Characteristics of the Lamb vector in the outer region of a turbulent boundary layer

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x_1	Streamwise direction	x_3	Spanwise direction
x_2	Wall-normal direction	U_1	Mean streamwise velocity (m/s)
U_3	Mean spanwise velocity (m/s)	U_2	Mean wall-normal velocity (m/s)
u_1	Fluctuating streamwise velocity (m/s)	u_2	Fluctuating wall-normal velocity (m/s)
u_3	Fluctuating spanwise velocity (m/s)	$u_1 u_2$	Reynolds shear stress
U_∞	Freestream streamwise velocity (m/s)	U_τ	skin friction velocity (m/s)
δ	Boundary layer thickness	θ	Momentum thickness
ω_1	Fluctuating streamwise vorticity (1/s)	ω_2	Fluctuating wall-normal vorticity (1/s)
ω_3	Fluctuating spanwise vorticity (1/s)	$[\vec{u} \times \vec{\omega}]$	Lamb vector (m/s ²)

I. Introduction

A characteristic feature of wall-turbulence is the presence of ‘sweeps’, $u_1 > 0$, $u_2 < 0$, Q4 events, (where high momentum fluid is transported to the wall) and ‘ejections’, $u_1 < 0$, $u_2 > 0$, Q2 events (low momentum fluid transported away from the wall). These sweep and ejection events are found to make substantial contributions to turbulence production.^{1–4} In other words, these Reynolds shear stress generating events act as a source for kinetic energy to sustain turbulence. In this study, the streamwise, wall-normal and spanwise directions are along the x_1 , x_2 and x_3 axes respectively. The mean and fluctuating velocity components along those three directions are represented by (U_1, U_2, U_3) and (u_1, u_2, u_3) , respectively.

The flow fields that contribute to Reynolds shear stress are well documented in the literature. Theodorsen⁵ and Offen & Kline⁶ explained the presence of sweeps and ejections using a hairpin model. Bandyopadhyay⁷ and Head & Bandyopadhyay⁸ found visual evidence for the existence of large structures with characteristic inclined interface. Adrian *et al.*⁹ called these large scale structures as hairpin packets that comprise a group of individual hairpin type vortices, whereas Na *et al.*¹⁰ classified large-scale low-momentum regions that produce Reynolds stress as superbursts. Ganapathisubramani *et al.*¹¹ found that a signature consistent with hairpin packets, contributes close to 25% of Reynolds shear stress and yet occupy only a fraction of the area. Guala *et al.*¹² found that very large scale motions (which can be understood as a collection of hairpin packets¹³) are very active and contain more than half the Reynolds shear stress.

While the focus on Reynolds shear stress is warranted, inspection of the mean momentum equation indicates that in fact it is the gradient of Reynolds shear stress that is of primary importance to the rate of change of momentum. The mean momentum equation along the streamwise direction (equation 1) for a zero pressure gradient turbulent boundary layer shows that the wall-normal gradient of Reynolds shear stress (T) is a force that acts as a momentum source/sink.

$$U_1 \frac{U_1}{\partial x_1} + U_2 \frac{\partial U_1}{\partial x_2} = \nu \frac{\partial^2 U_1}{\partial x_2^2} + \bar{T} \quad (1)$$

$$\text{where, } T = \frac{\partial(-u_1 u_2)}{\partial x_2} \quad (2)$$

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where, ν is the kinematic viscosity and $\overline{T}$ is the mean value of the wall-normal gradient of Reynolds shear stress.

Examination of wall-normal variation of Reynolds shear stress in wall-bounded flows reveal a single peak position (x_{2p}) for $\overline{-u_1 u_2}$, which appear to be Reynolds number dependent. At the peak, the gradient of Reynolds shear stress is zero. However, the term T acts as a momentum source for $x_2 < x_{2p}$ (where $\partial(\overline{-u_1 u_2})/\partial x_2 > 0$) and as a sink for $x_2 > x_{2p}$ (where $\partial(\overline{-u_1 u_2})/\partial x_2 < 0$). Wei *et al.*¹⁴ explored the mean momentum balance based on experimental and DNS data and proposed a four layer structure for wall-bounded flows where the wall-normal extent of the layers depend on the relative magnitude of T compared to the viscous term. Therefore, there is great interest in exploring the structure of T to understand the motions that contribute to the mean-momentum balance.

The wall-normal gradient of Reynolds shear stress in an incompressible flow can be written as follows:¹⁵

$$T = u_2 \omega_3 - u_3 \omega_2 + \frac{1}{2} \frac{\partial}{\partial x_1} (u_1^2 - u_2^2 - u_3^2) \quad (3)$$

where, ω_2, ω_3 are the fluctuating vorticity components along x_2 and x_3 respectively.

Klewicki¹⁶ documented the individual terms that appear in equation 3 and concluded that the wall-normal gradient of $\overline{-u_1 u_2}$ is dominated by the velocity-vorticity correlation terms. The streamwise gradients of the mean-square fluctuations (the term C) were generally over three order of magnitude smaller than the velocity-vorticity correlations. Therefore, equation 3 can be approximated as,

$$T \approx u_2 \omega_3 - u_3 \omega_2 = [u \times \omega]_1 \quad (4)$$

where $[u \times \omega]_1$ is the streamwise component of the Lamb vector.

Therefore, the gradient of Reynolds shear stress arises through the interactions between velocity and vorticity. These interactions produce motions that are capable of acting as either momentum sources/sinks. A few researchers have investigated the motions that contribute to the velocity-vorticity correlation. Rajagopal & Antonia¹⁷ studied the structure of velocity field associated with the spanwise vorticity field and reported cross correlations of $u_2 \omega_3$ and $u_1 \omega_3$. They concluded that the velocity signature is consistent with the presence of internal shear layers, inclined to the wall. Klewicki *et al.*¹⁸ investigated the $u_2 \omega_3$ term in the near wall region and concluded that the existence of positive spanwise vorticity for $x_2^+ > 12$ is consistent with the presence of a ring-like eddy.¹⁹ Priyadharshana *et al.*²⁰ compared the velocity-vorticity correlations over a wide range of Reynolds numbers and found that the interaction between velocity and vorticity depends on the Reynolds number and wall-roughness. They proposed a model where approximately uniform momentum zones (consistent with VLSMs) are intermittently segregated by narrow fissures of highly vortical flow and that the interaction between these vorticity fissure and velocity zones create momentum sources and sinks. However, the spatial distribution of these momentum source/sink like motions and their relationship with the Reynolds shear stress events is not documented.

In the current study, the spatial structure of momentum sources/sinks is explored by using dual-plane particle image velocimetry data that were obtain in streamwise-spanwise planes of a turbulent boundary layer. The primary goal is to determine the spatial structure of the streamwise component of the Lamb vector ($[u \times \omega]_1$) and classify the motions that act as momentum sources and sinks.

II. Experimental facility and Details

The experimental datasets analysed in this study were obtained by²³ by performing dual-plane particle image velocimetry measurements in a suction-type boundary layer wind tunnel at the department of Aerospace Engineering and Mechanics, University of Minnesota. Measurements were performed in a zero-pressure-gradient flow with freestream velocity $U_\infty = 5.9 \text{ m s}^{-1}$ and $Re_\tau = 1200$ ($Re_\tau = \delta U_\tau / \nu$, where δ is the boundary layer thickness, u_τ is the skin friction velocity and ν is the kinematic viscosity). All quantities presented in this paper are non-dimensionalised using u_τ and ν . The Reynolds number based on the momentum thickness Re_θ was 2800 and the value of δ in the region of the measurement planes was 70 mm. Measurements were obtained at $x_2^+ = 110$ ($x_2/\delta = 0.1$, log region) and at $x_2^+ = 575$ ($x_2/\delta = 0.5$, outer wake region).

The experiments use a unique three-camera polarization-based dual-plane PIV system to measure the complete velocity gradient tensor. A schematic of the experimental setup is shown in figure 1. The setup consists of two independent PIV systems that captures images of light scatter by olive oil droplets of size $1 \mu\text{m}$.

Figure 1. Schematic of the dual-plane PIV experimental setup. Figure reproduced from Ganapathisubramani et al.^{21,22}

The first system is a stereoscopic setup, which provided all three velocity components within a $x_1 - x_3$ plane, while the second system was a conventional PIV system that measured the in-plane velocity components in a neighbouring $x_1 - x_3$ plane located 20 wall units above in the wall-normal direction. Simultaneous measurements were performed in the two planes by using the polarization property of laser light sheets to isolate one camera system from the other. The images were analysed and velocity vectors were computed in both planes. The resolution of the resulting vector fields was about 12×12 wall units, and the total field size was $1.1\delta \times 1.1\delta$.

The instantaneous complete velocity gradient tensor was computed using the velocity data from both planes. A second order central difference method was used to compute all possible in-plane gradients while a first order forward difference was used to compute the wall-normal gradients. Further details of the experimental set up and validation of the velocity gradient and vorticity data are given in Ganapathisubramani et al.^{21,22}

III. Results

Quantitative analysis of the Lamb vector can be performed by computing probability density functions of $[u \times \omega]_1^+$ (superscript + indicates normalisation by ν and u_τ). Figures 2(a) and 2(b) show the probability density functions (*pdfs*) for $[u \times \omega]_1^+$ at $x_2^+ = 110$ ($x_2/\delta = 0.1$) and 575 ($x_2/\delta = 0.5$), respectively. The *pdfs* are plotted in linear-log axes. Both figures indicate that the *pdfs* are skewed towards negative values of $[u \times \omega]_1^+$. The tails of the distribution exhibit exponential decay as evidenced by the straight lines fitted to the positive and negative tails in both figures. The constants that were used for the exponential fit are indicated in the figures. It is clear from the figures that the range of $[u \times \omega]_1^+$ values decrease with increasing wall-normal location. The slope of the exponential decay is found to be steeper away from the wall as evidenced by the value of the slope k . (where $k = 15$ and 21 for $x_2^+ = 110$ and 575, respectively). The *pdfs* deviate from the exponential decay for large negative values of T^+ indicating a preference for large negative values of $[u \times \omega]_1^+$ at both wall-normal locations.

The location of the peak in the wall-normal profile of the Reynolds shear stress is found to be Reynolds number dependent. Sreenivasan & Sahay²⁴ presented a curve fit for the location of the peak based on the Karman number: $x_{2p}^+ \approx 2\sqrt{Re_\tau}$. Alternately, Fernholz & Finley²⁵ examined experimental data over a wide range of Reynolds numbers and found that the peak location varies as $x_{2p}^+ \approx Re_\theta^{0.61}$. Substituting the relevant values in the functional form, the peak in the mean Reynolds shear stress profile for the boundary layer in the current study is expected to be located between 70 wall-units to 100 wall-units. This indicates that both measurement planes are likely to be located beyond the peak location. This is reflected in the mean values of $[u \times \omega]_1^+$, which are -0.006 and -0.008 at $x_2^+ = 110$ and 575, respectively. These values are consistent with the values reported in previous studies in the literature.^{16,20}

Figure 2. Probability density functions of $[u \times \omega]_1^+$ at (a) $x_2^+ = 110$ and (b) $x_2^+ = 575$. The lines show the exponential decay of the values.

Figure 3. Instantaneous contours of (a) streamwise component of the Lamb vector ($[u \times \omega]_1^+$), and (b) Reynolds shear stress ($[u_1 u_2]^+$).

Figures 3(a) shows contours of the streamwise component of the Lamb vector ($[u \times \omega]_1^+$) in a streamwise-spanwise plane at $x_2^+ = 110$. The contours indicate that $[u \times \omega]_1^+$ has large quiescent zones with intermittent areas of intense values. The intense regions possess both positive and negative values indicating the presence of both momentum source- and sink-like behaviour. These momentum sources and sinks appear to be located adjacent to each other. Figures 3(b) reveal instantaneous contours of Reynolds shear stress ($[u_1 u_2]^+$) at the same instant as in figure 3(a). The figure shows that intense values of Reynolds shear stress is also intermittent. A comparison between figures 3(a) and 3(b) indicates that the momentum sources and sinks appear in areas where the Reynolds shear stress is negative. This indicates that the momentum sources/sinks are prevalent in ejection ($u_1 < 0$ and $u_2 > 0$) and sweep ($u_1 > 0$ and $u_2 < 0$) regions.

Figure 4. Probability of occurrence of momentum sinks in the vicinity of momentum sources (a) $x_2^+ = 110$ and (b) $x_2^+ = 575$.

The structure of $[u \times \omega]_1^+$ can be further explored by computing the probability of occurrence of momentum source/sink events in the vicinity of each other. Figures 4(a) shows the probability distribution of locating a momentum sink in the neighbourhood of a momentum source at $x_2^+ = 110$. The figure shows that the highest probability of locating a momentum sink is in a region that is approximately 60 wall-units downstream of a momentum source. Additionally, the probability is also high in a region that is 75 wall-units away in the either spanwise directions. However, the spanwise extent of these high probability regions are small. In fact, the figure reveals that the region of high probability possesses a ‘boomerang’ shape and momentum sources are encompassed by momentum sinks in all directions except in the upstream direction.

Figure 4(b) shows the probability distribution of locating a momentum sink in the neighbourhood of a momentum source at $x_2^+ = 575$. This figure shows that the highest probability of locating a momentum sink in the vicinity of a momentum source is at a downstream distance of approximately 60 wall-units. This suggests that the spatial separation between momentum sources/sinks does not change with wall-normal location. However, the shape of the high probability region appears to be different in this outer location. The highest probability is confined to the downstream direction.

The relationship between the Lamb vector (i.e. Momentum sources/sinks) and Reynolds shear stress can be explored by computing the joint pdf between the two quantities. In fact, the covariance between these two quantities will shed insights in to their relationship. The covariance between any two variables is derived from the joint pdf using the following equation,

$$COV(A, B) = \int AB P(A, B) dA dB \quad (5)$$

where, $P(A, B)$ is the joint pdf between the variables A and B .

Figures 5(a) and 5(b) show the covariance maps between the Lamb vector and Reynolds shear stress at $x_2^+ = 110$ and 575 , respectively. Both figures show strong correlation between $[\vec{u} \times \vec{\omega}]_1 < 0$ and $u_1 u_2 < 0$

Figure 5. Covariance between the Lamb vector and Reynolds shear stress at (a) $x_2^+ = 110$ and (b) $x_2^+ = 575$.

in both wall-normal locations. This suggests that momentum sources and sinks predominantly occur in regions of sweeps (Q4 events) and ejections (Q2 events). This is consistent with the observation based on instantaneous fields in figure 3. Although the covariance maps indicate the relationship between the two quantities, the characteristics of Reynolds shear stress in the vicinity of momentum source/sinks cannot be inferred from these plots. Therefore, an alternate diagnostic is required to derive this information.

Probability of occurrence distribution between the Lamb vector and Reynolds shear stress will be able to provide further insights in to the flow fields that act as momentum sources and sinks. These occurrence probabilities must be calculated for both momentum sources and sinks and must account for the four independent quadrants of Reynolds shear stress.

Figure 6(a)-6(d) shows the probability of occurrence of Q1, Q2, Q3 and Q4 events in the vicinity of momentum source at $x_2^+ = 110$. Figure 6(a) indicates that the highest probability of locating Q1 events in the vicinity of momentum sources is in a pear-shaped region around the momentum source. Beyond this region, the probability of locating a Q1 events appears to be homogeneous in all directions. Figure 6(b) indicates that Q2 events possess the highest probability of occurrence in the neighbourhood of momentum sources. The figure shows that this region extend in both upstream and downstream of the origin along the streamwise direction (to distances over 200 wall-units on either side). The figure also indicates that the spanwise width of this region is approximately 300 wall-units at this wall-normal location. The occurrence probability sharply decreases in the spanwise direction at a distance of approximately 200 wall-units. Figure 6(d) indicates that Q4 events possess the highest probability occurrence in the area where a sharp decrease in Q2 activity is observed. Finally, outward interactions (Q3 events) does not reveal any preference in its location as seen in figure 6(c). Similar distributions are observed in the outer wall-normal location ($x_2^+ = 575$, not shown here).

Figure 7(a)-7(d) shows the probability of occurrence of Q1, Q2, Q3 and Q4 events in the vicinity of momentum sinks at $x_2^+ = 110$. Figure 7(a) shows that Q1 events do not have a preference to appear in the vicinity of momentum sinks. In fact, the probability decreases near the origin (i.e. at the same point as a momentum sink). This is consistent with the observations based on the covariance maps in figure 5(a). Ejections events reveal a strong preference to be located in the neighbourhood of momentum sink as seen in figure 7(b). The region of highest probability possesses a wedge shape that extends approximately 100 wall-units in the upstream and downstream directions. The spanwise width of this wedge-shaped region decreases along the streamwise direction from 400 wall-units at $\Delta x_1^+ = -100$ to 100 wall-units at $\Delta x_1^+ = 100$. Figure 7(c) reveals that the probability of finding a concurrent Q3 event-momentum sink is low. However, the Q3 events appear to be located upstream and downstream of momentum sinks. These regions of high probability are located at a distance of ± 50 wall-units along the streamwise direction. The spanwise

Figure 6. Probability of occurrence of Reynolds shear stress events in the vicinity of momentum sources at $x_2^+ = 110$. (a) Q1 (inward interactions), (b) Q2 (ejections), (c) Q3 (outward interactions) and (d) Q4 (sweeps).

Figure 7. Probability of occurrence of Reynolds shear stress events in the vicinity of momentum sources at $x_2^+ = 110$. (a) Q1 (inward interactions), (b) Q2 (ejections), (c) Q3 (outward interactions) and (d) Q4 (sweeps).

width of these regions is approximately 100 wall-units. Finally, figure 7(d) shows that sweep events possesses highest probability in a region that extends along the streamwise direction.

IV. Summary and Conclusions

The spatial organisation of the Lamb vector is examined in the outer region of a turbulent boundary layer. The streamwise component of the Lamb vector acts as a momentum source in areas where its value is greater than zero and as a momentum sink in regions where its value is less than zero. Inspection of instantaneous fields indicates that momentum-sources and sinks reveal an intermittent character and they appear adjacent to each other. Conditional probability analyses indicates that momentum source-like events predominantly occurs upstream of momentum sink-like events in both wall-normal locations examined in this study.

The spatial layout and the relationship between momentum sources/sinks and Reynolds shear stress is explored by computing covariance maps and two-dimensional probability of occurrence distribution. The results indicate that momentum sources/sinks appear to in areas of negative Reynolds stresses. This suggests that turbulence production sites (i.e. areas of $u_1 u_2 < 0$) contains both momentum sources and sinks. Within these regions, momentum sources largely lie within ejection-type events (Q2 events), while sweep events occur adjacent to momentum sources. Both inward and outward interactions do not display any directional or location preference in the vicinity of momentum sources. Momentum sinks on the other hand, reveals comparable association with both sweep and ejection events. The region in the vicinity of momentum sinks also appears to be sites of outward interactions (Q3 events). This suggests that momentum sinks appears in areas where rapid quadrant transitions take place and this is consistent with previous studies that has explored the nature of momentum sources and sinks.²⁶

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