EXPONENTIAL STABILIZATION OF WELL-POSED SYSTEMS BY COLOCATED FEEDBACK*

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Abstract. We consider well-posed linear systems whose state trajectories satisfy $\dot{x} = Ax + Bu$, where u is the input and A is an essentially skew-adjoint and dissipative operator on the Hilbert space X . This means that the domains of A^* and A are equal and $A^* + A = -Q$, where $Q \geq 0$ is bounded on X . The control operator B is possibly unbounded, but admissible and the observation operator of the system is B^* . Such a description fits many wave and beam equations with colocated sensors and actuators, and it has been shown for many particular cases that the feedback $u = -\kappa y + v$, with $\kappa > 0$, stabilizes the system, strongly or even exponentially. Here, y is the output of the system and v is the new input. We show, by means of a counterexample, that if B is sufficiently unbounded, then such a feedback may be unsuitable: the closed-loop semigroup may even grow exponentially. (Our counterexample is a simple regular system with feedthrough operator zero.) However, we prove that if the original system is exactly controllable and observable and if κ is sufficiently small, then the closed-loop system is exponentially stable.

Key words. well-posed linear system, regular linear system, positive-real transfer function, output feedback, exact controllability and observability, skew-adjoint operator, colocated sensors and actuators, exponential stability

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1. Introduction and the main result. In this paper, we consider the stabilization of a special class of well-posed linear systems, as described below. To specify our terminology and notation, we recall that for any well-posed linear system Σ with input space U , state space X and output space Y , all Hilbert spaces, the state trajectories $z \in C([0, \infty), X)$ are described by the differential equation

$$(1.1) \quad \dot{z}(t) = Az(t) + Bu(t),$$

where $u \in L^2_{loc}([0, \infty), U)$ is the input function. The operator $A : \mathcal{D}(A) \rightarrow X$ is the generator of a strongly continuous semigroup of operators $\mathbb{T}$ on X and the (possibly unbounded) operator B is an admissible control operator for $\mathbb{T}$. In general, the output function y is in $L^2_{loc}([0, \infty), Y)$. If $u = 0$ and $z(0) \in \mathcal{D}(A)$, then y is given by

$$y(t) = Cz(t) \quad \forall t \geq 0,$$

where $C : \mathcal{D}(A) \rightarrow Y$ is an admissible observation operator for $\mathbb{T}$. B is called the control operator of Σ and C is called the observation operator of Σ . If $z(0) = 0$, then the input and output functions u and y are related by the formula

$$(1.2) \quad \hat{y}(s) = \mathbf{G}(s)\hat{u}(s),$$

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FIG. 1. The open-loop system Σ with negative output feedback via κ . If the number $\kappa > 0$ is sufficiently small, then this is a new well-posed linear system Σ^κ , called the closed-loop system, which is input-output stable. If Σ is exactly controllable and exactly observable, then Σ^κ is exponentially stable.

where a hat denotes the Laplace transform and $\mathbf{G}$ is the transfer function of Σ . The formula (1.2) holds for all $s \in \mathbb{C}$ with $\operatorname{Re} s$ sufficiently large. We refer to sections 3 and 4 for more details and references on admissibility and on well-posed linear systems. Now we specify the special class of systems studied in this paper.

Assumption ESAD. The operator A is *essentially skew-adjoint and dissipative*, which means that $\mathcal{D}(A) = \mathcal{D}(A^*)$ and there exists a $Q \in \mathcal{L}(X)$ with $Q \geq 0$ such that

$$(1.3) \quad Ax + A^*x = -Qx \quad \forall x \in \mathcal{D}(A).$$

This implies that $\mathbb{T}$ is a contraction semigroup. Note that A is a bounded perturbation of the skew-adjoint operator $A + \frac{1}{2}Q$. Such a model is often used to describe the dynamics of oscillating systems, such as waves or flexible structures (often $Q = 0$, so that $\mathbb{T}$ is unitary); for a literature survey see section 2.

Assumption COL. $Y = U$ and $C = B^*$.

In the literature on the stabilization of flexible structures, a very popular way of implementing actuators and sensors is through colocated pairs, i.e., an actuator and sensor pair act at the same physical position. This often leads to assumption COL being satisfied, often with a finite-dimensional U .

Our aim is to show that for certain numbers $\kappa > 0$, the static output feedback law $u = -\kappa y + v$ stabilizes the system, where v is the new input function. The closed-loop system Σ^κ is shown as a block diagram in Figure 1 (see section 4 for some background on output feedback). The system Σ^κ is called *input-output stable* if its transfer function $\mathbf{G}^\kappa = \mathbf{G}(I + \kappa\mathbf{G})^{-1}$ is uniformly bounded on the open right half-plane where $\operatorname{Re} s > 0$. It is called *exponentially stable* if the growth bound of its semigroup is negative. Our main result is the following theorem.

THEOREM 1.1. *Suppose that Σ is a well-posed linear system which is exactly controllable, exactly observable and satisfies assumptions ESAD and COL. Then there exists a $\kappa_0 > 0$ (possibly $\kappa_0 = \infty$) such that for all $\kappa \in (0, \kappa_0)$, the feedback law $u = -\kappa y + v$ (where u and y are the input and the output of Σ) leads to a closed-loop system Σ^κ which is well-posed and exponentially stable.*

In fact, this result is only a corollary of Theorem 5.8, in which the assumptions are weaker than exact controllability and exact observability. We also give a formula for κ_0 based on the transfer function $\mathbf{G}$, see Theorem 5.8.

In all the published examples that we are aware of, the feedback $u = -\kappa y + v$ is stabilizing for all $\kappa > 0$, at least in the input-output sense, and often strongly or exponentially. In section 5 we give an example of a simple open-loop system Σ which fits into our framework, and it is regular with feedthrough operator zero (see section 4 for definitions), but for which the feedback $u = -\kappa y + v$ is only exponentially stabilizing for sufficiently small $\kappa > 0$. For too large a κ , the closed-loop semigroup $\mathbb{T}^\kappa$ will have a positive growth rate.

In section 6 we introduce the theoretical framework for discussing colocated feedback for systems described by second order differential equations in time, with a suitable version of Theorem 1.1. As an illustration, we outline the problem of stabilizing a Rayleigh beam with two sensors located in one point, which fits into the theory developed in this section. With the actuators designed such that $B = C^*$ and with proportional output feedback with not too high feedback gain, the closed-loop system is exponentially stable, for any position of the two sensors.

Exact controllability and exact observability are very restrictive conditions, especially if U is finite-dimensional, see, for example, the discussion in Rebarber and Weiss [29]. There is a rich literature dealing with various specific linear systems or classes of systems that satisfy ESAD and COL and that are approximately controllable and observable (or satisfy other related assumptions). In these papers, the main conclusion is usually the weak or strong stability of the closed-loop system (and various nonexponential decay rates of the energy). This area will be examined in a sequel to this paper, which will contain a unified theory of the strong stabilization of systems satisfying ESAD and COL.

2. Comments on the literature and a self-contained presentation of the finite-dimensional case. Many models of controlled flexible structures satisfy assumptions ESAD and COL. The feedback $u = -\kappa y + v$ is very simple to implement and it is often used in the stabilization of these structures. Our results may be regarded as an abstract unifying theory for colocated exponential stabilization of flexible structures. However, it must be pointed out that not all such examples in the literature satisfy all our assumptions. In particular, the open-loop system is not always well posed.

Much of the early work on the stabilization of flexible structures concerned finite-dimensional systems, see, for example, Benhabib et al. [9] and Joshi [18] and the references therein. They used the feedback $u = -\kappa B^*z + v$ because of its simplicity and its nice robustness properties. Indeed, it works for all systems with a positive-real transfer function, as we shall explain below (see also Desoer and Vidyasagar [12]). A valuable recent source for the finite-dimensional theory is Gawronski [13]. For a survey on the stabilization of finite-dimensional linear systems by static output feedback we refer to Syrmos et al. [39] (see also Zeheb and Hertz [54]).

We think that it will be instructive to give here a short self-contained presentation of the finite-dimensional version of our main result (Theorem 1.1), as well as of some related results, without any claims to novelty. Our infinite-dimensional arguments will go along the same lines, but with many more technicalities.

Recall that a square matrix-valued transfer function $\mathbf{G}$, analytic on the open right half-plane $\mathbb{C}_0$, is called *positive-real* if $\overline{\mathbf{G}(s)} = \mathbf{G}(\bar{s})$ and

$$(2.1) \quad \mathbf{G}(s)^* + \mathbf{G}(s) \geq 0 \quad \forall s \in \mathbb{C}_0.$$

Positive-real transfer functions were introduced in electrical network theory, but they have strong connections with systems theory formulated in state space, see Anderson and Vongpanitlerd [3]. In particular, if the real square matrix A is dissipative, then for any real matrix B of appropriate dimensions, $\mathbf{G}(s) = B^*(sI - A)^{-1}B$ is positive-real. Such transfer functions often occur as models of flexible structures with colocated actuators and sensors, see [9, 13].

Now consider an arbitrary (but square) $m \times m$ matrix-valued transfer function $\mathbf{G}$. If the closed-loop system with transfer function $\mathbf{G}^\kappa$ is obtained from the open-loop system with transfer function $\mathbf{G}$ via the feedback $u = -\kappa y + v$, as in Figure 1, then

$\mathbf{G}^\kappa = \mathbf{G}(I + \kappa\mathbf{G})^{-1}$. The following lemma gives a simple sufficient condition for $\mathbf{G}^\kappa$ to be bounded on $\mathbb{C}_0$, a fact which is written as $\mathbf{G}^\kappa \in H^\infty$.

PROPOSITION 2.1. *Suppose that $cI + \mathbf{G}$ is positive-real for some $c \geq 0$, and denote $\kappa_0 = \frac{1}{c}$ (for $c = 0$, take $\kappa_0 = \infty$). Then for any $\kappa \in (0, \kappa_0)$, $\mathbf{G}^\kappa \in H^\infty$.*

Proof. Denoting $a = \frac{1}{\kappa}$ and $T(s) = aI + \frac{1}{2}(\mathbf{G}(s)^* + \mathbf{G}(s))$, we have $T(s) \geq (a - c)I > 0$. Hence, for every $v \in \mathbb{C}^m$ with $\|v\| = 1$ and for all $s \in \mathbb{C}_0$,

$$\|(aI + \mathbf{G}(s))v\| \geq \operatorname{Re} \langle (aI + \mathbf{G}(s))v, v \rangle = \langle T(s)v, v \rangle \geq a - c,$$

hence

$$(2.2) \quad \|(aI + \mathbf{G}(s))^{-1}\| \leq \frac{1}{a - c}$$

so that $(aI + \mathbf{G})^{-1} \in H^\infty$. We rewrite $\mathbf{G}^\kappa$ in the form

$$\mathbf{G}^\kappa = a[I - a(aI + \mathbf{G})^{-1}],$$

which shows that $\mathbf{G}^\kappa \in H^\infty$ (a bound will be given below). $\square$

PROPOSITION 2.2. *With the assumptions of Proposition 2.1, denote $c^\kappa = \frac{c}{1 - \kappa c}$ so that $c^\kappa \geq 0$. Then $c^\kappa I + \mathbf{G}^\kappa$ is positive-real.*

Proof. Notice that $\kappa c \in [0, 1)$. Introduce the transfer function

$$\mathbf{H} = \frac{\kappa}{1 - \kappa c} (cI + \mathbf{G})$$

so that clearly $\mathbf{H}$ is positive-real. It is readily verified that

$$(2.3) \quad c^\kappa I + \mathbf{G}^\kappa = \frac{1}{\kappa(1 - \kappa c)} \mathbf{H}(I + \mathbf{H})^{-1}.$$

This implies that for all $s \in \mathbb{C}_0$,

$$\begin{aligned} & \kappa(1 - \kappa c) \left[(c^\kappa I + \mathbf{G}^\kappa(s))^* + (c^\kappa I + \mathbf{G}^\kappa(s)) \right] \\ &= (I + \mathbf{H}(s)^*)^{-1} \left[\mathbf{H}(s)^* + \mathbf{H}(s) + 2\mathbf{H}(s)^*\mathbf{H}(s) \right] (I + \mathbf{H}(s))^{-1}. \end{aligned}$$

It is easy to see that the right-hand side of the above equation is nonnegative. Since clearly $\overline{\mathbf{G}^\kappa(s)} = \mathbf{G}^\kappa(\bar{s})$, it follows that $c^\kappa I + \mathbf{G}^\kappa$ is positive-real. $\square$

LEMMA 2.3. *If H is an $m \times m$ matrix such that $\operatorname{Re} \langle Hv, v \rangle \geq 0$ for all $v \in \mathbb{C}^m$, then $I + H$ is invertible and $\|H(I + H)^{-1}\| \leq 1$.*

Proof. The invertibility is clear. Denote $T = H(I + H)^{-1}$; then it is easy to see that $H(I - T) = T$. If in $\operatorname{Re} \langle Hv, v \rangle \geq 0$ we choose $v = (I - T)z$, then we get

$$0 \leq \operatorname{Re} \langle Tz, (I - T)z \rangle = \operatorname{Re} \langle Tz, z \rangle - \|Tz\|^2$$

so that $\|Tz\|^2 \leq \operatorname{Re} \langle Tz, z \rangle$. Taking $\|z\| = 1$ and using the Cauchy inequality, we obtain $\|Tz\|^2 \leq \|Tz\|$, which implies that $\|Tz\| \leq 1$ for all z with $\|z\| = 1$. $\square$

It is easy to see that, with the above notation, we also have $\|(I + H)^{-1}\| \leq 1$.

PROPOSITION 2.4. *With the assumptions of Proposition 2.1, we have*

$$(2.4) \quad \|c^\kappa I + \mathbf{G}^\kappa(s)\| \leq \frac{1}{\kappa(1 - \kappa c)} \quad \forall s \in \mathbb{C}_0.$$

Proof. Since $\mathbf{H}$ is positive-real, the last lemma implies that

$$\|\mathbf{H}(s)(I + \mathbf{H}(s))^{-1}\| \leq 1 \quad \forall s \in \mathbb{C}_0.$$

Now (2.4) follows from this estimate and (2.3). $\square$

The bound (2.4) is quite sharp: in the scalar case, as s approaches a pole of $\mathbf{G}$ on the imaginary axis, (2.4) tends to an equality. Note that in Propositions 2.1–2.4, $\mathbf{G}$ was not assumed to be rational.

Now consider a finite-dimensional system Σ described by the equations

$$\begin{cases} \dot{z}(t) &= Az(t) + Bu(t), \\ y(t) &= B^*z(t) + Du(t), \end{cases}$$

where A, B and D are real matrices of appropriate dimensions and $A^* + A = -Q \leq 0$ so that assumptions ESAD and COL are satisfied. If the number κ is such that $I + \kappa D$ is invertible, then the feedback $u = -\kappa y + v$ leads to the closed-loop system Σ^κ described by the equations

$$(2.5) \quad \begin{cases} \dot{z}(t) &= (A - B\kappa(I + \kappa D)^{-1}B^*)z(t) + B(I + \kappa D)^{-1}v(t), \\ y(t) &= (I + \kappa D)^{-1}B^*z(t) + D(I + \kappa D)^{-1}v(t). \end{cases}$$

We are interested in conditions that guarantee that the matrix

$$A^\kappa = A - B\kappa(I + \kappa D)^{-1}B^*$$

appearing in (2.5) is stable, i.e., its eigenvalues are in the open left half-plane. The finite-dimensional version of Theorem 1.1 reads as follows.

PROPOSITION 2.5. *With the above notation, if (A, B) is controllable and (A, B^*) is observable, then there exists a $\kappa_0 > 0$ such that for all $\kappa \in (0, \kappa_0)$, A^κ is stable.*

Like Theorem 1.1, this is only a corollary of a stronger result (Proposition 2.6 below) which is a little more complicated to state. This stronger result is the finite-dimensional counterpart of Theorem 5.8. To state it, we introduce the notation

$$(2.6) \quad \kappa_0 = \frac{1}{c}, \quad \text{where } c = \|E^+\|, \quad E = -\frac{1}{2}(D^* + D).$$

Here, E^+ denotes the positive part of E , i.e., $E^+ = EP^+$ where P^+ is the spectral projector corresponding to all the positive eigenvalues of E (hence $E^+ \geq E$ but $\|E^+\| \leq \|E\|$). Note that if $E \leq 0$, then $c = 0$, and then we put $\kappa_0 = \infty$.

PROPOSITION 2.6. *With the above notation, if (A, B) is stabilizable and (A, B^*) is detectable, then for all $\kappa \in (0, \kappa_0)$, A^κ is stable.*

Proof. The transfer function of Σ is $\mathbf{G}(s) = B^*(sI - A)^{-1}B + D$ and we have

$$\mathbf{G}(s)^* + \mathbf{G}(s) \geq D^* + D \quad \forall s \in \mathbb{C}_0.$$

This implies that, denoting $E = -\frac{1}{2}(D^* + D)$, $\mathbf{G} + E$ is positive-real, and hence $\mathbf{G} + E^+$ is positive-real. Then for $c = \|E^+\|$, $cI + \mathbf{G}$ is positive-real. (A simpler but more restrictive choice would be $c = \|E\|$.) The transfer function of the closed-loop system Σ^κ from (2.5) is $\mathbf{G}^\kappa$. By Proposition 2.1, we have $\mathbf{G}^\kappa \in H^\infty$ for all $\kappa \in (0, \kappa_0)$. It is easy to show that Σ^κ is stabilizable and detectable (because these properties are preserved by output feedback). It is well known that if a finite-dimensional system is stabilizable, detectable and input-output stable, then it is stable. $\square$

We now return to the discussion of the infinite-dimensional case. In this paper we replace the concept of positive-real transfer function with the more general concept of a positive transfer function. An analytic $\mathcal{L}(U)$ -valued function on $\mathbb{C}_0$ is called a *positive transfer function* if (2.1) holds. Note that for simplicity we have dropped the condition concerning complex conjugates, since it is not needed in our arguments: in particular, it is not needed in Proposition 2.1. (Defining the complex conjugate of an operator is a bit awkward and not necessary.) In the finite-dimensional case, this slight generalization amounts to dropping the requirement that the system matrices should be real. Note that if the generator A is dissipative and $B \in \mathcal{L}(U, X)$ (i.e., B is bounded), then $\mathbf{G}(s) = B^*(sI - A)^{-1}B$ is a positive transfer function. This is not always true for unbounded B , as we shall show.

The first PDE examples fitting into our framework assumed that A is dissipative, B is bounded and the open-loop transfer function is $\mathbf{G}(s) = B^*(sI - A)^{-1}B$ (i.e., $D = 0$); see Bailey and Hubbard [5], Balakrishnan [6, 7], Russell [30], and Slemrod [33, 34]. In this case, $\mathbf{G}$ is positive and (by Proposition 2.1) the feedback $u = -\kappa y + v$ stabilizes in an input-output sense. Of course, the most desirable type of stability is exponential stability and for this, in the special case $A^* + A = 0$, we need the system to be exactly controllable (or equivalently, exactly observable). This is the setup studied in Haraux [15], Liu [24] and others.

Many examples of flexible beams, plates and hybrid structures with colocated control and observation have been shown to be exponentially stabilizable by static output feedback; see, for example, Chen [10, 11], Rebarber [27], Triggiani [40], Tucsnak and Weiss [41], Luo, Guo and Morgul [25], Guo and Luo [14], and Ammari and Tucsnak [2]. In many of these examples the approach is a classical Lyapunov one, with the key step being the appropriate PDE formulation so that the energy of the system can play the role of a Lyapunov functional. If one examines these examples carefully, one can recognize that they fit into our framework (the assumptions ESAD and COL are satisfied) and they use the feedback $u = -\kappa y + v$ for stabilization. There are also examples in the literature where the open-loop system is not well posed, but application of the static feedback results in a well-posed exponentially stable closed-loop system, see Rebarber [27], Rebarber and Townley [28], Weiss [49], Lasiecka and Triggiani [20]. These examples are not covered by the theory in this paper (except for some partial results in Remarks 5.3 and 5.4).

The recent paper [20] is interesting because it also gives an abstract framework for treating exponential stabilization by colocated feedback. The assumptions are ESAD (with $Q = 0$), COL, $0 \in \rho(A)$ and $A^{-\frac{1}{2}}B \in \mathcal{L}(U, X)$. The last two assumptions (and also the fact that $Q = 0$) are more restrictive than in our framework, but on the other hand, they do not require the open-loop system to be well posed. We note that our examples in sections 5 and 6 are not covered by the theory in [20], because they do not satisfy $A^{-\frac{1}{2}}B \in \mathcal{L}(U, X)$. The main thrust of [20] is to give examples of non-well-posed systems satisfying the assumptions mentioned earlier, for which $A - BB^*$ is exponentially stable. (See also the corrections to [20] in [21].)

As mentioned at the end of section 1, there is also a rich literature dealing with weak or strong stabilization by colocated feedback, which we shall discuss elsewhere.

3. Admissible control and observation operators. In this section we gather, for easy reference, some basic facts about admissible control and observation operators and various controllability and observability concepts. For proofs and for more details we refer to the literature.

We assume that X is a Hilbert space and $A : \mathcal{D}(A) \rightarrow X$ is the generator of a strongly continuous semigroup $\mathbb{T}$ on X . We define the Hilbert space X_1 as $\mathcal{D}(A)$ with the norm $\|z\|_1 = \|(\beta I - A)z\|$, where $\beta \in \rho(A)$ is fixed (this norm is equivalent to the graph norm). The Hilbert space X_{-1} is the completion of X with respect to the norm $\|z\|_{-1} = \|(\beta I - A)^{-1}z\|$. This space is isomorphic to $\mathcal{D}(A^*)^*$, the dual of $\mathcal{D}(A^*)$ with respect to the pivot space X , and we have the continuous embeddings

$$(3.1) \quad X_1 \subset X \subset X_{-1}.$$

$\mathbb{T}$ extends to a semigroup on X_{-1} , denoted by the same symbol. The generator of this extended semigroup is an extension of A , whose domain is X , so that $A : X \rightarrow X_{-1}$, see Weiss [42]. We denote by $\omega(\mathbb{T})$ the growth bound of $\mathbb{T}$. The semigroup $\mathbb{T}$ is called *exponentially stable* if $\omega(\mathbb{T}) < 0$.

We assume that U is a Hilbert space and $B \in \mathcal{L}(U, X_{-1})$ is an *admissible control operator* for $\mathbb{T}$, defined as in [42]. This means that if z is the solution of $\dot{z}(t) = Az(t) + Bu(t)$, as in (1.1), which is an equation in X_{-1} , with $z(0) = z_0 \in X$ and $u \in L^2([0, \infty), U)$, then $z(t) \in X$ for all $t \geq 0$. In this case, z is a continuous X -valued function of t . We have that for all $t \geq 0$,

$$(3.2) \quad z(t) = \mathbb{T}_t z_0 + \Phi_t u,$$

where $\Phi_t \in \mathcal{L}(L^2([0, \infty), U), X)$ is defined by

$$(3.3) \quad \Phi_t u = \int_0^t \mathbb{T}_{t-\sigma} B u(\sigma) d\sigma.$$

The above integration is done in X_{-1} , but the result is in X . The Laplace transform of z is

$$\hat{z}(s) = (sI - A)^{-1} [z_0 + B\hat{u}(s)].$$

B is called *bounded* if $B \in \mathcal{L}(U, X)$ (and unbounded otherwise). If B is an admissible control operator for $\mathbb{T}$, then for every $\omega > \omega(\mathbb{T})$ there exists a positive constant δ such that

$$(3.4) \quad \|(sI - A)^{-1}B\|_{\mathcal{L}(U, X)} \leq \frac{\delta}{\sqrt{\operatorname{Re} s}} \quad \forall \operatorname{Re} s > \omega.$$

If $\dim U < \infty$ and $\mathbb{T}$ is normal or contractive, then (3.4) implies the admissibility of B , see Jacob and Partington [16] and Weiss [47, 48]. Similarly, if $\mathbb{T}$ is left-invertible, then again (3.4) implies the admissibility of B , see [47]. If $\mathbb{T}$ is invertible and $\tilde{\mathbb{T}}$ is the inverse semigroup (i.e., $\tilde{\mathbb{T}}_t = (\mathbb{T}_t)^{-1}$), then B is admissible for $\tilde{\mathbb{T}}$ if and only if it is admissible for $\mathbb{T}$, see, for example, [42].

The *degree of unboundedness* of an operator $B \in \mathcal{L}(U, X_{-1})$, denoted $\alpha(B)$, is the infimum of those $\alpha \geq 0$ for which there exist positive constants δ, ω such that

$$(3.5) \quad \|(\lambda I - A)^{-1}B\|_{\mathcal{L}(U, X)} \leq \frac{\delta}{\lambda^{1-\alpha}} \quad \forall \lambda \in (\omega, \infty).$$

It is clear from (3.4) that for any admissible $B \in \mathcal{L}(U, X_{-1})$ we have $\alpha(B) \leq \frac{1}{2}$, and if B is bounded then $\alpha(B) = 0$ (see [29] for further comments on $\alpha(B)$).

We assume that Y is another Hilbert space and $C \in \mathcal{L}(X_1, Y)$ is an *admissible observation operator* for $\mathbb{T}$, defined as in Weiss [43]. This means that for every $T > 0$ there exists a $K_T \geq 0$ such that

$$(3.6) \quad \int_0^T \|C\mathbb{T}_t z_0\|^2 dt \leq K_T^2 \|z_0\|^2 \quad \forall z_0 \in \mathcal{D}(A).$$

C is called *bounded* if it can be extended such that $C \in \mathcal{L}(X, Y)$.

We regard $L_{loc}^2([0, \infty), Y)$ as a Fréchet space with the seminorms being the L^2 norms on the intervals $[0, n]$, $n \in \mathbb{N}$. Then the admissibility of C means that there is a continuous operator $\Psi : X \rightarrow L_{loc}^2([0, \infty), Y)$ such that

$$(3.7) \quad (\Psi z_0)(t) = C\mathbb{T}_t z_0 \quad \forall z_0 \in \mathcal{D}(A).$$

The operator Ψ is completely determined by (3.7), because $\mathcal{D}(A)$ is dense in X . We introduce an extension of C , called the Λ -extension of C , defined by

$$(3.8) \quad C_\Lambda z_0 = \lim_{\lambda \rightarrow +\infty} C\lambda(\lambda I - A)^{-1} z_0,$$

whose domain $\mathcal{D}(C_\Lambda)$ consists of all $z_0 \in X$ for which the limit exists. We shall also use the *weak Λ -extension* of C , $C_{\Lambda w}$. It is defined as in (3.8), but replacing the strong limit by the weak limit. Thus, $C_{\Lambda w}$ is an extension of C_Λ to an even larger subspace of X , denoted by $\mathcal{D}(C_{\Lambda w})$. If we replace C by C_Λ , formula (3.7) becomes true for all $z_0 \in X$ and for almost every $t \geq 0$. If $y = \Psi z_0$, then its Laplace transform is

$$(3.9) \quad \hat{y}(s) = C(sI - A)^{-1} z_0.$$

The following duality result holds: if $\mathbb{T}$ is a semigroup on X with generator A , then $B \in \mathcal{L}(U, X_{-1})$ is an admissible control operator for $\mathbb{T}$ if and only if $B^* : \mathcal{D}(A^*) \rightarrow U$ is an admissible observation operator for the dual semigroup $\mathbb{T}^*$.

The dual version of (3.4) is as follows: if C is an admissible observation operator for $\mathbb{T}$, then for every $\omega > \omega(\mathbb{T})$ there exists a positive constant δ such that

$$(3.10) \quad \|C(sI - A)^{-1}\|_{\mathcal{L}(X, Y)} \leq \frac{\delta}{\sqrt{\operatorname{Re} s}} \quad \forall \operatorname{Re} s > \omega.$$

The *degree of unboundedness* of C , denoted $\alpha(C)$, is defined similarly as $\alpha(B)$. We have $\alpha(C) = \alpha(C^*)$, where C^* is regarded as a control operator for $\mathbb{T}^*$.

DEFINITION 3.1. *Let A be the generator of a strongly continuous semigroup $\mathbb{T}$ on X and let $B \in \mathcal{L}(U, X_{-1})$ be an admissible control operator for $\mathbb{T}$. The pair (A, B) is exactly controllable in time $T > 0$, if for every $x_0 \in X$ there exists a $u \in L^2([0, T], U)$ such that*

$$\Phi_T u = \int_0^T \mathbb{T}_{T-\sigma} B u(\sigma) d\sigma = x_0.$$

(A, B) is exactly controllable if it is exactly controllable in some finite time $T > 0$.

It will be useful to note that if A is the generator of a strongly continuous group (i.e., $-A$ is also a semigroup generator), then (A, B) is exactly controllable in time T if and only if $(-A, B)$ is exactly controllable in time T .

We introduce observability concepts via duality. Suppose that A is the generator of the strongly continuous semigroup $\mathbb{T}$ on X and $C \in \mathcal{L}(X_1, Y)$ is an admissible

observation operator for $\mathbb{T}$. Of course, this is equivalent to C^* being an admissible control operator for the dual semigroup $\mathbb{T}^*$. We say that (A, C) is *exactly observable (in time T)* if (A^*, C^*) is exactly controllable (in time T).

For more details on exact controllability in an operator-theoretic setting we refer also to Avdonin and Ivanov [4], Guo and Luo [14], Jacob and Zwart [17], Miller [26], Rebarber and Weiss [29], Russell and Weiss [31] and the references therein. In the PDE setting, the relevant literature on controllability is overwhelming, and we mention the books of Lions [23], Lagnese and Lions [22] and Komornik [19] and the paper of Bardos, Lebeau and Rauch [8].

4. Some background on well-posed linear systems. In this section we collect some basic facts about well-posed and regular linear systems, their transfer functions, output feedback and closed-loop systems. Only one result (Proposition 4.1) is new, and of course its proof is included.

By a *well-posed linear system* we mean a linear time-invariant system such that on any finite time interval, the operator from the initial state and the input function to the final state and the output function is bounded. The input, state and output spaces are Hilbert spaces, and the input and output functions are of class L^2_{loc} . To express this more clearly, let us denote by U the input space, by X the state space and by Y the output space of a well-posed linear system Σ . The input and output functions u and y are locally L^2 functions with values in U and in Y . The state trajectory z is an X -valued function. The boundedness property mentioned earlier means that for every $\tau > 0$ there is a $c_\tau \geq 0$ such that

$$(4.1) \quad \|z(\tau)\|^2 + \int_0^\tau \|y(t)\|^2 dt \leq c_\tau^2 \left(\|z(0)\|^2 + \int_0^\tau \|u(t)\|^2 dt \right)$$

(with c_τ independent of $z(0)$ and of u). For the detailed definition, background and examples we refer to Salamon [32], Staffans [35, 36, 37], Weiss [45, 46], Weiss and Rebarber [51] and Weiss, Staffans and Tucsnak [52].

We recall some necessary facts about well-posed linear systems. Let Σ be such a system, with input space U , state space X and output space Y . Then there are operators A, B, C satisfying the assumptions in the previous section, which are related to Σ in the following way. First of all, the state trajectories of Σ satisfy the equation (1.1), so that they are given by (3.2). $\mathbb{T}$ is called the *semigroup* of Σ , A is called its *semigroup generator*, the family $\Phi = (\Phi_t)_{t \geq 0}$ is called the *input maps* of Σ , and B is called the *control operator* of Σ . If u is the input function of Σ , z_0 is its initial state, and y is the corresponding output function, then

$$(4.2) \quad y = \Psi z_0 + \mathbb{F}u.$$

Here, Ψ is an operator as in (3.7), called the (extended) *output map* of Σ and C is called the *observation operator* of Σ . $\mathbb{F}$ is a continuous linear operator from $L^2_{\text{loc}}([0, \infty), U)$ to $L^2_{\text{loc}}([0, \infty), Y)$, called the (extended) *input-output map* of Σ . The formal definition of Σ in [44, 45] is via $\mathbb{T}, \Phi, \Psi$ and $\mathbb{F}$ so that $\Sigma = (\mathbb{T}, \Phi, \Psi, \mathbb{F})$.

The operator $\mathbb{F}$ appearing above is easiest to represent using Laplace transforms. To state this in precise terms, we introduce the notation $\mathbb{C}_\alpha$ for the open right half-plane consisting of those $s \in \mathbb{C}$ for which $\text{Re } s > \alpha$. An operator-valued analytic function $\mathbf{G}$ is called *well posed* (or *proper*) if there exists $\alpha \in \mathbb{R}$ such that the domain of $\mathbf{G}$ contains $\mathbb{C}_\alpha$ and $\mathbf{G}$ is uniformly bounded on $\mathbb{C}_\alpha$. We do not distinguish between two well-posed functions if one is a restriction of the other (to a smaller domain in

(C). If $\mathbb{F}$ is the input-output map of a well-posed system, then there exists a unique $\mathcal{L}(U, Y)$ -valued well-posed function $\mathbf{G}$, called the *transfer function* of Σ , which determines $\mathbb{F}$ as follows. If $u \in L^2([0, \infty), U)$ and $y = \mathbb{F}u$, then y has a Laplace transform $\hat{y}$ and, for $\operatorname{Re} s > \max\{\omega(\mathbb{T}), 0\}$ (recall that $\omega(\mathbb{T})$ is the growth bound of $\mathbb{T}$),

$$\hat{y}(s) = \mathbf{G}(s)\hat{u}(s).$$

This determines $\mathbb{F}$, since $L^2([0, \infty), U)$ is dense in $L^2_{loc}([0, \infty), U)$. $\mathbf{G}$ is uniformly bounded on every half-plane $\mathbb{C}_\omega$ with $\omega > \omega(\mathbb{T})$. Because of the identification of well-posed functions mentioned earlier, by a transfer function we mean in fact an equivalence class of analytic functions. For all $s, \beta \in \mathbb{C}_{\omega(\mathbb{T})}$ we have

$$(4.3) \quad \mathbf{G}(s) - \mathbf{G}(\beta) = C[(sI - A)^{-1} - (\beta I - A)^{-1}]B.$$

This shows that $\mathbf{G}$ is determined by A, B and C up to an additive constant operator. We call (A, B, C) the *generating triple* of Σ . In the time domain, the output function y corresponding to the input function u and state trajectory z is given by

$$(4.4) \quad y(t) = C_\Lambda [z(t) - (\beta I - A)^{-1}Bu(t)] + \mathbf{G}(\beta)u(t),$$

valid for almost every $t \geq 0$, if $\beta \in \mathbb{C}_{\omega(\mathbb{T})}$. Thus, the system Σ is completely determined (via (1.1) and (4.4)) by its generating triple (A, B, C) and by the value of its transfer function at one point.

The well-posed linear system Σ is called *regular* if the limit

$$(4.5) \quad \lim_{\lambda \rightarrow +\infty} \mathbf{G}(\lambda)v = Dv$$

exists for every $v \in U$, where λ is real (see [44, 45]). In this case, the operator $D \in \mathcal{L}(U, Y)$ is called the *feedthrough operator* of Σ . Regularity is equivalent to the fact that the product $C_\Lambda(sI - A)^{-1}B$ makes sense, for some (hence, for every) $s \in \rho(A)$. In this case, the formula for $\mathbf{G}$ looks like the finite-dimensional one:

$$(4.6) \quad \mathbf{G}(s) = C_\Lambda(sI - A)^{-1}B + D.$$

Moreover, the function y from (4.2) satisfies, for almost every $t \geq 0$,

$$(4.7) \quad y(t) = C_\Lambda z(t) + Du(t),$$

where z is the state trajectory of the system (compare this with (4.4)). The operators A, B, C, D are called the *generating operators* of Σ , because they determine Σ via (1.1) and (4.7). In particular, if A is a generator, one of B and C is admissible and the other is bounded, then for any $D \in \mathcal{L}(U, Y)$, the operators A, B, C and D are the generating operators of a regular linear system.

The well-posed linear system Σ is called *weakly regular* if the limit in (4.5) exists in the weak sense, see [37]. In this case, the weak limit still defines an operator $D \in \mathcal{L}(U, Y)$, called the *feedthrough operator* of Σ . Weak regularity is equivalent to the fact that the product $C_{\Lambda w}(sI - A)^{-1}B$ makes sense, for some (hence, for every) $s \in \rho(A)$. In this case, the formulas (4.6) and (4.7) become valid after we replace C_Λ by $C_{\Lambda w}$. This slight generalization of regular systems (to weakly regular ones) is sometimes needed because the dual of a regular linear system is not regular, in general, but it is weakly regular. Note that if Y is finite-dimensional, then the regularity of Σ is equivalent to its weak regularity.

PROPOSITION 4.1. *Let Σ be a well-posed linear system with control operator B , observation operator C and transfer function $\mathbf{G}$. If*

$$(4.8) \quad \alpha(B) + \alpha(C) < 1,$$

then Σ is regular. In fact, $\lim_{\lambda \rightarrow +\infty} \mathbf{G}(\lambda)$ exists in the operator norm.

Proof. From (4.3) it follows that $\mathbf{G}'(s) = C(sI - A)^{-2}B$ for all $s \in \mathbb{C}_{\omega(\mathbb{T})}$. So for all $\lambda > 0$ sufficiently large, we have

$$\|\mathbf{G}'(\lambda)\|_{\mathcal{L}(U,Y)} \leq \|C(\lambda I - A)^{-1}\|_{\mathcal{L}(X,Y)} \cdot \|(\lambda I - A)^{-1}B\|_{\mathcal{L}(U,X)}.$$

Hence, for every $\alpha_1 > \alpha(C)$ and $\alpha_2 > \alpha(B)$ we can find $\delta_1, \delta_2 > 0$ such that, for all $\lambda > 0$ sufficiently large,

$$\|\mathbf{G}'(\lambda)\|_{\mathcal{L}(U,Y)} \leq \frac{\delta_1}{\lambda^{1-\alpha_1}} \cdot \frac{\delta_2}{\lambda^{1-\alpha_2}}.$$

If (4.8) holds, then α_1 and α_2 can be chosen such that $\alpha_1 + \alpha_2 = 1 - \varepsilon$, with $\varepsilon > 0$. Denoting $\delta = \delta_1 \delta_2$, we obtain that for all sufficiently large $\lambda > 0$,

$$\|\mathbf{G}'(\lambda)\|_{\mathcal{L}(U,Y)} \leq \frac{\delta}{\lambda^{1+\varepsilon}}, \quad \varepsilon > 0.$$

By integration we obtain that for $\lambda_1 < \lambda_2$ sufficiently large,

$$\|\mathbf{G}(\lambda_1) - \mathbf{G}(\lambda_2)\| \leq \int_{\lambda_1}^{\lambda_2} \|\mathbf{G}'(\lambda)\| d\lambda \leq \frac{\delta}{\varepsilon} \left| \frac{1}{\lambda_1^\varepsilon} - \frac{1}{\lambda_2^\varepsilon} \right|$$

so that $\lim_{\lambda \rightarrow +\infty} \mathbf{G}(\lambda)$ exists (which is stronger than (4.5)). $\square$

It is often useful to introduce the space

$$(4.9) \quad Z = X_1 + (\beta I - A)^{-1}BU,$$

where $\beta \in \rho(A)$ (the space Z does not depend on the choice of β). Z is a Hilbert space with the following factor space norm:

$$\|z\|_Z = \inf_{z=x+(\beta I-A)^{-1}Bv} (\|x\|_1^2 + \|v\|_U^2)^{\frac{1}{2}}.$$

It is easy to see that $X_1 \subset Z \subset X$ with continuous embeddings, but X_1 need not be dense in Z . It was shown in [37, section 3] that for any well-posed system, C can be extended to an operator $\overline{C} \in \mathcal{L}(Z; Y)$ (this extension may be nonunique). For every $s \in \rho(A)$, the product $\overline{C}(sI - A)^{-1}B$ exists. For every such extension $\overline{C}$ there exists a unique $D \in \mathcal{L}(U, Y)$ such that $\mathbf{G}(s) = \overline{C}(sI - A)^{-1}B + D$. (For regular systems, we may take $\overline{C} = C_\Lambda$ and then D becomes the feedthrough operator of the system.) If $u \in \mathcal{H}_{loc}^1(0, \infty; U)$ and $Az(0) + Bu(0) \in X$, then for all $t \geq 0$ we have $z(t) \in Z$, the equation $\dot{z}(t) = Az(t) + Bu(t)$ holds for every $t \geq 0$, the output function y is continuous, and it is given for all $t \geq 0$ by

$$(4.10) \quad y(t) = \overline{C}z(t) + Du(t).$$

Let Σ be a well-posed linear system with generating triple (A, B, C) and transfer function $\mathbf{G}$. An operator $K \in \mathcal{L}(Y, U)$ is called an *admissible feedback operator* for Σ

(or for $\mathbf{G}$) if $I - \mathbf{G}K$ has a well-posed inverse (equivalently, if $I - K\mathbf{G}$ has a well-posed inverse). If this is the case, then the system with output feedback $u = Ky + v$ (see Figure 1, but with $-K$ in place of κ) is well posed (its input is v , its state and output are the same as for Σ), see [46]. This new system is called the *closed-loop system* corresponding to Σ and K , and it is denoted by Σ^K . Its transfer function is $\mathbf{G}^K = \mathbf{G}(I - K\mathbf{G})^{-1} = (I - \mathbf{G}K)^{-1}\mathbf{G}$. We have that $-K$ is an admissible feedback operator for Σ^K and the corresponding closed-loop system is Σ . Let us denote by (A^K, B^K, C^K) the generating triple of Σ^K . Then for every $x_0 \in \mathcal{D}(A^K)$ and for every $z_0 \in \mathcal{D}(A)$,

$$(4.11) \quad A^K x_0 = (A + BK C^K) x_0, \quad A z_0 = (A^K - B^K K C) z_0.$$

Note that $B^K \in \mathcal{L}(U, X_{-1}^K)$ and $C^K \in \mathcal{L}(X_1^K, Y)$, where X_{-1}^K and X_1^K are the counterparts of X_{-1} and X_1 for Σ^K . Any interconnection of finitely many well-posed linear systems can be thought of as a closed-loop system in the above sense.

The following invariance result is taken from [46, section 6].

PROPOSITION 4.2. *Let Σ be a well-posed linear system, let K be an admissible feedback operator for Σ , and let Σ^K be the corresponding closed-loop system. We denote by (A, B, C) the generating triple of Σ and by (A^K, B^K, C^K) the generating triple of Σ^K . Then the following holds:*

- (a) (A, B) is exactly controllable if and only if (A^K, B^K) has the same property,
- (b) (A, C) is exactly observable if and only if (A^K, C^K) has the same property.

We recall some concepts that are used in the optimal control literature (under different names), following the formulation in Weiss and Rebarber [51].

DEFINITION 4.3. *Let A be the generator of a strongly continuous semigroup $\mathbb{T}$ on X and suppose that $B \in \mathcal{L}(U, X_{-1})$ is an admissible control operator for $\mathbb{T}$. Then (A, B) is optimizable if for every $z_0 \in X$ there exists a $u \in L^2([0, \infty), U)$ such that the state trajectory z defined in (3.2) is in $L^2([0, \infty), X)$.*

Let $C \in \mathcal{L}(X_1, Y)$ be an admissible observation operator for $\mathbb{T}$. Then (A, C) is estimatable if (A^, C^*) is optimizable.*

Estimatability can be formulated also directly, without using adjoints, see [51]. Optimizability is one possible generalization of the concept of stabilizability from finite-dimensional systems to infinite-dimensional ones. Similarly, estimatability is one possible generalization of detectability. It is clear that exact controllability implies optimizability, and exact observability implies estimatability. It was shown in [51] that optimizability and estimatability are invariant under output feedback (just like exact controllability and exact observability).

PROPOSITION 4.4. *With the notation from Proposition 4.2, (A, B) is optimizable if and only if (A^K, B^K) is optimizable and (A, C) is estimatable if and only if (A^K, C^K) is estimatable.*

We quote from [51] the following characterization of exponential stability.

THEOREM 4.5. *A well-posed linear system is exponentially stable if and only if it is optimizable, estimatable and input-output stable.*

It follows from this theorem that a well-posed linear system is exponentially stable if it is exactly controllable, exactly observable, and input-output stable.

5. Positivity and exponential stabilization. In this section we prove a rather technical result (Theorem 5.2) about the positivity of the transfer function $\mathbf{G} + E$, where $\mathbf{G}$ is the transfer function of the system Σ satisfying ESAD and COL, and E is given by a certain limit, see (5.4). If B is not too unbounded (more precisely, if

$\alpha(B) < \frac{1}{2}$), then Σ is regular and

$$(5.1) \quad E = -\frac{1}{2}(D^* + D),$$

where D is the feedthrough operator of Σ . Formula (5.1) may hold even if $\alpha(B) = \frac{1}{2}$, but we show by a counterexample that (5.1) may fail if $\alpha(B) = \frac{1}{2}$. If $0 \in \rho(A)$ and $Q = 0$, then it turns out that E is related to the direct current (DC)-gain of the system, $\mathbf{G}(0)$:

$$E = -\frac{1}{2}[\mathbf{G}(0)^* + \mathbf{G}(0)],$$

see (5.5). For this formula to hold, Σ does not need to be regular.

We also prove a result (Proposition 5.7) about the stabilization of systems whose transfer function $\mathbf{G}$ is such that for some $c \geq 0$, $cI + \mathbf{G}$ is a positive transfer function. These two results imply our main result about exponential stabilization, Theorem 5.8 (which is stronger than Theorem 1.1). We consider the general class of well-posed linear systems, but we also explain some consequences for the smaller but simpler class of weakly regular linear systems.

Notation and assumptions. We consider a well-posed system Σ with input and output space U , state space X , semigroup $\mathbb{T}$, and transfer function $\mathbf{G}$. Σ satisfies the assumptions ESAD and COL from section 1, and the operators A, Q and B are as in section 1. The space X_{-1} is defined as in section 3.

It follows from ESAD that the growth bound of $\mathbb{T}$ satisfies $\omega(\mathbb{T}) \leq 0$ so that $\mathbf{G}$ is defined on $\mathbb{C}_0$. Assumption ESAD also implies that $\mathbb{T}$ is invertible, so that $\sigma(A)$ is contained in a vertical strip in the closed left half-plane. Thus, $\rho(A)$ includes a left half-plane. We now introduce a natural extension of $\mathbf{G}$ to $\rho(A)$.

LEMMA 5.1. *With the above notation and assumptions, there exists a unique extension of $\mathbf{G}$ to $\rho(A)$ which satisfies*

$$(5.2) \quad \mathbf{G}(s) - \mathbf{G}(\beta) = B^*[(sI - A)^{-1} - (\beta I - A)^{-1}]B \quad \forall s, \beta \in \rho(A).$$

This extension is analytic, and it is bounded on any right half-plane $\mathbb{C}_\varepsilon$ with $\varepsilon > 0$, as well as on some left half-plane.

Note that if $\rho(A)$ is connected, then the above extension of $\mathbf{G}$ coincides with its analytic continuation to $\rho(A)$.

Proof. It is clear that the original $\mathbf{G}$ satisfies (5.2) on $\mathbb{C}_0$, because (5.2) is just (4.3) with $C = B^*$. This implies that $\mathbf{G}(s)$ can be defined for $s \in \rho(A)$ by (5.2) with $\beta \in \mathbb{C}_0$, and the definition is independent of the choice of β . It is also clear that the extended $\mathbf{G}$ is analytic, and it satisfies (5.2) if at least one of the numbers s, β is in $\mathbb{C}_0$. We have to check that (5.2) also holds if neither s nor β is in $\mathbb{C}_0$. To show this, we choose $z \in \mathbb{C}_0$ and we decompose

$$\mathbf{G}(s) - \mathbf{G}(\beta) = [\mathbf{G}(s) - \mathbf{G}(z)] + [\mathbf{G}(z) - \mathbf{G}(\beta)].$$

Now (5.2) follows from the fact that it holds for each of the two terms. It follows from the theory in section 4 that $\mathbf{G}$ is bounded on $\mathbb{C}_\varepsilon$ for any $\varepsilon > 0$, since $\omega(\mathbb{T}) \leq 0$.

Finally, we have to show that the extended $\mathbf{G}$ is bounded on some left half-plane. Choose $\mu > 0$ such that $\sigma(A)$ is contained in the vertical strip, where $-\mu \leq \operatorname{Re} s \leq 0$. Then for $\operatorname{Re} s > \mu$, (5.2) with the resolvent identity implies that

$$\mathbf{G}(s) - \mathbf{G}(-\bar{s}) = 2(\operatorname{Re} s)B^*(sI - A)^{-1}(\bar{s}I + A)^{-1}B.$$

B is an admissible control operator for the inverse semigroup $\tilde{\mathbb{T}}_t = (\mathbb{T}_t)^{-1}$ (see section 3, after (3.4)). It follows from (3.4) rewritten for $\tilde{\mathbb{T}}$ that for some $\delta^1, \omega^1 > 0$,

$$\|(sI + A)^{-1}B\|_{\mathcal{L}(U, X)} \leq \frac{\delta^1}{\sqrt{\operatorname{Re} s}} \quad \forall \operatorname{Re} s > \omega^1.$$

Applying this estimate and (3.10), we obtain that for $\operatorname{Re} s$ sufficiently large,

$$\|\mathbf{G}(s) - \mathbf{G}(-\bar{s})\| \leq 2\delta\delta^1.$$

Since $\mathbf{G}$ is bounded on $\mathbb{C}_\varepsilon$ (for any $\varepsilon > 0$), this estimate shows that it is also bounded on some left half-plane. $\square$

The last lemma can be derived also as a consequence of the theory of time-invertible systems in Staffans and Weiss [38].

THEOREM 5.2. *With the above notation and assumptions, there exist operators $E = E^* \in \mathcal{L}(U)$ such that $\mathbf{G} + E$ is a positive transfer function, i.e.,*

$$(5.3) \quad \mathbf{G}(s)^* + \mathbf{G}(s) + 2E \geq 0 \quad \forall s \in \mathbb{C}_0.$$

One such operator E is given by

$$(5.4) \quad E = -\frac{1}{2} \lim_{\lambda \rightarrow +\infty} [\mathbf{G}(\lambda)^* + \mathbf{G}(-\lambda)].$$

If $0 \in \rho(A)$, then the same E is also given by

$$(5.5) \quad E = -\frac{1}{2} [\mathbf{G}(0)^* + \mathbf{G}(0)] + B^*(A^*)^{-1}QA^{-1}B.$$

If Σ is weakly regular, with feedthrough operator D , then E is also given by

$$(5.6) \quad \langle Ev, v \rangle = -\frac{1}{2} \langle (D^* + D)v, v \rangle + \lim_{\lambda \rightarrow +\infty} \lambda \|(\lambda I - A)^{-1}Bv\|^2 \quad \forall v \in U.$$

Note that the limit in (5.4) exists in the operator norm, even if Σ is not regular.

Proof. We shall use the following identity (a consequence of ESAD):

$$(5.7) \quad (\bar{s}I - A^*)^{-1} + (sI - A)^{-1} = (\bar{s}I - A^*)^{-1} [2(\operatorname{Re} s)I + Q] (sI - A)^{-1}.$$

Using the formulas (4.3) and (5.7), we calculate

$$\begin{aligned} & [\mathbf{G}(s)^* + \mathbf{G}(s)] - [\mathbf{G}(\beta)^* + \mathbf{G}(\beta)] \\ &= B^*[(sI - A)^{-1} - (\beta I - A)^{-1}]B + B^*[(\bar{s}I - A^*)^{-1} - (\bar{\beta}I - A^*)^{-1}]B \\ &= B^*[(\bar{s}I - A^*)^{-1} + (sI - A)^{-1}]B - B^*[(\bar{\beta}I - A^*)^{-1} + (\beta I - A)^{-1}]B \\ &= B^*(\bar{s}I - A^*)^{-1} [2(\operatorname{Re} s)I + Q] (sI - A)^{-1} B \\ &\quad - B^*(\bar{\beta}I - A^*)^{-1} [2(\operatorname{Re} \beta)I + Q] (\beta I - A)^{-1} B, \end{aligned}$$

for all $s, \beta \in \mathbb{C}_0$. Rearranging the above formula, we obtain

$$\begin{aligned} & [\mathbf{G}(s)^* + \mathbf{G}(s)] - B^*(\bar{s}I - A^*)^{-1} [2(\operatorname{Re} s)I + Q] (sI - A)^{-1} B \\ &= [\mathbf{G}(\beta)^* + \mathbf{G}(\beta)] - B^*(\bar{\beta}I - A^*)^{-1} [2(\operatorname{Re} \beta)I + Q] (\beta I - A)^{-1} B. \end{aligned}$$

Thus, both sides are equal to a bounded, self-adjoint operator on U , which depends neither on s nor on β . We denote this operator by $-2E$. Now since

$$B^*(\bar{s}I - A^*)^{-1} [2(\operatorname{Re} s)I + Q] (sI - A)^{-1} B \geq 0 \quad \forall s \in \mathbb{C}_0,$$

we deduce that (5.3) holds. For all $\lambda > 0$ we have

$$(5.8) \quad 2E = -[\mathbf{G}(\lambda)^* + \mathbf{G}(\lambda)] + B^*(\lambda I - A^*)^{-1}[2\lambda I + Q](\lambda I - A)^{-1}B.$$

Since $-A^* = A + Q$, we have

$$(\lambda I - A^*)^{-1} = (\lambda I + A)^{-1} - (\lambda I - A^*)^{-1}Q(\lambda I + A)^{-1}.$$

Substituting this into (5.8), we obtain

$$\begin{aligned} 2E &= -[\mathbf{G}(\lambda)^* + \mathbf{G}(\lambda)] + 2\lambda B^*(\lambda I + A)^{-1}(\lambda I - A)^{-1}B \\ &+ B^*(\lambda I + A)^{-1}Q(\lambda I - A)^{-1}B - B^*(\lambda I - A^*)^{-1}Q(\lambda I + A)^{-1}[2\lambda I + Q](\lambda I - A)^{-1}B. \end{aligned}$$

We want to examine what happens in this rather long formula when $\lambda \rightarrow +\infty$. Since B^* is an admissible observation operator for the inverse semigroup generated by $-A$, by (3.10) $\|B^*(\lambda I + A)^{-1}\|$ decays like $1/\sqrt{\lambda}$ for large λ . Similarly, by (3.4), the factors $\|(\lambda I - A)^{-1}B\| = \|B^*(\lambda I - A^*)^{-1}\|$ decay like $1/\sqrt{\lambda}$ for large λ . The factor $(\lambda I + A)^{-1}[2\lambda I + Q]$ remains bounded as $\lambda \rightarrow +\infty$. It follows from these considerations that the last two terms in the long formula (the ones containing a factor Q) tend to zero as $\lambda \rightarrow +\infty$. Hence,

$$2E = \lim_{\lambda \rightarrow +\infty} [-\mathbf{G}(\lambda)^* - \mathbf{G}(\lambda) + 2\lambda B^*(\lambda I + A)^{-1}(\lambda I - A)^{-1}B].$$

On the other hand, we know from (5.2) and the resolvent identity that for $\lambda > 0$ sufficiently large, using the extension of $\mathbf{G}$ to $\rho(A)$, the following holds:

$$\mathbf{G}(\lambda) - \mathbf{G}(-\lambda) = 2\lambda B^*(\lambda I + A)^{-1}(\lambda I - A)^{-1}B.$$

Substituting this into the previous formula, we get (5.4).

To prove (5.5), assume that $0 \in \rho(A)$. Taking $\lambda \rightarrow 0$ in (5.8), we obtain (5.5).

To prove (5.6), suppose now that Σ is weakly regular with feedthrough operator D . Then, applying (5.8) to $v \in U$, taking the scalar product with v and taking limits as $\lambda \rightarrow +\infty$, we obtain

$$\begin{aligned} 2\langle Ev, v \rangle &= -\lim_{\lambda \rightarrow +\infty} \langle (\mathbf{G}(\lambda)^* + \mathbf{G}(\lambda))v, v \rangle + \lim_{\lambda \rightarrow +\infty} 2\lambda \|(\lambda I - A)^{-1}Bv\|^2 \\ &+ \lim_{\lambda \rightarrow +\infty} \langle Q(\lambda I - A)^{-1}Bv, (\lambda I - A)^{-1}Bv \rangle \\ &= -\langle (D + D^*)v, v \rangle + 2 \lim_{\lambda \rightarrow +\infty} \lambda \|(\lambda I - A)^{-1}Bv\|^2, \end{aligned}$$

since the inequality (3.4) shows that the limit containing Q is zero. $\square$

Remark 5.3. It follows from the last theorem that $cI + \mathbf{G}$ is a positive transfer function for $c = \|E^+\|$, where E^+ is the positive part of the self-adjoint operator E from (5.4). Indeed, this follows from $cI \geq E^+ \geq E$ (for finite-dimensional systems this was in the proof of Proposition 2.6). Note that the proof of (5.3), (5.4) and (5.5) did not use the well posedness of Σ , but only the admissibility of B . Thus, if A satisfies ESAD, B is an admissible control operator for the semigroup generated by A , and $C = B^*$, then for any function $\mathbf{G}$ satisfying (4.3) there exists a $c \geq 0$ such that $cI + \mathbf{G}$ is a positive transfer function. This $\mathbf{G}$ need not be well posed.

Remark 5.4. In the case that $Q = 0$, the conclusions of the previous remark remain valid without assuming that B is admissible; we only need the fact that

$B \in \mathcal{L}(U, X_{-1})$. Thus, in this case, (5.3), (5.4) and (5.5) still hold without assuming well posedness or admissibility. Moreover, (5.4) and (5.5) can be replaced by

$$E = -\frac{1}{2}[\mathbf{G}(\lambda)^* + \mathbf{G}(-\lambda)] \quad \forall \lambda > 0.$$

Recall that $\alpha(B)$ denotes the degree of unboundedness of B , introduced in (3.5). In the following proposition we discuss the consequences of B being less than maximally unbounded, i.e., $\alpha(B) < \frac{1}{2}$, in the context of Theorem 5.2.

PROPOSITION 5.5. *With the notation and assumptions of this section, suppose that $\alpha(B) < \frac{1}{2}$. Then Σ is regular and E from (5.4) is given by (5.1). Moreover, in this case, E given by (5.1) is the smallest self-adjoint operator in $\mathcal{L}(U)$ satisfying (5.3).*

By the minimality of E we mean the following: If $F = F^* \in \mathcal{L}(U)$ is such that $\mathbf{G} + F$ is a positive transfer function, then $E \leq F$.

Proof. First we note that the regularity of Σ follows from Proposition 4.1 (since $\alpha(B^*) \leq \frac{1}{2}$). The formula (5.1) follows from (5.6), where the second term is now zero. Let $F = F^*$ be such that $\mathbf{G} + F$ is a positive transfer function so that

$$2F + [\mathbf{G}(\lambda)^* + \mathbf{G}(\lambda)] \geq 0 \quad \forall \lambda > 0.$$

Taking limits as $\lambda \rightarrow +\infty$, we obtain that $2F + D + D^* \geq 0$. From here, using (5.1) we obtain $2F - 2E \geq 0$, i.e., E is minimal. $\square$

By Theorem 5.2 $\mathbf{G} + E$ is a positive transfer function. If Σ is regular and (5.1) holds, then this implies that $\mathbf{G} - D$ is also a positive transfer function. From (4.6) with $C = B^*$ we then obtain that $B_\lambda^*(sI - A)^{-1}B$ is a positive transfer function.

The following example is a regular linear system satisfying ESAD and COL, with feedthrough operator $D = 0$, for which the operator E from (5.4) or (5.6) is nonzero. For this example, $B_\lambda^*(sI - A)^{-1}B$ is *not* a positive transfer function and the feedback $u = -\kappa y + v$ is exponentially destabilizing if κ is too large.

Example 5.6. Consider the usual realization of a delay line of length h , $h > 0$, as given, e.g., on p. 831 of [45]. The state space of this system Σ_0 is $X = L^2[-h, 0]$, the semigroup is the left shift operator with zero entering from the right, with the generator

$$A_0 = \frac{d}{dx}, \quad \mathcal{D}(A_0) = \{z_0 \in \mathcal{H}^1(-h, 0) \mid z_0(0) = 0\}.$$

The control operator is $B = \delta_0$ and the observation operator is $C = \delta_{-h}^*$, which means that $Cz_0 = z_0(-h)$ for $z_0 \in \mathcal{D}(A_0)$. The feedthrough operator is zero and the transfer function of this system is $\mathbf{G}_0(s) = e^{-hs}$. We call the input w and the output y . (For an isomorphic system described by the wave equation we refer to [53, section 7]. We close a positive unity feedback loop around this delay line, meaning that $w = y + u$, where u is the new input function. This leads to a new well-posed linear system Σ with the transfer function

$$(5.9) \quad \mathbf{G}(s) = \frac{e^{-hs}}{1 - e^{-hs}}.$$

The semigroup $\mathbb{T}$ of this new system is the periodic left shift semigroup on X , which is unitary. The generating operators of Σ can be computed directly, or using the formulas from [46, section 7]. The generator of $\mathbb{T}$ is again $A = \frac{d}{dx}$, but now the domain is

$$\mathcal{D}(A) = \{z_0 \in \mathcal{H}^1(-h, 0) \mid z_0(0) = z_0(-h)\}.$$

It is easy to see that this A is skew-adjoint. The operators B and C remain basically the same, but of course the new C is defined on the new $\mathcal{D}(A)$ and this results in $C = B^*$. Thus, the system Σ fits into the framework of this paper (it satisfies ESAD and COL). Moreover, Σ is regular and its feedthrough operator is zero. It is not difficult to check that B is maximally unbounded, i.e., $\alpha(B) = \frac{1}{2}$.

We remark that the space Z from (4.9) is now $Z = \mathcal{H}^1(-h, 0)$ and for every $z_0 \in Z$ we have $C_\Lambda z_0 = z_0(-h)$. It follows from what we said at (4.10) that if $u \in \mathcal{H}^1(0, \infty)$ and if z_0 , the initial state of Σ satisfies $z_0 \in \mathcal{H}^1(-h, 0)$ and $z_0(0) = z_0(-h) + u(0)$ (equivalently, $Az_0 + Bu(0) \in X$), then for all $t \geq 0$ we have

$$z(\cdot, t) \in \mathcal{H}^1(-h, 0), \quad z(0, t) = z(-h, t) + u(t),$$

and the output function y is given by $y(t) = z(-h, t)$.

The spectrum $\sigma(A)$ now consists of the poles of $\mathbf{G}$, so that $\rho(A)$ is a connected set. The extension of $\mathbf{G}$ to $\rho(A)$, as introduced in Lemma 5.1, is the analytic continuation of $\mathbf{G}$, still given by (5.9). Formula (5.4) gives $E = \frac{1}{2}$. It is not difficult to verify that

$$\mathbf{G}(i\omega)^* + \mathbf{G}(i\omega) = -1 \quad \forall \omega \in \mathbb{R}$$

so that $\mathbf{G}$ is not positive-real, but of course $\mathbf{G} + E = \mathbf{G} + \frac{1}{2}$ is positive-real. Note that E is minimal in the sense of Proposition 5.5, even though $\alpha(B) = \frac{1}{2}$.

If we close a negative feedback loop around Σ by putting $u = -\kappa y + v$, where v is the new input, then we get the closed-loop transfer function

$$\mathbf{G}^\kappa(s) = \frac{e^{-hs}}{1 - (1 - \kappa)e^{-hs}}.$$

This transfer function is bounded on some right half-plane, since

$$|\mathbf{G}^\kappa(s)| \leq e^{-h\sigma} / |1 - (1 - \kappa)e^{-h\sigma}|, \quad \sigma = \operatorname{Re} s,$$

and so the closed-loop system is well posed. $\mathbf{G}^\kappa$ is stable for $0 < \kappa < 2$, but for $\kappa \geq 2$ the transfer function has unstable poles. This shows that the closed-loop semigroup becomes unstable. Moreover, the larger κ becomes, the more unstable the closed-loop system becomes (its poles move to the right).

If a system with transfer function $\mathbf{G}$ is such that $cI + \mathbf{G}$ is positive, we show that sufficiently small output feedbacks stabilize the system in an input-output sense. If the system is optimizable and estimatable (for example, if it is exactly controllable and exactly observable), then we can conclude exponential stability. During the following proposition (and its proof) we suspend the standing notation and assumptions introduced at the beginning of this section. Thus, for example, the system Σ is not required to satisfy assumptions ESAD and COL.

PROPOSITION 5.7. *Suppose that Σ is a well-posed linear system with input and output space U . Assume that its transfer function $\mathbf{G}$ is such that for some $c \geq 0$, $cI + \mathbf{G}$ is a positive transfer function. Denote $\kappa_0 = \frac{1}{c}$ (for $c = 0$, take $\kappa_0 = \infty$). Then for every $\kappa \in (0, \kappa_0)$ the operator $K = -\kappa I$ is an admissible feedback operator for Σ , and the corresponding closed-loop system Σ^κ is input-output stable, i.e., $\mathbf{G}^\kappa = \mathbf{G}(I + \kappa\mathbf{G})^{-1} \in H^\infty(\mathcal{L}(U))$. Moreover, if Σ is optimizable and estimatable, then the closed-loop system is exponentially stable.*

Proof. Denote $a = \frac{1}{\kappa}$. The proof of the fact that for all $v \in U$ with $\|v\| = 1$,

$$(5.10) \quad \|(aI + \mathbf{G}(s))v\| \geq a - c \quad \forall s \in \mathbb{C}_0,$$

is exactly like in the proof of Proposition 2.1. However, since U may now be infinite-dimensional, boundedness from below is not enough to conclude the invertibility of $aI + \mathbf{G}(s)$. By a similar argument, we have that $\|(aI + \mathbf{G}(s))^*v\| \geq a - c$, and this together with (5.10) implies the desired invertibility and also that (2.2) holds. From here we easily get $\mathbf{G}^\kappa \in H^\infty$, as in the proof of Proposition 2.1.

Now applying Theorem 4.5, we can conclude exponential stability if Σ^κ is optimizable and estimatable. If Σ is optimizable and estimatable, then Σ^κ inherits these properties, according to Proposition 4.4. $\square$

The above proof appears to be short and simple, but it relies on Proposition 4.4 and Theorem 4.5, and the proof of the latter (in [51]) is rather involved.

THEOREM 5.8. *With the notation and assumptions from the beginning of this section, let $c = \|E^+\|$, where E^+ is the positive part of E from (5.4) and denote $\kappa_0 = \frac{1}{c}$ (for $c = 0$, take $\kappa_0 = \infty$). Then for every $\kappa \in (0, \kappa_0)$, $K = -\kappa I$ is an admissible feedback operator for Σ , leading to the closed-loop system Σ^κ which is input-output stable. If Σ is optimizable and estimatable, then Σ^κ is exponentially stable.*

Proof. According to Remark 5.3, $cI + \mathbf{G}$ is a positive transfer function for $c = \|E^+\|$. Now the theorem follows from Proposition 5.7 using this c . $\square$

The following proposition may be useful for checking if a specific system fits into the framework used in this paper.

PROPOSITION 5.9. *Suppose that A satisfies ESAD and $B \in \mathcal{L}(U, X_{-1})$. Then B is an admissible control operator for $\mathbb{T}$ if and only if B^* is an admissible observation operator for $\mathbb{T}$. Moreover, we have $\alpha(B) = \alpha(B^*)$.*

Proof. Denote $A_0 = A + \frac{1}{2}Q$ so that A_0 is skew-adjoint. Take $s \in \mathbb{C}_0$. From

$$(5.11) \quad (sI - A_0)^{-1} - (sI - A)^{-1} = \frac{1}{2}(sI - A_0)^{-1}Q(sI - A)^{-1},$$

we see that the norms $\|x\|_{-1} = \|(sI - A)^{-1}x\|$ and $\|x\|_{-1}^0 = \|(sI - A_0)^{-1}x\|$ are equivalent. Hence, the space X_{-1} for A and for A_0 is the same. Multiplying the last identity with B from the right and taking norms, we can see that $\alpha(B)$ with respect to A and A_0 is the same. We denote by $\mathbb{S}$ the (unitary) semigroup generated by A_0 , and by $\mathbb{S}^*$ its adjoint (or inverse) semigroup, generated by $-A_0$. It is easy to see that B is an admissible control operator for $\mathbb{T}$ if and only if it is for $\mathbb{S}$ (because the perturbation $\frac{1}{2}Q$ is bounded). Similarly, we can show that $\alpha(B^*)$ with respect to A and A_0 is the same and B^* is an admissible observation operator for $\mathbb{T}$ if and only if it is for $\mathbb{S}$. Thus, so far we have shown that for our purposes, there is no difference between A and A_0 (i.e., between $\mathbb{T}$ and $\mathbb{S}$).

It is easy to see that the admissibility of B for $\mathbb{S}$ and for $\mathbb{S}^*$ are equivalent (see the text after (3.4)). On the other hand, the admissibility of B for $\mathbb{S}^*$ is equivalent to the admissibility of B^* for $\mathbb{S}$ (see section 3). Hence, B is an admissible control operator for $\mathbb{S}$ if and only if B^* is an admissible observation operator for $\mathbb{S}$. It remains to prove that $\alpha(B) = \alpha(B^*)$. This follows from

$$\begin{aligned} \|(\lambda I - A_0)^{-1}B\|^2 &= \|B^*(\lambda I + A_0)^{-1}(\lambda I - A_0)^{-1}B\| \\ &= \|B^*(\lambda I - A_0)^{-1}(\lambda I + A_0)^{-1}B\| = \|B^*(\lambda I - A_0)^{-1}\|^2. \quad \square \end{aligned}$$

6. A class of undamped second order systems. In this section we introduce a class of undamped second order systems satisfying ESAD and COL. For these systems we derive an explicit expression for E from (5.4) that shows clearly that E^+ is not always zero. Hence, as we have seen in section 5, the range of exponentially stabilizing feedback gains is bounded in general.

Let U_0, U_1 and H be Hilbert spaces and let $A_0 : \mathcal{D}(A_0) \rightarrow H$ be positive and boundedly invertible on H . Consider the system described by the following second order differential equation and two output equations:

$$(6.1) \quad \begin{cases} \ddot{q} + A_0 q &= C_0^* u_0 + A_0^{-1} C_1^* \dot{u}_1, \\ y_0 &= \overline{C_0} \dot{q}, \quad y_1 = \overline{C_1} q, \end{cases}$$

for sufficiently smooth signals u_0, u_1 and compatible initial conditions $q(0)$ and $\dot{q}(0)$. The input signals u_0, u_1 and the output signals y_0, y_1 are such that $u_0(t) \in U_0$, $u_1(t) \in U_1$, $y_0(t) \in U_0$, $y_1(t) \in U_1$. To formulate these equations in the form (1.1) and (4.10), which describe a well-posed linear system for sufficiently smooth u and compatible $z(0)$, we need to introduce various spaces and operators, and to make some assumptions. For every $\mu > 0$, we define $H_\mu = \mathcal{D}(A_0^\mu)$, with the norm $\|\varphi\|_\mu = \|A_0^\mu \varphi\|_H$, and we define $H_{-\mu} = H_\mu^*$ (duality with respect to the pivot space H). We denote $H_0 = H$ and $\|\varphi\|_0 = \|\varphi\|_H$. We assume that

$$C_0 \in \mathcal{L}(H_{\frac{1}{2}}, U_0), \quad C_1 \in \mathcal{L}(H_1, U_1).$$

We identify U_0 and U_1 with their duals so that $C_0^* \in \mathcal{L}(U_0, H_{-\frac{1}{2}})$, $C_1^* \in \mathcal{L}(U_1, H_{-1})$.

We assume that C_0 and C_1 have extensions $\overline{C_0}$ and $\overline{C_1}$ such that the operators

$$D_0 = \overline{C_0} A_0^{-1} C_1^* \in \mathcal{L}(U_1, U_0), \quad D_1 = \overline{C_1} A_0^{-1} C_0^* \in \mathcal{L}(U_0, U_1)$$

exist. We introduce the input (and output) space U and the state space X :

$$(6.2) \quad U = U_0 \times U_1, \quad X = H_{\frac{1}{2}} \times H.$$

Now the equations of the system Σ from (6.1) can be rewritten in the form

$$(6.3) \quad \begin{cases} \dot{z} &= Az + Bu, \\ y &= \overline{C}z + Du, \end{cases}$$

where

$$(6.4) \quad z = \begin{bmatrix} q \\ w \end{bmatrix} \in X, \quad u = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \in U, \quad y = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \in U,$$

$$(6.5) \quad A : \mathcal{D}(A) \rightarrow X, \quad A = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix},$$

$$X_1 = \mathcal{D}(A) = H_1 \times H_{\frac{1}{2}}, \quad X_{-1} = H \times H_{-\frac{1}{2}}, \quad A^* = -A,$$

$$(6.6) \quad B = \begin{bmatrix} 0 & A_0^{-1} C_1^* \\ C_0^* & 0 \end{bmatrix} \in \mathcal{L}(U, X_{-1}), \quad C = B^* = \begin{bmatrix} 0 & C_0 \\ C_1 & 0 \end{bmatrix} \in \mathcal{L}(X_1, U),$$

$$(6.7) \quad \overline{C} = \begin{bmatrix} 0 & \overline{C_0} \\ \overline{C_1} & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & D_0 \\ 0 & 0 \end{bmatrix}.$$

We shall now prove that for sufficiently smooth u , (6.1) is equivalent to (6.3). We also show how the condition of “compatible initial conditions,” $Az(0) + Bu(0) \in X$, can be expressed in terms of the functions and operators appearing in (6.1). The significance of this condition was explained in section 4 (around (4.10)).

PROPOSITION 6.1. For $u \in \mathcal{H}_{loc}^1(0, \infty; U)$, the equations (6.1) (for a specific $t \geq 0$) are equivalent to (6.3) (for the same t) with z, u, y as in (6.4) and with

$$w(t) = \dot{q}(t) - A_0^{-1}C_1^*u_1(t).$$

The first part of (6.1) is regarded as an equation in $H_{-\frac{1}{2}}$, and the first part of (6.3) is regarded as an equation in X_{-1} . Moreover, for any $t \geq 0$, the conditions

$$(6.8) \quad A_0q(t) - C_0^*u_0(t) \in H, \quad \dot{q}(t) \in H_{\frac{1}{2}}$$

are equivalent to $Az(t) + Bu(t) \in X$.

Proof. We rewrite the first part of (6.3):

$$\frac{d}{dt} \begin{bmatrix} q(t) \\ w(t) \end{bmatrix} = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix} \begin{bmatrix} q(t) \\ w(t) \end{bmatrix} + \begin{bmatrix} 0 & A_0^{-1}C_1^* \\ C_0^* & 0 \end{bmatrix} \begin{bmatrix} u_0(t) \\ u_1(t) \end{bmatrix},$$

or, equivalently,

$$\dot{q}(t) = w(t) + A_0^{-1}C_1^*u_1(t), \quad \dot{w}(t) = -A_0q(t) + C_0^*u_0(t).$$

By differentiating the first formula, we obtain the first equation in (6.1). To derive the last two equations in (6.1), we compute, starting from (6.3),

$$\begin{aligned} \begin{bmatrix} y_0(t) \\ y_1(t) \end{bmatrix} &= \begin{bmatrix} 0 & \overline{C_0} \\ \overline{C_1} & 0 \end{bmatrix} \begin{bmatrix} q(t) \\ w(t) \end{bmatrix} + \begin{bmatrix} 0 & \overline{C_0}A_0^{-1}C_1^* \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_0(t) \\ u_1(t) \end{bmatrix} \\ &= \begin{bmatrix} \overline{C_0}(\dot{q}(t) - A_0^{-1}C_1^*u_1(t)) + \overline{C_0}A_0^{-1}C_1^*u_1(t) \\ \overline{C_1}q(t) \end{bmatrix} = \begin{bmatrix} \overline{C_0}\dot{q}(t) \\ \overline{C_1}q(t) \end{bmatrix}. \end{aligned}$$

To obtain (6.3) from (6.1), we just reverse the order of the computations. It is easy to verify that the conditions (6.8) are equivalent to $Az(t) + Bu(t) \in X$. $\square$

Clearly, Σ satisfies ESAD and COL, but it need not be well posed. The optimal control of systems of this type, but with $C_1 = 0$, has been studied in [49] without assuming well posedness. Here, we do assume that our system Σ is well posed (but not necessarily regular). Then, the equations (6.3) correspond to the representation of well-posed systems via (1.1) and (4.10).

As explained in section 4 (before (4.10)), the transfer function of Σ is given by $\mathbf{G}(s) = \overline{C}(sI - A)^{-1}B + D$, which is easy to compute in terms of A_0, C_0 and C_1 : for $-s^2 \in \rho(A_0)$,

$$(6.9) \quad \mathbf{G}(s) = \begin{bmatrix} C_0s \\ \overline{C_1} \end{bmatrix} (s^2I + A_0)^{-1} [C_0^* \ A_0^{-1}C_1^*s].$$

Note the curious fact that $\mathbf{G}$ does not depend on the extended operator $\overline{C_0}$. This extended operator appears in the representation (6.3) of Σ , but (6.9) shows that the system Σ is in fact independent of the choice of the extension $\overline{C_0}$.

In particular, we see from (6.9) that

$$\mathbf{G}(0) = \begin{bmatrix} 0 & 0 \\ D_1 & 0 \end{bmatrix}.$$

According to Theorem 5.2, we have $E = -\frac{1}{2}[\mathbf{G}(0)^* + \mathbf{G}(0)] + B^*(A^*)^{-1}QA^{-1}B$ so that (since $Q = 0$)

$$(6.10) \quad E = -\frac{1}{2} \begin{bmatrix} 0 & D_1^* \\ D_1 & 0 \end{bmatrix}.$$

Such systems, but with $C_1 = 0$, have been considered in [2, 14, 49].

PROPOSITION 6.2. *Suppose that the well-posed system Σ from (6.3) is exactly controllable (or equivalently, exactly observable). Let $E \in \mathcal{L}(U)$ be the operator from (6.10), let E^+ be the positive part of E , and put $c = \|E^+\|$, $\kappa_0 = \frac{1}{c}$. Then for every $\kappa \in (0, \kappa_0)$, $K = -\kappa I$ is an admissible feedback operator for Σ and the resulting closed-loop system Σ^κ is exponentially stable.*

Proof. The equivalence of exact controllability and exact observability follows by duality, since $A^* = -A$ (see the comments after Definition 3.1). It is clear that exact controllability implies optimizability, and (by duality) exact observability implies estimatability. Thus, we can apply Theorem 5.8 to obtain this proposition. $\square$

Example 6.3. We describe a well-posed system which fits into the above framework, so that proportional output feedback can exponentially stabilize it. Since the computations are rather long, they are the subject of a separate paper [50].

The physical system that we are modeling consists of a hinged elastic beam with two sensors: one measures the angular velocity of the beam at a point ξ and the other measures the bending (curvature) of the beam at the same point. These two measurements are advantageous because they make the open-loop system exactly observable. Our aim is to design the actuators and the feedback law in order to exponentially stabilize this system. Using Proposition 6.2, we shall design the actuators such that they are colocated, meaning that $B = C^*$, and then the open-loop system is described by equations of the form (6.1). Here, C_0 will be the operator corresponding to the measurement of the angular velocity at ξ , and C_1 will be the operator corresponding to the measurement of the bending at ξ . It turns out that the actuators cause a discontinuity of the bending exactly at ξ . As in the preceding theory, we are forced to use an extension of the operator C_1 , which means that we have to decide if the corresponding sensor measures the left or the right limit of the bending at ξ , or a combination of the lateral limits.

We model the open-loop system as a homogenous Rayleigh beam situated along the interval $[0, \pi]$, with the two sensors located at $\xi \in (0, \pi)$. This is an extension of an example discussed in [1] and [52], and these papers contain further references to Rayleigh beam models. The equations describing the open-loop system are

$$(6.11) \quad \ddot{q}(x, t) - \alpha \frac{\partial^2 \dot{q}}{\partial x^2}(x, t) + \frac{\partial^4 q}{\partial x^4}(x, t) = \mathcal{U}(x, t),$$

$$(6.12) \quad q(0, t) = q(\pi, t) = 0, \quad \frac{\partial^2 q}{\partial x^2}(0, t) = \frac{\partial^2 q}{\partial x^2}(\pi, t) = 0, \quad t \geq 0,$$

$$(6.13) \quad y_0(t) = \frac{\partial \dot{q}}{\partial x}(\xi, t), \quad y_1(t) = \frac{\partial^2 q}{\partial x^2}(\xi, t).$$

Here $q(x, t)$ represents the transverse displacement of the beam ($x \in [0, \pi]$ and $t \geq 0$), and $\alpha > 0$ is a constant, proportional to the moment of inertia of the cross section of the beam. In (6.11), $\mathcal{U}$ denotes the control terms which are to be designed.

In order to fit this system into the framework of (6.1), (6.2), we denote $H = \mathcal{H}_0^1(0, \pi)$, $V = \mathcal{H}^2(0, \pi) \cap \mathcal{H}_0^1(0, \pi)$. On H we define the inner product such that

$$\langle \varphi, \psi \rangle_H = \left\langle \left(I - \alpha \frac{d^2}{dx^2} \right) \varphi, \psi \right\rangle_{L^2} \quad \forall \varphi, \psi \in V.$$

We introduce the operator $\mathcal{R} : L^2(0, \pi) \rightarrow V$ defined by

$$\mathcal{R} = \left(I - \alpha \frac{d^2}{dx^2} \right)^{-1}.$$

It is easy to see that $\mathcal{R} > 0$ when regarded as an operator from $L^2(0, \pi)$ to $L^2(0, \pi)$. We also define the linear operator $A_0 : \mathcal{D}(A_0) \rightarrow H$ by

$$\mathcal{D}(A_0) = \left\{ \varphi \in \mathcal{H}^3(0, \pi) \mid \varphi(0) = \varphi(\pi) = 0, \frac{d^2\varphi}{dx^2}(0) = \frac{d^2\varphi}{dx^2}(\pi) = 0 \right\},$$

$$A_0\varphi = \frac{d^4}{dx^4}(\mathcal{R}\varphi) \quad \forall \varphi \in \mathcal{D}(A_0).$$

The set $\{\varphi_k \mid k \in \mathbb{N}\}$ with

$$\varphi_k(x) = \frac{1}{\sqrt{1 + \alpha k^2}} \cdot \sqrt{\frac{2}{\pi}} \sin kx \quad \forall x \in [0, \pi]$$

is an orthonormal basis in H , and we have

$$A_0\varphi_k = \frac{k^4}{1 + \alpha k^2} \varphi_k, \quad \mathcal{R}\varphi_k = \frac{1}{1 + \alpha k^2} \varphi_k$$

so that, in particular, A_0 is self-adjoint, strictly positive and it commutes with $\mathcal{R}$. Defining H_μ and $\|\varphi\|_\mu$ for $\mu \in \mathbb{R}$ by fractional powers of A_0 and duality, as explained after (6.1), we have $H_0 = H = \mathcal{H}_0^1(0, \pi)$ and

$$H_1 = \mathcal{D}(A_0), \quad H_{\frac{1}{2}} = V, \quad H_{-\frac{1}{2}} = L^2(0, \pi), \quad H_{-1} = \mathcal{H}^{-1}(0, \pi), \quad H_{-1.5} = V',$$

the norm $\|\cdot\|_{-\frac{1}{2}}$ being equivalent to the L^2 -norm. Here, V' denotes the dual of V with respect to the pivot space $L^2(0, \pi)$. For every $\mu \in \mathbb{R}$, A_0 and $\mathcal{R}$ have extensions (or restrictions) to H_μ and we have $A_0 \in \mathcal{L}(H_\mu, H_{\mu-1})$, $\mathcal{R} \in \mathcal{L}(H_\mu, H_{\mu+1})$.

The spaces U_0 and U_1 from (6.2) are $U_0 = U_1 = \mathbb{C}$, so that $U = \mathbb{C}^2$. Corresponding to the two measurements in (6.13), we define the operators $C_0 \in \mathcal{L}(H_{\frac{1}{2}}, \mathbb{C})$ and $C_1 \in \mathcal{L}(H_1, \mathbb{C})$ by

$$C_0\varphi = \frac{d\varphi}{dx}(\xi), \quad C_1\varphi = \frac{d^2\varphi}{dx^2}(\xi).$$

Then $C_0^* \in H_{-\frac{1}{2}}$ and $C_1^* \in H_{-1}$ (the adjoints of C_0 and C_1 with respect to the pivot space H) are

$$C_0^* = -\mathcal{R} \frac{d}{dx} \delta_\xi, \quad C_1^* = \mathcal{R} \frac{d^2}{dx^2} \delta_\xi,$$

where δ_ξ is the Dirac mass at the point ξ .

We remark that according to (6.2), the state space of our system is $X = \mathcal{H}^2(0, \pi) \cap \mathcal{H}_0^1(0, \pi) \times \mathcal{H}_0^1(0, \pi)$ with the norm

$$\left\| \begin{bmatrix} q \\ w \end{bmatrix} \right\|^2 = \int_0^\pi \left| \frac{d^2 q(x)}{dx^2} \right|^2 dx + \int_0^\pi |w(x)|^2 dx + \alpha \int_0^\pi \left| \frac{dw(x)}{dx} \right|^2 dx.$$

Assuming that the terms of (6.11) are in $H_{-1.5} = V'$ and applying $\mathcal{R}$ to these terms, we obtain

$$(6.14) \quad \ddot{q} + A_0 q = \mathcal{R} \mathcal{U} \quad \text{in } H_{-\frac{1}{2}} = L^2(0, \pi).$$

Now the first equation in (6.1) shows that the control terms, represented above by $\mathcal{RU}$, should be

$$(6.15) \quad \mathcal{RU} = C_0^* u_0 + A_0^{-1} C_1^* \dot{u}_1 = -u_0 \mathcal{R} \frac{d}{dx} \delta_\xi - \dot{u}_1 b,$$

where $b = -A_0^{-1} C_1^* \in H$ is given by

$$b(x) = \begin{cases} \frac{ax}{\xi} & \text{for } x \leq \xi, \\ \frac{a\pi - ax}{\pi - \xi} & \text{for } x > \xi, \end{cases} \quad \text{where} \quad \frac{1}{a} = \frac{1}{\xi} + \frac{1}{\pi - \xi}.$$

The graph of b consists of two straight lines, with a peak at $x = \xi$. If we apply $\mathcal{R}^{-1}$ to the terms of (6.14), with $\mathcal{RU}$ as in (6.15), we obtain that in $H_{-1.5}$,

$$(6.16) \quad \ddot{q} - \alpha \frac{\partial^2 \ddot{q}}{\partial x^2} + \frac{\partial^4 q}{\partial x^4} = -u_0 \frac{d}{dx} \delta_\xi - \dot{u}_1 [\alpha \delta_\xi + b].$$

This equation must also hold in the sense of distributions on $(0, \pi)$. Together with (6.12) and (6.13), it defines our open-loop system with the “designed” actuators. However, since in general $\frac{\partial^2 q}{\partial x^2}$ will have a discontinuity at ξ , the second part of (6.13) has to be replaced by $y_1(t) = \overline{C}_1 q$, where $\overline{C}_1$ is an extension of C_1 given by

$$\overline{C}_1 \chi = \gamma \chi''(\xi-) + (1 - \gamma) \chi''(\xi+).$$

PROPOSITION 6.4. *For $u \in \mathcal{H}_{loc}^1(0, \infty; \mathbb{C}^2)$, the equations (6.16) (in $H_{-1.5}$) and (6.13) (with C_1 extended as above) are equivalent to*

$$\dot{z} = Az + Bu \quad (\text{in } X_{-1}), \quad y = \overline{C}z + Du,$$

with z, u, y as in (6.4) and with $w = \dot{q} - A_0^{-1} C_1^* u_1 = \dot{q} + bu_1$. Here, A is as in (6.5), B is as in (6.6), and $\overline{C}, D$ are as in (6.7). These equations determine a well-posed linear system Σ with input and output space $U = \mathbb{C}^2$ and state space $X = H_{\frac{1}{2}} \times H$. This system is exactly controllable and exactly observable.

For the proof of this proposition we refer to our paper [50].

PROPOSITION 6.5. *Let Σ be the system from Proposition 6.4. Denote*

$$D_1 = \overline{C}_1 A_0^{-1} C_0^*, \quad \kappa_0 = \frac{2}{|D_1|}.$$

Then for every $\kappa \in (0, \kappa_0)$, $K = -\kappa I$ is an admissible feedback operator for Σ and the resulting closed-loop system Σ^κ is exponentially stable.

Proof. We know from Proposition 6.4 that the system Σ is well posed, exactly controllable, and exactly observable. Let $E \in \mathcal{L}(\mathbb{C}^2)$ be the 2×2 matrix from (6.10). Its eigenvalues are $\pm \frac{1}{2}|D_1|$ so that E^+ (the positive part of E) is a matrix of rank one with the eigenvalues $\frac{1}{2}|D_1|$ and 0. Hence, $c = \|E^+\| = \frac{1}{2}|D_1|$, $\kappa_0 = \frac{1}{c} = \frac{2}{|D_1|}$. According to Proposition 6.2, for every $\kappa \in (0, \kappa_0)$, $K = -\kappa I$ is an admissible feedback operator for Σ and Σ^κ is exponentially stable. $\square$

In [50] we derive a simple formula for the number D_1 appearing in the last proposition: $D_1 = \frac{\xi}{\pi} - \gamma$. Note that choosing $\gamma = \frac{\xi}{\pi}$ leads to $\kappa_0 = \infty$.

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