An Analytical Solution to the Heat Transfer Problem in Thick-walled Hunt Flow

Michael J Bluck, Michael J Wolfendale

Mechanical Engineering Dept. Imperial College, London

Abstract

The flow of a liquid metal in a rectangular duct, subject to a strong transverse magnetic field is of interest in a number of applications. An important application of such flows is in the context of coolants in fusion reactors, where heat is transferred to a lead-lithium eutectic. It is vital, therefore, that the heat transfer mechanisms are understood. Forced convection heat transfer is strongly dependent on the flow profile. In the hydrodynamic case, Nusselt numbers and the like, have long been well characterised in duct geometries. In the case of liquid metals in strong magnetic fields (magnetohydrodynamics), the flow profiles are very different and one can expect a concomitant effect on convective heat transfer. For fully developed laminar flows, the magnetohydrodynamic problem can be characterised in terms of two coupled partial differential equations. The problem of heat transfer for perfectly electrically insulating boundaries (Shercliff case) has been studied previously [1]. In this paper, we demonstrate corresponding analytical solutions for the case of conducting hartmann walls of arbitrary thickness. The flow is very different from the Shercliff case, exhibiting jets near the side walls and core flow suppression which have profound effects on heat transfer.

Keywords:

Magnetohydrodynamics, Nusselt number, Heat transfer

1. Introduction

The flow in a rectangular duct, subject to strong transverse magnetic fields, is of significant interest in fusion applications due to the use of liquid metal coolants employed in some fusion blanket designs. Depending on the circumstances, this magnetohydrodynamic problem may be simplified by assuming a laminar fully-developed flow. The problem then reduces to two coupled partial differential equations, whose solution was first obtained by Shercliff [2] for the case of perfectly insulating walls. Shercliff obtained explicit analytical solutions for the velocity and magnetic field profiles for this case and his work was subsequently extended to the case of imperfectly and perfectly conducting walls by Hunt [3, 4]. In this paper we consider the ‘Hunt-II’ case, with conducting Hartmann walls and insulating side walls.

In the context of fusion blankets, of equal or greater importance is the concomitant heat transfer, as the extraction of heat is one of the main roles of the blanket itself. Analytical solutions for the temperature profile have been obtained for the Shercliff case [1]. Such solutions also exist for flow between parallel plates and flows in circular channels [5, 6], and for 1-D heat transfer [7]. In this article the solutions in [1] are extended to the case of the flow in a rectangular duct with electrically conducting Hartmann walls and insulating side walls, one of the cases considered by Hunt. However, our analysis will be based on a more recent result which is valid for arbitrary Hartmann wall thickness. Not only is this of broader validity, this recent solution takes the form of a sum of the Shercliff solution (insulating walls) and a ‘Hunt’ term, (ie. $u = u^{sh} + u^h$), as indicated in Appendix A. This, as will be shown, allows the heat

transfer analysis to follow a method very similar to that given in [1].

In section 2 we will formulate the problem and state the thick-walled Hunt solution (Appendix). The analytical solution to the heat transfer problem is developed in section 3 for both uniform peripheral temperature (H_1) and uniform peripheral heat flux (H_2) cases. This is followed by a demonstration of the results in terms of temperature profiles and Nusselt number calculations in section 4.

2. Formulation

Referring to Fig. 1, the momentum equation in a fully developed MHD flow in a rectangular duct of size $-ad_h \leq X \leq ad_h$ and $-bd_h \leq Y \leq bd_h$, subject to an applied X -directed magnetic field B_x^0 is given by

$$\nu \left(\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2} \right) - \frac{1}{\rho} \frac{\partial p}{\partial Z} + \frac{1}{\rho} \frac{1}{\mu} \frac{\partial B_z}{\partial X} B_x^0 = 0 \quad (1)$$

The flow of conducting fluid generates an induced magnetic field B_z , satisfying

$$\frac{1}{\mu\sigma} \left(\frac{\partial^2 B_z}{\partial X^2} + \frac{\partial^2 B_z}{\partial Y^2} \right) + B_x^0 \frac{\partial U}{\partial X} = 0 \quad (2)$$

Where U is the velocity, ν is the kinematic viscosity, μ the magnetic permeability, ρ the density and σ the electrical conductivity of the fluid. In the case considered here, the magnetic field satisfies a Laplace equation within the Hartmann walls which are of thickness $d_h\delta$. The induced magnetic field is assumed to vanish at the boundary of the domain shown in Figure 1.

Nomenclature

δ	Dimensionless Hartmann wall thickness.
Γ	Wetted perimeter (m).
μ	Fluid magnetic permeability (Hm^{-1}).
ν	Kinematic viscosity (m^2s^{-1}).
ρ	Fluid density (kgm^{-3}).
σ	Fluid electrical conductivity (Sm^{-1}).
A	Duct cross-sectional area (m^2).
a	Duct half-width (m).
b	Duct half-height (m).
B_x^0	Applied x-directed magnetic field (Wb m^{-2}).
B_z	Induced magnetic field profile (Wb m^{-2}).
d_h	Hydraulic diameter (m).
h	Dimensionless magnetic field profile.
Ha	Hartmann number.

Hg	Hagen number.
Nu	Nusselt number.
p	Pressure (Pa).
Pr	Prandtl number.
q''	Heat flux (Wm^{-2}).
Re	Reynolds number.
T	Temperature profile (K).
t	Dimensionless temperature profile.
T_m	Bulk temperature (K).
T_w	Wall temperature (K).
U	Velocity profile (ms^{-1}).
u	Dimensionless velocity profile.
U_m	Mean velocity (ms^{-1}).

Figure 1: Duct coordinate system

Non-dimensionalising, by setting

$$x = \frac{X}{d_h}, y = \frac{Y}{d_h}, z = \frac{Z}{d_h} \quad (3)$$

$$u = \frac{U}{U_m} \quad (4)$$

where

$$U_m = \frac{1}{A} \int_A U dA \quad (5)$$

and

$$h = \frac{1}{\mu} \frac{1}{\sqrt{\rho\nu\sigma}} \frac{1}{U_m} B_z \quad (6)$$

we obtain

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + Ha \frac{\partial h}{\partial x} = \frac{Hg}{Re} \quad (7)$$

where

$$Ha = B_x^0 d_h \sqrt{\frac{\sigma}{\rho\nu}} \quad (8)$$

and the Hagen number is defined as

$$Hg = \frac{(\partial p / \partial Z) d_h^3}{\rho\nu^2} \quad (9)$$

The no-slip condition requires that $u = 0$ at the wall. The fluid magnetic field h satisfies

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + Ha \frac{\partial u}{\partial x} = 0 \quad (10)$$

in the fluid region. For the Hunt II problem considered here, the induced magnetic field vanishes at the side-walls, is continuous at the Hartmann wall-fluid interface and vanishes at the external boundary. The solution to this problem for square ducts is known [10], and a generalisation to rectangular ducts is given in the appendix.

In the following we consider the energy equation, which in steady state, fully developed flow, can be written as

$$\frac{\nu}{Pr} \left(\frac{\partial^2 T}{\partial X^2} + \frac{\partial^2 T}{\partial Y^2} \right) = U \frac{\partial T}{\partial Z} \quad (11)$$

3. Analytical Solution

The solution to the heat transfer problem takes the velocity profile as its input. In this case the profile is of Hunt type, and this profile, in the thin-wall approximation case, is well known [3, 4]. However, in the following, our analysis will be based on a more recent result which is valid for arbitrary Hartmann wall thickness. Not only is this of broader validity, this recent solution takes the form of a sum of the Shercliff solution (insulating walls) and a 'Hunt' term, ie. $u = u^{sh} + u^h$. The precise form of the Shercliff solution is given in [1] and the Hunt II solution is

given in Appendix A which is a generalization of the solution in [10] to non-square ducts. As a result of this decomposition, the heat transfer analysis follows a method very similar to that given in [1].

3.1. H_1 Heat transfer case

The H_1 transfer case describes circumstances where the heat flux is uniform in the axial direction and the wall temperature T_w is uniform in the peripheral direction. Under the conditions of fully developed flow, it can be assumed that

$$\frac{\partial T}{\partial Z} = \frac{dT_m}{dZ} = \text{const} \quad (12)$$

where the bulk temperature T_m is defined as

$$T_m = \frac{\int_A UT dA}{\int_A U dA} \quad (13)$$

We non-dimensionalize as in [1], with the non-dimensional temperature profile $t(x, y)$ being defined by

$$t = \frac{T}{(dT_w/dZ) d_h} \quad (14)$$

Inserting the non-dimensional forms into equation (11) and defining $u = u^{sh} + u^h$, gives

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} = (RePr)(u^{sh} + u^h) \quad (15)$$

We now proceed to determine the non-dimensional temperature profile $t(x, y)$ by decomposing the solution into a particular integral and a general solution. It is straightforward to obtain the following particular integral, which satisfies (15).

$$t_p(x, y) = HgPr \sum_{n=1}^{\infty} (f_n^{sh}(x) + f_n^h(x)) \cos \lambda_n y \quad (16)$$

where

$$f_n^{sh}(x) = \frac{k_n}{\lambda_n^2 b} \left(-\frac{1}{\lambda_n^2} - \frac{\sinh p_{n2} a \cosh p_{n1} x}{(p_{n1}^2 - \lambda_n^2) \sinh 2a\eta} + \frac{\sinh p_{n1} a \cosh p_{n2} x}{(p_{n2}^2 - \lambda_n^2) \sinh 2a\eta} \right)$$

and

$$f_n^h(x) = P_n \left(\frac{\cosh p_{n2} a \cosh p_{n1} x}{(p_{n1}^2 - \lambda_n^2) \cosh 2a\eta} - \frac{\cosh p_{n1} a \cosh p_{n2} x}{(p_{n2}^2 - \lambda_n^2) \cosh 2a\eta} \right) \quad (17)$$

where p_{n1} , p_{n2} and λ_n are defined in the appendix. This expression vanishes on the upper and lower walls ($y = \pm b$), but does not vanish at $x = \pm a$.

It can be easily shown that there is a general solution to the Laplace equation on the same geometry, which vanishes on $y = \pm b$, but is non-zero on $x = \pm a$:

$$t_l(x, y) = HgPr \sum_{n=1}^{\infty} (a_n \sinh \lambda_n (x - a) + b_n \sinh \lambda_n (x + a)) \cos \lambda_n y \quad (18)$$

for constants a_n and b_n . We will now require that $t_p + t_l$ vanishes on $x = \pm a$. Using these expressions we can calculate a_n and b_n as

$$a_n = \frac{f_n^{sh}(-a) + f_n^h(-a)}{\sinh 2\lambda_n a} \quad (19)$$

$$b_n = -\frac{f_n^{sh}(a) + f_n^h(a)}{\sinh 2\lambda_n a} \quad (20)$$

Note also that $f_n^{sh}(-a) = f_n^{sh}(a)$ and $f_n^h(-a) = f_n^h(a)$.

Given these results, we can re-dimensionalize and obtain the following solution to equation (11) satisfying $T = T_w$ at the boundary:

$$T(x, y) = T_w + d_h HgPr \frac{dT_m}{dZ} \sum_{n=1}^{\infty} g_n(x) \cos \lambda_n y \quad (21)$$

where

$$g_n(x) = f_n^{sh}(x) + f_n^h(x) + (f_n^{sh}(-a) + f_n^h(-a)) \frac{\sinh \lambda_n (x - a)}{\sinh 2\lambda_n a} - (f_n^{sh}(a) + f_n^h(a)) \frac{\sinh \lambda_n (x + a)}{\sinh 2\lambda_n a} \quad (22)$$

from which the local Nusselt number Nu_n for a wall with unit normal $\mathbf{n}$ as

$$Nu_n = \frac{d_h \mathbf{n} \cdot \nabla T}{(T_w - T_m)} \quad (23)$$

and the overall mean Nusselt number as

$$\overline{Nu} = \frac{d_h \int_{\partial A} \mathbf{n} \cdot \nabla T ds}{\Gamma (T_w - T_m)} \quad (24)$$

where Γ is the wetted perimeter.

3.2. H_2 Heat transfer case

The H_2 transfer case describes circumstances where the heat flux is uniform in the axial direction and is also uniform in the peripheral direction. As in [1], we follow an analysis similar to [8]. The same magnetohydrodynamic conditions arise as in the H_1 case, but the treatment of the energy equation differs somewhat. Due to the uniform peripheral and axial heat flux q'' we can perform an energy balance:

$$q'' \Gamma dZ = \rho c_p A U_m dT \quad (25)$$

from which it follows that

$$\frac{\partial T}{\partial Z} = \frac{dT_m}{dZ} = \frac{q'' \Gamma}{\rho c_p A U_m} = \frac{4q''}{\rho c_p d_h U_m} \quad (26)$$

In this case we define the non-dimensional temperature profile $t(x, y)$ as

$$t = \frac{k}{q'' d_h} (T - T_m) \quad (27)$$

Inserting these into equation (11) gives

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} = 4(u^{sh} + u^h) \quad (28)$$

The boundary conditions at the wall are

$$k \frac{\partial T}{\partial n} = q'' \quad (29)$$

which take the non-dimensional form

$$\frac{\partial t}{\partial n} = 1 \quad (30)$$

Following [8] we homogenise this equation by defining $\theta(x, y)$ as

$$\theta(x, y) = t(x, y) - \left(\frac{x^2}{2a} + \frac{y^2}{2b} \right) \quad (31)$$

Substituting for t into equation (28) and considering only one quarter of the duct due to symmetry we obtain

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 4(u^{sh} + u^h - 1) \quad (32)$$

$$\frac{\partial \theta}{\partial x} = 0, \text{ on } x = 0 \quad (33)$$

$$\frac{\partial \theta}{\partial x} = 0, \text{ on } x = a \quad (34)$$

$$\frac{\partial \theta}{\partial y} = 0, \text{ on } y = 0 \quad (35)$$

$$\frac{\partial \theta}{\partial y} = 0, \text{ on } y = b \quad (36)$$

To solve this equation, we begin by considering the the homogeneous form and decomposing via a separation of variables approach. As shown in [8], the solution to (32) is of the form

$$\theta(x, y) = X_0(x) + \sum_{n=1}^{\infty} X_n(x) \cos\left(\frac{n\pi y}{b}\right) \quad (37)$$

Substituting this expression into equation (32) gives

$$X_0''(x) + \sum_{n=1}^{\infty} \left(X_n''(x) - \left(\frac{n\pi}{b} \right)^2 X_n(x) \right) \cos\left(\frac{n\pi y}{b}\right) = 4(u^{sh} + u^h - 1) \quad (38)$$

Considered as a Fourier series, the zeroth order term is the mean value of the RHS of (38), so

$$X_0''(x) = \frac{4}{b} \int_0^b (u^{sh} + u^h - 1) dy \quad (39)$$

which can be evaluated to give

$$X_0''(x) = -4 \left(1 + \frac{1}{b} \frac{Hg}{Re} \sum_{n=1}^{\infty} \frac{(-1)^n}{\lambda_n} (u_n^{sh}(x) + u_n^h(x)) \right) \quad (40)$$

Following two successive integrations with respect to x , we obtain

$$\begin{aligned} X_0(x) &= -2x^2 + c_1 x + c_2 \\ &+ \frac{8}{b^2} \frac{Hg}{Re} \sum_{n=1}^{\infty} \frac{1}{\lambda_n^4} \left(\frac{x^2}{2} - \frac{\sinh p_{n_2} a \cosh p_{n_1} x}{p_{n_1}^2 \sinh 2a\eta} \right. \\ &+ \left. \frac{\sinh p_{n_1} a \cosh p_{n_2} x}{p_{n_2}^2 \sinh 2a\eta} \right) \\ &- \frac{4}{b} \frac{Hg}{Re} \sum_{n=1}^{\infty} \frac{(-1)^n P_n}{\lambda_n} \left(\frac{\cosh p_{n_2} a \cosh p_{n_1} x}{p_{n_1}^2 \cosh 2a\eta} \right. \\ &- \left. \frac{\cosh p_{n_1} a \cosh p_{n_2} x}{p_{n_2}^2 \cosh 2a\eta} \right) \end{aligned} \quad (41)$$

It now remains to compute the X_n terms, which is achieved by applying an orthogonality relation to equation (38) which gives

$$X_m''(x) - \left(\frac{m\pi}{b} \right)^2 X_m(x) = \frac{8}{b} \int_0^b (u^{sh} + u^h - 1) \cos\left(\frac{m\pi y}{b}\right) dy \quad (42)$$

which can be written as

$$X_m''(x) - \left(\frac{m\pi}{b} \right)^2 X_m(x) = \frac{8}{b} \frac{Hg}{Re} \sum_{n=1}^{\infty} \gamma_{mn} (u_n^{sh}(x) + u_n^h(x)) \quad (43)$$

where

$$\gamma_{mn} = \frac{1}{2} (-1)^{m+n-1} \left(\frac{1}{\lambda_n - \frac{m\pi}{b}} + \frac{1}{\lambda_n + \frac{m\pi}{b}} \right) \quad (44)$$

The solution to this equation is decomposed into the sum of a particular integral X_m^p and a general solution, X_m^g , to the homogeneous form of (43). The particular solution is given by

$$X_m^p(x) = \sum_{n=1}^{\infty} (a_{mn} + b_{mn} \cosh p_{n_1} x + c_{mn} \cosh p_{n_2} x) \quad (45)$$

Substituting this into (43) and equating coefficients gives

$$a_{mn} = -\frac{8}{m^2 \pi^2} \frac{Hg}{Re} \frac{k_n}{\lambda_n^2} \gamma_{mn} \quad (46)$$

$$\begin{aligned} b_{mn} &= \frac{8}{b} \frac{Hg}{Re} \gamma_{mn} \left\{ \frac{P_n \cosh p_{n_2} a}{\left(p_{n_1}^2 - \left(\frac{m\pi}{b} \right)^2 \right) \cosh 2a\eta} \right. \\ &- \left. \frac{k_n \sinh p_{n_2} a}{\lambda_n^2 b \left(p_{n_1}^2 - \left(\frac{m\pi}{b} \right)^2 \right) \sinh 2a\eta} \right\} \end{aligned} \quad (47)$$

$$\begin{aligned} c_{mn} &= -\frac{8}{b} \frac{Hg}{Re} \gamma_{mn} \left\{ \frac{P_n \cosh p_{n_1} a}{\left(p_{n_2}^2 - \left(\frac{m\pi}{b} \right)^2 \right) \cosh 2a\eta} \right. \\ &- \left. \frac{k_n \sinh p_{n_1} a}{\lambda_n^2 b \left(p_{n_2}^2 - \left(\frac{m\pi}{b} \right)^2 \right) \sinh 2a\eta} \right\} \end{aligned} \quad (48)$$

The homogeneous solution of equation (43) is given by

$$X_m^g(x) = d_m \sinh \frac{m\pi x}{b} + e_m \cosh \frac{m\pi x}{b} \quad (49)$$

Substituting equation (49), (45) and (41) into (37), we obtain $\theta(x, y)$. Finally we apply the remaining boundary conditions, giving at $x = 0$,

$$c_1 + \sum_{n=1}^{\infty} d_n \left(\frac{n\pi}{b} \right) \cos \left(\frac{n\pi y}{b} \right) = 0 \Rightarrow c_1 = d_n = 0 \quad (50)$$

and at $x = a$, we have

$$0 = \frac{\partial X_0}{\partial x}(a) + \sum_{n=1}^{\infty} \left(e_n \left(\frac{n\pi}{b} \right) \sinh \left(\frac{n\pi a}{b} \right) + \sum_{m=1}^{\infty} b_{nm} p_{m_1} \sinh p_{m_1} a + c_{nm} p_{m_2} \sinh p_{m_2} a \right) \cos \left(\frac{n\pi y}{b} \right) \quad (51)$$

Note that by construction, the mean value of $u = u^{sh} + u^h$ is 1. As a result, from (39),

$$\frac{\partial X_0}{\partial x}(a) - \frac{\partial X_0}{\partial x}(0) = \frac{4}{b} \int_{y=0}^b \int_{x=0}^a (u^{sh} + u^h - 1) dx dy = 0 \quad (52)$$

Since $\frac{\partial X_0}{\partial x}(0) = 0$ it follows that $\frac{\partial X_0}{\partial x}(a) = 0$. It then necessarily follows that

$$e_n = -\frac{b}{n\pi} \frac{\sum_{m=1}^{\infty} b_{nm} p_{m_1} \sinh p_{m_1} a + c_{nm} p_{m_2} \sinh p_{m_2} a}{\sinh \left(\frac{n\pi a}{b} \right)} \quad (53)$$

Combining these results we obtain the following expression for t in terms of the coefficients:

$$\begin{aligned} t(x, y) = & c_2 + \left(\frac{x^2}{2a} + \frac{y^2}{2b} \right) - 2x^2 \\ & + \frac{8}{b^2} \frac{Hg}{Re} \sum_{n=1}^{\infty} \frac{1}{\lambda_n^4} \left(\frac{x^2}{2} - \frac{\sinh p_{n_2} a \cosh p_{n_1} x}{p_{n_1}^2 \sinh 2a\eta} \right. \\ & + \left. \frac{\sinh p_{n_1} a \cosh p_{n_2} x}{p_{n_2}^2 \sinh 2a\eta} \right) \\ & - \frac{4}{b} \frac{Hg}{Re} \sum_{n=1}^{\infty} \frac{(-1)^n P_n}{\lambda_n} \left(\frac{\cosh p_{n_2} a \cosh p_{n_1} x}{p_{n_1}^2 \cosh 2a\eta} \right. \\ & - \left. \frac{\cosh p_{n_1} a \cosh p_{n_2} x}{p_{n_2}^2 \cosh 2a\eta} \right) \\ & + \sum_{n=1}^{\infty} \left[e_n \cosh \frac{n\pi x}{b} \right. \\ & + \sum_{m=1}^{\infty} (a_{nm} + b_{nm} \cosh p_{m_1} x \\ & + \left. c_{nm} \cosh p_{m_2} x) \right] \cos \left(\frac{n\pi y}{b} \right) \end{aligned} \quad (54)$$

It can be shown that as the Hartmann number tends to zero, this expression tends to that given in [8]. The constant c_2 is obtained, as in [1], by application of the constraint:

$$\int_A u t dA = 0 \quad (55)$$

The local Nusselt number at the wall can then be determined from

$$Nu_{\mathbf{n}} = \frac{d_{\mathbf{h}} \mathbf{n} \cdot \nabla T}{(T_w - T_m)} = \frac{d_{\mathbf{h}} q''}{k(T_w - T_m)} = \frac{1}{t_w} \quad (56)$$

where t_w is the non-dimensional temperature evaluated at the wall. In accordance with [9] and others, we compute the mean Nusselt number as the reciprocal of the weighted mean wall temperature.

4. Results

The Hunt-type profiles differ significantly from the Shercliff case. In particular, as Ha increases, as well as the narrowing of the Hartmann layer near the conducting walls, the core flow is suppressed and strong jets develop in regions parallel to the side layers. This has a considerable influence on the heat transfer relative to the Shercliff case. The normalized velocity profiles for a square duct and various Hartmann numbers are indicated in figure 2. The suppression of the core flow and development of the side-wall jets is clear.

4.1. H_1 Case

The summations in (16) and (18) are truncated to 20 terms with a normalized truncation error of less than 10^{-6} . The overall mean Nusselt numbers are shown for this case of uniform axial heat flux and uniform peripheral temperature for a range of Hartmann numbers and aspect ratios in Figures 3 and 4. As expected, for low Hartmann numbers, the results converge to the non-MHD case with a value of 3.608 [9, 8]. Contributions to heat transfer increase significantly as the electromagnetic forces begin to dominate the viscous forces ($Ha > 1$). This is partly due to the increased shear stresses in the Hartmann and side layers. Note that in the Shercliff case the narrower Hartmann layer always dominates this process [1]. In this Hunt case, however, the initial dominance of the Hartmann layer is eventually suppressed as a result of the reduced core flow and the side layer jets. For $Ha > 10^2 - 10^3$ the heat transfer rate saturates, in general agreement with Blum et al. [5] for the case of a circular duct. Ultimately the jets give rise to much improved heat transfer over the Shercliff case [1]. For aspect ratios with long Hartmann walls ($a < b$) in Figure 3, this effect is reduced as the jets make increasingly modest overall contribution. Conversely, for aspect ratios with long side walls ($a > b$) in Figure 4, this effect is enhanced.

To illuminate the processes involved, the temperature profiles (relative to the wall temperature) for a range of Hartmann numbers and aspect ratios are considered. In all cases $\frac{\partial p}{\partial z} = 2 \times 10^{-6} Pa$, $\rho = 1000 kg/m^{-3}$, $\nu = 10^{-6}$, $\sigma_f = \sigma_w$, $\delta = 0.05$ and $d_h = 0.1m$. In Fig. 5 we see the temperature profiles in a square duct ($a/b = 1$), for a selection of Hartmann numbers. As the Hartmann number increases, the improved heat transfer overall leads to an increased core temperature and slightly reduced side-wall boundary layer temperatures due to the jets.

Figure 6 shows the corresponding profiles for $a/b = 0.25$ which has a short side wall and a long Hartmann wall. Again, as the Hartmann number increases, improved heat transfer overall leads to an increased core temperature. This effect is limited due to the relatively modest contribution of the short side-wall jets.

Figure 2: u profiles for a range of Ha , $a/b = 1$. The evolution of the jets near the side-walls and the suppression of the core flow is clear.

Figure 7 shows the corresponding profiles for $a/b = 4$ which has a long side wall and a short Hartmann wall. The temperature gradient near the long side-wall jets is clearly enhanced and makes a significant overall contribution to the overall heat transfer.

Figure 3: Overall mean H_1 Nusselt number.

4.2. H_2 Case

All summations in (54) are truncated to 30 terms with a normalized truncation error of less than 10^{-6} . The overall mean Nusselt numbers are shown for this case of uniform axial and peripheral heat flux for a range of Hartmann numbers and aspect ratios in Figures 8 and 9.

For a square duct the Nusselt number diminishes beyond $Ha > 10$. For aspect ratios with long Hartmann walls ($a < b$) in Figure 8, there is an initial increase in Nusselt number, followed by a reduction. For aspect ratios with long side walls ($a > b$) in Figure 9, the Nusselt number reduces with increasing Ha and increasing aspect ratio.

The mechanism for this behaviour can be deduced from considering the flow profiles of the kind seen in Figure 2 and the temperature profiles (relative to the bulk temperature) for a range of Hartman numbers and aspect ratios in Figures 10-12. From Figure 2, and in general, the core flow is suppressed as Ha increases and jets develop at the side walls. As a result, the increased shear stress at the Hartmann walls improves heat transfer at least initially, however, ultimately the suppressed core flow counteracts this effect leading to essentially heat conduction dominated heat transfer from the Hartmann wall and a corresponding reduction in heat transfer. The degree to which

Figure 4: Overall mean H_1 Nusselt number.

these competing effects contribute to the overall heat transfer is dependent on the effect of the development of the side-wall jets and aspect ratio.

Consider the extreme case in Figure 11, showing the profiles for $a/b = 0.25$, which have a short side-wall and a long Hartmann wall. In this fixed heat flux case, we would expect the long Hartmann wall to dominate the heat transfer. As the Hartmann number increases, improved heat transfer due to increased Hartmann wall shear stress and modestly suppressed core flow leads to a reduction in Hartmann wall temperatures. The developing jets aid this by forming cool regions near the side-walls, allowing an increased temperature gradient in these regions (eg. at $Ha = 100$). Ultimately ($Ha = 1000$), however, the core flow stagnates and the jets narrow, resulting in more localized cool regions near the short side-walls. The cool regions are then insufficient in extent to aid the Hartmann wall heat transfer and the overall Nusselt number diminishes.

At the other extreme, in Figure 12, is shown the profiles for $a/b = 4$. These have a long side-wall and a short Hartmann wall. In this fixed heat flux case, we would expect the long side-wall to dominate the heat transfer. As the Hartmann number increases, any improvement in heat transfer due to increased Hartmann wall shear stress is insignificant, and is suppressed as the core stagnates. The developing jets again form cool regions near the side-walls, and as Ha increases, these jets extract the majority of heat from the side-walls. As a result, these side-wall regions exhibit essentially adiabatic behaviour, shown most clearly for $Ha = 1000$. The Hartmann wall temperatures are very large as a result of the poor heat transfer in the stag-

Figure 5: Temperature profiles for a range of Ha , $a/b = 1$.

Figure 6: Temperature profiles for a range of Ha , $a/b = 0.25$.

Figure 7: Temperature profiles for a range of Ha , $a/b = 4$.

nated core flow.

Figure 8: Overall mean H_2 Nusselt number.

5. Conclusions

The heat transfer problem for MHD flow in rectangular ducts with electrically insulated side-walls and arbitrarily thick conducting Hartmann walls, subject to a transverse magnetic field is solved analytically for both H_1 and H_2 heat transfer cases. To the authors knowledge these results are novel. In particular the magnetic field is seen to have a profound influence on heat transfer, and, whereas aspect ratio has a minor impact on convective heat transfer in the non-MHD case, the case for MHD is very different:

In the H_1 case, increasing Hartmann number (magnetic field) results in a monotonic increase in Nusselt number. This is despite the suppression of the core flow, the negative effects of which are ultimately dominated by the jets near the side walls. The aspect ratio also has a strong effect; the Nusselt number decreasing with increasing Hartmann wall length for large magnetic fields.

The results for the H_2 case are quite different; they indicate an asymptotic *reduction* in Nusselt number for large magnetic fields. In this heat flux controlled case, the impact of the suppressed core flow dominates the heat transfer despite the contribution of the jets. The Nusselt number in the H_2 case is also strongly dependent on aspect ratio. The Nusselt number *increases* with increasing Hartmann wall length and for sufficiently long Hartmann walls, there is an initial increase in heat

Figure 9: Overall mean H_2 Nusselt number.

transfer with increasing magnetic field, reaching a peak, and decreasing thereafter.

From this analysis, it is clear that duct geometry and orientation relative to the magnetic field is likely to have significant effects on the effectiveness of heat transfer. These results should also prove useful for the validation of numerical methods and for systems code analyses where heat transfer effects are important - particularly in the design of liquid metal coolant fusion blankets.

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Appendix A. The Thick walled Hunt II Solution

The following thick-walled Hunt II solution is a generalization of the result in [10] to a rectangular duct $-a \leq x \leq a$ and $-b \leq y \leq b$ with non-conducting side-walls and conducting Hartmann walls of thickness δ .

$$u(x, y) = u^{sh}(x, y) + \frac{Hg}{Re} \sum_{n=1}^{\infty} u_n^h(x) \cos \lambda_n y \quad (\text{A.1})$$

$$h(x, y) = h^{sh}(x, y) + \frac{Hg}{Re} \sum_{n=1}^{\infty} h_n^h(x) \cos \lambda_n y \quad (\text{A.2})$$

Ha = 10

Ha = 100

Ha = 1000

Figure 10: Temperature profiles for a range of Ha, $a/b = 1$.

Figure 11: Temperature profiles for a range of Ha, $a/b = 0.25$.

Figure 12: Temperature profiles for a range of Ha, $a/b = 4$.

$$u_n^h(x) = \frac{P_n}{\cosh 2a\eta} (\cosh p_{n_2}a \cosh p_{n_1}x - \cosh p_{n_1}a \cosh p_{n_2}x) \quad (\text{A.3})$$

$$h_n^h(x) = p_n^L \frac{\cosh \lambda(a+x) \sinh \eta(a-x)}{\cosh 2a\eta} + p_n^R \frac{\cosh \lambda(a-x) \sinh \eta(a+x)}{\cosh 2a\eta} \quad (\text{A.4})$$

$$\eta^2 = \lambda^2 + \lambda_n^2, p_{n_2} - p_{n_1} = 2\eta \quad (\text{A.5})$$

where

$$P_n = p_n^L = \frac{d_n^{sh}}{a_n + b_n}, p_n^R = -\frac{d_n^{sh}}{a_n + b_n}, \quad (\text{A.6})$$

wherein

$$a_n = \eta + \frac{\sigma_f}{\sigma_w} \lambda_n \tanh 2a\eta \coth \lambda_n \delta \quad (\text{A.7})$$

$$b_n = \frac{\eta \cosh 2a\lambda}{\cosh 2a\eta} \quad (\text{A.8})$$

$$d_n^{sh} = \frac{k_n}{\lambda_n^2 b} \left(\frac{p_{n_2} \sinh p_{n_1}a \cosh p_{n_2}a - p_{n_1} \sinh p_{n_2}a \cosh p_{n_1}a}{\sinh 2a\eta} \right) \quad (\text{A.9})$$

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