Decentralized Coordination of Large Populations of Flexible Electrical Appliances through Demand Saturation

Antonio De Paola, David Angeli and Goran Strbac

Abstract—This paper presents a novel decentralized control strategy for integration of price-responsive loads in the electricity market. Previous work has shown that, by approximating the devices population as a continuum, it is possible to provide necessary and sufficient conditions for the existence of a Nash equilibrium (no device has unilateral interest in changing its scheduling when considering the resulting profile of aggregate demand). These results are now extended by introducing a time varying proportional constraint on the maximum power consumption of the appliances. This allows to saturate the flexible demand and obtain a Nash equilibrium for a much wider range of scenarios. The performance of the proposed control technique, which also minimizes the task time of all appliances, is tested in simulation.

I. INTRODUCTION

In recent years power systems have been undergoing significant transformations, gradually moving from a traditional centralized model to the distributed paradigm of the smart grid [1]. One of the driving factors in this transformation is the introduction of new loads, such as electric vehicles and smart appliances, that accommodate flexibility in their power consumption. It is expected that an increased penetration of these devices in the network would correspond to significant advantages [2]: customers would benefit from cheaper electricity bills with a minimum impact on their comfort while power systems could achieve an increased reliability and a more efficient utilization of their current assets. To obtain these results, it is crucial to incorporate the flexible loads in the electricity market through a proper coordination of the individual appliances. Many different approaches have been proposed in such sense, including centralized techniques [3] and distributed mechanisms that use stochastic optimization [4], adaptive strategies [5] or Lagrange relaxation [6]. This paper follows the results presented in [7] where, approximating the flexible appliances population as a continuum and adopting a game theory framework, necessary and sufficient conditions are provided for convergence (in one step) to a Nash equilibrium. This means that all price-responsive appliances, having scheduled their power consumption according to a broadcast demand/price signal, have no incentive in changing their strategy when the resulting aggregate demand is considered. We now focus in determining additional control actions that induce an equilibrium when the aforementioned conditions are not fulfilled. In practice, this would cause “rebound peaks” [8]: the flexible appliances,

trying to operate when energy is cheap, introduce a new peak demand, potentially higher than the original one. Significant research activity has investigated this problem, proposing discontinuous [9] and randomized [10] price functions or advocating aggregators between the market and the customers [11]. In this work, similarly to what is suggested in [6], a time-varying proportional constraint (equal for all devices) is imposed on their maximum power consumption. The demand variation introduced by the appliances is limited at critical time instants, determining the corresponding power constraint through the resolution of a Cauchy problem. Under very general assumptions, it is possible to converge to a Nash equilibrium while minimizing the task time of all appliances. The possibility of coordinating flexible appliances (in particular electric vehicles) by inducing a Nash equilibrium is not novel. Similar results have been obtained in [12], [13] by introducing an additional quadratic term in the cost function of the agents, and in [14], [15] by specifying some conditions on the population size and on the electricity price. In this work, by imposing power consumption constraints, one-step convergence to an equilibrium can be obtained for any penetration level of flexible demand, with no additional terms in the objective function of the devices. Such result does not require detailed knowledge of the energy price function. The decentralized technique is tested in simulations, considering future scenarios of the UK grid with high penetration of price-responsive demand. A more detailed description of the proposed approach, with proofs of the theoretical results and additional properties, is presented in [16].

II. MODELLING FRAMEWORK FOR FLEXIBLE APPLIANCES AND ENERGY MARKET

The characterization of the flexible loads and the description of their interactions with the power system follows the modelling choices presented in [7]. For the purposes of this work, we consider interruptible appliances that can continuously modulate their power consumption and have complete time availability. This hypothesis, albeit restrictive, can accommodate realistic scenarios (e.g. charging of EVs during night-time) and constitutes the starting point for the design of a more extensive framework. Each appliance is characterized by two parameters: the total energy E_{tot} required to complete its task and the minimum time t_{min} that ensures task completion if the device operates at constant rated power $P_r = E_{\text{tot}}/t_{\text{min}}$. The sets of distinct values within the population for E_{tot} and t_{min} are denoted by $\mathcal{E}$ and $\mathcal{T}$, respectively. Flexible demand can then be characterized by the unnormalized distribution m of the devices parameters. Given

A. De Paola, D. Angeli and G. Strbac are with the Department of Electrical and Electronic Engineering, Imperial College London, London, SW7 2AZ UK (e-mail: ad5709@imperial.ac.uk; d.angeli@imperial.ac.uk; g.strbac@imperial.ac.uk).

$E_1, E_2 \in \mathcal{E}$ and $t_1, t_2 \in \mathcal{T}$, the integral $\int_{t_1}^{t_2} \int_{E_1}^{E_2} m(t, E) dE dt$ corresponds to the number of devices with $E_1 \leq E_{\text{tot}} \leq E_2$ and $t_1 \leq t_{\min} \leq t_2$. We assume that the appliances population is sufficiently large to be described by a continuum and the distribution m corresponds to an integrable function. For a compact description of the flexible devices, we introduce $f: \mathcal{T} \rightarrow \mathbb{R}^+$ and its derivative f' :

$$f(t) := \int_0^t \int_{\mathcal{E}} E \cdot m(\tau, E) dE d\tau \quad f'(t) := \int_{\mathcal{E}} E \cdot m(t, E) dE \quad (1)$$

where $f(t)$ is the total energy required by loads with $t_{\min} \leq t$.

Remark 1: The power density f' is the only description of the flexible appliances required to implement the proposed control strategy. Ideally, it can be centrally determined if each appliance communicates its parameters $t_{\min}$ and E_{tot} . In the considered framework, with a continuum of agents, *incentive compatibility* is guaranteed: no device has interest in providing false information as its effect on f' is negligible. In real applications, it should be possible to estimate f' with a reasonable approximation, using historical data. Alternatively, aggregators could collect local information on groups of devices, preserving the privacy of the individual users. All the results presented in this paper hold if f and f' are well-defined. Nevertheless, for a more compact exposition, the following assumption is introduced:

Assumption 1: The function f' has compact support:

$$\text{supp}(f') = [q_{\min}, q_{\max}] \subset (0, T].$$

Remark 2: Since $f'(t)$ quantifies the amount of energy required by devices with $t_{\min} = t$, its support is equal to the set $\mathcal{T}$ of distinct broadcast parameters $t_{\min}$.

The flexible devices interact with the power system through the electricity market, abstracted by the monotone increasing function $\Pi: [0, +\infty) \rightarrow [0, +\infty)$. For a certain power demand $D_a(t)$, the corresponding energy price is given by $p(t) = \Pi(D_a(t))$. In turn, the aggregate demand D_a is the sum of two components: the inflexible demand D_i , corresponding to the other loads in the system and known a priori, and the total power consumption D_f of the flexible devices, defined later on. All flexible loads are supposed to be price-responsive: given a demand signal D received from the system operator (or the corresponding price $\Pi(D)$), they schedule their power consumption u in order to complete their task at minimum cost. The equivalent optimization problem is:

$$\begin{aligned} \min_{u(\cdot)} \quad & \int_0^T \Pi(D(t)) \cdot u(t) dt \\ \text{s. t.} \quad & 0 \leq u(t) \leq \frac{E_{\text{tot}}}{t_{\min}} \\ & \int_0^T u(t) dt = E_{\text{tot}} \end{aligned} \quad (2)$$

We denote by $u_D^*(\cdot, s, x)$ the scheduled power profile of a single device with parameters $E_{\text{tot}} = x$ and $t_{\min} = s$ which is optimal for (2) when D is broadcast. The resulting flexible demand D_f and aggregate $D_{a,D}$ are straightforward to derive:

$$\begin{aligned} D_f(t) &= \int_{\mathcal{T}} \int_{\mathcal{E}} u_D^*(t, s, x) m(s, x) dx ds \\ D_{a,D}(t) &= D_i(t) + D_f(t). \end{aligned} \quad (3)$$

A. Analysis through measures of sublevel sets

The equilibrium conditions presented in the next subsection rely on the study of the sublevel sets of the broadcast demand D , described by the *cumulative distribution* $Q_D: [d_{\min}, d_{\max}] \rightarrow [0, T]$. The quantity $Q_D(d)$ corresponds to the amount of time “spent” by the signal D below the value d . Denoted by μ the Lebesgue measure, it holds:

$$Q_D(d) := \mu(\{\tau \in [0, T] : D(\tau) \leq d\}). \quad (4)$$

Assumption 2: The broadcast signal $D: [0, T] \rightarrow [0, +\infty)$ is a continuous function with no level sets of positive measure. For any $d \in \text{Im}(D) = [d_{\min}, d_{\max}]$, it must hold:

$$\mu(\{\tau \in [0, T] : D(\tau) = d\}) = 0. \quad (5)$$

This assumption generally holds for typical profiles of inflexible demand D_i , considered in the rest of our analysis as candidate solutions for the broadcast D in the appliances coordination problem. Moreover, condition (5) guarantees that $Q_D(d)$ in (4) is continuous and strictly monotone increasing. It is possible to introduce a change of variable and, instead of time t , consider the measure $q = Q_D(D(t))$. A bar accent denotes quantities in the new variable. For any $D(t)$ fulfilling Assumption 2, the corresponding $\bar{D}(q)$ is defined as:

$$\bar{D}(q) := Q_D^{-1}(q). \quad (6)$$

This function returns the demand value which induces a sublevel set of the broadcast signal D with measure q or, equivalently, such that $Q_D(\bar{D}(q)) = q$. Its relationship with the corresponding quantity in time is:

$$D(t) = \bar{D}(Q_D(D(t))) \quad \forall t \in [0, T].$$

Such representation can be extended to any $\rho: [0, T] \rightarrow \mathbb{R}$ for which it holds:

$$\rho(t_1) = \rho(t_2) \quad \forall t_1, t_2 \in [0, T] : D(t_1) = D(t_2). \quad (7)$$

In this case, $\bar{\rho}(q)$ is equal to the function ρ evaluated at any time $t \in [0, T]$ such that $Q_D(D(t)) = q$. The following relationship holds between ρ and $\bar{\rho}$:

$$\rho(t) = \bar{\rho}(Q_D(D(t))) \quad \forall t \in [0, T].$$

B. Nash Equilibria with price-based strategy

The considered decentralized scheme coordinates the appliances through the broadcast of a suitable demand profile D (or the corresponding price $\Pi(D)$), letting each appliance greedily schedule its power consumption in order to minimize its individual energy cost. The main objective is to achieve a Nash equilibrium in the system. This means that each device has no interest in changing its scheduled power consumption u_D^* when the broadcast demand D in its cost function (2) is replaced by the corresponding aggregate demand $D_{a,D}$ in (3), resulting from the application of u_D^* :

$$\begin{aligned} \int_0^T \Pi(D_{a,D}(t)) u_D^*(t, s, x) dt &= \min_{u(\cdot)} \int_0^T \Pi(D_{a,D}(t)) u(t) dt \\ \text{s. t.} \quad & 0 \leq u(t) \leq \frac{x}{s} \\ & \int_0^T u(t) dt = x \end{aligned} \quad (8)$$

Important results in this respect have been provided in [7, Theorem 1-2], showing that a Nash equilibrium can be obtained (by broadcasting $D = D_i$) if and only if:

$$\Lambda_{D_i}(q) \leq \Lambda_f(q) \quad \forall q \in \text{supp}(f') \quad (9)$$

where the *negotiable valley capacity* $\Lambda_D(q) := \frac{d}{dq} Q_D^{-1}(q)$ and the *power density of task durations* $\Lambda_f(q) := f'(q)/q$ are two quantities that characterize respectively the broadcast signal D and the appliances population.

III. SATURATION OF FLEXIBLE DEMAND

Additional control actions are now considered in order to achieve a Nash equilibrium also when condition (9) does not hold. In these scenarios, it has been established that the devices parameters and the penetration level of flexible demand are such that an equilibrium cannot be induced by simply broadcasting a price signal. For any choice of the communicated price, the strategies of the appliances will introduce a new peak demand, causing suboptimality of their originally scheduled power consumption. To overcome this problem, we propose to saturate the flexible demand at critical time instants. This is done by broadcasting to the population an additional signal $\alpha : [0, T] \rightarrow [0, 1]$ (time-varying and equal for all devices) which constitutes a proportional constraint on their maximum power consumption.

Remark 3: In practical implementations, one can assume that the constraint α can be directly enforced on the appliances. Alternatively, discontinuous price functions can be introduced, consistently penalizing all devices that exceed their prescribed fraction of maximum power consumption. The cost minimization problem solved by each device to schedule its power when α is introduced becomes:

$$\begin{aligned} \min_{u(\cdot)} \quad & \int_0^T \Pi(D(t)) \cdot u(t) dt \\ \text{s. t.} \quad & 0 \leq u(t) \leq \alpha(t) \frac{E_{\text{tot}}}{t_{\min}} \\ & \int_0^T u(t) dt = E_{\text{tot}} \end{aligned} \quad (10)$$

On the basis of the previous results, we choose $D = D_i$ and we aim at designing α which induces a Nash equilibrium in the system or, equivalently, fulfils the following condition for all $t_{\min} = s \in \mathcal{T}$ and $E_{\text{tot}} = x \in \mathcal{E}$:

$$\begin{aligned} \int_0^T \Pi(D_{a,D_i}(t)) u_D^*(t, s, x) dt &= \min_{u(\cdot)} \int_0^T \Pi(D_{a,D_i}(t)) u(t) dt \\ \text{s. t.} \quad & 0 \leq u(t) \leq \alpha(t) \frac{x}{s} \\ & \int_0^T u(t) dt = x \end{aligned} \quad (11)$$

Note that a Nash equilibrium constitutes a stable solution for the individual devices. It follows that “rebound peaks” are avoided, as these would correspond to suboptimality of their operation strategies (loads would be operating when demand and therefore prices are higher). The resulting flattened demand profile is also advantageous for the system operator, as it guarantees lower generation costs and increased reliability

of operations. The equilibrium analysis will be conducted by considering the scheduling problem in the measure variable $q = Q_D(D(t))$, which is equivalent to (10):

$$\begin{aligned} \min_{\bar{u}(\cdot)} \quad & \int_0^T \Pi(\bar{D}(q)) \cdot \bar{u}(q) dq \\ \text{s. t.} \quad & 0 \leq \bar{u}(q) \leq \bar{\alpha}(q) \frac{E_{\text{tot}}}{t_{\min}} \\ & \int_0^T \bar{u}(q) dq = E_{\text{tot}} \end{aligned} \quad (12)$$

As a preliminary step, we determine the optimal solution for problem (12), calculating the power $\bar{u}_D^*$ scheduled by the devices when the signals D and α are broadcast. In this respect, considering that Π and $\bar{D}$ defined in (6) are both monotone increasing functions, optimality is achieved by operating at maximum feasible rate at the lowest values of q , until the task is completed. Denoting by $\bar{\alpha}_l(q) = \int_0^q \bar{\alpha}(\tau) d\tau$ the (monotone increasing) integral of $\bar{\alpha}$ over $[0, q]$, we have:

$$\bar{u}_D^*(q, s, x) = \begin{cases} \bar{\alpha}(q) \frac{x}{s} & \text{if } \bar{\alpha}_l(q) \leq s \\ 0 & \bar{\alpha}_l(q) > s \end{cases} \quad (13)$$

Having calculated the scheduled power consumption $\bar{u}_D^*$, it is straightforward to derive the corresponding flexible demand as a function of the measure q :

$$\begin{aligned} \bar{D}_f(q) &= \int_{\mathcal{E}} \int_{\mathcal{T}} \bar{u}_D^*(q, s, x) m(s, x) ds dx \\ &= \bar{\alpha}(q) \int_{\mathcal{E}} \int_{\bar{\alpha}_l(q)}^T \frac{x}{s} m(s, x) ds dx \\ &= \bar{\alpha}(q) \int_{\bar{\alpha}_l(q)}^T \frac{1}{s} \int_{\mathcal{E}} x m(s, x) dx ds = \bar{\alpha}(q) \int_{\bar{\alpha}_l(q)}^T \frac{f'(s)}{s} ds \end{aligned} \quad (14)$$

Remark 4: For the proportional constraint $\bar{\alpha}$, scheduled power consumption $\bar{u}_D^*$ and flexible demand $\bar{D}_f$ as functions of the measure q , it is straightforward to calculate the corresponding quantities in time:

$$u_D^*(t, s, x) = \bar{u}_D^*(Q_D(D(t)), s, x) \quad (15)$$

$$\alpha(t) = \bar{\alpha}(Q_D(D(t))) \quad D_f(t) = \bar{D}_f(Q_D(D(t)))$$

A similar expression holds for the aggregate demand:

$$D_{a,D}(t) = D_i(t) + D_f(t) = D_i(t) + \bar{D}_f(Q_D(D(t))) \quad (16)$$

Note that, when the inflexible demand is broadcast ($D = D_i$), the aggregate demand D_{a,D_i} depends exclusively on the current broadcast demand value $d = D_i(t)$:

$$\begin{aligned} D_{a,D_i}(t) &= K(d) = d + \bar{D}_f(Q_{D_i}(d)) \\ &= d + \bar{\alpha}(Q_{D_i}(d)) \int_{\bar{\alpha}_l(Q_{D_i}(d))}^T \frac{f'(s)}{s} ds \end{aligned} \quad (17)$$

IV. DESIGN OF THE PROPORTIONAL CONSTRAINT

The main objective in the design of α is to fulfill (11) and induce a Nash equilibrium in the system. To this end, equilibrium conditions in the measure q can be obtained (formal proof provided in [16]):

Proposition 1: Given a constraint signal $\bar{\alpha} : [0, T] \rightarrow [0, 1]$ and the broadcast demand $D = D_i$ fulfilling Assumption 2, the equilibrium condition (11) holds if:

$$\bar{D}'_i(q) + \bar{D}'_f(q) \geq 0 \quad \forall q \in [0, T] \quad (18)$$

where $\bar{D}_i(q)$ is the inflexible demand as a function of the measure $q = Q_{D_i}(D_i(t))$, as specified in (6) with $D = D_i$, $\bar{D}_f$ is the flexible demand profile defined in (14) and primes denote derivation with respect to q .

This result is used in Section IV-A to approach the problem as a reshape of flexible demand, calculating the power constraint through the resolution of a Cauchy problem. Exploiting the monotonicity of the corresponding dynamical system in the opposite sense of integration, as explained in Section IV-B, we provide in Section IV-C the definitive solution.

A. Shaping of flexible demand

It is not possible to directly apply the results of Proposition 1 and determine a constraint function $\bar{\alpha}$ that induces an equilibrium since its relationship with $\bar{D}'_f$ is not instantaneous:

$$\bar{D}'_f(q) = \bar{\alpha}'(q) \int_{\bar{\alpha}_l(q)}^T \frac{f'(s)}{s} ds - \bar{\alpha}^2(q) \frac{f'(\bar{\alpha}_l(q))}{\bar{\alpha}_l(q)}. \quad (19)$$

For this reason, an indirect approach is followed: a reference $\bar{D}_r$ is calculated for the flexible demand so that (18) holds for $\bar{D}_f = \bar{D}_r$ and then $\bar{\alpha}$ is determined in order the shape the behaviour of the flexible appliances and generate a total power consumption equal to $\bar{D}_r$. The following expression is chosen for the flexible demand reference:

$$\bar{D}_r(q) = \int_q^T \bar{F}(\tau) d\tau. \quad (20)$$

Considering $\bar{F}(\cdot)$ as a control input, it is possible to define the following dynamical system with states $\bar{\alpha}_l$ and $\bar{F}_l$:

$$\begin{aligned} \dot{\bar{\alpha}}_l(q) &= \bar{\alpha}(q) = \frac{\int_q^T \bar{F}(\tau) d\tau}{\int_{\bar{\alpha}_l(q)}^T \frac{f'(\tau)}{\tau} d\tau} = \frac{F_{\text{tot}} - \bar{F}_l(q)}{\int_{\bar{\alpha}_l(q)}^T \frac{f'(\tau)}{\tau} d\tau} & \dot{\bar{F}}_l(q) &= \bar{F}(q) \\ \bar{\alpha}_l(0) &= 0 & \bar{F}_l(0) &= 0 \end{aligned} \quad (21)$$

where $F_{\text{tot}} = \bar{F}_l(T) = \int_0^T \bar{F}(\tau) d\tau$ is the integral of $\bar{F}$ over the interval $[0, T]$. For a fixed control signal $\bar{F}(\cdot)$, the unique solution of (21) is denoted by $(\bar{\varphi}_\alpha(\cdot), \bar{\varphi}_F(\cdot))$. Note that the expression of $\bar{\alpha}(q)$ in (21) guarantees that $\bar{D}_r$ in (20) and $\bar{D}_f$ in (14) are equal. It follows that condition (18) for Nash equilibrium becomes:

$$\bar{F}(q) \leq \bar{D}'_i(q) \quad \forall q \in [0, T]. \quad (22)$$

Rather than directly calculating $\bar{\alpha}$, we determine $\bar{F}$ which satisfies (22) (and therefore guarantees an equilibrium), obtaining the corresponding $\bar{\alpha}$ through (21). In this respect, an additional constraint must be taken into account. Since $\bar{\alpha}(q)$ represents a proportional reduction in the maximum power of the devices, a state $(\bar{\alpha}_l(q), \bar{F}_l(q))$ will be feasible only if:

$$0 \leq \bar{\alpha}(q) = \frac{F_{\text{tot}} - \bar{F}_l(q)}{\int_{\bar{\alpha}_l(q)}^T \frac{f'(\tau)}{\tau} d\tau} \leq 1. \quad (23)$$

Although the main objective in the design of the control $\bar{F}$ (and corresponding power constraint $\bar{\alpha}$) is to induce a Nash equilibrium, we also aim at optimizing some global quantity in the system. In particular, we want to minimize the total time T_{end} required for all devices to

complete their tasks. From (13) and Remark 4, the interval of power consumption for an appliance with $t_{\min} = s$ is given by $\mathcal{S}_D(s) = \{t \in [0, T] : \bar{\alpha}_l(Q_D(D(t))) < s\}$. Since $\mathcal{S}_D(s_1) \subset \mathcal{S}_D(s_2)$ when $s_1 < s_2$, from Remark 2 and Assumption 1 it holds:

$$T_{\text{end}} = \min \{q : \bar{\alpha}_l(q) \geq q_{\max}\}.$$

The function $\bar{F}$ and corresponding constraint $\bar{\alpha}$ can then be calculated by solving the following optimization problem:

$$\begin{aligned} \min_{\bar{F}(\cdot), F_{\text{tot}}, T_{\text{end}}} \quad & T_{\text{end}} \\ \text{s.t.} \quad & \bar{\alpha}_l(0) = 0 & \bar{F}_l(0) &= 0 \\ & \bar{\alpha}_l(T_{\text{end}}^-) = q_{\max} & \bar{F}_l(T_{\text{end}}^-) &= F_{\text{tot}} \\ & \dot{\bar{\alpha}}_l(q) = \frac{F_{\text{tot}} - \bar{F}_l(q)}{\int_{\bar{\alpha}_l(q)}^T \frac{f'(\tau)}{\tau} d\tau} & \dot{\bar{F}}_l(q) &= \bar{F}(q) \\ & 0 \leq \dot{\bar{\alpha}}_l(q) \leq 1 & \bar{F}(q) &\leq \bar{D}'_i(q) \\ & \forall q \in [0, T_{\text{end}}] \end{aligned} \quad (24)$$

Once the optimal $\bar{F}$ for (24) is calculated, the corresponding proportional constraint is equal to $\bar{\alpha}(q) = \dot{\bar{\alpha}}_l(q)$. The values of α as a function of t can be obtained by applying expression (15) in Remark 4.

B. Backward-integrated Dynamical System

It is in general difficult to solve (24) with standard optimal control techniques as the final value F_{tot} of the state $\bar{F}_l$ (not known a priori) appears in the dynamics of $\bar{\alpha}_l$. For this reason, the same dynamics are preliminarily considered in the opposite direction of integration (adopting a tilde accent):

$$\begin{aligned} \dot{\tilde{\alpha}}_l^\varepsilon(q) &= \frac{\tilde{F}_l^\varepsilon(q)}{\int_{q_{\max} - \tilde{\alpha}_l^\varepsilon(q)}^{q_{\max}} \frac{f'(\tau)}{\tau} d\tau} = \frac{\tilde{F}_l^\varepsilon(q)}{h(\tilde{\alpha}_l^\varepsilon(q))} & \dot{\tilde{F}}_l^\varepsilon(q) &= \tilde{F}(q) \\ \tilde{\alpha}_l^\varepsilon(0) &= \varepsilon & \tilde{F}_l^\varepsilon(0) &= 0 \end{aligned} \quad (25)$$

where the function $h(x) = \int_{q_{\max} - x}^{q_{\max}} \frac{f'(\tau)}{\tau} d\tau$ is used for a more compact expression of $\dot{\tilde{\alpha}}_l^\varepsilon$. Note that (25) defines a family of Cauchy problems parameterized by the initial value ε of the state variable $\tilde{\alpha}_l^\varepsilon$. Fixed a control profile $\tilde{F}(\cdot)$, the unique solution of (25) will be denoted by $(\tilde{\varphi}_\alpha^\varepsilon(\cdot), \tilde{\varphi}_F^\varepsilon(\cdot))$. Taking into account that $\tilde{\alpha}_l^\varepsilon$ is not well defined when $(\tilde{\alpha}_l^0(0), \tilde{F}_l^0(0)) = (0, 0)$, we consider decreasing values of ε , letting $(\tilde{\varphi}_\alpha(\cdot), \tilde{\varphi}_F(\cdot))$ express the limit of solutions of (25) for ε which tends to zero. Existence and uniqueness of such limit are guaranteed since equation (25) defines a cooperative system (the solution is monotone with respect to initial conditions). The lemmas presented next show the strong relationships between the solution $(\tilde{\varphi}_\alpha(\cdot), \tilde{\varphi}_F(\cdot))$ of the forward system (21) and the limiting solution $(\tilde{\varphi}_\alpha(\cdot), \tilde{\varphi}_F(\cdot))$ of the backward system (25).

Lemma 1: Consider a feasible function $\bar{F} : [0, \hat{T}] \rightarrow \mathbb{R}^+$ for (21) such that, for the corresponding solution $(\tilde{\varphi}_\alpha(\cdot), \tilde{\varphi}_F(\cdot))$, it holds:

$$\tilde{\varphi}_\alpha(\hat{T}^-) = q_{\max} \quad \tilde{\varphi}_F(\hat{T}^-) = F_{\text{tot}} \quad (26)$$

If the control $\tilde{F}(q) = \bar{F}(\hat{T} - q)$ is applied to (25), for the resulting limiting solution $(\tilde{\varphi}_\alpha(\cdot), \tilde{\varphi}_F(\cdot))$ we have:

$$\begin{aligned}\tilde{\varphi}_\alpha(q) &= q_{\max} - \tilde{\varphi}_\alpha(\hat{T} - q) \\ \tilde{\varphi}_F(q) &= F_{\text{tot}} - \tilde{\varphi}_F(\hat{T} - q)\end{aligned}\quad \forall q \in (0, \hat{T}). \quad (27)$$

A similar relationship also holds in the opposite sense:

Lemma 2: Consider any $\tilde{F} : [0, \hat{T}] \rightarrow \mathbb{R}^+$ which is feasible for (25) and such that, for the corresponding limiting solution $(\tilde{\varphi}_\alpha(\cdot), \tilde{\varphi}_F(\cdot))$, it holds:

$$\tilde{\varphi}_\alpha(\hat{T}) = q_{\max} \quad \tilde{\varphi}_F(\hat{T}) = F_{\text{tot}} \quad (28)$$

Given the solution $(\bar{\varphi}_\alpha(\cdot), \bar{\varphi}_F(\cdot))$ of (21), obtained applying $\bar{F}(q) = \tilde{F}(\hat{T} - q)$, we have:

$$\begin{aligned}\bar{\varphi}_\alpha(q) &= q_{\max} - \bar{\varphi}_\alpha(\hat{T} - q) \\ \bar{\varphi}_F(q) &= F_{\text{tot}} - \bar{\varphi}_F(\hat{T} - q)\end{aligned}\quad \forall q \in [0, \hat{T}). \quad (29)$$

A formal proof of Lemmas 1 and 2 is provided in [16] but the results can be easily verified by comparing the values of $\varphi_F(q)$ and $\varphi_I(q) = \int_0^q \varphi_F(\tau) d\tau$ in the two cases (with bar and tilde accent). Having established a correspondence between the two dynamical systems, problem (24) can be solved by operating with the backward system (25) and exploiting its monotonicity properties:

Proposition 2: The dynamical system described by (25) is cooperative [17].

Proof: It is sufficient to consider the sign of the following partial derivatives:

$$\begin{aligned}\frac{\partial \dot{\tilde{\alpha}}_I^\varepsilon}{\partial \tilde{F}_I^\varepsilon} &= \frac{1}{h(\tilde{\alpha}_I^\varepsilon)} \geq 0 & \frac{\partial \dot{\tilde{F}}_I^\varepsilon}{\partial \tilde{F}_I^\varepsilon} &= 0 & \frac{\partial \dot{\tilde{\alpha}}_I^\varepsilon}{\partial \tilde{F}} &= 0 & \frac{\partial \dot{\tilde{F}}_I^\varepsilon}{\partial \tilde{F}} &= 1 > 0.\end{aligned} \quad (30)$$

C. Task-time Minimizing Solution

The sets of admissible states $(\tilde{\alpha}_I^\varepsilon, \tilde{F}_I^\varepsilon)$ and controls $\tilde{F}(\cdot)$ for system (25), respectively $\mathcal{X}$ and $\mathcal{U}_{\hat{T}}$, are defined as:

$$\begin{aligned}\mathcal{X} &= \{(\tilde{\alpha}_I^\varepsilon, \tilde{F}_I^\varepsilon) : \tilde{F}_I^\varepsilon \leq h(\tilde{\alpha}_I^\varepsilon)\} \\ \mathcal{U}_{\hat{T}} &= \{\tilde{F}(\cdot) : \tilde{F}(q) \in [0, \bar{D}_I'(\hat{T} - q)] \quad \forall q \in [0, \hat{T}]\}.\end{aligned} \quad (31)$$

From Proposition 2, system (25) is monotone for the orders induced from the positive orthants in the state and control space. This means that its state components can be maximized, at each q , by applying the maximum feasible control $\tilde{F}$. Each value of $\hat{T}$ induces a corresponding maximizing solution. With the proper choice of the parameter $\hat{T}$, it is then possible to satisfy (28) and apply Lemma 2, extending the same result to the forward system (21) and allowing to solve the original optimization problem (24). To do so, reminding that $h(x) = \int_{q_{\max}-x}^{q_{\max}} \frac{f'(\tau)}{\tau} d\tau$, the following feedback law as a function of q and current states $\tilde{\alpha}_I$ and $\tilde{F}_I$ is introduced:

$$\tilde{F}^*(q, \tilde{\alpha}_I, \tilde{F}_I) = \begin{cases} \bar{D}_I'(\hat{T} - q) & \text{if } \tilde{F}_I < h(\tilde{\alpha}_I) \\ \min(\bar{D}_I'(\hat{T} - q), h'(\alpha_I)) & \text{if } \tilde{F}_I = h(\tilde{\alpha}_I) \end{cases} \quad (32)$$

We denote by $\Phi^{\hat{T}}(x_0, q)$ the solution of the ODEs in (25) at ‘time’ q and with initial conditions x_0 ($q = 0$) when the (time-varying and discontinuous) feedback law $\tilde{F}^*$ is applied. The

single components of Φ are denoted with α and F subscripts. Consider now the final value of the limiting solution, defined as follows:

$$\gamma(\hat{T}) := \lim_{\varepsilon \rightarrow 0} \Phi_\alpha^{\hat{T}}([\varepsilon, 0], \hat{T}). \quad (33)$$

Given the monotonicity of system (25) from Proposition 2 and since $\tilde{F}^*$ is the maximum feasible control at any time instant and current state, for any other control signal $\tilde{F} \in \mathcal{U}_{\hat{T}}$ and corresponding limiting solution $(\tilde{\varphi}_\alpha(\cdot), \tilde{\varphi}_F(\cdot))$, it holds:

$$\tilde{\varphi}_\alpha(\hat{T}) \leq \gamma(\hat{T}). \quad (34)$$

Definition 1: Assuming that problem (24) is feasible, we define $T^* \in [0, T]$ as the unique time value for which it holds:

$$\gamma(T^*) = q_{\max} \quad \gamma(\hat{T}) < q_{\max} \quad \forall \hat{T} < T^*. \quad (35)$$

Existence and uniqueness of T^* follow from the positivity and boundedness of the state derivatives in (25). The main result of this section can now be provided.

Theorem 1: Denote by $\tilde{\psi}^*(q)$ the following signal:

$$\tilde{\psi}^*(q) = \lim_{\varepsilon \rightarrow 0} \tilde{F}^* \left(q, \Phi^{T^*}([\varepsilon, 0], q) \right). \quad (36)$$

The control $\bar{F}^*$ defined below is feasible and optimal for (24):

$$\bar{F}^*(q) = \tilde{\psi}^*(T^* - q) \quad \forall q \in [0, T^*]. \quad (37)$$

Proof: The feasibility of $\bar{F}^*$ is initially assessed, reminding that $\bar{\varphi}^*$ denotes the solution of (21) when $\bar{F}^*$ is applied and $\tilde{\varphi}^*$ is the limiting solution of system (25) when $\tilde{F} = \tilde{F}^*$ and ε tends to zero. The final state condition $\bar{\alpha}_I(T_{\text{END}}^-) = q_{\max}$ in (24) becomes $\bar{\varphi}_\alpha^*(T^*) = q_{\max}$. Since $\gamma(T^*) = \bar{\varphi}_\alpha^*(T^*) = q_{\max}$, this can be proven by applying Lemma 2 with $\hat{T} = T^*$ and $q = \hat{T}^-$. For the constraints of (24) on state derivatives, considering (32) and (36), we have:

$$\tilde{\psi}^*(q) \leq \bar{D}_I'(T^* - q) \quad \frac{\tilde{\varphi}_F^*(q)}{h(\tilde{\varphi}_\alpha^*(q))} \leq 1 \quad \forall q \in (0, T^*].$$

To verify the second inequality, given the initial conditions of the system, it is sufficient to note that $\tilde{\psi}^*(q) = \tilde{\varphi}_F^*(q) \leq h'(\tilde{\varphi}_\alpha^*(q))$ when $\tilde{\varphi}_F^*(q) = h(\tilde{\varphi}_\alpha^*(q))$. Using Lemma 2 on the solution $\tilde{\varphi}^*$ of system (21) when $\tilde{F}^*$ is applied, the following holds at any $q \in [0, T^*]$:

$$\begin{aligned}\bar{F}^*(q) &= \tilde{\psi}^*(T^* - q) \leq \bar{D}_I'(q) \\ 0 \leq \tilde{\varphi}_\alpha^*(q) &= \frac{F_{\text{tot}} - \tilde{\varphi}_F^*(q)}{\int_{\tilde{\varphi}_\alpha^*(q)}^{q_{\max}} \frac{f'(\tau)}{\tau} d\tau} = \frac{\tilde{\varphi}_F^*(T^* - q)}{h(\tilde{\varphi}_\alpha^*(T^* - q))} \leq 1\end{aligned} \quad (38)$$

Having established the feasibility of $\bar{F}^*$ for (24), we now prove its optimality. Assume that there exists another feasible control $\bar{F}^\diamond$ such that, for the corresponding state trajectory $\bar{\varphi}$, it holds $\bar{\varphi}_\alpha(T^\diamond) = q_{\max}$ with $T^\diamond < T^*$. We denote by $\bar{F}^\diamond(q) = \bar{F}^\diamond(T^\diamond - q)$ the corresponding control for system (25). Applying Lemma 1, from (27) evaluated at $q = \hat{T} = T^\diamond$, we have:

$$\bar{\varphi}_\alpha(T^\diamond) = q_{\max} = \gamma(T^*). \quad (39)$$

Conversely, given (34) and (35) for $\hat{T} = T^\diamond$, it must hold:

$$\bar{\varphi}_\alpha(T^\diamond) \leq \gamma(T^\diamond) < \gamma(T^*) = q_{\max}.$$

This contradicts (39), proving that $\bar{F}^\diamond$ as specified above does not exist and therefore $\bar{F}^*$ is optimal for (24). ■

Theorem 1 shows that the task-time minimizing profile of flexible demand in the q variable $\bar{D}_f(q)$ (and the function $\bar{\alpha}^*(q)$ that induces it when broadcast to the devices) can be calculated operating “backward”. Fixed a final instant $\hat{T}$, the feedback $\bar{F}^*$ for the backward system (25) is evaluated, maximizing the final state $\bar{\alpha}_I(\hat{T})$. There exists $\hat{T}$ (equal to T^* in Definition 1) such that $\bar{\alpha}_I(\hat{T}) = q_{\max}$. This guarantees a correspondence with the forward system, as specified by Lemma 2. By applying the equivalent control $\bar{F}^*(q) = \bar{F}^*(\hat{T} - q)$ forward in time, we solve the optimization problem (24). This means that the resulting value $\bar{\alpha}^*(q) = \bar{\alpha}_I^*(q)$ in system (21) is the proportional constraint in the variable q that, when broadcast to the devices, achieves a Nash equilibrium and minimizes the total task time of the population.

V. SIMULATION RESULTS

The decentralized coordination strategy for flexible demand and the optimal control law presented in the previous sections are now tested in a simulative framework, considering a typical 24h UK demand profile [18] for D_i . The population of flexible appliances is characterized by a function f' which is given by the sum of two truncated gaussian centered at $3h$ and $7h$, while the total energy required for task completion of all devices amounts to $55GWh$. In the present case, condition (9) for existence of an unconstrained Nash equilibrium, as established in [7], is not fulfilled. This can be seen in Fig. 1, where the negotiable valley capacity $\Lambda_D(q) = \frac{d}{dq} Q_D^{-1}(q)$ and the power density of task durations $\Lambda_f(q) = f'(q)/q$ are compared. It follows that, under the current assumptions, it is not

Fig. 1. Negotiable valley capacity Λ_{D_i} (blue trace) and power density of task durations Λ_f (red trace) for the chosen function f' and inflexible demand profile D_i . The green dashed line represents the optimal control $\bar{F}^*$ for the total task time minimization problem (24).

possible to successfully coordinate the appliances by simply broadcasting a demand/price signal. Therefore, it is also necessary to specify a proportional limitation α on their maximum power consumption, limiting flexible demand at critical time instants. The signal α is obtained by numerical

integration of system (21) when the optimal control $\bar{F}^*$ is applied, as discussed in the previous section. In order to determine $\bar{F}^*$, the time instant T^* introduced in Definition 1 and fulfilling (35) is preliminarily calculated. Considering the monotonicity of $\gamma(\hat{T})$ in (33), it is sufficient to apply a bisection algorithm until convergence is achieved within a suitable error margin. In the considered scenario, using a time step $\Delta t = 0.01h$ for the integration of the differential equations in (25), we obtain the final value $T^* = 11.25h$. It is now possible to numerically calculate $\bar{\Psi}^*$ and $\bar{F}^*$ from (36) and (37), respectively. The latter, which from Theorem 1 is the optimal solution for the task time minimization (24), is shown as a dashed green trace in Fig. 1.

If we apply $\bar{F} = \bar{F}^*$ in the dynamical system (21), we obtain the corresponding optimal proportional constraint $\bar{\alpha}^*$. This is shown in Fig. 2, together with the resulting profiles of inflexible, flexible and aggregate demand as functions of the measure q . Note that, since Q_D in (4) is monotone increasing, we expect the same to hold for the inflexible demand profile $\bar{D}_i(q) = Q_{D_i}^{-1}(q)$. The opposite is true for $\bar{D}_f(q) = \bar{D}_r(q)$ since this corresponds to the integral of the positive quantity $\bar{F}^*(\cdot)$ over the interval $[q, T]$. If, as in the present case, the aggregate demand $\bar{D}_a(q) = \bar{D}_i(q) + \bar{D}_f(q)$ is also monotone increasing, we can conclude that (18) is fulfilled and therefore, from Proposition 1, a Nash equilibrium is achieved in the energy market. A qualitative description of the proposed control can be obtained by analysing Fig. 1 and Fig. 2 over different time intervals. For $q > T^*$ we see that $\bar{F}^*$ and the flexible demand $\bar{D}_f$ are equal to zero as all appliances have already completed their tasks. In the interval which goes from approximately $8h$ to $T^* = 11.25h$, the input $\bar{F}^*$ corresponds to the function $\Lambda_f(q - \Delta)$ with Δ equal to about $1h$. In this case the constraint $\bar{\alpha}^*$ is equal to 1 and the resulting demand profile corresponds to the one obtained if all task times were increased by Δ . At about $q = 8h$ the function $\bar{F}^*$ intersects Λ_{D_i} , implying that the maximum power consumption of the appliances must be limited by setting $\bar{\alpha}^*(q) < 1$. This is done by imposing $\bar{F}^*(q) = \Lambda_{D_i}(q) = \bar{D}_i'(q)$, corresponding to a flat profile of aggregate demand in the q variable. Two similar intervals, characterized by $\bar{\alpha}^* = 1$ and by constant $\bar{D}_a$, respectively, can be identified for $q < 6.9h$. The demand profiles and the constraint $\bar{\alpha}^*$ across time are shown in Fig. 3. The shaded blue areas represent the intervals of power consumption of devices with $t_{\min}$ equal to $2h$, $4.5h$ and $8h$. Note that they correspond to the lowest values of $D_a(t)$ (and therefore of energy price $\Pi(D_a(t))$), confirming that a Nash equilibrium is indeed achieved. This result can be compared with the alternative scenario in Fig. 4, where no constraint α is applied. Having established that (9) does not hold, we do not expect any Nash equilibrium in this case. This is confirmed graphically if one considers, for example, the smallest shaded blue area, which correspond to the scheduled interval of power consumption for devices with $t_{\min} = 2h$. Note how these appliances could reduce their energy cost by operating on the small valleys that appear in the aggregate demand around $t = 2h$ and $t = 5.5h$.

Fig. 2. Profiles of inflexible (blue dashed trace), flexible (blue area) and aggregate demand (red trace) as functions of the measure q when the proportional constraint $\bar{\alpha}^*(q)$ (magenta trace) is imposed.

Fig. 3. Profiles of inflexible (blue dashed trace), flexible (blue area) and aggregate demand (red trace) as functions of time when $D = D_i$ and the proportional constraint $\alpha^*(t)$ (magenta trace) are broadcast. The shaded blue areas represent the intervals of power consumption of appliances with parameter $t_{\min}$ equal to $2h$, $4.5h$ and $8h$.

Fig. 4. Demand profiles when $D = D_i$ is broadcast and no constraint α is applied (inflexible demand: blue dashed trace - flexible demand: blue area - aggregate demand: red trace). The shaded blue areas represent the intervals of power consumption of appliances with $t_{\min}$ equal to $2h$, $4.5h$ and $8h$.

VI. CONCLUSIONS

A novel decentralized control strategy for integration of large populations of flexible appliances in the power system is proposed in this paper. On the basis of previous equilibrium results, by introducing a proportional constraint on maximum power consumption, it is possible to achieve a Nash equilibrium in the system for any penetration level of flexible demand. The proposed technique, which also guarantees minimization of the total task time of the appliances, has been tested in simulations of realistic future scenarios of the Great Britain power system.

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