Example 10: Let $(a_0, a_1, a_2, a_3, a_4) = (0, 0, 1, 1, 0)$.

Application of the above algorithm yields:

$$\begin{aligned}\Delta_0 &= 0, & L_0 &= 0, & R_0 &= \begin{bmatrix} 1 & 0 \\ 0 & s \end{bmatrix} R_{-1} = \begin{bmatrix} 1 & 0 \\ 0 & s \end{bmatrix} \\ \Delta_1 &= 0, & L_1 &= 0, & R_1 &= \begin{bmatrix} 1 & 0 \\ 0 & s \end{bmatrix} R_0 = \begin{bmatrix} 1 & 0 \\ 0 & s^2 \end{bmatrix} \\ \Delta_2 &= 1, & L_2 &= 2, & R_2 &= \begin{bmatrix} 1 & -1 \\ s & 0 \end{bmatrix} R_1 = \begin{bmatrix} 1 & -s^2 \\ s & 0 \end{bmatrix} \\ \Delta_3 &= 1, & L_3 &= 2 \\ R_3 &= \begin{bmatrix} 1 & -1 \\ 0 & s \end{bmatrix} R_2 = \begin{bmatrix} 1-s & -s^2 \\ s^2 & 0 \end{bmatrix} \\ \Delta_4 &= -1, & L_4 &= 2 \\ R_4 &= \begin{bmatrix} 1 & 1 \\ -s & 0 \end{bmatrix} R_3 = \begin{bmatrix} 1-s+s^2 & -s^2 \\ -s+s^2 & s^3 \end{bmatrix}.\end{aligned}$$

As a result

$$\begin{bmatrix} 1 - \sigma + \sigma^2 & -\sigma^2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0$$

represents a C-MCUM for $\{b\}$. Taking the reciprocal row vector, we get the (unique) minimal partial realization $1/(s^2 - s + 1)$.

V. CONCLUSIONS

The minimal partial realization problem has been considered as an instance of exact modeling of a behavior on a half-axis, as in [3]. Solutions within this framework are based on polynomials rather than Hankel matrices. A central role is played by behaviors that are the span of a finite number of trajectories and thus do not have a transfer function. It is for this reason that the notion of a behavior rather than a transfer function is essential to the approach. We put the theory to work in deriving an efficient and constructive iterative solution for the scalar case: the celebrated Berlekamp–Massey algorithm. An interesting feature of the algorithm is that its efficiency is enhanced by the update at each step of four polynomials rather than two. It is a topic of future research to put this idea to work for identification purposes, in the context of approximate modeling.

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A Descriptor Solution to a Class of Discrete Distance Problems

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Abstract—Hankel norm and Nehari-type approximation problems arise in model reduction and $\mathcal{H}_\infty$ -control theory. Existing solutions to the discrete-time version of these problems may be derived using a standard state-space framework, but the resulting solution formulas require an invertible A -matrix. As a further complication, the D -matrix in the representation formula for all solutions becomes unbounded in the optimal case. The aim of this paper is to show that both these complications may be removed by analyzing these problems in a descriptor framework.

Index Terms—Descriptor systems, discrete-time Nehari problem, H_∞ control, model reduction.

I. INTRODUCTION

It is known that many model reduction and $\mathcal{H}_\infty$ -control problems may be transformed into the following distance problem: let $R(z)$ be a stable real rational transfer matrix with McMillan degree n . Then for any $\gamma > 0$ and any integer $k < n$, find all transfer matrices $Q(z)$, with at most k poles inside the unit disc, that satisfy $\|R(z) + Q(z)\|_\infty \leq \gamma$ [1], [6]. A necessary and sufficient condition for the existence of a solution requires $\gamma \geq (k+1)$ st Hankel singular value of $R(z)$ [1], [4]. The discrete-time version of this problem has received less attention than its continuous-time counterpart. Although the discrete problem can be tackled using a standard state-space approach, this approach breaks down if $R(z)$ has poles at the origin [7], [8]. This difficulty may be traced to the fact that the conjugation operation cannot be carried out in a standard state-space framework because $R^*(z)$ is

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anticausal. As an added complication, the direct feedthrough matrix in the solution formulas becomes unbounded as $\gamma \rightarrow \sigma_{k+1}(R)$ (even if $R(z)$ has no poles at the origin). One can argue that the discrete-time results can be obtained from those of continuous time using a bilinear transformation. While this procedure is feasible, one would prefer to have a self-contained discrete algorithm since many control problems are inherently discrete-time problems. It is also desirable, for numerical conditioning purposes, that the solution depends in a simple and direct way on the original data of the problem [7], [8]. This paper gives a general solution in a descriptor framework which allows the model reduction of discrete-time systems containing time delays (singular A -matrix) and which avoids the problems associated with unbounded D -matrices in the solution representation formula in the optimal case. The advantage of a descriptor framework is that it is closed with respect to conjugation and system inversion as well as the usual operations of addition, subtraction, and multiplication. The solution procedure closely follows that in [4] and [5].

Section II describes the notation, and Section III contains some preliminaries on the representation of systems in a descriptor framework. Section IV gives a characterization of all solutions to the Nehari approximation problem. Three examples are given in Section V, and the conclusions appear in Section VI.

II. NOTATION

This is fairly standard and is reproduced here for convenience.

$\mathcal{R}^{p \times m}, \mathcal{C}^{p \times m}$ are real (complex) matrices.

$\mathcal{D}_+, \mathcal{D}_-, \mathcal{D}_o$ $\{z \in \mathcal{C} : |z| > 1\}, \{z \in \mathcal{C} : |z| < 1\}, \{z \in \mathcal{C} : |z| = 1\}$.

$\lambda_k(A), A'$ is the k th largest eigenvalue of A , complex conjugate transpose of A .

$I_c(A) = [\pi_c(A), \nu_c(A), \delta_c(A)]$ represents the inertia of A , the number of eigenvalues in the open right half-plane, left half-plane, and on the imaginary axis, respectively.

$I_d(E, A) = [\pi_d(E, A), \nu_d(E, A), \delta_d(E, A)]$ is the inertia of the pencil $zE - A$, the number of eigenvalues in $\mathcal{D}_+, \mathcal{D}_-$, and $\mathcal{D}_o$, respectively.

$\mathcal{L}_{\infty}^{p \times m}$ represents the space of $p \times m$ transfer matrices with entries bounded on $\mathcal{D}_o$.

$\|\cdot\|_{\infty}$ is the $\mathcal{L}_{\infty}$ -norm of matrices in $\mathcal{L}_{\infty}$.

$\mathcal{H}_{\infty}^{+p \times m}(\mathcal{H}_{\infty}^{-p \times m})$ is the subspace of $\mathcal{L}_{\infty}^{p \times m}$, matrices analytic and bounded in $\mathcal{D}_+(\mathcal{D}_-)$.

$\mathcal{B}_{\gamma} \mathcal{H}_{\infty}^{-p \times m}$ represent the γ -ball of $\mathcal{H}_{\infty}^{-p \times m}$, the matrices in $\mathcal{H}_{\infty}^{-p \times m}$ which satisfy $\|\cdot\|_{\infty} \leq \gamma$.

$G^{\sim}(z)$ denotes $G'(1/\bar{z})\sigma_k(G)$ is the k th-largest Hankel singular value of G .

$\rho(A)$ is the spectral radius of (complex) matrix A .

Prefix $\mathcal{R}$ denotes real rational.

Transfer matrices will be represented by uppercase boldface type and with the dependence on z suppressed. Matrix dimensions of spaces will also be occasionally suppressed.

III. DESCRIPTOR ALLPASS SYSTEMS

Consider the descriptor system of equations [3]

$$Ex_{k+1} = Ax_k + Bu_k, \quad y_k = Cx_k + Du_k, \quad k = 0, 1, \dots \quad (1)$$

where $E, A \in \mathcal{R}^{n \times n}, B \in \mathcal{R}^{n \times m}, C \in \mathcal{R}^{p \times n}$, and $D \in \mathcal{R}^{p \times m}$. Taking z -transforms

$$zEx = Ax + Bu, \quad y = Cx + Du. \quad (2)$$

The notations

$$G \triangleq (E, A, B, C, D) \quad \text{and} \quad \mathbf{G} \triangleq \left[\begin{array}{c|c} E & A \\ \hline * & C \end{array} \middle| \begin{array}{c} B \\ D \end{array} \right]$$

are used to represent the descriptor system in (2). A change of basis means the transformation

$$\mathbf{G} \xrightarrow{T_l, T_r} (T_l E T_r, T_l A T_r, T_l B, C T_r, D)$$

for nonsingular T_l and T_r . If D is nonsingular

$$\mathbf{G}^{-1} \triangleq (E, A - BD^{-1}C, BD^{-1}, -D^{-1}C, D^{-1}).$$

The expression

$$\begin{aligned} \mathcal{F}_l \left(\left[\begin{array}{cc} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{21} & \mathbf{H}_{22} \end{array} \right], \mathbf{U} \right) \\ = \mathbf{H}_{11} + \mathbf{H}_{12} \mathbf{U} (\mathbf{I} - \mathbf{H}_{22} \mathbf{U})^{-1} \mathbf{H}_{21} \end{aligned}$$

defines a lower linear fractional transformation provided $(\mathbf{I} - \mathbf{H}_{22} \mathbf{U})^{-1}$ exists. If $\mathbf{U}$ is a set, $\mathcal{F}_l(\mathbf{H}, \mathbf{U})$ denotes the set $\{\mathcal{F}_l(\mathbf{H}, \mathbf{U}) : \mathbf{U} \in \mathbf{U}\}$. If $\mathbf{G}^{-1} = \gamma^{-2} \mathbf{G}^{\sim}$, then $\mathbf{G}$ is γ -allpass and satisfies $\mathbf{G} \mathbf{G}^{\sim} = \mathbf{G}^{\sim} \mathbf{G} = \gamma^2 \mathbf{I}$. A descriptor system is called *stable* if all the eigenvalues of $(zE - A)$ are in $\mathcal{D}_-$ and *antistable* if all these eigenvalues are in $\mathcal{D}_+$. System (1) is called *causal* if its state x_l is completely determined by the initial state x_0 and inputs $u_0, u_1, \dots, u_l$; otherwise, it is termed *anticausal*. Any anticausal system must have a descriptor realization [3]. These are needed since it is known that the Hankel approximation of a given stable system may be anticausal. We begin by giving an allpass lemma for the descriptor system (2). Apart from dealing with the case of a possibly singular E , we cater to the case of $(zE - A)$ singular for all z .

Lemma 3.1: Suppose there exist $P = P'$ and X such that

$$EPE' - APA' - X'A' - AX = BB', \quad EX = 0. \quad (3)$$

Then we have the following.

- 1) If $(AP + X')$ is nonsingular and

$$BD' + (AP + X')C' = 0, \quad DD' + CPC' = \gamma^2 I \quad (4)$$

(2) defines a unique γ -allpass transfer function $\mathbf{H} = D + C(zE - A)^{\#} B$. The symbol $(\cdot)^{\#}$ denotes a generalized inverse (defined in (6) below).

- 2) If $X = 0$, and $(E + A)$ and P are nonsingular, $\nu_d(E, A) \leq \pi_c(P)$ and $\pi_d(E, A) \leq \nu_c(P)$.
- 3) Suppose that the descriptor system has the special form

$$\left[\begin{array}{c|c} E & A \\ \hline * & C \end{array} \middle| \begin{array}{c} B \\ D \end{array} \right] = \left[\begin{array}{ccc|cc} I_{n_1} & 0 & A_{11} & A_{12} & B_1 \\ 0 & 0_{n_2} & A_{21} & A_{22} & B_2 \\ \hline * & * & C_1 & C_2 & D \end{array} \right] \quad (5)$$

then the dimension of the system can be reduced to $n_1 + [n_2 - \text{rank}(A_{22})]$. Furthermore if A_{22} is nonsingular, then the system has a standard realization of order n_1 .

Proof:

- 1) Let $zE - A = \mathbf{NFM}$, be a Smith diagonalization where $\mathbf{F} = \text{diag}(\mathbf{F}_1, 0)$, $\mathbf{N}$ and $\mathbf{M}$ are unimodular polynomial matrices and $\mathbf{F}_1$ has full normal rank. Define

$$(zE - A)^{\#} := \mathbf{M}^{-1} \text{diag}(\mathbf{F}_1^{-1}, 0) \mathbf{N}^{-1}. \quad (6)$$

Now, (3) gives

$$\begin{aligned} (zE - A)P(z^{-1}E' - A') + (zE - A)(PA' + X) \\ + (AP + X')(z^{-1}E' - A') = BB' \end{aligned} \quad (7)$$

$$\Rightarrow \mathbf{FMPM}^{\sim} \mathbf{F}^{\sim} + \mathbf{FM}(PA' + X)(\mathbf{N}^{\sim})^{-1} + \mathbf{N}^{-1}(AP + X')\mathbf{M}^{\sim} \mathbf{F}^{\sim} = \mathbf{N}^{-1}BB'(\mathbf{N}^{\sim})^{-1} \quad (8)$$

$$\Rightarrow \mathbf{JN}^{-1} = 0 \quad (9)$$

where $J = [0 \ I]$. This shows that $\text{range}(B) \subset \text{range}(zE - A)$, and so (2) has a solution for x given any u . A simple verification shows that x solves (2) if and only if $x = (zE - A)^\# Bu + M^{-1}J'w$ for some w . Multiplying (8) from the left by J and using (9)

$$\begin{aligned} JN^{-1}(AP + X')M^\sim F^\sim N^\sim \\ = 0 \Rightarrow N^{-1}(AP + X')M^\sim = \begin{bmatrix} * & * \\ 0 & Z^\sim \end{bmatrix} \end{aligned}$$

since $JF = 0$ and F_1 has full normal rank. The $*$ denotes an expression irrelevant for the present purpose. Z is full rank since N, M , and $AP + X'$ are full rank by assumption. Hence, (4) implies

$$DB'(N^\sim)^{-1}J' + CM^{-1}J'Z = 0 \Rightarrow CM^{-1}J' = 0 \quad (10)$$

by (9) and the nonsingular nature of Z . Thus

$$\begin{aligned} y &= Du + C(zE - A)^\# Bu + CM^{-1}J'w \\ &= \{D + C(zE - A)^\# B\}u = Hu. \end{aligned}$$

Equations (6) and (10) imply that $C(zE - A)^\#(zE - A) = CM^{-1}\text{diag}(I, 0)M = C$. Now we conclude from (4) and (7) that

$$\begin{aligned} HH^\sim &= DD' + CPC' + C(zE - A)^\# \\ &\quad \cdot (APC' + X'C' + BD') \\ &\quad + (CPA' + CX + DB')(z^{-1}E' - A') = \gamma^2 I. \end{aligned}$$

2) Since $X = 0$, (3) can be written as

$$\begin{aligned} (E + A)P(E' - A') + (E - A)P(E' + A') \\ = 2BB' \Rightarrow \hat{A}' + \hat{A} = 2BB' \geq 0 \end{aligned}$$

where $\hat{A} = (E - A)P(E' + A')$. Since $\delta_c(P) = 0$, [4, Corollary 3.2] implies that $\pi_c(\hat{A}\hat{H}) \leq \pi_c(\hat{H})$ and $\nu_c(\hat{A}\hat{H}) \leq \nu_c(\hat{H})$, where $\hat{H} = (E' + A')^{-1}P^{-1}(E + A)^{-1}$. Since $\pi_c(\hat{H}) = \pi_c(P)$

$$\begin{aligned} \pi_c[(E - A)(E + A)^{-1}] &\leq \pi_c(P) \\ \nu_c[(E - A)(E + A)^{-1}] &\leq \nu_c(P). \end{aligned} \quad (11)$$

A calculation using the bilinear transformation $s = (z - 1/z + 1)$ shows that $\pi_d(E, A) = \pi_c[(E - A)(E + A)^{-1}]$ and $\nu_d(E, A) = \nu_c[(E - A)(E + A)^{-1}]$, which, together with (11), proves the result.

3) Let A_{22} have a singular value decomposition, $U'A_{22}V = \text{diag}(A_{221}, 0_{n_2-r})$, where A_{221} is nonsingular and U, V are orthogonal. Applying the change of basis $T_l = \text{diag}(I, U')$, $T_r = \text{diag}(I, V)$ to (5) and partitioning conformably gives, after an appropriate adjustment of indexes

$$H \stackrel{s}{=} \left[\begin{array}{ccc|ccc|c} I & 0 & 0 & A_{11} & A_{12} & A_{13} & B_1 \\ 0 & 0 & 0 & A_{21} & A_{22} & 0 & B_2 \\ 0 & 0 & 0 & A_{31} & 0 & 0 & B_3 \\ \hline * & * & * & C_1 & C_2 & C_3 & D \end{array} \right].$$

Applying the basis change

$$T_l = \text{diag}\left(\left[\begin{array}{cc} I & -A_{12}A_{22}^{-1} \\ 0 & I \end{array}\right], I\right)$$

and

$$\begin{aligned} T_r &= \text{diag}\left(\left[\begin{array}{cc} I & 0 \\ -A_{22}^{-1}A_{21} & I_{21} \end{array}\right], I\right) \\ H &\stackrel{s}{=} \left[\begin{array}{ccc|ccc|c} I & 0 & 0 & A_{11} & 0 & A_{13} & B_1 \\ 0 & 0 & 0 & 0 & A_{22} & 0 & B_2 \\ 0 & 0 & 0 & A_{31} & 0 & 0 & B_3 \\ \hline * & * & * & C_1 & C_2 & C_3 & D \end{array} \right] \\ &\stackrel{s}{=} \left[\begin{array}{ccc|ccc|c} I & 0 & & A_{11} & & A_{13} & B_1 - A_{12}A_{22}^{-1}B_2 \\ 0 & 0 & & A_{31} & & 0 & B_3 \\ \hline * & * & & C_1 - C_2A_{22}^{-1}A_{21} & & C_3 & D - C_2A_{22}^{-1}B_2 \end{array} \right]. \end{aligned}$$

If A_{22} in (5) is nonsingular, $n_2 = r$, and H has a standard realization of order n_1 . $\square$

Remark 3.1: The (numerically ill-conditioned) Smith form is used in the proof of Lemma 3.1 to establish some system theoretic properties (existence and allpass character). This form will, however, not be used to calculate the solution of the Nehari approximation problem considered in this paper.

Remark 3.2: Equations (3) and (4) will be referred to as the descriptor allpass equations. If $E = I$, (3) implies $X = 0$. In this case, these equations reduce to the familiar allpass equations [7] $P - APA' = BB'$, $BD' + APC' = 0$, $DD' + CPC' = \gamma^2 I$.

IV. THE NEHARI APPROXIMATION PROBLEM

This section solves the suboptimal and optimal discrete-time Nehari approximation problem.

Problem 4.1: Suppose $R \in \mathcal{RH}_\infty^{p \times m}$ has McMillan degree n . Then for all integer $k < n$ and all $\sigma_k(R) > \gamma \geq \sigma_{k+1}(R)$, find the set of all error systems, $\mathcal{S} := \{E = R + Q; Q \in \mathcal{H}_\infty^{-p \times m}(k); \|E\|_\infty \leq \gamma\}$.

Our first result gives conditions under which an allpass embedding of R will act as a generator of the set of all error systems $\mathcal{S}$ in the suboptimal case $\sigma_k(R) > \gamma > \sigma_{k+1}(R)$.

Lemma 4.1 [5]: For R and k defined in Problem 4.1, let $s_k(R) > \gamma > \sigma_{k+1}(R)$ and suppose there exists a γ -allpass embedding $H \in \mathcal{RL}_\infty$ such that

$$\begin{aligned} H &= \begin{bmatrix} R + Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \\ Q_a &:= \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \in \mathcal{RH}_\infty^-(k), \quad Q_{12}^{-1}, Q_{21}^{-1} \in \mathcal{RH}_\infty^-. \end{aligned} \quad (12)$$

Then the set of all suboptimal error systems associated with R is given by

$$\mathcal{S} = \mathcal{F}_l(H, \mathcal{B}_{\gamma^{-1}} \mathcal{H}_\infty^{-p \times m}). \quad (13)$$

The next theorem gives such an allpass embedding of R in a descriptor framework.

Theorem 4.2: Let $R \in \mathcal{RH}_\infty^{p \times m}$ have a minimal realization given by $R \stackrel{s}{=} (A, B, C, 0)$ where $\rho(A) < 1$, $A \in \mathcal{R}^{n \times n}$, $B \in \mathcal{R}^{n \times m}$, $C \in \mathcal{R}^{p \times n}$, and $D \in \mathcal{R}^{p \times m}$. Let $P_1 = P_1' > 0$ and $Q_1 = Q_1' > 0$ be the controllability and the observability Grammians satisfying

$$P_1 - AP_1A' - BB' = 0, \quad Q_1 - A'Q_1A - C'C = 0. \quad (14)$$

Then $\sigma_i(\mathbf{R}) = \sqrt{\lambda_i(P_1 Q_1)}$. If $\sigma_k(\mathbf{R}) > \gamma > \sigma_{k+1}(\mathbf{R})$, Problem 4.1 implies is solved by (13) with

$$\mathbf{Q}_a \stackrel{s}{=} \left[\begin{array}{cc|cc} E_Q & A_Q & B_{Q_1} & B_{Q_2} \\ * & C_{Q_1} & D_{11} & D_{12} \\ * & C_{Q_2} & D_{21} & D_{22} \end{array} \right] \stackrel{s}{=} \left[\begin{array}{ccc|cc} I & 0 & -I & -Z' & 0 & 0 \\ 0 & 0 & I & A_o & Q_1 A_i B & \gamma A_i' C' \\ * & * & 0 & C A_i P_1 & C A_i B & \gamma I \\ * & * & 0 & \gamma B' A_i' & \gamma I & 0 \end{array} \right] \quad (15)$$

where $A_i = (I + A)^{-1}$, $A_o := Q_1 A_i P_1 - \gamma^2 A' A_i'$ and $Z := P_1 Q_1 - \gamma^2 I$. Furthermore, the order of the realization of $\mathbf{Q}_a$ can be made equal to $2n - \text{rank}(A_o)$. Finally if A_o is nonsingular, $\mathbf{Q}_a$ has a standard realization of order n .

Proof: Since $\sigma_k(\mathbf{R}) > \gamma > \sigma_{k+1}(\mathbf{R})$, Z is nonsingular and

$$\pi_c(P_1 Z^{-1}) = \pi_c(Q_1 Z) = k. \quad (16)$$

Also, since $\rho(A) < 1$, A_i exists. Define $\mathbf{H} \in \mathcal{RL}_\infty$ by

$$\mathbf{H} = D_{\mathcal{H}} + C_{\mathcal{H}}(zE_{\mathcal{H}} - A_{\mathcal{H}})^{\#} B_{\mathcal{H}} \stackrel{s}{=} \left[\begin{array}{ccc|cc} I & 0 & A & 0 & B & 0 \\ 0 & E_Q & 0 & A_Q & B_{Q_1} & B_{Q_2} \\ * & * & C & C_{Q_1} & D_{11} & D_{12} \\ * & * & 0 & C_{Q_2} & D_{21} & D_{22} \end{array} \right]. \quad (17)$$

We use Lemma 3.1 to prove that $\mathbf{H}$ satisfies the conditions of Lemma 4.1. Let

$$T_l = \begin{bmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & I & I \end{bmatrix}, \quad T_r = \begin{bmatrix} 0 & 0 & I \\ I & 0 & 0 \\ 0 & I & T_{33} \end{bmatrix}$$

$$P_{\mathcal{H}} = \begin{bmatrix} P_1 & Z & -I \\ Z' & Q_1 Z & 0 \\ -I & 0 & 0 \end{bmatrix}, \quad X_{\mathcal{H}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -I & 0 & -Q_1 \end{bmatrix}$$

where $T_{33} = A_i A P_1$. A routine calculation verifies that $\det[T_l(A_{\mathcal{H}} P_{\mathcal{H}} + X_{\mathcal{H}}) T_r] = -\gamma^2 \det(Q_1 Z) \neq 0$, since Z and Q_1 are nonsingular. Hence $A_{\mathcal{H}} P_{\mathcal{H}} + X_{\mathcal{H}}$ is nonsingular. It can be verified that the descriptor allpass equations (3) and (4) are satisfied so that (17) defines a unique γ -allpass transfer function from Lemma 3.1. We show that (15) has the correct stability properties by demonstrating that $\nu_d(E_Q, A_Q) \leq k$. Setting

$$T_r = \begin{bmatrix} A_o & -I \\ -I & 0 \end{bmatrix}, \quad \nu_d(E_Q, A_Q) = \nu_d(E_Q T_r, A_Q T_r) = \nu_d(A_o, A_1)$$

where $A_1 := Z' - A_o$. Using the (3, 3) block of (3) partitioned conformably with $P_{\mathcal{H}}$, $A_o P_1 Z^{-1} A_o' - A_1 P_1 Z^{-1} A_1' = Q_1 A_i B B' A_i' Q_1 + \gamma^2 A_i' C' C A_i$. Hence, it follows from (16) and part 2) of Lemma 3.1 that $\nu_d(E_Q, A_Q) \leq k$, since $A_o + A_1 = Z'$ and $P_1 Z^{-1}$ are nonsingular. Next, we establish the properties in (12). Observe that $\mathbf{Q}_{12} \stackrel{s}{=} (E_Q, A_Q, B_{Q_2}, C_{Q_1}, \gamma I)$. It follows that

$$\mathbf{Q}_{12}^{-1} \stackrel{s}{=} \left[\begin{array}{c|c} \hat{E} & \hat{A} \\ \hline * & \hat{C} \end{array} \right] \stackrel{s}{=} \left[\begin{array}{ccc|cc} I & 0 & -I & -Z' & 0 \\ 0 & 0 & I & A_i' A' Z' & A_i' C' \\ * & * & 0 & \gamma^{-1} C A_i P_1 & \gamma^{-1} I \end{array} \right].$$

Multiplying $\hat{E}$ and $\hat{A}$ from the right by

$$\hat{T}_r = \begin{bmatrix} A_i' A' Z' & -I \\ -I & 0 \end{bmatrix}$$

$$\nu_d(\hat{E}, \hat{A}) = \nu_d(\hat{E} \hat{T}_r, \hat{A} \hat{T}_r) = \nu_d(A_i' A' Z', A_i' Z') = \nu_d(A', I).$$

Hence, it follows from (14) and part 2) of Lemma 3.1 that $\nu_d(A', I) \leq \pi_c(-Q_1) = 0$. This proves that $\mathbf{Q}_{12}^{-1} \in \mathcal{RH}_\infty^-$. A similar argument proves that $\mathbf{Q}_{21}^{-1} \in \mathcal{RH}_\infty^-$. Finally, the order of the realization of $\mathbf{Q}_a$ follows from part 3) of Lemma 3.1. $\square$

Next, we treat the optimal case when $\gamma = \sigma_{k+1}(\mathbf{R})$. Complications arise because Z in Theorem 4.2 becomes singular. The next result gives conditions under which an allpass embedding of $\mathbf{R}$ will act as a generator of $\mathcal{S}$.

Lemma 4.3 [5]: For $\mathbf{R}$ and k defined in Problem 4.1, suppose that $\gamma = \sigma_{k+1}(\mathbf{R})$ has multiplicity r and that there exists a γ -allpass embedding $\mathbf{H}^o \in \mathcal{RL}_\infty$ such that

$$\mathbf{H}^o = \begin{matrix} m & p-l \\ p & \left[R + Q_{11}^o \quad Q_{12}^o \right] \\ m-l & \left[Q_{21}^o \quad Q_{22}^o \right] \end{matrix}$$

$$\mathbf{Q}_a^o = \begin{bmatrix} Q_{11}^o & Q_{12}^o \\ Q_{21}^o & Q_{22}^o \end{bmatrix} \in \mathcal{RH}_\infty^-(k), (Q_{12}^o)^l, (Q_{21}^o)^r \in \mathcal{RH}_\infty^-$$

where $l \leq r$ and where $(\cdot)^l$ and $(\cdot)^r$ denote left and right inverses, respectively. Then

$$\mathcal{S} = \mathcal{F}_l(\mathbf{H}^o, \mathcal{B}_{\gamma^{-1}} \mathcal{H}_\infty^{-(p-l) \times (m-l)}). \quad (18)$$

In order to give a construction for such an allpass embedding of $\mathbf{R}$, we need the following result which gives some properties of balanced realizations of discrete-time systems.

Lemma 4.4:

- 1) For $\mathbf{R}$ and γ defined in Lemma 4.3 there exists a balanced realization

$$\mathbf{R} \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array} \right] = \begin{matrix} n-r & r & m \\ r & \left[\begin{array}{ccc} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ C_1 & C_2 & 0 \end{array} \right] \\ p & & \end{matrix}$$

such that

$$P_1 = Q_1 = \text{diag}(\Sigma_1, \gamma I_r) \\ Z = \text{diag}(Z_1, 0_r) := \text{diag}(\Sigma_1^2 - \gamma^2 I_{n-r}, 0_r) \quad (19)$$

with $\delta_c(\Sigma_1 - \gamma I_{n-r}) = 0$. Furthermore, $\rho(A_{11}) < 1$ and $\rho(A_{22}) < 1$.

- 2) Assume that part 1) is satisfied and introduce the following partitions into (15):

$$A_o = Q_1 A_i P_1 - \gamma^2 A' A_i' = A_i' (A' Z' + C' C A_i P_1) \\ =: \begin{matrix} n-r & r \\ n-r & \left[\begin{array}{cc} A_{11}^o & A_{12}^o \\ A_{21}^o & A_{22}^o \end{array} \right] \end{matrix} \quad (20)$$

$$B_o = \begin{matrix} m & p \\ n & [Q_1 A_i B \quad \gamma A_i' C'] \end{matrix} =: \begin{matrix} m & p \\ n-r & \left[\begin{array}{cc} B_{11} & B_{12} \\ B_{21} & B_{22} \end{array} \right] \end{matrix}$$

$$C_o = \begin{matrix} n \\ p & \left[\begin{array}{cc} C A_i P_1 \\ \gamma B' A_i' \end{array} \right] \end{matrix} =: \begin{matrix} n-r & r \\ p & \left[\begin{array}{cc} C_{11} & C_{12} \\ C_{21} & C_{22} \end{array} \right] \end{matrix}. \quad (21)$$

Then

$$\begin{aligned}\gamma A_i B B' A_i' &= \gamma^{-1} \begin{bmatrix} C_{21}' C_{21} & C_{21}' C_{22} \\ C_{22}' C_{21} & C_{22}' C_{22} \end{bmatrix} \\ &= \begin{bmatrix} * & A_{21}^{o'} \\ A_{21}^o & A_{22}^o \end{bmatrix} \geq 0 \\ \gamma A_i' C' C A_i &= \gamma^{-1} \begin{bmatrix} B_{12} B_{12}' & B_{12} B_{22}' \\ B_{22} B_{12}' & B_{22} B_{22}' \end{bmatrix} \\ &= \begin{bmatrix} * & A_{12}^o \\ A_{12}^{o'} & A_{22}^o \end{bmatrix} \geq 0 \\ \gamma^{-1} B_{21} B_{21}' &= \gamma^{-1} C_{12}' C_{12} = \gamma^{-1} B_{22} B_{22}' \\ &= \gamma^{-1} C_{22}' C_{22} = A_{22}^o \geq 0.\end{aligned}\quad (22)$$

Proof: See [9] for part 1). Part 2) follows from direct calculations. $\square$

Remark 4.1: It can be easily shown, using the semidefinite character of (22) to (23), that without loss of generality we can assume that A_{22}^o in (20) is nonsingular. It follows from (23) that B_{21} and C_{12} have full rank. By invoking a simple scaling argument, it can be assumed that they are parts of orthogonal matrices. Hence there exist matrices $B_{\perp} \in \mathcal{R}^{(m-r) \times m}$ and $C_{\perp} \in \mathcal{R}^{p \times (p-r)}$ such that $[B_{21}' B_{\perp}']'$ and $[C_{12} C_{\perp}]$ are orthogonal.

The solution of the optimal problem will proceed as follows. First we let $\gamma \rightarrow \sigma_{k+1}(\mathbf{R})$ in the embedding of Theorem 4.2. This introduces pole-zero cancellations at ∞ and a loss of rank in B_{Q_2} and C_{Q_2} .

Theorem 4.5: Let $\mathbf{R}, k, \gamma$ and all symbols be as defined in Lemma 4.4. Assume that A_{22}^o is nonsingular. Then Problem 4.1 has the solution (18) with (w), as shown at the bottom of the page. The order of the realization of $\mathbf{Q}_a^o$ can be chosen equal to $2(n-r) - \text{rank}(A_{11}^o - A_{12}^o A_{22}^{o-1} A_{21}^o)$. Finally, if $(A_{11}^o - A_{12}^o A_{22}^{o-1} A_{21}^o)$ is nonsingular, $\mathbf{Q}_a^o$ has a standard realization of order $n-r$.

Proof: Since $\delta_c(\Sigma_1 - \gamma I_{n-r}) = 0$, Z_1 is nonsingular and

$$\pi_c(\Sigma_1 Z_1^{-1}) = \pi_c(\Sigma_1 Z_1) = k. \quad (24)$$

Substituting (19)–(21) into (15) and removing the nonminimal modes corresponding to $z = -1$

$$\mathbf{Q}_a^o \stackrel{s}{=} \begin{bmatrix} I & 0 & 0 & -I & -Z_1' & 0 & 0 & 0 \\ 0 & 0 & 0 & I & A_{11}^o & A_{12}^o & B_{11} & B_{12} \\ 0 & 0 & 0 & 0 & A_{21}^o & A_{22}^o & B_{21} & B_{22} \\ * & * & * & 0 & C_{11} & C_{12} & D_{11} & \gamma I \\ * & * & * & 0 & C_{21} & C_{22} & \gamma I & 0 \end{bmatrix}.$$

Using a procedure similar to that employed in part 3) of Lemma 3.1 and applying a basis change

$$\begin{aligned}T_l &= \text{diag} \left(I, \begin{bmatrix} I & -A_{12}^o A_{22}^{o-1} \\ 0 & I \end{bmatrix} \right) \\ T_r &= \text{diag} \left(I, \begin{bmatrix} I & 0 \\ -A_{22}^{o-1} A_{21}^o & I \end{bmatrix} \right)\end{aligned}$$

and we have (x), also shown at the bottom of the page.

We will now show that this defines a unique γ -allpass transfer function. Let

$$\begin{aligned}P_{\mathcal{H}} &= \left[\begin{array}{cc|cc} \Sigma_1 & 0 & Z_1 & -I \\ 0 & \gamma & 0 & 0 \\ \hline Z_1' & 0 & \Sigma_1 Z_1 & 0 \\ -I & 0 & 0 & 0 \end{array} \right] \\ X_{\mathcal{H}} &= \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -I & 0 & 0 & -\Sigma_1 \end{array} \right].\end{aligned}\quad (25)$$

A simple calculation verifies that $\det(A_{\mathcal{H}} P_{\mathcal{H}} + X_{\mathcal{H}}') = -\gamma^2 \det(\Sigma_1 Z_1) \det(I + A_{11}) \det(A_i') \neq 0$, since, Σ_1, Z_1 and $(I + A_{11})$ are nonsingular. Let

$$\begin{aligned}H &= \left[\begin{array}{cc|cc} \mathbf{R} + \mathbf{Q}_{11} & \mathbf{Q}_{12} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} \end{array} \right] \\ &\stackrel{s}{=} \left[\begin{array}{cccc|cc} I & 0 & A & 0 & B & 0 \\ 0 & E_Q & 0 & A_Q & B_{Q_1} & B_{Q_2} \\ * & * & C & C_{Q_1} & D_{11} & D_{12} \\ * & * & 0 & C_{Q_2} & D_{21} & D_{22} \end{array} \right] \\ &= \left[\begin{array}{cccc|cc} I & 0 & 0 & A & 0 & 0 \\ 0 & I & 0 & 0 & -I & -Z_1' \\ 0 & 0 & 0 & 0 & I & A_o \\ * & * & * & C & 0 & C_1^o \\ * & * & * & 0 & 0 & C_2^o \end{array} \right] \left[\begin{array}{cc} B & 0 \\ 0 & 0 \\ B_1^o & B_2^o \\ D_{11}^o & D_{12}^o \\ D_{21}^o & D_{22}^o \end{array} \right].\end{aligned}$$

It can be verified that the descriptor allpass (3) and (4) are satisfied so that this defines a unique γ -allpass transfer function from Lemma 3.1. We prove that $\mathbf{Q}_a^o \in \mathcal{RH}_{\infty}^-$ by showing that $\nu_d(E_Q, A_Q) \leq k$. Setting $T_r = \begin{bmatrix} A^o & -I \\ -I & 0 \end{bmatrix}$

$$\begin{aligned}\nu_d(E_Q, A_Q) &= \nu_d(E_Q T_r, A_Q T_r) \\ &= \nu_d \left(\begin{bmatrix} A^o & -I \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_1 & I \\ 0 & -I \end{bmatrix} \right) = \nu_d(A^o, A_1)\end{aligned}$$

where $A_1 := Z_1' - A^o$. Using the (3, 3) block of (3) verifies that $A^o \Sigma_1 Z_1^{-1} A^o - A_1 \Sigma_1 Z_1^{-1} A_1' = B_1^o B_1^{o'} + B_2^o B_2^{o'}$. It follows

$$\mathbf{Q}_a^o \stackrel{s}{=} \left[\begin{array}{cc|cc} E_Q^o & A_Q^o & B_{Q_1}^o & B_{Q_2}^o \\ * & C_{Q_1}^o & D_{11}^o & D_{12}^o \\ * & C_{Q_2}^o & D_{21}^o & D_{22}^o \end{array} \right] = \left[\begin{array}{cccc|cc} I & 0 & -I & -Z_1' & 0 & 0 \\ 0 & 0 & I & A_{11}^o - A_{12}^o A_{22}^{o-1} A_{21}^o & B_{11} - A_{12}^o A_{22}^{o-1} B_{21} & B_{12} - A_{12}^o A_{22}^{o-1} B_{22} \\ * & * & 0 & C_{11} - C_{12} A_{22}^{o-1} A_{21}^o & D_{11} - C_{12} A_{22}^{o-1} B_{21} & \gamma C_{\perp} \\ * & * & 0 & B_{\perp} C_{21} & \gamma B_{\perp} & 0 \end{array} \right] \quad (w)$$

$$\begin{aligned}\mathbf{Q}_a^o &\stackrel{s}{=} \left[\begin{array}{cccc|cc} I & 0 & -I & -Z_1' & 0 & 0 \\ 0 & 0 & I & A_{11}^o - A_{12}^o A_{22}^{o-1} A_{21}^o & B_{11} - A_{12}^o A_{22}^{o-1} B_{21} & B_{12} - A_{12}^o A_{22}^{o-1} B_{22} \\ * & * & 0 & C_{11} - C_{12} A_{22}^{o-1} A_{21}^o & D_{11} - C_{12} A_{22}^{o-1} B_{21} & \gamma I - C_{12} A_{22}^{o-1} B_{22} \\ * & * & 0 & C_{21} - C_{22} A_{22}^{o-1} A_{21}^o & \gamma I - C_{22} A_{22}^{o-1} B_{21} & -C_{22} A_{22}^{o-1} B_{22} \end{array} \right] \\ &\stackrel{s}{=} \left[\begin{array}{cccc|cc} I & 0 & -I & -Z_1' & 0 & 0 \\ 0 & 0 & I & A^o & B_1^o & B_2^o \\ * & * & 0 & C_1^o & D_{11}^o & D_{12}^o \\ * & * & 0 & C_2^o & D_{21}^o & D_{22}^o \end{array} \right] \quad (x)\end{aligned}$$

from (24) and part 2) of Lemma 3.1 that $\nu_d(E_Q, A_Q) \leq k$, since $A^\circ + A_1 = Z_1'$ and $\Sigma_1 Z_1'^{-1}$ are nonsingular. A simple calculation using (20) and (21) shows that $B_{21}Q_{22}C_{12} = -\gamma I_r$. Consider the identity

$$\begin{bmatrix} I & 0 \\ 0 & B_\perp \\ 0 & B_{21} \end{bmatrix} \begin{bmatrix} R + Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & C_\perp & C_{12} \end{bmatrix} \\ = \begin{bmatrix} R + Q_{11} & Q_{12}C_\perp & Q_{12}C_{12} \\ B_\perp Q_{21} & B_\perp Q_{22}C_\perp & B_\perp Q_{22}C_{12} \\ B_{21}Q_{21} & B_{21}Q_{22}C_\perp & -\gamma I_r \end{bmatrix}.$$

Since the left-hand side is γ -allpass, it follows that

$$H^\circ := \begin{bmatrix} R + Q_{11} & Q_{12}C_\perp \\ B_\perp Q_{21} & B_\perp Q_{22}C_\perp \end{bmatrix} =: \begin{bmatrix} R + Q_{11}^\circ & Q_{12}^\circ \\ Q_{21}^\circ & Q_{22}^\circ \end{bmatrix} \quad (26)$$

is γ -allpass. Note that (3) and (4) are still satisfied with P_H and X_H defined in (25). It only remains for us to show that Q_{12}° and Q_{21}° have left and right inverses, respectively, in $\mathcal{RH}_\infty^-$. It can be verified, using (26) and the allpass equations, that we have (y), shown at the bottom of the page, since $C_\perp' C_\perp = I$. Multiplying $\hat{E}$ and $\hat{A}$ on the right by

$$\begin{aligned} \hat{T}_r &= \begin{bmatrix} (I + A'_{11})^{-1} A'_{11} Z' & -I \\ -I & 0 \end{bmatrix} \\ \nu_d(\hat{E}, \hat{A}) &= \nu_d(\hat{E}\hat{T}_r, \hat{A}\hat{T}_r) \\ &= \nu_d((I + A'_{11})^{-1} A'_{11} Z', (I + A'_{11})^{-1} Z') \\ &= \nu_d(A'_{11}, I) \end{aligned}$$

since Z_1' and $I + A'_{11}$ are nonsingular. It now follows from part 1) of Lemma 4.4 that $(Q_{12}^\circ)' \in \mathcal{RH}_\infty^-$. A similar procedure can be used to prove that $(Q_{21}^\circ)' \in \mathcal{RH}_\infty^-$. The order of the realization Q_a° follows from part 3) of Lemma 3.1.

V. EXAMPLES

This section gives three examples to illustrate the algorithms presented earlier. The first shows that a causal system with poles at the origin does not necessarily lead to an anticausal generator. Let

$$R = \frac{1}{z} \stackrel{s}{=} \left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array} \right]$$

and let $\gamma > \sigma_1(R) = 1$. By applying the algorithm of Theorem 4.2

$$Q_a \stackrel{s}{=} \left[\begin{array}{ccc|c} 1 & 0 & -1 & \gamma^2 - 1 \\ 0 & 0 & 1 & 1 \\ * & * & 0 & 1 \\ * & * & 0 & \gamma \end{array} \middle| \begin{array}{c} 0 \\ 1 \\ 1 \\ \gamma \end{array} \right].$$

Since $A_o = 1$ we use the procedure of part 3) of Lemma 3.1 to obtain the following causal generator:

$$Q_a \stackrel{s}{=} \left[\begin{array}{c|cc} -\gamma^2 & 1 - \gamma^2 & \gamma(1 - \gamma^2) \\ \hline -1 & 0 & 0 \\ -\gamma & 0 & -\gamma^2 \end{array} \right] \\ = \frac{1}{z + \gamma^2} \left[\begin{array}{cc|c} \gamma^2 - 1 & \gamma(\gamma^2 - 1) & \\ \hline \gamma(\gamma^2 - 1) & -\gamma^2(z + 1) & \end{array} \right] \in \mathcal{RH}_\infty^-.$$

Note that at optimality ($\gamma = 1$), $Q = 0$ is the unique Nehari extension of R .

The next example shows that a singular A_o leads to an anticausal generator. Let

$$R = \frac{3}{z(z - \frac{1}{2})} \stackrel{s}{=} \left[\begin{array}{cc|c} .5 & 3 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 0 & 0 \end{array} \right], \quad \gamma = 2\sqrt{6} > \sigma_1(R).$$

It can be checked that $A_o = \begin{bmatrix} 8/3 & -32 \\ -2/3 & 8 \end{bmatrix}$ is singular. Applying the algorithm of Theorem 4.2

$$Q_a = \left[\begin{array}{cc|c} -6/(z + 4) & -2\sqrt{6}(z - 2)/(z + 4) & \\ \hline -2\sqrt{6}(z - 2)/(z + 4) & -12z(z + 1)/(z + 4) & \end{array} \right].$$

In the final example, we illustrate the solution of the optimal problem. Let

$$R = \left[\begin{array}{cc|c} \sqrt{3}/z(z - 0.5) & \sqrt{2}/z & \\ \hline -\sqrt{3}/z(z - 0.5) & \sqrt{2}/z & \end{array} \right] \\ \stackrel{s}{=} \left[\begin{array}{ccc|cc} 0 & 0 & 0 & 0 & 2 \\ 0 & \frac{1}{2} & 3 & 0 & 0 \\ 0 & 0 & 0 & 1/\sqrt{6} & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array} \right], \quad \gamma = 2 = \sigma_1(R).$$

Here

$$A_o = \begin{bmatrix} 24 & 0 & 0 \\ 0 & 8/3 & -2/3 \\ 0 & -32 & 8 \end{bmatrix}$$

is singular. Applying the algorithm of Theorem 4.5, we have (z), shown at the top of the page.

VI. CONCLUSIONS

A general algorithm is presented for the solution of the one-block discrete-time Nehari approximation problem. The form of the solution depends on the parameters of the problem in a simple and direct way. An approach using properties of descriptor systems is used, and it is shown that this approach lends itself naturally to the solution of the discrete-time Nehari approximation problem. In the development of the solution we have made no assumptions regarding the poles of R .

$$(Q_{12}^\circ)' \stackrel{s}{=} \left[\begin{array}{cc|c} \hat{E} & \hat{A} & \hat{B} \\ \hline * & \hat{C} & \hat{D} \end{array} \right] = \left[\begin{array}{ccc|cc} I & 0 & -I & -Z_1' & 0 \\ 0 & 0 & I & (I + A'_{11})^{-1} A'_{11} Z_1' & -\gamma(I + A'_{11})^{-1} C_1' \\ * & * & 0 & \gamma^{-1} C_\perp' C_{Q_1} & \gamma^{-1} C_\perp' \end{array} \right] \quad (y)$$

$$Q_a = \left[\begin{array}{cc|c} \sqrt{2} \left[z - \left(2 + \frac{\sqrt{3}}{\sqrt{2}} \right) \right] / (z + 4) & -2\sqrt{3} \left[z^2 + \left(1 - \frac{1}{\sqrt{6}} \right) z + \frac{2}{\sqrt{6}} \right] / (z + 4) & \\ \hline \sqrt{2} \left[z - \left(2 - \frac{\sqrt{3}}{\sqrt{2}} \right) \right] / (z + 4) & -2\sqrt{3} \left[z^2 + \left(1 + \frac{1}{\sqrt{6}} \right) z - \frac{2}{\sqrt{6}} \right] / (z + 4) & \end{array} \right] \quad (z)$$

In particular, the algorithm applies even if $\mathbf{R}$ has poles at the origin. As Theorems 4.2 and 4.5 show, the generator will have standard state-space realization if A_o is nonsingular.

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On an Open Problem Related to the Strict Local Minima of Multilinear Objective Functions

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Abstract—This paper gives a combinatorial proof of a "yes" answer to an open question presented in [1], stated as follows: "given a multilinear polynomial $E(\mathbf{x}): [0, 1]^n \rightarrow \mathbb{R}$, is it true that $E_{\mathbf{b}}(\mathbf{x}) = E(\mathbf{x}) - \mathbf{b}^t \mathbf{x}$ has a strict local minimum over the discrete set $\{0, 1\}^n$ for almost all $\mathbf{b}$ of sufficiently small norm?" The given combinatorial proof is completed directly by providing a sufficient condition for a conjecture on the strict local minima of multilinear polynomials also postulated in [1] to hold. In addition, a simple counterexample is presented to demonstrate that the conjecture may be not true if the provided sufficient condition is not satisfied.

Index Terms—Analog neural networks, discrete optimization problems, multilinear polynomials, objective functions.

I. INTRODUCTION

In [1] the author has given a strict and elegant analysis of minimum-seeking properties of analog neural networks with multilinear objective functions over the discrete set $\{0, 1\}^n$. These analysis results provided a theoretical foundation of the analog and neural approach to discrete optimization problem. A typical discrete optimization problem is to minimize $E(\mathbf{x})$ as $\mathbf{x} = (x_1, x_2, \dots, x_n)^t$

varies over the discrete set $\{0, 1\}^n$, where $E: [0, 1]^n \rightarrow \mathbb{R}$ is a multilinear polynomial. An interesting problem is as follows: under what conditions does E have a strict local minimum over $\{0, 1\}^n$, which is important for the optimization network with E as its objective function to work well [1]? It has been demonstrated in [1] by the example $E(\mathbf{x}) = x_1 x_2 \cdots x_n$ that a multilinear polynomial need not have any strict local minima. To overcome this difficulty, a bias vector $\mathbf{b} = (\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n)^t$ was introduced, and a modified multilinear function $E_{\mathbf{b}}(\mathbf{x}) = E(\mathbf{x}) - \mathbf{b}^t \mathbf{x}$ was used as the augmented objective function [1]. Therefore, an interesting open question was proposed as follows [1]: "given a multilinear polynomial $E(\mathbf{x}): [0, 1]^n \rightarrow \mathbb{R}$, is it true that $E_{\mathbf{b}}(\mathbf{x})$ has a strict local minimum over the discrete set $\{0, 1\}^n$ for almost all $\mathbf{b}$ of sufficiently small norm?" As was pointed out in [1], the answer to this question as stated is "yes," which can be proved from the obtained analysis results in [1]. The proof, however, is very indirect and unsatisfactory. The question is basically combinatorial in nature, and the answer should therefore have a combinatorial proof [1]. In this paper, we will give a combinatorial proof as expected. Interestingly, the given combinatorial proof can be completed directly by providing a sufficient condition for a conjecture on the strict local minima of multilinear polynomials also postulated in Appendix B of [1] to hold, which was regarded as going slightly beyond the above open question in [1]. It will be demonstrated by a simple counterexample that the conjecture may be not true if the provided sufficient condition is not satisfied.

II. A COMBINATORIAL PROOF OF THE OPEN QUESTION

We need the definitions and lemmas stated as follows.

Definition 2.1: A function $E: \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be a multilinear polynomial if it can be written as the form

$$\begin{aligned} E(\mathbf{x}) = & a_0 + \sum_{i_1=1}^n a_{i_1} x_{i_1} + \sum_{i_1 < i_2=1}^n a_{i_1 i_2} x_{i_1} x_{i_2} \\ & + \cdots + \sum_{i_1 < \cdots < i_{n-1}=1}^n a_{i_1 \cdots i_{n-1}} x_{i_1} \cdots x_{i_{n-1}} \\ & + a_{1 \cdots n} x_1 \cdots x_n \end{aligned}$$

where $a_0, a_{i_1}, a_{i_1 i_2}, \dots, a_{i_1 \cdots i_{n-1}}$ and $a_{1 \cdots n}$ are real numbers.

Definition 2.2: A vector $\mathbf{x} \in \{0, 1\}^n$ is said to be a local minimum of the objective function E if

$$E(\mathbf{x}) \leq E(\mathbf{y}), \quad \text{for all } \mathbf{y} \in N(\mathbf{x})$$

where $N(\mathbf{x})$ denotes the set of all vectors in $\{0, 1\}^n$, lying at a Hamming distance of one from $\mathbf{x}$. $\mathbf{x}$ is said to be a *strict* local minimum of E if

$$E(\mathbf{x}) < E(\mathbf{y}), \quad \text{for all } \mathbf{y} \in N(\mathbf{x}).$$

Definition 2.3: A sequence $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k\}$ is said to be a chain in $\{0, 1\}^n$ if $\mathbf{x}_{i+1} \in N(\mathbf{x}_i)$ for $i = 1, \dots, k-1$. A set $S \subseteq \{0, 1\}^n$ is said to be connected if there is a chain between every pair of points in S .

Lemma 2.1: Let $M \subseteq \{0, 1\}^n$ be the set of local minima of the objective function E . Then M can be divided into its connected

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