Hierarchical Optimization in $\mathcal{H}_\infty$

G. D. Halikias and I. M. Jaimoukha

Abstract—This paper considers a hierarchical optimal control problem which involves an optimal $\mathcal{H}_\infty$ -norm cost in the primary problem and an $\mathcal{H}_\infty$ -norm type or a quadratic secondary objective. Using allpass dilation techniques and results from superoptimal interpolation theory, it is shown that the problem can be reduced to a multidisk minimization in terms of a free parameter of reduced dimensions. Convex programming techniques may then be employed to obtain a numerical solution to the problem.

Index Terms— H_∞ superoptimization, mixed H_2/H_∞ control, multidisk optimization, robust control.

I. INTRODUCTION

Hierarchical optimization of a sequence of objective functions $\{f_i(\cdot)\}, i = 1, 2, \dots$, (indexed in order of priority) involves a sequential optimization procedure in which nonuniqueness arising from the optimization of $f_k(\cdot)$ is exploited to optimize $f_{k+1}(\cdot)$. It may be successfully applied provided: 1) the design objectives can be ranked in order of importance and 2) nonunique optimizers exist for the high-priority objectives, and the corresponding solution sets are parameterized in a simple way. In control applications, it is sometimes possible to rank design objectives in order of importance. For example, low interaction in multivariable design is often regarded as less essential compared to robust stability. Moreover, the mere complexity of the underlying optimization problem may enforce the hierarchical approach. In the context of control, a natural hierarchical optimization problem involves the minimization of a performance objective within a family of controllers which guarantee a certain level of robustness.

Using the Youla parameterization of all stabilizing compensators, the two-level suboptimal hierarchical optimization problem may be written in the form: $\inf\{\|L_2(Q)\|_\alpha : Q \in \mathcal{H}_\infty, \|L_1(Q)\|_\infty \leq a_1\}$ where $\alpha \in \{2, \infty\}$ and $L_1(\cdot)$ and $L_2(\cdot)$ are affine. Such problems are described as multidisk, and a number of techniques have been developed for their solution [3]. These methods rely on the fact that both objective functions are affine. In the optimal case considered here

$$\inf\{\|L_2(Q)\|_2 : Q \in \operatorname{argmin}\{\|L_1(Q)\|_\infty : Q \in \mathcal{H}_\infty\}\} \quad (1)$$

the multidisk algorithms break down. For example, following an ϵ -suboptimal approach with respect to the primary objective using the method in [13] results in a convex program in the secondary objective of arbitrarily high dimension (see Section V). A more natural procedure for solving (1) is to parameterize all optimal solutions in the primary problem and use this parameterization in the subsequent optimization. Unfortunately, linearity is no longer retained when the standard parameterization is employed; instead, this is in

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the form of a linear fractional map of the set of stable contractions [4]–[6], which complicates the solution of the second-layer problem.

To obtain a problem tractable via multidisk techniques, it is vital to transform the derived parameterization into an affine, suboptimally constrained setting. In the one-block case, this was achieved via a diagonalization procedure with the help of the maximal Schmidt vectors of a Hankel operator defined from the interpolation data [8]. In the four-block case considered here, an alternative approach is followed based on an allpass dilation technique [7]. Using this method, no need arises to construct Schmidt vectors, the corresponding diagonalizing pair being obtained directly via the off-diagonal blocks of the allpass dilated system [11]. Under a mild assumption (inertia-type optimality), an affine parameterization of the optimal solution set in the primary problem is derived. The secondary objective is subsequently formulated as a multidisk optimization in terms of a matrix parameter of reduced dimensions which may be tackled via standard convex programming techniques [3], [10]. A short version of this paper has appeared in [9].

II. NOTATION AND PROBLEM STATEMENT

$\mathcal{R}$ and $\mathcal{C}$ denote the sets of real and complex numbers. $\mathcal{C}_+$ ($\bar{\mathcal{C}}_+$) and $\mathcal{C}_-$ ($\bar{\mathcal{C}}_-$) denote the open (closed) right half-plane and open (closed) left half-plane, respectively. $\mathcal{R}(s)$ is the field of proper, real-rational functions of s . $\mathcal{L}_\infty$ is the space of matrix functions, analytic and bounded on the extended $j\omega$ axis, with corresponding norm $\|\cdot\|_\infty$. $\mathcal{H}_\infty$ ($\mathcal{H}_\infty^-$) denotes the subspace of $\mathcal{L}_\infty$ consisting of all matrix functions, analytic and bounded in $\mathcal{C}_+$ ($\mathcal{C}_-$). The γ -ball of $\mathcal{H}_\infty$, that is, the set $\{G \in \mathcal{H}_\infty : \|G_\infty\| \leq \gamma\}$, is denoted by $\mathcal{B}_\gamma \mathcal{H}_\infty$. Transfer function dependence on s is mostly suppressed.

The general-distance problem refers to the minimization $\inf\{\|E\|_\infty : Q \in \mathcal{H}_\infty\}$ where

$$E = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} + Q \end{bmatrix} \quad \text{and} \quad R = (R_{ij}) \in \mathcal{RH}_\infty^-$$

where the prefix $\mathcal{R}$ denotes real-rational. A superoptimal approximation denotes the $\mathcal{H}_\infty$ function Q_{so} which minimizes the sequence $\{\sup_{\omega \in \mathcal{R}} \sigma_1[E(j\omega)], \dots, \sup_{\omega \in \mathcal{R}} \sigma_q[E(j\omega)]\}$ with respect to lexicographic ordering, where $\sigma_i(\cdot)$ denotes the i th largest singular value and q is the smallest dimension of R_{22} . (Under certain assumptions made in Section IV, Q_{so} is unique.) The sequence of singular values which is optimal in this sense is referred to as the superoptimal levels of R .

G is called γ -allpass if $G^\sim G = GG^\sim = \gamma^2 I$ where $G^\sim(s)$ is the para-Hermitian conjugate of $G(s)$. If $G \in \mathcal{H}_\infty$ satisfies $G^\sim G = I$ ($GG^\sim = I$), it is called inner (co-inner). $G \in \mathcal{H}_\infty^{p \times p}$ is called a unit of $\mathcal{H}_\infty^{p \times p}$ if $G^{-1} \in \mathcal{H}_\infty^{p \times p}$. A lower linear fractional map of $X = [X_{ij}]_{i,j \in \{1,2\}}$ and K is defined as $\mathcal{F}_l(X, K) = X_{11} + X_{12}K(I - X_{22}K)^{-1}X_{21}$, provided $I - X_{22}(\infty)K(\infty)$ is nonsingular. Similarly, an upper linear fractional map is defined as $\mathcal{F}_u(X, K) = X_{22} + X_{21}K(I - X_{11}K)^{-1}X_{12}$, provided $I - X_{11}(\infty)K(\infty)$ is nonsingular. If $\mathcal{U}$ is a set, then $\mathcal{F}_l(X, \mathcal{U})$ is taken to denote the set $\{\mathcal{F}_l(X, U) : U \in \mathcal{U}\}$.

Next, we give a formal statement of the hierarchical optimization problem, together with some simplifying assumptions.

Problem II.1: Consider the generalized regulator in Fig. 1. Find

$$\gamma_o = \inf\{\|T_{z_1 w}(K)\|_\alpha : K \in \operatorname{argmin}\{\|T_{z_2 w}(K)\|_\infty, K \in \mathcal{S}\}\}$$

where $\mathcal{S}$ is the set of all internally stabilizing compensators of P , $T_{z_i w}$ denotes the closed-loop transfer function from w to z_i , $i = 1, 2$ and $\alpha \in \{2, \infty\}$.

Fig. 1. Generalized regulator.

Assumptions II.1: Let P in Fig. 1 have a state-space realization

$$P = \begin{bmatrix} p_0 & p_1 & p_2 \\ p_1 & p_2 & p_2 \end{bmatrix} \begin{bmatrix} m_1 & m_2 \\ m_2 & m_2 \end{bmatrix} \begin{bmatrix} A & B_1 & B_2 \\ C_0 & D_{01} & D_{02} \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} \in \mathcal{R}(s)$$

(the partitioning being induced by the signals in Fig. 1), with $A \in \mathcal{R}^{n \times n}$. It is assumed that

- 1) the triple (A, B_2, C_2) is stabilizable and detectable;
- 2) $\text{rank}(D_{12}) = m_2$, $\text{rank}(D_{21}) = p_2$, $\text{rank}(\begin{bmatrix} A-j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}) = n + p_2$, $\text{rank}(\begin{bmatrix} A-j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}) = n + m_2$, $\forall \omega \in \mathcal{R}$;
- 3) $\mathcal{H}_\infty$ optimality in the primary problem is of inertia-type (this will be clarified in Section IV);
- 4) $D_{01} = 0$ and $D_{02} = 0$ (in the case $\alpha = 2$ only).
- 5) the largest Hankel singular value of R_a (defined in Theorem IV.1) is simple.

Remark II.1: Assumption 1 ensures there exists a stabilizing controller, and thus that the problem is nontrivial, while Assumption 2 justifies an implicit assumption in the problem formulation (i.e., that the infimum in the primary problem is actually attained) [4]. Assumption 3 is essential to ensure the existence of the affine parameterization derived in Section IV. The requirements on D_{01} and D_{02} in Assumption 4 ensure finite cost in the secondary objective. Finally, Assumption 5 is made for ease of exposition/notational purposes and may be removed in a standard way [7].

III. GENERAL DISTANCE PROBLEM FORMULATION

This section uses the Youla parameterization of all internally stabilizing compensators in a standard procedure to reduce the $\mathcal{H}_\infty$ -norm constraint in the primary problem into a model matching, and subsequently, a general distance constraint. The same procedure is used to transform the secondary optimization problem into a model matching problem.

Let P_{22} have right and left coprime factorizations $P_{22} = N_r D_r^{-1} = D_l^{-1} N_l$, respectively, with $N_r, N_l, D_r, D_l \in \mathcal{RH}_\infty$ and define $V_r, U_r, V_l, U_l \in \mathcal{RH}_\infty$ via the double Bezout identity

$$\begin{bmatrix} V_r & U_r \\ -N_l & D_l \end{bmatrix} \begin{bmatrix} D_r & -U_l \\ N_r & V_l \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Then the set of all internally stabilizing compensators of P is given by $\mathcal{S} = \{-(U_l + D_r Q)(V_l - N_r Q)^{-1} : Q \in \mathcal{H}_\infty\}$ [4]. Further, both closed-loop transfer functions are affine in Q : $T_{z_1 w} = T_{01} + T_{02} Q T_{21}$ and $T_{z_2 w} = T_{11} + T_{12} Q T_{21}$ where

$$\begin{bmatrix} T_{01} \\ T_{11} \end{bmatrix} := \begin{bmatrix} P_{01} - P_{02} U_l D_l P_{21} \\ P_{11} - P_{12} U_l D_l P_{21} \end{bmatrix} \quad \begin{bmatrix} T_{02} \\ T_{12} \end{bmatrix} := \begin{bmatrix} P_{02} \\ P_{12} \end{bmatrix} D_r, \text{ and } T_{21} := D_l P_{21} \text{ are all in } \mathcal{RH}_\infty. \quad (2)$$

Note that in the $\alpha = 2$ case, $T_{z_1 w}(\infty) = 0$ due to Assumption II.1.4 and hence, provided a stabilizing compensator is chosen, the $\mathcal{H}_2$ -cost associated with the secondary objective is always finite.

Using the parameterization of the closed-loop transfer matrices, Problem II.1 reduces to

$$\gamma_o = \inf \{ \|T_{01} + T_{02} Q T_{21}\|_\alpha : Q \in \arg\min \{ \|T_{11} + T_{12} Q T_{21}\|_\infty : Q \in \mathcal{H}_\infty \} \}. \quad (3)$$

This may be transformed in a standard way [4], [1], to the general distance problem

$$\inf_{Q \in \mathcal{H}_\infty} \left\| \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} + Q \end{bmatrix} \right\|_\infty$$

where

$$R = \begin{bmatrix} l & m \\ q & p \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = \begin{bmatrix} T_{11}^\sim & \\ & T_{21}^\sim \end{bmatrix} T_{11} [\hat{T}_1^\sim \quad \hat{T}_2^\sim] \in \mathcal{RH}_\infty^- \quad (4)$$

with $p = m_2, q = p_1 - m_2, m = p_2$, and $l = m_1 - p_2$ (Assumption II.1.2 ensures l and q are nonnegative), and where $T_\perp$ and $\hat{T}_\perp$ are inner and co-inner complements to T_{12} and T_{21} , respectively. Hence problem (3) reduces to

$$\gamma_o = \inf \left\{ \|T_{01} + T_{02} Q T_{21}\|_\alpha : Q \in \arg\min \left\{ \left\| \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} + Q \end{bmatrix} \right\|_\infty : Q \in \mathcal{H}_\infty \right\} \right\}. \quad (5)$$

IV. AN AFFINE PARAMETERIZATION OF ALL OPTIMAL SOLUTIONS IN THE HIGH-PRIORITY OBJECTIVE

It is known that the optimization problem in (3) cannot be solved using current multidisk algorithms [3], [13]. The main problem is the optimality required in the primary $\mathcal{H}_\infty$ -norm constraint. These techniques can be applied to the present problem only by relaxing the upper bound from its optimal level. However, this is unsatisfactory since it is known that near optimality, the suboptimal $\mathcal{H}_\infty$ problem is ill-conditioned [7]. Another major problem with this approach is outlined in Section V. Both problems are circumvented by deriving an affine parameterization of all optimal solutions in a suboptimally constrained setting.

The optimal $\mathcal{H}_\infty$ -norm in the primary problem satisfies

$$s_1(R) := \inf_{Q \in \mathcal{H}_\infty^{p \times m}} \left\| \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} + Q \end{bmatrix} \right\|_\infty \geq \max \left\{ \| \begin{bmatrix} R_{11} & R_{12} \end{bmatrix} \|_\infty, \left\| \begin{bmatrix} R_{11} \\ R_{21} \end{bmatrix} \right\|_\infty \right\} =: \gamma_p \quad (6)$$

where the inequality follows from Parrott's theorem [1]. Assumption II.1.3 can now be defined more precisely; it states that the inequality in (6) is strict, which corresponds to the so-called inertia-type optimality: If Q was allowed to vary over $\mathcal{L}_\infty$, the infimum would be γ_p [1] and thus strict inequality indicates the inertia of Q is an active constraint.

The optimal level $s_1(R)$ can be calculated to any desired degree of accuracy using the so-called γ -iteration [7]. The following theorem summarizes the required properties of the solution of the four-block optimal distance problem.

Theorem IV.1: Let $R \in \mathcal{RH}_\infty^-$ be as given in (4) and assume that $s_1 := s_1(R) > \gamma_p$ (Assumption II.1.3). Then:

- 1) there exists an s_1 -allpass embedding of R of the form

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} & 0 \\ R_{21} & R_{22} + Q_{22} & R_{23} + Q_{23} & Q_{24} \\ R_{31} & R_{32} + Q_{32} & R_{33} + Q_{33} & Q_{34} \\ 0 & Q_{42} & Q_{43} & Q_{44} \end{bmatrix} \in \mathcal{RL}_\infty$$

with

$$Q_a := [Q_{ij}]_{i,j \in \{2,3,4\}} \in \mathcal{RH}_\infty \\ R_a := [R_{ij}]_{i,j \in \{1,2,3\}} \in \mathcal{RH}_\infty^- \quad \text{and} \quad \|H_{22}\|_\infty < s_1 \quad (7)$$

- 2) the set of all four-block optimal approximations of R is given by

$$S(R, s_1) := \left\{ Q \in \mathcal{H}_\infty^{p \times m} : \left\| \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} + Q \end{bmatrix} \right\|_\infty = s_1 \right\} \\ = \mathcal{F}_l \left(\begin{bmatrix} Q_{22} & Q_{24} \\ Q_{42} & Q_{44} \end{bmatrix}, \mathcal{B}_{s_1^{-1}} \mathcal{H}_\infty^{(p-1) \times (m-1)} \right) \quad (8)$$

while the corresponding set of all 4-block optimal error functions is given by

$$\mathcal{E}(R, s_1) := \left\{ E = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} + Q \end{bmatrix} : Q \in \mathcal{H}_\infty^{p \times m}, \right. \\ \left. \|E\|_\infty = s_1 \right\} = \mathcal{F}_l(H, \mathcal{U})$$

where

$$\mathcal{U} = \begin{bmatrix} 0_{q \times l} & 0_{q \times (m-1)} \\ 0_{(p-1) \times l} & \mathcal{B}_{s_1^{-1}} \mathcal{H}_\infty \end{bmatrix}.$$

Proof: See [7], [2], and [6]. The fact that $\|H_{22}\|_\infty < s_1$ generalizes a result in [6] and follows from the fact that, under Assumption II.1.3, H_{12} and H_{21} have full column and row ranks, respectively, on the extended $j\omega$ -axis. $\square$

Remark IV.1: It is known that under Assumption II.1.3, there is a dimension reduction (of at least the multiplicity of the largest Hankel singular value of R_a) in the optimal generator compared with the suboptimal generator (Q_{44} has dimensions smaller than $m \times p$ [7, eqs. (4.37) and (4.50)]). To avoid an intricate indexing system we assume the dimension drop is 1; hence Assumption II.1.5.

Remark IV.2: Theorem IV.1 gives a complete parameterization of all optimal solutions to the primary problem which could be used, in principle, to minimize the secondary objective. Note, however, that the resulting optimization problem would be highly complex since the parameterization is given in a linear fractional (bilinear) form

$$\gamma_o = \inf_{U \in \mathcal{B}_{s_1^{-1}} \mathcal{H}_\infty} \left\| T_{01} + T_{02} \mathcal{F}_l \left(\begin{bmatrix} Q_{22} & Q_{24} \\ Q_{42} & Q_{44} \end{bmatrix}, U \right) T_{21} \right\|_\alpha.$$

An affine parameterization can be recovered using results from superoptimal interpolation theory [11], [12], [14].

We can regard H as an s_1 -allpass embedding of H_{22} . The next theorem uses standard spectral factorization techniques to construct another s_1 -allpass embedding of H_{22} that has reduced dimension. Note that we cannot use directly the allpass completion formula in [5], since we require a special structure for the system; in particular the off-diagonal blocks of the dilated system must be block-triangular and have specific inertia properties.

Theorem IV.2: Let all variables be as in Theorem IV.1. There exists an s_1 -allpass embedding of H_{22} of the form

$$\hat{H} = \begin{bmatrix} \hat{H}_{11} & \hat{H}_{12} \\ \hat{H}_{21} & \hat{H}_{22} \end{bmatrix} = \begin{bmatrix} \hat{R}_{11} & \hat{R}_{12} & \hat{R}_{13} & 0 \\ \hat{R}_{21} & \hat{E}_{22} & \hat{E}_{23} & \hat{Q}_{24} \\ \hat{R}_{31} & \hat{E}_{32} & E_{33} & Q_{34} \\ 0 & \hat{Q}_{42} & Q_{43} & Q_{44} \end{bmatrix} \\ = \begin{matrix} & l & m-1 & q & p-1 \\ \begin{matrix} q \\ p-1 \\ l \\ m-1 \end{matrix} & \begin{bmatrix} \hat{R}_{11} & \hat{R}_{12} & \hat{R}_{13} & 0 \\ \hat{R}_{21} & \hat{R}_{22} + \hat{Q}_{22} & \hat{R}_{23} + \hat{Q}_{23} & \hat{Q}_{24} \\ \hat{R}_{31} & \hat{R}_{32} + \hat{Q}_{32} & R_{33} + Q_{33} & Q_{34} \\ 0 & \hat{Q}_{42} & Q_{43} & Q_{44} \end{bmatrix} \end{matrix} \in \mathcal{RL}_\infty$$

with

$$\hat{R}_{ij} \in \mathcal{RH}_\infty^-, \quad \hat{Q}_{ij} \in \mathcal{RH}_\infty, \quad \text{and} \quad \hat{Q}_{24}^{-1}, \hat{Q}_{42}^{-1} \in \mathcal{RH}_\infty. \quad (9)$$

Furthermore, the set of all four-block s_1 -suboptimal approximations of $\hat{H}$ is given by

$$S(\hat{H}, s_1) := \left\{ Q \in \mathcal{H}_\infty^{(p-1) \times (m-1)} : \left\| \begin{bmatrix} \hat{R}_{11} & \hat{R}_{12} \\ \hat{R}_{21} & \hat{R}_{22} + Q \end{bmatrix} \right\|_\infty \leq s_1 \right\} \\ = \mathcal{F}_l \left(\begin{bmatrix} \hat{Q}_{22} & \hat{Q}_{24} \\ \hat{Q}_{42} & Q_{44} \end{bmatrix}, \mathcal{B}_{s_1^{-1}} \mathcal{H}_\infty^{(p-1) \times (m-1)} \right) \quad (10)$$

while the corresponding set of all four-block s_1 -suboptimal error functions is given by

$$\mathcal{E}(\hat{H}, s_1) := \left\{ \hat{E} = \begin{bmatrix} \hat{R}_{11} & \hat{R}_{12} \\ \hat{R}_{21} & \hat{R}_{22} + Q \end{bmatrix} : Q \in \mathcal{H}_\infty^{(p-1) \times (m-1)}, \|\hat{E}\|_\infty \leq s_1 \right\} \\ = \mathcal{F}_l(\hat{H}, \mathcal{U}). \quad (11)$$

Proof: The fact that $\|H_{22}\|_\infty < s_1$ implies that there exist spectral factors $\hat{Q}_{24}$ and $\hat{Q}_{42}$ satisfying (9) such that

$$\hat{Q}_{42} \hat{Q}_{42}^\sim = s_1^2 I - Q_{43} Q_{43}^\sim - Q_{44} Q_{44}^\sim = Q_{42} Q_{42}^\sim$$

and

$$\hat{Q}_{24} \hat{Q}_{24}^\sim = s_1^2 I - Q_{34}^\sim Q_{34} - Q_{44}^\sim Q_{44} = Q_{24}^\sim Q_{24}. \quad (12)$$

Since $s_1^2 I - H_{22} H_{22}^\sim > 0$, a Schur-type argument shows that

$$s_1^2 I - \hat{E}_{32} \hat{E}_{32}^\sim - E_{33} E_{33}^\sim - Q_{34} Q_{34}^\sim > 0$$

with

$$\hat{E}_{32} := -(E_{33} Q_{43}^\sim + Q_{34} Q_{44}^\sim) \hat{Q}_{42}^{-\sim} \in \mathcal{RL}_\infty \quad (13)$$

where $(\cdot)^{-\sim}$ denotes $[(\cdot)^\sim]^{-1}$. Hence (13) may be spectrally factored as $\hat{R}_{31} \hat{R}_{31}^\sim = s_1^2 I - \hat{E}_{32} \hat{E}_{32}^\sim - E_{33} E_{33}^\sim - Q_{34} Q_{34}^\sim$ with $\hat{R}_{31} \in \mathcal{RH}_\infty^-$. Thus $s_1^2 I - H_{22} H_{22}^\sim$ may be factored as

$$s_1^2 I - H_{22} H_{22}^\sim = \begin{bmatrix} \hat{R}_{31} & \hat{E}_{32} \\ 0 & \hat{Q}_{42} \end{bmatrix} \begin{bmatrix} \hat{R}_{31} & \hat{E}_{32} \\ 0 & \hat{Q}_{42} \end{bmatrix}^\sim \\ =: \hat{H}_{21} \hat{H}_{21}^\sim > 0 \quad (14)$$

with $\hat{H}_{21} \in \mathcal{RL}_\infty$. A dual argument leads to a factorization

$$s_1^2 I - H_{22} H_{22}^\sim = \begin{bmatrix} \hat{R}_{13} & 0 \\ \hat{E}_{23} & \hat{Q}_{24} \end{bmatrix} \begin{bmatrix} \hat{R}_{13} & 0 \\ \hat{E}_{23} & \hat{Q}_{24} \end{bmatrix}^\sim \\ =: \hat{H}_{12} \hat{H}_{12}^\sim > 0$$

with $\hat{H}_{12} \in \mathcal{RL}_\infty$. Choosing $\hat{H}_{11} = -\hat{H}_{12} H_{22}^\sim \hat{H}_{21}^{-\sim} = -\hat{H}_{12}^\sim H_{22}^\sim \hat{H}_{21}$ [6] ensures that $\hat{H}$ is s_1 -allpass. Moreover

$$\hat{H}_{11} = \begin{bmatrix} \hat{R}_{13} (Q_{43}^\sim \hat{Q}_{42}^{-\sim} \hat{E}_{32}^\sim - E_{33}^\sim) \hat{R}_{31}^\sim & -\hat{R}_{13} Q_{43}^\sim \hat{Q}_{42}^{-\sim} \\ -\hat{Q}_{24}^\sim Q_{34}^\sim \hat{R}_{31} & -(\hat{E}_{23} Q_{43}^\sim + \hat{Q}_{24} Q_{44}^\sim) \hat{Q}_{42}^{-\sim} \end{bmatrix} \\ =: \begin{bmatrix} \hat{R}_{11} & \hat{R}_{12} \\ \hat{R}_{21} & \hat{E}_{22} \end{bmatrix} \in \mathcal{RL}_\infty$$

from which it follows that $\hat{R}_{12}, \hat{R}_{21} \in \mathcal{RH}_\infty^-$. If $\hat{R}_{11} \in \mathcal{RH}_\infty^-$, we are done; otherwise, we can always find an allpass $X \in \mathcal{RH}_\infty^-$ such that $\hat{R}_{11}X \in \mathcal{RH}_\infty^-$. Replacing $\hat{R}_{11}, \hat{R}_{21}, \hat{R}_{31}$ by $\hat{R}_{11}X, \hat{R}_{21}X, \hat{R}_{31}X$, respectively, will then ensure the correct inertia properties without affecting the s_1 -allpass character of $\hat{H}$. The fact that s_1 is suboptimal follows from (14) since $\hat{H}$ allpass and $\hat{H}_{21}$ full rank implies that $\|\hat{H}_{11}\|_\infty < s_1$. Finally, (10) and (11) follow from (9) and [7]. $\square$

An inspection of (8) and (10) together with the fact that both H and $\hat{H}$ are s_1 -allpass completions of H_{22} suggests an implicit relation between $\mathcal{S}(R, s_1)$ and $\mathcal{S}(\hat{R}, s_1)$. The next result shows that this relation is affine and can be regarded as an alternative parameterization of the optimal approximation set (8).

Theorem IV.3: Let all variables be as defined in Theorem IV.2. Then $\mathcal{S}(R, s_1) = M_{11} + M_{12}\mathcal{S}(\hat{R}, s_1)M_{21}$ with $M_{12} = Q_{24}\hat{Q}_{24}^{-1}$, $M_{21} = \hat{Q}_{42}^{-1}Q_{42}$ and $M_{11} = Q_{22} - M_{12}\hat{Q}_{22}M_{21}$. Furthermore, $M_{11}, M_{12}, M_{21} \in \mathcal{RH}_\infty$ with $M_{12}^*M_{12} = I_{p-1}$ and $M_{21}M_{21}^* = I_{m-1}$.

Proof: We can write $\mathcal{S}(R, s_1) = Q_{22} + Q_{24}\mathcal{X}(I - Q_{44}\mathcal{X})^{-1}Q_{42}$ and $\mathcal{S}(\hat{R}, s_1) = \hat{Q}_{22} + \hat{Q}_{24}\mathcal{X}(I - Q_{44}\mathcal{X})^{-1}\hat{Q}_{42}$ with $\mathcal{X} = \mathcal{B}_{s_1-1}\mathcal{H}_\infty$. A simple manipulation then gives the required expression for $\mathcal{S}(R, s_1)$. Equations (7) and (9) prove that $M_{ij} \in \mathcal{RH}_\infty$, while (12) proves that $M_{12}^*M_{12} = I_{p-1}$ and $M_{21}M_{21}^* = I_{m-1}$. $\square$

In the notation of Theorem IV.1, the optimization problem in (5) can now be written as

$$\begin{aligned} \gamma_o &= \inf_{\{Q \in \mathcal{H}_\infty^{p \times m}: \|T_{11} + T_{12}Q T_{21}\|_\infty = s_1\}} \|T_{01} + T_{02}Q T_{21}\|_\alpha \\ &= \inf_{Q \in \mathcal{S}(R, s_1)} \|T_{01} + T_{02}Q T_{21}\|_\alpha. \end{aligned} \quad (15)$$

Using Theorem IV.3 gives the following equivalent formulation of Problem II.1.

Theorem IV.4: Let all variables be as defined in Theorem IV.3. Define

$$\begin{aligned} \hat{K}_{11} &:= [I - \hat{R}_{21}(s_1^2 I - \hat{R}_{11}^* \hat{R}_{11})^{-1} \hat{R}_{21}^*]^{-\frac{1}{2}} \mathcal{F}_u(s_1^{-1} \hat{R}, s_1^{-1} \hat{R}_{11}) \\ &\quad \times [I - \hat{R}_{12}^*(s_1^2 I - \hat{R}_{11} \hat{R}_{11}^*)^{-1} \hat{R}_{12}]^{-\frac{1}{2}} \end{aligned} \quad (16)$$

$$\hat{K}_{12} := s_1^{-1} [I - \hat{R}_{21}(s_1^2 I - \hat{R}_{11}^* \hat{R}_{11})^{-1} \hat{R}_{21}^*]^{-\frac{1}{2}} \quad (17)$$

$$K_{21} := [I - \hat{R}_{12}^*(s_1^2 I - \hat{R}_{11} \hat{R}_{11}^*)^{-1} \hat{R}_{12}]^{-\frac{1}{2}} \quad (18)$$

where $(\cdot)^{\frac{1}{2}}$ denotes a spectral factor and $(\cdot)^{-\frac{1}{2}}$ denotes $[(\cdot)^{\frac{1}{2}}]^{-1}$. Let $\hat{K}_{11}$ have a left coprime factorization $\hat{K}_{11} = D_{11}^{-1} K_{11}$ with D_{11} inner. Then Problem II.1 is equivalent to

$$\gamma_o = \inf \{ \|J_{11} + J_{12}Q J_{21}\|_\alpha : Q \in \mathcal{H}_\infty^{(p-1) \times (m-1)}, \|K_{11} + K_{12}Q K_{21}\|_\infty \leq 1 \} \quad (19)$$

where $J_{11} := T_{01} + T_{02}M_{11}T_{21}$, $J_{12} := T_{02}M_{12}$, $J_{21} := M_{21}T_{21}$, $K_{12} := D_{11}^{-1}\hat{K}_{12}$, K_{11} and K_{21} are in $\mathcal{RH}_\infty$. Furthermore

$$\inf \{ \|K_{11} + K_{12}Q K_{21}\|_\infty : Q \in \mathcal{H}_\infty^{(p-1) \times (m-1)} \} < 1. \quad (20)$$

Proof: To show that (16)–(18) are well defined and that $\hat{K}_{12}$ and $\hat{K}_{21}$ may be assumed to be in $\mathcal{RH}_\infty$, note first that $\|\hat{H}_{11}\|_\infty < s_1$ implies that $(s_1^2 I - \hat{R}_{11} \hat{R}_{11}^*)^{-1}$, $(s_1^2 I - \hat{R}_{11}^* \hat{R}_{11})^{-1} \in \mathcal{RL}_\infty$. Further

$$\begin{aligned} \|[\hat{R}_{11} \quad \hat{R}_{12}]\|_\infty < s_1 &\Rightarrow s_1^2 I - \hat{R}_{11} \hat{R}_{11}^* - \hat{R}_{12} \hat{R}_{12}^* > 0 \\ &\Rightarrow I - \hat{R}_{12}^*(s_1^2 I - \hat{R}_{11} \hat{R}_{11}^*)^{-1} \hat{R}_{12} > 0 \end{aligned}$$

and thus the spectral factor in (18) [and by a dual argument (17)] is well defined and can be chosen a unit of $\mathcal{RH}_\infty$. This also shows that the linear fractional map in (16) is well-posed and, since $\hat{R} \in \mathcal{RH}_\infty^-$,

we have $\hat{K}_{11} \in \mathcal{RL}_\infty$. The existence of the required factorization of $\hat{K}_{11}$ follows from [1, Th. 5.1.2]. Using (15) and Theorem IV.3 gives

$$\begin{aligned} \gamma_o &= \inf_{Q \in \mathcal{S}(\hat{R}, s_1)} \|T_{01} + T_{02}(M_{11} + M_{12}Q M_{21})T_{21}\|_\alpha \\ &= \inf_{Q \in \mathcal{S}(\hat{R}, s_1)} \|J_{11} + J_{12}Q J_{21}\|_\alpha \end{aligned}$$

in which $J_{11}, J_{12}, J_{21} \in \mathcal{RH}_\infty$ from (2). $\mathcal{H}_\infty$ -optimality in the primary problem is now captured by the inequality

$$\left\| \begin{bmatrix} \hat{R}_{11} & \hat{R}_{12} \\ \hat{R}_{21} & \hat{R}_{22} + Q \end{bmatrix} \right\|_\infty \leq s_1$$

which is equivalent to $\|\hat{K}_{11} + \hat{K}_{12}Q K_{21}\|_\infty \leq 1$ [1, Th. 4.3]. Since D_{11} is square inner, this is equivalent to $\|K_{11} + K_{12}Q K_{21}\|_\infty \leq 1$ with $K_{11}, K_{12}, K_{21} \in \mathcal{RH}_\infty$. Finally, $\|\hat{H}_{11}\|_\infty < s_1 \Rightarrow \|K_{11} + K_{12}\hat{Q}_{22}K_{21}\|_\infty < 1$ proving (20). $\square$

Theorems IV.3 and IV.4 define a parameterization in terms of $Q \in \mathcal{H}_\infty^{(p-1) \times (m-1)}$ which satisfies a constraint of the form $\|K_{11} + K_{12}Q K_{21}\|_\infty \leq 1$. This may be interpreted as the limiting case of the suboptimal problem

$$\left\| \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} + Q \end{bmatrix} \right\|_\infty \leq \gamma \quad (\gamma > s_1) \quad (21)$$

as γ approaches s_1 : In the suboptimal case, the available degrees of freedom arise from two sources: 1) suboptimality (i.e., the larger the value of γ , the larger the set of admissible Q 's) and 2) the matrix character of the problem. As $\gamma \rightarrow s_1$ the degrees of freedom coming from the first source are exhausted and we are only left with the degrees of freedom in 2). The results of this section show that in this limiting case it is possible to recapture the remaining degrees of freedom into a suboptimal setting which is essentially equivalent to (21) by deflating the problem. In this form, the degrees of freedom in the primary objective become transparent and are more suitable for the subsequent optimization.

A final observation refers to approximate solutions: It can be shown that if the largest t superoptimal levels of R are equal, the dimensionality of the affine parameterization can be further reduced. Consider the case when the t largest superoptimal levels are clustered together, i.e., $s_1(R) - s_t(R) < \delta$ where $\delta > 0$ is small. The t th level superoptimal set

$$\begin{aligned} \mathcal{S}_t(R) &:= \left\{ Q \in \mathcal{S}_{t-1}(R) : \sup_{\omega \in \mathcal{R}} \sigma_i \left(\begin{bmatrix} R_{11}(j\omega) & R_{12}(j\omega) \\ R_{21}(j\omega) & R_{22}(j\omega) + Q(j\omega) \end{bmatrix} \right) \right. \\ &\quad \left. = s_i(R), \quad i = 1, 2, \dots, t \right\} \quad \text{with } \mathcal{S}_0 = \mathcal{H}_\infty \end{aligned}$$

where $s_i(R)$ denotes the i th superoptimal level, is then a good approximation to (its superset) $\mathcal{S}_1(R)$, and thus it is reasonable to expect that restricting Q to $\mathcal{S}_t(R)$ would result in a small deviation from optimality in the subsequent optimization. Note that this effects only the secondary objective; optimality in the primary problem remains in force.

Theorem IV.5: Consider the general distance problem (4) and define the following matrix functions recursively for $k = 0, 1, \dots, t$, $t < \min(p, m)$: $\hat{R}_0 = R$, $\hat{R}^k \in \mathcal{RH}_\infty^-$ obtained by applying the construction in Theorem IV.2 to $\hat{R}^{k-1}$. Assume that at each level k

$$\begin{aligned} s_1(\hat{R}^k) &:= \inf_{Q \in \mathcal{H}_\infty^{(p-k-1) \times (m-k-1)}} \left\| \begin{bmatrix} \hat{R}_{11}^k & \hat{R}_{12}^k \\ \hat{R}_{21}^k & \hat{R}_{22}^k + Q \end{bmatrix} \right\|_\infty \\ &> \max \left\{ \left\| \begin{bmatrix} \hat{R}_{11}^k & \hat{R}_{12}^k \end{bmatrix} \right\|_\infty, \left\| \begin{bmatrix} \hat{R}_{11}^k \\ \hat{R}_{21}^k \end{bmatrix} \right\|_\infty \right\} \end{aligned}$$

in which $\hat{R}_{ij}^k$ denote the partitions of $\hat{R}^k$ with $\hat{R}_{11}^k \in \mathcal{RH}_\infty^{-q \times l}$ and $\hat{R}_{22}^k \in \mathcal{RH}_\infty^{-(p-k-1) \times (m-k-1)}$. Then the problem, $\inf \{\|T_{01} + T_{02}QT_{21}\|_\alpha : Q \in \mathcal{S}_t(R)\}$ is equivalent to the problem, $\inf \{\|J_{11}^t + J_{12}^tQJ_{21}^t\|_\alpha : \|K_{11}^t + K_{12}^tQK_{21}^t\|_\infty \leq 1\}$ where J_{ij}^t, K_{ij}^t are well-defined functions in $\mathcal{RH}_\infty$ with $(J_{12}^t)^\sim J_{12}^t = I_{p-t}$ and $J_{21}^t(J_{21}^t)^\sim = I_{m-t}$.

Proof: The proof follows by the repeated application of Theorems IV.1 and IV.2 and is omitted. $\square$

Remark IV.3: The result shows that the parameterization is affine in an $\mathcal{H}_\infty$ function of dimension $(p-t) \times (m-t)$. This dimensionality reduction leads to a further simplification in the numerical solution of the second level problem.

V. SOLUTION OF THE HIERARCHICAL PROBLEM VIA n -GRID APPROXIMATION METHODS

In Section IV, Problem II.1 was transformed to a suboptimally constrained setting (Theorem IV.3) which may be tackled via convex programming optimization techniques. The resulting optimization is, in general, infinite dimensional. In [13], multi-objective problems of the type considered in this paper are transformed to a sequence of finite-dimensional problems with guaranteed exponential convergence. The technique, essentially suboptimal in character, could, in principle, be applied to the hierarchical problem considered here by slightly relaxing the upper bound in the primary objective from its optimal level. It is shown next that such an approach is impractical and would result in optimization problems with arbitrarily high dimensions. This can be circumvented by using the alternative parameterization of Section IV.

Given an open subset U of $\mathcal{C}$, denote by $\mathcal{H}_\infty^{p \times q}(U)$, the Banach space of all $p \times q$ matrix-valued functions analytic and bounded in U and equipped with the standard norm $\|F\|_U := \sup \{\|F(z)\| : z \in U\}$. Given an open set V such that $U \subseteq V \subseteq \mathcal{C}$, we seek to approximate uniformly $\mathcal{B}_\rho \mathcal{H}_\infty^{p \times q}(V)$, the ball of $\mathcal{H}_\infty^{p \times q}(V)$ of radius ρ , by a finite-dimensional subspace of the form $S_\nu(f) = \{\sum_{j=1}^\nu A_j f_j : A_j \in \mathcal{C}^{p \times q}\}$, where $f = (f_1, f_2, \dots, f_\nu) \in \mathcal{H}_\infty(U)$, in the $\mathcal{H}_\infty^{p \times q}(U)$ -norm. The least achievable error over all $pq\nu$ -dimensional subspaces of $\mathcal{H}_\infty^{p \times q}(U)$

$$d(\nu, \rho, U, V) := \inf_{f \in \mathcal{H}_\infty(U)} \sup_{F \in \mathcal{B}_\rho \mathcal{H}_\infty^{p \times q}(V)} \inf_{X \in S_\nu(f)} \|X - F\|_U \quad (22)$$

is the ν -width of $\mathcal{B}_\rho \mathcal{H}_\infty^{p \times q}(V)$ in $\mathcal{H}_\infty^{p \times q}(U)$. We identify U with $\mathcal{C}_+$, $P = \{s : \operatorname{Re} s > 0\}$ and V with P_r , the complement of the closed disk center $(1+r^2)/(1-r^2)$ and radius $2r/(r^2-1)$, ($r > 1$). From [13, Th. 3], $d(\nu, \rho, P, P_r) = \rho r^{-\nu}$ and $f_j^*(s) = [(s-1)/(s+1)]^{j-1}$, $j = 1, \dots, \nu$ is optimal in the sense of (22). Consider the suboptimal version of Problem II.1

$$\inf \{\|T_{01} + T_{02}QT_{21}\|_\alpha : Q \in \mathcal{H}_\infty, \|T_{11} + T_{12}QT_{21}\|_\infty < s_1(R) + \epsilon\} \quad (23)$$

where ϵ is a small positive number. Let $Q_0 \in \mathcal{S}(R, s_1)$ so that $s_1(R) = \|T_{11} + T_{12}Q_0T_{21}\|_\infty$. Since Q_0 can be chosen in $\mathcal{R}(s)$, we can find constants $\rho_0 > 0$ and $r_0 > 1$ such that $Q_0 \in \mathcal{B}_{\rho_0} \mathcal{H}_\infty^{p \times q}(P_{r_0})$. Let $L = \|T_{12}\|_\infty \|T_{21}\|_\infty$. Then for all ν with

$$\nu \geq \nu_0(\epsilon) := \frac{\ln(\rho_0 L) - \ln(\epsilon)}{\ln(r_0)} \quad (24)$$

there exists $Q \in S_\nu(f^*)$ with $\|T_{11} + T_{12}QT_{21}\|_\infty < s_1(R) + \epsilon$. Moreover, the sequence of convex optimization problems

$$\beta_\nu := \inf \{\|T_{01} + T_{02}QT_{21}\|_\alpha : Q \in S_\nu(f^*), \|T_{11} + T_{12}QT_{21}\|_\infty < s_1(R) + \epsilon\}$$

in $\nu, \nu+1, \dots$, where ν is the smallest integer satisfying (24) converges to the optimum level (23) as $\nu \rightarrow \infty$ [13]. Since $\nu_0(\epsilon) \rightarrow \infty$ as $\epsilon \rightarrow 0$, it is clear that this approach is not appropriate for the solution to the optimal hierarchical optimization problem. The following result, however, shows that the problem can be circumvented by using the affine parameterization of Theorem IV.4. Note also that the dimensionality of the convex program is further reduced due to the dimension deflation discussed previously.

Theorem V.1: Take $\hat{Q} \in \mathcal{S}_2(R)$ and let $\tilde{\rho} > 0, \tilde{r} > 1$ be such that $\tilde{Q} \in \mathcal{B}_{\tilde{\rho}} \mathcal{H}_\infty(P_{\tilde{r}})$. Then:

1) for all ν with

$$\nu \geq \frac{\ln(\tilde{\rho}L) - \ln[s_1(R) - s_2(R)]}{\ln(\tilde{r})} \quad (25)$$

in which $L = \|K_{12}\|_\infty \|K_{21}\|_\infty$ (see Theorem IV.4), $S_\nu(f^*) \cap \mathcal{S}_1(R)$ is a nonempty set;

2) let $\hat{R}$ be as defined in Theorem IV.2 and set $s_2 := s_2(R) = s_1(\hat{R})$. Then

$$L \leq s_1 / (s_1^2 - s_2^2) \quad (26)$$

3) let $\beta_* = \inf_{Q \in \mathcal{S}_1(R)} \|T_{01} + T_{02}QT_{21}\|_\alpha$ and define the sequence of finite-dimensional convex optimization problems

$$\beta_\nu = \inf \{\|T_{01} + T_{02}QT_{21}\|_\alpha : Q \in S_\nu(f^*) \cap \mathcal{S}_1(R)\}.$$

Then for every ν satisfying (25), β_ν is finite and $\beta_\nu \geq \beta_{\nu+1} \geq \dots \rightarrow \beta_*$ as $\nu \rightarrow \infty$.

Proof: Parts 1) and 3) are a direct consequence of [13, Ths. 1 and 3] on using the parameterization in Theorem IV.4. To establish the bound on L in part 2) note that

$$\begin{aligned} & (K_{12}^{-1})(K_{12}^{-1})^\sim \\ &= s_1^2 [I - \hat{R}_{21}(s_1^2 I - \hat{R}_{11}^\sim \hat{R}_{11})^{-1} \hat{R}_{21}^\sim] \\ &\Rightarrow \|K_{12}\|_\infty = 1 / [s_1 \sqrt{1 - \|\hat{R}_{21}(s_1^2 I - \hat{R}_{11}^\sim \hat{R}_{11})^{-1}\|_\infty^2}]. \end{aligned} \quad (27)$$

Now

$$\begin{aligned} s_2 = s_1(\hat{R}) &:= \inf_{Q \in \mathcal{H}_\infty} \left\| \begin{bmatrix} \hat{R}_{11} & \hat{R}_{12} \\ \hat{R}_{21} & \hat{R}_{22} + Q \end{bmatrix} \right\|_\infty \\ &\geq \max \left\{ \left\| \begin{bmatrix} \hat{R}_{11} & \hat{R}_{12} \end{bmatrix} \right\|_\infty, \left\| \begin{bmatrix} \hat{R}_{11} \\ \hat{R}_{21} \end{bmatrix} \right\|_\infty \right\}. \end{aligned}$$

Hence $\hat{R}_{11}^*(j\omega) \hat{R}_{11}(j\omega) + \hat{R}_{21}^*(j\omega) \hat{R}_{21}(j\omega) \leq s_2^2 I = s_1^2 I - (s_1^2 - s_2^2)I, \forall \omega \in \mathcal{R}$, which implies that

$$\begin{aligned} & [s_1^2 I - \hat{R}_{11}^*(j\omega) \hat{R}_{11}(j\omega)]^{-\frac{1}{2}} \hat{R}_{21}^*(j\omega) \hat{R}_{21}(j\omega) \\ & \times [s_1^2 I - \hat{R}_{11}^*(j\omega) \hat{R}_{11}(j\omega)]^{-\frac{1}{2}} \\ & \leq I - (s_1^2 - s_2^2) [s_1^2 I - \hat{R}_{11}^*(j\omega) \hat{R}_{11}(j\omega)]^{-1} \end{aligned}$$

for all $w \in \mathcal{R}$, and thus

$$\begin{aligned} \sigma_1[\hat{R}_{21}(j\omega) [s_1^2 I - \hat{R}_{11}^*(j\omega) \hat{R}_{11}(j\omega)]^{-\frac{1}{2}}] &\leq \sqrt{1 - \frac{s_1^2 - s_2^2}{s_1^2}} \\ &= \frac{s_2}{s_1} \Rightarrow \|\hat{R}_{21}(s_1^2 I - \hat{R}_{11}^\sim \hat{R}_{11})^{-\frac{1}{2}}\|_\infty \leq \frac{s_2}{s_1}. \end{aligned}$$

Using (27) gives $\|K_{12}\|_\infty \leq 1/\sqrt{s_1^2 - s_2^2}$. A dual argument gives $\|K_{21}\|_\infty \leq s_1/\sqrt{s_1^2 - s_2^2}$ and (26) follows immediately. $\square$

Remark V.1: Theorem V.1 shows that the technique in [13] can be applied to the hierarchical problem considered in this paper (strict optimality in the primary objective). The bound on L in (26) implies that the resulting sequence of finite-dimensional convex optimization problems are of low order [see (25)] which makes our technique implementable, in contrast to the ϵ -suboptimal approach in the primary objective outlined earlier. This means that if a suboptimal approach is taken with respect to the *secondary objective*, the results in [13] (adapted to the present case) imply that the minimization can be carried out over a finite-dimensional subspace of fixed dimension.

VI. CONCLUSION

This paper outlined a solution to a hierarchical optimal control problem which combines an $\mathcal{H}_\infty$ cost in the primary problem with an $\mathcal{H}_\infty$ - or an $\mathcal{H}_2$ -type secondary objective. Using an allpass dilation technique and results from superoptimal interpolation theory, the problem was formulated as a multidisk minimization in terms of a free parameter of reduced dimension which can be tackled using standard convex programming algorithms. Although the results assume a single vector of external disturbances w (see Fig. 1), an extension to the more general case of two independent external inputs is straightforward.

The results presented in this paper can deal with (strict) optimality in the primary objective, in contrast to existing multiple-objective techniques. These methods could be applied to the problem considered here, by relaxing the upper bound from its optimal level. Compared to this approach, our solution offers a number of advantages: 1) It is known that near optimality the suboptimal $\mathcal{H}_\infty$ problem is numerically ill-conditioned [7]. Optimal approaches on the other hand, circumvent this problem by reducing the dimensionality of the optimal generator; 2) The present method makes the available degrees of freedom at optimality more transparent and leads to a dimensionality reduction in the convex optimization problem associated with the secondary objective [see (15) and (19)]; and 3) If a suboptimal approach is taken with respect to the secondary objective, our method reduces to a convex optimization over a finite-dimensional subspace of fixed dimension (see Remark V.1).

REFERENCES

- [1] J. C. Doyle and C. C. Chu, "Robust control of multivariable and large-scale systems," Honeywell Systems and Res. Center Rep., 1986.
- [2] J. C. Doyle, K. Glover, P. P. Khargonekar, and B. A. Francis, "State-space solutions to the standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems," *IEEE Trans. Automat. Contr.*, vol. 34, pp. 831–847, 1989.
- [3] P. Dorato, "A survey of Robust multi-objective design techniques," in *Control of Uncertain Dynamic Systems*, S. P. Bhattacharyya and L. H. Keel, Eds. CRC, 1991.
- [4] B. A. Francis, *A Course in $\mathcal{H}_\infty$ Control Theory*. New York: Springer Verlag, 1987.
- [5] K. Glover, "All optimal Hankel-norm approximations of linear multivariable systems and their $\mathcal{L}_\infty$ error bound," *Int. J. Contr.*, vol. 39, pp. 1115–1193, 1984.
- [6] —, *A Tutorial on Hankel-Norm Approximation, From Data to Model*, J. C. Willems, Ed. New York: Springer-Verlag, 1989.
- [7] K. Glover, D. J. N. Limebeer, J. C. Doyle, E. M. Kasenally, and M. G. Safonov, "A characterization of all solutions to the four block general distance problem," *SIAM J. Contr. Optim.*, vol. 29, pp. 283–324, 1991.
- [8] G. D. Halikias, "An affine parametrization of all one-block $\mathcal{H}_\infty$ -optimal matrix interpolating functions," *Int. J. Contr.*, vol. 57, pp. 1421–1441, 1993.
- [9] G. D. Halikias and I. M. Jaimoukha, "Hierarchical optimization in $\mathcal{H}_\infty$," in *Proc. 34th Conf. Decision Contr.*, 1995, pp. 3664–3669.

- [10] G. D. Halikias, I. M. Jaimoukha, and D. A. Wilson, "A numerical solution to the matrix $\mathcal{H}_\infty/\mathcal{H}_2$ optimal control problem," *Int. J. Robust and Nonlinear Contr.*, vol. 7, pp. 711–726, 1997.
- [11] I. M. Jaimoukha and D. J. N. Limebeer, "A state-space algorithm for the solution of the two-block superoptimal distance problem," *SIAM J. Contr. Optim.*, vol. 31, pp. 1115–1134, 1993.
- [12] D. J. N. Limebeer, G. D. Halikias, and K. Glover, "State-space algorithm for the computation of super-optimal matrix interpolating functions," *Int. J. Contr.*, vol. 50, pp. 2431–2466, 1989.
- [13] C. W. Scherer, "Multiobjective $\mathcal{H}_2/\mathcal{H}_\infty$ control," *IEEE Trans. Automat. Contr.*, vol. 40, pp. 1054–1062, 1995.
- [14] N. J. Young, "The Nevanlinna-Pick problem for matrix-valued functions," *J. Operator Theory*, vol. 15, pp. 239–265, 1986.

LMI Approach to an H_∞ -Control Problem with Time-Domain Constraints over a Finite Horizon

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Abstract—The discrete time H_∞ -control problem is considered with time-domain constraints over a finite horizon. It is shown that a solution of this problem can be obtained from a system of linear matrix inequalities (LMI's).

Index Terms— H_∞ control, LMI, time-domain constraints.

I. INTRODUCTION

In the last few years, problems of designing controllers capable of achieving both time- and frequency-domain performance specifications have attracted the attention of many authors. For instance, a convex optimization approach for solving such problems has been proposed in [1]–[3] (see also the references therein) by using the well-known Youla parameterization of stabilizing controllers. This approach has been further developed in [4]–[7]. There the time-domain specifications over a finite horizon were described by affine constraints on the first Markov parameters of the Youla parameter $Q(z)$ of the controller. Then, the exact fulfillment of the frequency-domain specification in [4]–[7] is accomplished by using the remaining freedom in the expression of $Q(z)$ and reducing the problem into a standard H_∞ -control problem [10].

The objective of the present paper is to combine the idea of [4]–[7] with the linear matrix inequality (LMI) approach [8] to the standard H_∞ control. It is shown that both the time- and the frequency-domain constraints can be treated entirely within the framework of LMI. As a result, the convexity structure of the problem becomes much clearer. Furthermore, computational effort to get the solution can be considerably reduced.

II. PROBLEM FORMULATION

We consider the standard feedback configuration illustrated in Fig. 1. Here P and K represent the plant and the controller, which are discrete time-invariant linear systems. As usual, it is assumed

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