustrate the various steps involved in the superoptimal algorithm. Let

$$R = \begin{bmatrix} R_{11} & R_{12} \end{bmatrix} = \left[\begin{array}{c|c} \frac{4.8}{s-5} & \frac{8}{s-1/2} \\ \hline \frac{-6.4}{s-5} & \frac{6}{s-1/2} \end{array} \middle| \begin{array}{c} \frac{9.6}{s-5} \\ \hline \frac{-12.8}{s-5} \end{array} \right] \in \mathbb{RH}_{\infty}^{-2 \times 3}.$$

Now

$$VR = \begin{bmatrix} .8 & .6 \\ .6 & -.8 \end{bmatrix} R = \left[\begin{array}{c|c} 0 & \frac{1}{s-1/2} \\ \hline \frac{8}{s-5} & 0 \end{array} \middle| \begin{array}{c} 0 \\ \hline \frac{16}{s-5} \end{array} \right],$$

where V is orthogonal. Thus, it follows that $s_i(R) = s_i(VR)$ for $i = 1, 2$. Hence,

$$s_1(R) = s_1(VR) = s_1\left(\left[\frac{8}{s-5} \middle| \frac{16}{s-5}\right]\right) = 2 \quad s_2(R) = s_1(VR) = s_1\left(\frac{1}{s-1/2}\right) = 1.$$

The superoptimal solution can be obtained using the following steps.

1. Using the results in [7], we have that $s_1(R) = 2$ and the generator H is given by

$$H = \left[\begin{array}{cc|c} R_{11} & R_{12} + Q_{12} & Q_{13} \\ R_{21} & R_{22} + Q_{22} & Q_{23} \\ 0 & Q_{32} & Q_{33} \end{array} \right] = \left[\begin{array}{cc|c} \frac{4.8}{s-5} & \frac{.8}{s-1/2} + \frac{4/15}{s+5/6} & \frac{9.6}{s-5} + 1.2 \\ -\frac{6.4}{s-5} & \frac{.6}{s-1/2} + \frac{.2}{s+5/6} & -\frac{12.8}{s-5} - 1.6 \\ 2\frac{s-3}{s-5} & 0 & \frac{8}{s-5} \end{array} \middle| \begin{array}{c} 1.6\frac{s+1/2}{s+5/6} \\ 1.2\frac{s+1/2}{s+5/6} \\ 0 \end{array} \right].$$

$$= \left[\begin{array}{cc|c} 2\frac{s-3}{s-5} & 0 & \frac{8}{s-5} \\ 0 & 2\frac{s+1/2}{s+5/6} & 0 \end{array} \middle| \begin{array}{c} 0 \\ \frac{4/3}{s+5/6} \end{array} \right].$$

2. Using Theorem 2, it can be verified that the all-pass embedding of H_{22} (at s_1) is given by

$$\bar{H} = \left[\begin{array}{cc|c} \hat{R}_{11} & \hat{R}_{12} + \bar{Q}_{12} & \bar{Q}_{13} \\ \bar{R}_{21} & \bar{R}_{22} + \bar{Q}_{22} & \bar{Q}_{23} \\ 0 & \bar{Q}_{32} & \bar{Q}_{33} \end{array} \right] = \left[\begin{array}{cc|c} 0 & \frac{1}{s-1/2} + \frac{1/3}{s+5/6} & 2\frac{s+1/2}{s+5/6} \\ 2 & 0 & 0 \\ 0 & 2\frac{s+1/2}{s+5/6} & \frac{4/3}{s+5/6} \end{array} \right].$$

3. The second-level system $\hat{R}$ is given by the antistable projection of $\bar{H}_{11}$ so that

$$\hat{R} = [\bar{H}_{11}]_- = \left[\begin{array}{c} 1 \\ s - 1/2 \end{array} \right],$$

which defines an effectively one-block problem since $\hat{R}_{11} = 0$. Then it follows that

$$s_1(\hat{R}) = s_1\left(\frac{1}{s - 1/2}\right) = 1,$$

and it can be checked using Corollary 1 that the set $\mathbb{S}_1(\hat{R})$ has only one element given by $\hat{Q}_{12} = 1$.

4. Finally, we use Theorem 3 to obtain the second superoptimal level $s_2(R)$ as $s_2(R) = s_1(\hat{R}) = 1$ and the (unique) second-level superoptimal approximation of R as

$$S_2(R) = Q_{12} + Q_{13}\bar{Q}_{13}^{-1}[\hat{Q}_{12} - \bar{Q}_{12}]\bar{Q}_{32}^{-1}Q_{32} = \left[\begin{array}{cc} .8 & 1.2 \\ .6 & -1.6 \end{array} \right].$$

Example 2. Let $R \in \mathbb{R}\mathcal{H}_\infty^{-2 \times 3}$ be given by

$$R = [R_{11} \quad R_{12}] = \left[\begin{array}{cc} \frac{2.5}{s-1.8} + \frac{0.5}{s-3.4} & \frac{12.5}{s-1.8} + \frac{0.5}{s-3.4} \\ \frac{0.5}{s-1.8} + \frac{3.5}{s-3.4} & \frac{2.5}{s-1.8} + \frac{3.5}{s-3.4} \end{array} \middle| \begin{array}{c} \frac{2.5}{s-1.8} + \frac{3.5}{s-3.4} \\ \frac{0.5}{s-1.8} + \frac{24.5}{s-3.4} \end{array} \right].$$

Then by carrying out the steps given in this paper, the superoptimal levels are given by $s_1(R) = 5.0$, $s_2(R) = 2.40157$, and the (unique) superoptimal approximation of R is obtained as

$$Q_{so} = \frac{1}{(s + 2.46868)} \left[\begin{array}{cc} 3.33701(s + 2.53702) & 1.24724(s + 2.66313) \\ 1.24724(s + 2.66313) & 4.06457(s + 2.53215) \end{array} \right].$$

A plot of the singular values of the superoptimal error system $[R_{11} \ R_{12} + Q_{so}]$ will verify that they are flat at the superoptimal levels. $\square$

9. Conclusion. Here we summarize the main contributions of this paper.

A new and simple algorithm is presented for the solution of the 2-block superoptimal distance problem. The algorithm reduces the superoptimal problem into a series of optimal problems, one for each extra singular value to be minimized. It is shown that the superoptimal solution is unique if each of these optimal problems has an optimal level satisfying (58).

An expression is obtained for the upper bound on the MacMillan degree of the superoptimal approximation. This gives a simple generalization of the bound already obtained for the 1-block problem [8], [15].

The superoptimal algorithm is shown to be truly recursive in that it can be stopped after minimizing a given number of singular values. In this case, there is a predictable saving in the computation and a corresponding reduction in the MacMillan degree of the approximation.

It is shown that the algorithm can be extended readily to the solution of the 4-block SODP.

Apart from providing a way of restoring uniqueness to the optimal approximations, superoptimal solutions can offer advantages over the optimal ones. Some work has already been done regarding the application of superoptimal approximations in control design problems. For example, in [17] superoptimality is justified for problems involving multiobjective disturbance rejection. In [9], the results in [12] are used to relate superoptimal approximations to robust stability $\mathcal{H}_\infty$ design. The authors are currently considering the potential applications of the superoptimal approximations to model reduction problems.

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