- [18] L. K. Jones, "A simple lemma on greedy approximation in Hilbert space and convergence rates for projection pursuit regression and neural network training," *Annals of Statistics*, vol. 20, pp. 608–613, 1992.
- [19] F. Girosi, "Regularization theory, radial basis functions and networks," in *From Statistics to Neural Networks. Theory and Pattern Recognition Applications*. ser. Subseries F, Computer and Systems Sciences, J. H. Friedman, V. Cherkassky, and H. Wechsler, Eds. Berlin, Germany: Springer-Verlag, 1993.
- [20] L. Breiman, "Hinging hyperplanes for regression, classification, and function approximation," *IEEE Trans. Inform. Theory*, vol. 39, pp. 993–1013, 1993.
- [21] H. J. Kushner and G. G. Yin, *Stochastic Approximation Algorithms and Applications*. New York: Springer-Verlag, 1997.
- [22] M. Deng and Y.-C. Ho, "An ordinal optimization approach to optimal control problems," *Automatica*, vol. 35, pp. 331–338, 1999.
- [23] J. T. Lee, E. Lau, and Y.-C. Ho, "The Witsenhausen counterexample: A hierarchical search approach for nonconvex optimization problems," *IEEE Trans. Automatic Control*, vol. 46, pp. 382–397, 2001, to be published.

Normalized $\mathcal{H}_\infty$ Controller Reduction With *A Priori* Error Bounds

Haitham M. H. El-Zobaidi, Imad M. Jaimoukha, and
David J. N. Limebeer

Abstract—This note describes an approach to the reduction of controllers for the normalized coprime factor robustness problem as well as the normalized $\mathcal{H}_\infty$ problem. It is shown that a relative error approximation of a coprime factor representation of any suboptimal controller leads to a stability guarantee and an upper bound on the performance degradation when the reduced order controller is implemented. When the approximation is performed on the controller generator, guaranteed *a priori* stability and performance bounds are obtained in terms of the synthesis Riccati equation solutions of the normalized $\mathcal{H}_\infty$ control problems.

Index Terms—Controller reduction, $\mathcal{H}_\infty$ control.

I. INTRODUCTION

This note considers the order reduction of controllers for the normalized $\mathcal{H}_\infty$ design techniques of [11] and [12]. We show that if the inverse-weighted balanced truncation technique of [15] is applied to the generator of all (suboptimal) normalized $\mathcal{H}_\infty$ controllers, one may induce a reduction on the coprime factors of the central controller. When this reduced order controller is implemented, we obtain guaranteed *a priori* stability and performance degradation bounds (*a priori* in the sense that they depend only on the performance level and the solutions of the synthesis Riccati equations). Despite the intense interest in obtaining reduced order $\mathcal{H}_\infty$ controllers with guaranteed *a priori* bounds (in the above sense) these efforts have hitherto only been partially successful.

Generally, linear quadratic Gaussian (LQG) and $\mathcal{H}_\infty$ controllers are of high order and are therefore impractical to implement [1] and

approximating such controllers is usually treated as an independent problem. The approximation approach is to apply one of the well known model reduction techniques to the controller, and then infer *a posteriori* the impact of the reduction process on closed-loop stability and performance. While the use of the synthesis Riccati equation solutions has already been suggested [10], [12], these ideas have not so far been translated into a method with an underpinning theoretical basis. For the normalized LQG and $\mathcal{H}_\infty$ problems, controller reduction based on balancing with respect to the synthesis Riccati equation solutions are known to work in practice, but no *a priori* bounds have been provided which justify the method.

This note is a generalization of the controller reduction procedure in [9] where the filtering and control Riccati equation solutions for the normalized LQG problem are shown to be equal to the controllability and observability Gramians of the left and right coprime factorizations of the plant and controller, respectively. The approach is based on solving a normalized LQG problem and then using an *a posteriori* bound on the $\mathcal{H}_\infty$ norm of the normalized closed-loop transfer matrix to obtain a condition for closed-loop stability and an upper bound on the performance degradation. A major contribution of [9] is to emphasize the relevance of relative controller reduction to closed-loop controller approximation when the controller is represented in coprime factor form.

Motivated by [9], model and controller reduction procedures were developed in [5] for multi-objective synthesis problems using linear matrix inequality (LMI) techniques. Guaranteed *a priori* bounds were obtained using augmented coprime factorizations of the model and controller that utilized the closed-loop Riccati inequalities. These bounds were used to derive conditions on stability and performance degradation similar to the ones developed here.

Related to this work is that in [4] and [8] where the emphasis is on performance preserving controller reduction, rather than bounding the performance degradation resulting from a controller reduction step. The coprime factor controller reduction problem was investigated in [16] where an additive reduction formulation was used to derive *a posteriori* conditions on the stability and performance degradation of the normalized closed-loop that are tighter than those in [11]. In contrast, our approach demonstrates that if the inverse-weighted balanced truncation technique is used, it is possible to obtain *a priori* conditions for stability and bounds on the performance degradation.

The notation used is standard and follows that of [17], except that real rational transfer functions will be represented by boldface letters with the dependence on s mostly suppressed.

II. PRELIMINARIES

In this section, we review the relative error model reduction problem [15]. For a transfer matrix G , the relative approximation of G is defined as $\hat{G} = (I + \Delta)G$ where Δ is the relative error function. This problem is a special case of the frequency-weighted model reduction problem [7] where the model is frequency weighted using its own inverse. In the sequel, we make extensive use of the following result, which is a generalization of the inverse-weighted model reduction method of [15], and gives *a priori* error bound for the relative model approximation.

Theorem II.1: Let $G \stackrel{s}{=} (A, B, C, D) \in \mathcal{RH}_\infty$ be square and minimum phase and assume also that $\det D \neq 0$ so that $G^{-1} \stackrel{s}{=} (A - BD^{-1}C, BD^{-1}, -D^{-1}C, D^{-1})$ is defined. Let $P = P' \geq 0$ and

Manuscript received August 10, 1999; revised June 26, 2000 and November 29, 2000. Recommended by Associate Editor S. Hara.

The authors are with the Center for Process Systems Engineering and the Department of Electrical and Electronic Engineering, Imperial College of Science, Technology and Medicine, London SW7 2BY, U.K. (e-mail: zobaidih@ps.ic.ac.uk; i.jaimoukha@ic.ac.uk; d.limebeer@ic.ac.uk).

Publisher Item Identifier S 0018-9286(01)08809-2.

Fig. 1. Normalized coprime factor feedback configuration.

$Q = Q' \geq 0$ be generalized controllability and observability Gramians of the realizations of G^{-1} and G , respectively, so that

$$\begin{aligned} (A - BD^{-1}C)P + P(A - BD^{-1}C)' \\ + (BD^{-1})(BD^{-1})' \leq 0 \\ A'Q + QA + C'C \leq 0. \end{aligned} \quad (1)$$

Suppose that the realization for G is inverse-weighted balanced, so that

$$\begin{aligned} P = Q = \text{diag}(\sigma_1 I_{s_1}, \dots, \sigma_r I_{s_r}, \sigma_{r+1} I_{s_{r+1}}, \dots, \sigma_N I_{s_N}) \\ = \text{diag}(\Sigma_1, \Sigma_2), \quad \sigma_1 > \dots > \sigma_N \geq 0 \end{aligned} \quad (2)$$

and that it is partitioned compatibly with Σ_1 and Σ_2 as

$$G \stackrel{s}{=} \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array} \right].$$

Then $\hat{G} \stackrel{s}{=} (A_{11}, B_1, C_1, D) = (I + \Delta)G$ is stable and minimum phase where $\Delta \in \mathcal{RH}_\infty$ satisfies $\|\Delta\|_\infty \leq \delta$ with

$$\delta := -1 + \prod_{i=r+1}^N \left(1 + 2\sigma_i \sqrt{1 + \sigma_i^2 + 2\sigma_i^2} \right) \quad (3).$$

Proof: The theorem is a modification of Lemma 2 in [15] (replacing (1) by equalities and G by G^T). ■

III. NORMALIZED COPRIME FACTOR $\mathcal{H}_\infty$ CONTROLLER REDUCTION

In this section, we derive guaranteed *a priori* stability and performance bounds when a normalized coprime factor $\mathcal{H}_\infty$ controller [11] is replaced by a reduced-order approximation.

Let G be a given plant and consider the normalized feedback configuration shown in Fig. 1 [11], [17]. The closed-loop transfer matrix from $[w_1^T w_2^T]^T$ to $[z_1^T z_2^T]^T$ is given by $\mathcal{H}(G, K) = [K^T I]^T (I - GK)^{-1} [-G I]$. For any $\gamma > \gamma_{\text{opt}} = \inf \{\|\mathcal{H}(G, K)\|_\infty : K \text{ stabilizes } G\}$, the suboptimal $\mathcal{H}_\infty$ control problem is to find all internally stabilizing controllers K such that $\|\mathcal{H}(G, K)\|_\infty \leq \gamma$. Any such K is called a γ -suboptimal controller. The next lemma summarizes the solution of the suboptimal problem [11], [17].

Lemma III.1: Let G have a minimal state-space realization $G \stackrel{s}{=} (A, B, C, 0)$. Then there exist unique, stabilizing, positive definite solutions X_2 and Y_2 to the algebraic Riccati equations

$$A'X_2 + X_2A - X_2BB'X_2 + C'C = 0 \quad (4)$$

$$AY_2 + Y_2A' - Y_2C'C Y_2 + BB' = 0 \quad (5)$$

respectively, and $\gamma_{\text{opt}} = \sqrt{1 + \lambda_{\max}(X_2 Y_2)} \geq 1$. For any $\gamma > \gamma_{\text{opt}} \geq 1$ define $\beta := \sqrt{1 - \gamma^{-2}}$ so that $0 < \beta \leq 1$ and $Z =$

$(I - \gamma^{-2}\beta^{-2}Y_2X_2)^{-1}$. Then all γ -suboptimal controllers are given by

$$\begin{aligned} K &= (\Theta_{11}Q + \Theta_{12})(\Theta_{21}Q + \Theta_{22})^{-1} \\ Q &\in \mathcal{RH}_\infty \quad \|Q\|_\infty < \gamma \end{aligned} \quad (6)$$

where

$$\begin{aligned} \Theta &= \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} \\ &\stackrel{s}{=} \left[\begin{array}{cc|cc} A - BB'X_2 & ZB & \beta^{-1}ZY_2C' & \\ -\beta^{-2}B'X_2 & I & 0 & \\ \hline -Y^{-2}C & 0 & \beta^{-1}I & \end{array} \right] \in \mathcal{RH}_\infty \quad (7) \\ \Theta^{-1} &\stackrel{s}{=} \left[\begin{array}{cc|cc} Z(A - Y_2C'C)Z^{-1} & ZB & ZY_2C' & \\ \hline \beta^{-2}B'X_2 & I & 0 & \\ -\beta^{-1}C & 0 & \beta I & \end{array} \right] \in \mathcal{RH}_\infty. \end{aligned} \quad (8)$$

Proof: The proof can be found in a slightly different form in [11]. The A -matrix for Θ^{-1} is obtained from the realization of Θ using (4) and (5) and the fact that $A - Y_2C'C = (I + Y_2X_2)(A - BB'X_2)(I + Y_2X_2)^{-1}$ [3]. ■

Remark III.1: Taking $Q = 0$ in (6) gives the central controller in the form of the right coprime factorization $K_{11} = \Theta_{12}\Theta_{22}^{-1} \stackrel{s}{=} (A - BB'X_2 - \beta^{-2}ZY_2C'C, ZY_2C', -\beta^{-2}B'X_2, 0)$.

The following result shows that we can perform a relative reduction on a coprime factorization of any γ -suboptimal controller K . It is similar to the one stated by [9], except that the coprime factorization of G is not required. It is also given in a form that allows a feedback interpretation of the relative error function.

Theorem III.1: Let K be a γ -suboptimal controller for G . Suppose that $K = UV^{-1}$ is a right coprime factorization of K and let $[\hat{U}^T \hat{V}^T]^T = (I + \Delta)[U^T V^T]^T$ be a relative approximation of $[U^T V^T]^T$. Define $\hat{K} = \hat{U}\hat{V}^{-1}$. Then

- 1) $\mathcal{H}(G, \hat{K}) = (I + \Delta)\mathcal{H}(G, K)[I + \Delta\mathcal{H}(G, K)]^{-1}$;
- 2) if $\Delta \in \mathcal{RH}_\infty$ and $\|\Delta\|_\infty < \gamma^{-1}$ then $\mathcal{H}(G, \hat{K}) \in \mathcal{RH}_\infty$ and $\|\mathcal{H}(G, \hat{K})\|_\infty < (\gamma/1 - \|\Delta\|_\infty\gamma)$.

Proof: This follows from [9] and the fact that

$$\begin{aligned} \mathcal{H}(G, K) - \mathcal{H}(G, \hat{K}) \\ &= \mathcal{H}(G, K) - (I + \Delta)\mathcal{H}(G, K)[I + \Delta\mathcal{H}(G, K)]^{-1} \\ &= [\mathcal{H}(G, K) - I]\Delta\mathcal{H}(G, K)[I + \Delta\mathcal{H}(G, K)]^{-1}. \end{aligned} \quad (9)$$

The effect of relative controller reduction on the closed loop is shown in Fig. 2. Note that the stability condition and the bound on performance degradations require the evaluation of $\|\Delta\|_\infty$. The following result gives an *a priori* bound on $\|\Delta\|_\infty$ by using the inverse-weighted balanced truncation method of Theorem II.1 on the controller generator Θ .

Theorem III.2: Let all variables be as defined in Lemma III.1. Let $P = P' \geq 0$ be the controllability Gramian of the realization of Θ^{-1} in (8) and let $Q = Q' \geq 0$ be the observability Gramian of the realization of Θ in (7) so that

$$\begin{aligned} [Z(A - Y_2C'C)Z^{-1}]P + P[Z(A - Y_2C'C)Z^{-1}]' \\ + ZBB'Z' + ZY_2C'C Y_2Z' = 0 \\ (A - BB'X_2)'Q + Q(A - BB'X_2) \\ + \beta^{-4}X_2BB'X_2 + \beta^{-4}C'C = 0. \end{aligned}$$

bottom of the page. The A -matrix for Θ^{-1} in (15) is obtained from the realization of Θ as

$$\begin{aligned} A - (1 - \gamma^{-2})BB'X_\infty + Z_\infty BB'X_\infty - Z_\infty Y_\infty C'C \\ = Z_\infty (A - (1 - \gamma^{-2})Y_\infty C'C) Z_\infty^{-1} \end{aligned}$$

where the second expression is obtained using the fact that

$$\begin{aligned} A - (1 - \gamma^{-2})^2 BB'X_\infty - Z_\infty Y_\infty C'C \\ = Z (A - BB'X_\infty Z_\infty - (1 - \gamma^{-2})Y_\infty C'C) Z_\infty^{-1}. \end{aligned}$$

■ Then

The implications of performing a relative reduction on a coprime factorization of any γ -suboptimal controller $\mathbf{K}$ is analyzed in the following theorem.

Theorem IV.1: Let $\mathbf{K}$ be any γ -suboptimal controller for $\mathbf{G}$. Suppose that $\mathbf{K} = \mathbf{U}\mathbf{V}^{-1}$ is a right coprime factorization of $\mathbf{K}$ and let $\begin{bmatrix} \hat{\mathbf{U}}^T \hat{\mathbf{V}}^T \end{bmatrix}^T = (\mathbf{I} + \Delta) [\mathbf{U}^T \mathbf{V}^T]^T$ be a relative approximation of $[\mathbf{U}^T \mathbf{V}^T]^T$. Define $\hat{\mathbf{K}} = \hat{\mathbf{U}}\hat{\mathbf{V}}^{-1}$. Then

- 1) $\check{\mathcal{H}}(\mathbf{G}, \hat{\mathbf{K}}) = [\check{\mathcal{H}}(\mathbf{G}, \mathbf{K}) - \check{\mathcal{J}}\Delta\mathcal{H}(\mathbf{G}, \mathbf{K})][\mathbf{I} + \Delta\mathcal{H}(\mathbf{G}, \mathbf{K})]^{-1}$ where $\check{\mathcal{J}} = \text{diag}(-\mathbf{I}, 0)$.
- 2) If $\|\Delta\|_\infty < \frac{(1 + \gamma)^{-1}}{(1 + \gamma)\|\Delta\|_\infty}$ and $\Delta \in \mathcal{RH}_\infty$ then $\check{\mathcal{H}}(\mathbf{G}, \hat{\mathbf{K}}) \in \mathcal{RH}_\infty$ and $\|\check{\mathcal{H}}(\mathbf{G}, \hat{\mathbf{K}})\|_\infty < (\gamma + (1 + \gamma)\|\Delta\|_\infty)/(1 - (1 + \gamma)\|\Delta\|_\infty)$.

Proof: The proof follows from Theorem III.1 on noting that $\check{\mathcal{H}}(\mathbf{G}, \mathbf{K}) = \mathcal{H}(\mathbf{G}, \mathbf{K}) + \mathbf{J}$ and

$$\begin{aligned} \check{\mathcal{H}}(\mathbf{G}, \mathbf{K}) - \check{\mathcal{H}}(\mathbf{G}, \hat{\mathbf{K}}) \\ = [\mathcal{H}(\mathbf{G}, \mathbf{K}) - \mathbf{I}]\Delta\mathcal{H}(\mathbf{G}, \mathbf{K})[\mathbf{I} + \Delta\mathcal{H}(\mathbf{G}, \mathbf{K})]^{-1} \\ = [\check{\mathcal{H}}(\mathbf{G}, \mathbf{K}) + \check{\mathcal{J}}]\Delta\mathcal{H}(\mathbf{G}, \mathbf{K})[\mathbf{I} + \Delta\mathcal{H}(\mathbf{G}, \mathbf{K})]^{-1}. \end{aligned}$$

■

For the remainder of this section, we assume that $\gamma > \max\{1, \gamma_{\text{opt}}\}$. This is not a severe restriction since $\gamma_{\text{opt}} < 1$ if and only if $\mathbf{G}$ is stable and has Hankel norm less than 1 [13]. The following result gives an *a priori* bound on $\|\Delta\|_\infty$ using the inverse-weighted balanced truncation method of Theorem II.1 on the generator Θ and is similar to Theorem III.2.

Theorem IV.2: Let all variables be as defined in Lemma IV.1. Suppose that $\gamma > \max\{1, \gamma_{\text{opt}}\}$ and define $\beta := \sqrt{1 - \gamma^{-2}}$ so that

$0 < \beta \leq 1$. Let $P = P' \geq 0$ be a generalized controllability Gramian of the realization of Θ^{-1} in (15) and $Q = Q' \geq 0$ be a generalized observability Gramian of the realization of Θ in (14) so that

$$\begin{aligned} [Z_\infty (A - \beta^2 Y_\infty C'C) Z_\infty^{-1}] P \\ + P [Z_\infty (A - \beta^2 Y_\infty C'C) Z_\infty^{-1}]' + Z_\infty BB' Z_\infty' \\ + Z_\infty Y_\infty C'C Y_\infty Z_\infty' \leq 0 \\ (A - \beta^2 BB'X_\infty)' Q + Q (A - \beta^2 BB'X_\infty) \\ + X_\infty BB'X_\infty + C'C \leq 0. \end{aligned}$$

- 1) the generalized Gramians P and Q can be taken as

$$P = \beta^{-2} Z_\infty Y_\infty Z_\infty' \quad (16)$$

$$Q = \beta^{-2} X_\infty; \quad (17)$$

- 2) suppose that the realization for Θ is inverse-weighted balanced, so that (2) holds, and that it is partitioned compatibly with Σ_1 and Σ_2 as (10). Then Θ in (11) is stable and minimum phase and $\hat{\Theta} = (\mathbf{I} + \Delta)\Theta$ where $\Delta \in \mathcal{RH}_\infty$ satisfies $\|\Delta\|_\infty \leq \delta$ and δ is defined in (3) with $\sigma_i^2 = \beta^{-4} \nu_i^2 (1 - \gamma^{-2} \nu_i^2)^{-2}$ where $\nu_1^2, \dots, \nu_N^2$ are the (distinct) eigenvalues of $X_\infty Y_\infty$.

Proof:

- 1) Rearranging (12) and multiplying by β^{-2} gives

$$\begin{aligned} (A - \beta^2 BB'X_\infty)' \beta^{-2} X_\infty + \beta^{-2} X_\infty (A - \beta^2 BB'X_\infty) \\ + X_\infty BB'X_\infty + \beta^{-2} C'C = 0. \end{aligned}$$

This proves (17) since $\beta^{-2} \geq 1$. Rearranging (13), pre- and post-multiplying by $\beta^{-1} Z_\infty$ and $\beta^{-1} Z_\infty'$, respectively

$$\begin{aligned} A_\infty \beta^{-2} Z_\infty Y_\infty Z_\infty' + \beta^{-2} Z_\infty Y_\infty Z_\infty' A_\infty' \\ + \beta^{-2} Z_\infty BB' Z_\infty' + Z_\infty Y_\infty C'C Y_\infty Z_\infty' = 0 \end{aligned}$$

where $A_\infty = Z_\infty (A - \beta^2 Y_\infty C'C) Z_\infty^{-1}$. This proves (16) since $\beta^{-2} \geq 1$.

- 2) This is a straightforward application of Theorem II.1. ■

Similar to Corollary III.1, we use Theorems III.1 and IV.2 to give a controller reduction procedure with guaranteed *a priori* stability and performance bounds.

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A - (1 - \gamma^{-2})BB'X_\infty & Z_\infty B & Z_\infty Y_\infty C' \\ \hline -B'X_\infty & I & 0 \\ C & 0 & I \end{array} \right] \in \mathcal{RH}_\infty \quad (14)$$

$$\Theta^{-1} \stackrel{s}{=} \left[\begin{array}{c|cc} Z_\infty (A - (1 - \gamma^{-2})Y_\infty C'C) Z_\infty^{-1} & Z_\infty B & Z_\infty Y_\infty C' \\ \hline B'X_\infty & I & 0 \\ -C & 0 & I \end{array} \right] \in \mathcal{RH}_\infty. \quad (15)$$

$$K_{11} = \Theta_{12} \Theta_{22}^{-1} \stackrel{s}{=} \left[\begin{array}{c|c} A - (1 - \gamma^{-2})BB'X_\infty - Z_\infty Y_\infty C'C & Z_\infty Y_\infty C' \\ \hline -B'X_\infty & 0 \end{array} \right].$$

TABLE I
 $\|\mathcal{H}(\mathbf{G}, \hat{\mathbf{K}})\|_\infty$: U -CLOSED-LOOP SYSTEM IS UNSTABLE

Order of $\hat{\mathbf{K}}$	5	4	3	2
$\ \mathbf{K} - \hat{\mathbf{K}}\ _\infty$	2.2963	2.2934	2.3707	1363.2
$\ [\Theta_{12}^T \ \Theta_{22}^T]^T - [\hat{\mathbf{U}}^T \ \hat{\mathbf{V}}^T]^T\ _\infty$	2.2952	2.2965	2.4130	16.8565
$\ [\hat{\Theta}_{21} \ \hat{\Theta}_{22}] - [\hat{\mathbf{U}} \ \hat{\mathbf{V}}]\ _\infty$	2.2952	2.2963	2.3780	14.7970
$\ (I - \mathbf{G}\mathbf{K})^{-1}\mathbf{G}(\mathbf{K} - \hat{\mathbf{K}})\ _\infty$	2.2963	2.2939	U	U
$\ [-\hat{\mathbf{N}} \ \hat{\mathbf{M}}]([\Theta_{12}^T \ \Theta_{22}^T]^T - [\hat{\mathbf{U}}^T \ \hat{\mathbf{V}}^T]^T)\ _\infty$	2.2951	2.2965	11.7788	7.8843
$\ ([\hat{\Theta}_{21} \ \hat{\Theta}_{22}] - [\hat{\mathbf{U}} \ \hat{\mathbf{V}}])[-\mathbf{N}^T \ \mathbf{M}^T]^T\ _\infty$	2.2952	2.2963	2.3792	35.7589
$\ \text{diag}(\gamma^{-1}I, I)\Theta^{-1}([\Theta_{12}^T \ \Theta_{22}^T]^T - [\hat{\mathbf{U}}^T \ \hat{\mathbf{V}}^T]^T)\ _\infty$	2.2952	2.2963	U	U
$\ ([\hat{\Theta}_{21} \ \hat{\Theta}_{22}] - [\hat{\mathbf{U}} \ \hat{\mathbf{V}}])\hat{\Theta}^{-1}\text{diag}(\gamma^{-1}I, I)\ _\infty$	2.2952	2.2963	U	U
$\ (\hat{\Theta} - \Theta)\Theta^{-1}\ _\infty$	2.2952	2.2963	U	U
$\ \Theta^{-1}(\hat{\Theta} - \Theta)\ _\infty$ (Our algorithm)	2.2963	2.2929	2.3592	7.5637

Corollary IV.1: Let all variables be as define in Theorem IV.2 and let $\mathbf{K}_{11} = \Theta_{12}\Theta_{22}^{-1}$ be the central controller. Suppose that T is an inverse-weighted balancing transformation for Θ so that

$$\begin{aligned} & \beta^{-2}T^{-1}Z_\infty Y_\infty Z_\infty' (T^{-1})' \\ &= \beta^{-2}T'X_\infty T \\ &= \text{diag}(\sigma_1 I_{s_1}, \dots, \sigma_r I_{s_r}, \sigma_{r+1} I_{s_{r+1}}, \dots, \sigma_N I_{s_N}) \\ &= \text{diag}(\Sigma_1, \Sigma_2) \end{aligned}$$

with $\sigma_1 > \dots > \sigma_N \geq 0$. Apply the similarity transformation T to $\mathbf{K}_{11}$ and partition compatibly with Σ_1 and Σ_2 as shown in the second equation at the bottom of the page. Define the reduced controller $\hat{\mathbf{K}}_{11} = \hat{\Theta}_{12}\hat{\Theta}_{22}^{-1} \triangleq (\hat{A}_{11}, \hat{B}_1, \hat{C}_1, 0)$. Then $\check{\mathcal{H}}(\mathbf{G}, \hat{\mathbf{K}}_{11}) \in \mathcal{RH}_\infty$ if $\delta < (1 + \gamma)^{-1}$ in which case

$$\|\check{\mathcal{H}}(\mathbf{G}, \hat{\mathbf{K}}_{11})\|_\infty < \frac{\gamma + (1 + \gamma)\delta}{1 - (1 + \gamma)\delta}.$$

Remark IV.1: There are two sources of conservatism in the stability condition and performance degradation bound for $\check{\mathcal{H}}(\mathbf{G}, \hat{\mathbf{K}})$ compared to $\mathcal{H}(\mathbf{G}, \hat{\mathbf{K}})$. First, the bounds that are derived are related to $(\gamma + 1)$ rather than γ . Second, the *a priori* bound on $\|\Delta\|_\infty$ uses the generalized Gramians P and Q which satisfy the inequalities of Theorem IV.2. However, in practice, that conservatism does not reflect on the quality of the controller approximation.

Remark IV.2: It is interesting to note that in the limit $\gamma \rightarrow \infty$, we have $X_\infty \rightarrow X_2, Y_\infty \rightarrow Y_2$ and $Z_\infty \rightarrow I$ [13]. Thus the normalized $\mathcal{H}_\infty$ control problem reduces to the normalized LQG problem. That is, $\mathbf{K}_{11}$ is the unique stabilizing controller that minimizes $\|\check{\mathcal{H}}(\mathbf{G}, \mathbf{K})\|_2$. When this controller is approximated using the method given in this

section, we can use Theorem IV.1 to write the stability condition as $1 + \|\check{\mathcal{H}}(\mathbf{G}, \mathbf{K}_{11})\|_\infty < \delta^{-1}$ and the performance bound as

$$\begin{aligned} & \|\check{\mathcal{H}}(\mathbf{G}, \hat{\mathbf{K}}_{11})\|_2 \\ & < \frac{\|\check{\mathcal{H}}(\mathbf{G}, \mathbf{K}_{11})\|_2 + \|\Delta\|_2(1 + \|\check{\mathcal{H}}(\mathbf{G}, \mathbf{K}_{11})\|_\infty)}{1 - (1 + \|\check{\mathcal{H}}(\mathbf{G}, \mathbf{K}_{11})\|_\infty)\delta} \end{aligned}$$

provided that $1 + \|\check{\mathcal{H}}(\mathbf{G}, \mathbf{K}_{11})\|_\infty < \delta^{-1}$, where $\|\check{\mathcal{H}}(\mathbf{G}, \mathbf{K}_{11})\|_\infty$ and $\|\Delta\|_2$ need to be calculated *a posteriori*. This can be considered as a justification for the LQG-based balance-and-truncate method of [10]. If $\|\mathcal{H}(\mathbf{G}, \mathbf{K}_{11})\|_\infty$ is calculated instead, then (9) shows that the stability condition becomes $\|\mathcal{H}(\mathbf{G}, \mathbf{K}_{11})\|_\infty < \delta^{-1}$ and the performance bound becomes

$$\|\check{\mathcal{H}}(\mathbf{G}, \hat{\mathbf{K}}_{11})\|_2 < \frac{\|\check{\mathcal{H}}(\mathbf{G}, \mathbf{K}_{11})\|_2 + \|\Delta\|_2\|\mathcal{H}(\mathbf{G}, \mathbf{K}_{11})\|_\infty}{1 - \|\mathcal{H}(\mathbf{G}, \mathbf{K}_{11})\|_\infty\delta}$$

which reproduces the controller reduction results of [9] (see [9, Th. 17]).

V. EXAMPLE

To demonstrate the effectiveness of the controller reduction procedure we consider synthesizing a reduced-order $\mathcal{H}_\infty$ controller for the HIMAT vehicle taken from the μ -Analysis and Synthesis Toolbox [2]. The model consists of two-input, two-output and four states that represents a scaled, linearized plant of a remotely piloted vehicle version of an advanced fighter. We used the same weighting functions for the loop shaping design suggested in [2].

For the normalized coprime factor design case we used $\gamma = 2.2952$ to synthesize a 6-th order $\mathcal{H}_\infty$ γ -suboptimal controller which is reduced using several methods and compared to the approach given here and the results are listed in Table I (see [4] and [17] for details of these

$$\mathbf{K}_{11} \triangleq \left[\begin{array}{c|c} T^{-1}(A - B^2 B B' X_\infty - Z_\infty Y_\infty C' C) T & T^{-1} Z_\infty Y_\infty C' \\ \hline -B' X_\infty T & 0 \end{array} \right] \triangleq: \left[\begin{array}{cc|c} \hat{A}_{11} & \hat{A}_{12} & \hat{B}_1 \\ \hat{A}_{21} & \hat{A}_{22} & \hat{B}_2 \\ \hline \hat{C}_1 & \hat{C}_2 & 0 \end{array} \right].$$

TABLE II
REDUCED ORDER CONTROLLER PERFORMANCE

Order of $\hat{K}$	5	4	3	2
$\ \Delta\ _\infty$	0.0005	0.0021	0.1961	0.9963
δ	0.0005	0.0026	0.2201	1.9340
Estimated $\ \mathcal{H}(G, \hat{K})\ _\infty$ using $\ \Delta\ _\infty$	2.2980	2.3065	4.1736	$\ \Delta\ _\infty \gamma > 1$
Estimated $\ \mathcal{H}(G, \hat{K})\ _\infty$ using δ	2.2980	2.3092	4.6382	$\gamma \delta > 1$
Actual $\ \mathcal{H}(G, \hat{K})\ _\infty$	2.2963	2.2929	2.3592	7.5637

TABLE III
 $\|\check{\mathcal{H}}(G, \hat{K})\|_\infty : U\text{-CLOSED-LOOP SYSTEM IS UNSTABLE}$

Order of $\hat{K}$	5	4	3	2
$\ K - \hat{K}\ _\infty$	2.3210	2.3182	2.3954	1146.52
$\ [\Theta_{12}^T \ \Theta_{22}^T]^T - [\hat{U}^T \ \hat{V}^T]^T\ _\infty$	2.3199	2.3212	2.4338	15.4278
$\ [\hat{\Theta}_{21} \ \hat{\Theta}_{22}] - [\hat{U} \ \hat{V}]\ _\infty$	2.3199	2.3210	2.4015	16.6340
$\ (I - GK)^{-1}G(K - \hat{K})\ _\infty$	2.3210	2.3186	U	U
$\ [-\hat{N} \ \hat{M}]([\Theta_{12}^T \ \Theta_{22}^T]^T - [\hat{U}^T \ \hat{V}^T]^T)\ _\infty$	2.3198	2.3212	U	7.4912
$\ ([\hat{\Theta}_{21} \ \hat{\Theta}_{22}] - [\hat{U} \ \hat{V}])[-N^T \ M^T]^T\ _\infty$	2.3199	2.3210	2.4031	69.6613
$\ \text{diag}(\gamma^{-1}I, I)\Theta^{-1}([\Theta_{12}^T \ \Theta_{22}^T]^T - [\hat{U}^T \ \hat{V}^T]^T)\ _\infty$	2.3199	2.3210	U	U
$\ ([\hat{\Theta}_{21} \ \hat{\Theta}_{22}] - [\hat{U} \ \hat{V}])\hat{\Theta}^{-1}\text{diag}(\gamma^{-1}I, I)\ _\infty$	2.3199	2.3209	U	U
$\ (\hat{\Theta} - \Theta)\Theta^{-1}\ _\infty$	2.3199	2.3210	U	U
$\ \Theta^{-1}(\hat{\Theta} - \Theta)\ _\infty$ (Our algorithm, using exact Gramians)	2.3210	2.3178	2.3869	8.1069
$\ \Theta^{-1}(\hat{\Theta} - \Theta)\ _\infty$ (Our algorithm, using generalized Gramians)	2.3210	2.3178	2.3843	8.1055

TABLE IV
REDUCED ORDER CONTROLLER PERFORMANCE

Order of $\hat{K}$	5	4	3	2
$\ \Delta\ _\infty$	0.0006	0.0026	0.2344	1.0844
δ	0.0006	0.0032	0.2686	2.3311
Estimated $\ \check{\mathcal{H}}(G, \hat{K})\ _\infty$ using $\ \Delta\ _\infty$	2.3272	2.3484	13.9601	$(\gamma + 1)\ \Delta\ _\infty > 1$
Estimated $\ \check{\mathcal{H}}(G, \hat{K})\ _\infty$ using δ	2.3272	2.3554	29.6997	$(\gamma + 1)\delta > 1$
Actual $\ \check{\mathcal{H}}(G, \hat{K})\ _\infty$	2.3210	2.3178	2.3843	8.1055

methods). Table I indicates that our approach seems to perform better than any of the other procedures. Table II gives $\|\Delta\|_\infty$, its upper bound δ , estimates for $\|\mathcal{H}(G, \hat{K})\|_\infty$ using $\|\Delta\|_\infty$ and δ , and compares the estimates with the actual value of $\|\mathcal{H}(G, \hat{K})\|_\infty$. The table demonstrates that the bounds suggested are good indicators for the quality of the truncation when the controller is moderately reduced. The higher the level of truncation, the more conservative the bound on performance degradation.

The design was repeated for the normalized $\mathcal{H}_\infty$ case with $\gamma = 2.32$ and Table III summarizes the results. Similar conclusions can be drawn as in the normalized coprime factors case for the quality of the controller reduction. Table IV indicates that when $\|\Delta\|_\infty$ or its bound δ is large, then the bound on performance degradation is quite conservative. Tables I and III show that, apart from providing *a priori* stability and error bounds, the quality of our approximation is at least as good as any of the other techniques. Using the energy propagation concept in-

roduced by [14], and applied to the closed-loop model reduction algorithm of [6], it can be shown that the eigenvalues of the Gramians used in our controller balancing and truncation have a controller input/output energy interpretation.

VI. CONCLUSION

It is shown that the Gramians (or generalized Gramians) for the normalized controller generator Θ are given in terms of the synthesis Riccati equation solutions. This is exploited to give an *a priori* upper bound on the relative error when the inverse-balanced truncation technique is used to approximate Θ . This approximation induces a relative reduction on the coprime factors of the normalized central controller. Combined with the bound on the relative approximation error, this gives guaranteed *a priori* bounds on closed-loop stability and performance degradation. A theoretical justification for controller reduction techniques based on balancing with respect to the synthesis Riccati solutions is thus established. The technique also provides an intimate link between the controller synthesis and controller reduction problems.

REFERENCES

- [1] B. D. O. Anderson and Y. Liu, "Controller reduction: Concepts and approaches," *IEEE Trans. Automat. Contr.*, vol. 34, pp. 802–812, Aug. 1989.
- [2] G. J. Balas, J. C. Doyle, K. Glover, A. Packard, and R. Smith, *μ -Analysis and Synthesis Toolbox*. Natick, MA, 1995.
- [3] R. Bucy, "The Riccati equation and its bounds," *J. Comput. Syst. Sci.*, vol. 6, pp. 343–353, 1972.
- [4] X. Chen and K. Zhou, " $\mathcal{H}_2$ and $\mathcal{H}_\infty$ controller reduction with guaranteed performance," in *Proc. IEEE Conf. Decision Control*, Kobe, Japan, 1996, pp. 903–904.
- [5] H. M. H. El-Zobaidi, "Model and Controller Reduction in Linear Robust Control," Ph.D. dissertation, Imperial College of Science, Technology, and Medicine, University of London, 1998.
- [6] H. M. H. El-Zobaidi and I. Jaimoukha, "Robust control and model and controller reduction of linear parameter varying systems," in *Proc. IEEE Conf. Decision Control*, 1998, pp. 3015–3020.
- [7] D. Enns, "Model reduction with balanced realizations: An error bound and a frequency weighted generalization," in *Proc. IEEE Conf. Decision Control*, Las Vegas, NV, Dec. 1984, pp. 127–132.
- [8] P. J. Goddard and K. Glover, "Controller approximation: Approaches for preserving $\mathcal{H}_\infty$ performance," *IEEE Trans. Automat. Contr.*, vol. 43, pp. 858–871, July 1998.
- [9] G. Gu, "Model reduction with relative/multiplicative error bounds and relations to controller reduction," *IEEE Trans. Automat. Contr.*, vol. 40, pp. 1478–1485, Aug. 1995.
- [10] E. A. Jonkheere and L. M. Silverman, "A new set of invariants for linear systems-application to reduced order compensator design," *IEEE Trans. Automat. Contr.*, vol. AC-28, pp. 953–964, Oct. 1983.
- [11] D. McFarlane and K. Glover, *Robust Controller Design Using Normalized Coprime Factor Plant Description*. Berlin, Germany: Springer-Verlag, 1990, vol. 138, Lecture Notes in Control and Information Sciences.
- [12] D. Mustafa and K. Glover, *Minimum Entropy $\mathcal{H}_\infty$ Control*. Berlin, Germany: Springer-Verlag, 1990, Number 146 in Lecture Notes in Control and Information Sciences.
- [13] —, "Controller reduction by $\mathcal{H}_\infty$ -balanced truncation," *IEEE Trans. Automat. Contr.*, vol. 36, pp. 668–682, June 1991.
- [14] S. Weiland, "Balanced representations and approximation of linear systems," in *Proc. IEEE Conf. Decision Control*, Tampa, FL, 1989, pp. 1334–1336.
- [15] K. Zhou, "Frequency-weighted $\mathcal{L}_\infty$ norm and optimal Hankel norm model reduction," *IEEE Trans. Automat. Control*, vol. 40, pp. 1687–1699, Oct. 1995.
- [16] K. Zhou and J. Chen, "Performance bounds for coprime factor controller reductions," *Syst. Control Lett.*, vol. 26, no. 2, pp. 119–127, 1995.
- [17] K. Zhou, J. C. Doyle, and K. Glover, *Robust and Optimal Control*. Upper Saddle River, NJ: Prentice-Hall, 1996.

An Improved Version of the Singularity-Induced Bifurcation Theorem

Yang Lijun and Tang Yun

Abstract—It has been shown recently that there is a new type of codimension one bifurcation, called the singularity-induced bifurcation (SIB) arising in parameter dependent differential-algebraic equations (DAEs) of the form $\dot{x} = f$ and $0 = g$, and which occurs generically when an equilibrium path of the DAE crosses the singular surface defined by $g = 0$ and $\det g_y = 0$. The SIB refers to a stability change of the DAE owing to some eigenvalue of a related linearization diverging to infinity when the jacobian g_y is singular. In this article an improved version (Theorem 1.1) of the SIB theorem with its simple proof is given, based on a decomposition theorem (Theorem 2.1) of parameter dependent polynomials.

Index Terms—Differential-algebraic equation (DAE), parameter dependent polynomial, singularity-induced bifurcation (SIB).

I. INTRODUCTION

Parameter dependent differential-algebraic equations (DAEs) of the form

$$\begin{cases} \dot{x} = f(x, y, \lambda) & f: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}^n \\ 0 = g(x, y, \lambda) & g: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}^m \end{cases} \quad (1.1)$$

are widely used to model the dynamics of physical systems, for example, power systems, nonlinear circuits, robotics and so on. It has been shown recently that there are generically three types of codimension one local bifurcation associated with the DAE (1.1), namely saddle-node, Hopf, and the singularity-induced bifurcation (SIB), see [2] and [3]. The last one is a new type of bifurcation and does not occur in usual ordinary differential equation (ODE) systems. Roughly speaking, the SIB refers to a stability change of the DAE (1.1) owing to some eigenvalues of a related linearization $f_x - f_y g_y^{-1} g_x$ diverging to infinity when the jacobian g_y is singular. One of the important consequences of the SIB is that it leads to relaxation oscillations of singularly perturbed ODEs [3], [5].

The SIB was first presented and analyzed by Venkatasubramanian *et al.* in [2] and [3]. They proved the SIB theorem by the approach of the center manifold theory applied to an ODE associated with the DAE (1.1). The proof is quite complicated so that it was only outlined in [2] and [3]. Beardmore devotes the fourth chapter of his dissertation [1] to a long elementary proof of the SIB theorem.

The aim of this note is to give an improved version, Theorem 1.1, of the SIB theorem and its simple proof. The improved version requires weaker assumptions however has stronger conclusions than that of the original SIB theorem [2, Th. 3, p. 1999], see Remark 1.2. We derive the improved version from a decomposition theorem (Theorem 2.1), while the latter easily follows from the implicit function theorem. Therefore, the proof presented here is much shorter and more transparent. In our opinion, the decomposition theorem reveals the nature of the SIB phenomenon quite well and is also interesting in its own right.

Now we introduce the improved version of the SIB theorem. Let (x_0, y_0, λ_0) be an equilibrium of the DAE (1.1). If the Jacobian g_y is non-

Manuscript received July 5, 2000; revised December 17, 2000. Recommended by Associate Editor K. Gu. This work was supported by the National Key Basic Research Special Fund under Grant G1998020309, and by the National Natural Science Foundation of China under Grant 19990510.

The authors are with the Department of Mathematical Sciences, Tsinghua University, Beijing 100084, P. R. China (e-mail: lyang@math.tsinghua.edu.cn; tangyun@mail.tsinghua.edu.cn).

Publisher Item Identifier S 0018-9286(01)08808-0.