Fig. 3. Difference between all possible integer measurement schedules and the optimal, unachievable schedule with non-integer measurements as given by (47). The star indicates the rounded version of n^* , as given by (48) which achieves the minimal feasible cost. The x -axis is the index of the cost, on a sorted basis.

VI. CONCLUSION

In this technical note, we have provided a collection of optimal measurement scheduling results for a class of linear, scalar Gauss-Markov systems using a terminal cost function. We have provided rigorous proofs for five different measurement scheduling scenarios. In each case we have shown that the resulting optimal scheduling policy is an *index* policy. As a consequence, this means that the optimal policy is easy to compute. Numerical examples have been given to illustrate the benefits of measurement scheduling for such systems in a number of scenarios. These results show that the gains due to optimal scheduling may be substantial. We have also shown that the optimal results can be used to derive suboptimal scheduling policies that are also easy to compute and provide acceptable performance.

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Control of Constrained Discrete-Time Systems With Bounded ℓ_2 Gain

Paul J. Goulart, Eric C. Kerrigan, and Teodoro Alamo

Abstract—We consider the problem of designing a control law for a constrained linear system with bounded disturbances that ensures constraint satisfaction over an infinite horizon, while also guaranteeing that the closed-loop system has bounded ℓ_2 gain. To this end, we propose a receding horizon control strategy based on the repeated calculation of optimal finite horizon feedback policies. We parameterize these policies such that the input at each time is an affine function of current and prior states, and minimize a worst-case quadratic cost where the disturbance energy is negatively weighted as in $\mathcal{H}_\infty$ control. We show that the resulting receding horizon controller has two advantages over previous results for this problem. First, the policy optimization problem to be solved at each time step can be rendered convex-concave, with a number of decision variables and constraints that grows polynomially with the problem size, thereby making its solution amenable to standard techniques in convex optimization. Second, the achievable ℓ_2 gain of the resulting closed-loop system is bounded and non-increasing with increasing control horizon. A numerical example is included to demonstrate the improvement in achievable ℓ_2 gain relative to existing methods.

Index Terms—Constrained systems, receding horizon control (RHC), robust control, robust optimization, uncertain systems.

I. INTRODUCTION

This technical note considers the control of linear discrete-time systems subject to bounded additive disturbances on the state, and subject to mixed convex constraints on the states and inputs. Formulation of computationally tractable control laws that guarantee stability and constraint satisfaction for such systems remains an important open problem in the literature [1]. This technical note is motivated in particular by the considerable recent interest in applications of robust predictive control to the problem of ℓ_2 gain minimization (i.e. $\mathcal{H}_\infty$ receding horizon control) for constrained linear [2]–[6] and nonlinear systems [7]–[9]. It is generally accepted that if disturbances are to be accounted for in the underlying finite horizon control problem, to be solved in a receding horizon scheme, then that problem must be formulated in terms of (generally nonlinear) *feedback policies*, rather than open-loop control sequences as in [4], [10, Chap. 9]. A standard approach to this problem is to employ methods in dynamic programming to compute a solution to the control problem over a finite horizon [11]. In general, these dynamic programming-based schemes or, alternatively, closed-loop schemes such as [12] based on finite-dimensional optimization, are computationally intractable because the size of the optimization problem required for their solution grows exponentially with the problem size.

We propose an alternative scheme based on a class of finite-horizon control policies where the input at each time instant is parameterized as an affine function of current and prior system states; such policies are

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P. J. Goulart and E. C. Kerrigan are with the Department of Aeronautics and Department of Electrical and Electronic Engineering, Imperial College London, London SW7 2AZ, U.K. (e-mail: p.goulart@imperial.ac.uk; e.kerrigan@imperial.ac.uk).

T. Alamo is with the Departamento de Ingeniería de Sistemas y Automática, Universidad de Sevilla, Sevilla 41092, Spain (e-mail: alamo@cartuja.us.es).

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therefore more general than pre-stabilization based schemes [5]–[7], or those based on online calculation of linear control laws [2], [3]. As shown in [13], this alternative scheme is attractive since the set of feasible policy parameters can be rendered convex via a suitable reparameterization, thereby allowing a constraint admissible policy from this set to be computed using newly available techniques in robust optimization [14]. Additionally, the size of the optimization problem required to find such a policy grows only polynomially with the problem size, making it a suitable candidate for online computation. By imposing additional convex constraints on the set of policies in this class and employing a worst-case quadratic cost function, where the disturbance is negatively weighted as in $\mathcal{H}_\infty$ control [15], [16], we show that one can make additional guarantees about the ℓ_2 gain of the resulting closed-loop system when these policies are used in the synthesis of receding horizon control (RHC) laws. The proposed method is particularly attractive since the min-max optimization problem to be solved at each time is convex-concave, and thus amenable to standard computational methods in convex optimization. In cases where the system constraints are polyhedral, we show that this min-max problem can be recast as a semidefinite program (SDP)—however, unlike proposals that employ ellipsoidal invariant sets [2], [3], the proposed scheme handles asymmetric constraints and a wider variety of disturbance classes.

The technical note is organized as follows. Section II discusses the problem of interest and outlines a number of standing assumptions and definitions. It also describes the class of control policies used throughout, and relates a number of results pertaining to the minimization of ℓ_2 gain over a finite horizon. Section III extends these results to the infinite horizon case; we provide conditions under which one can simultaneously guarantee constraint satisfaction for the closed-loop system for all time while ensuring bounded ℓ_2 gain, and show that the achievable bound is a decreasing function of the control horizon length. In Section IV we show how the proposed min-max optimization problem can be solved using semidefinite programming techniques for cases of practical interest. Some conclusions are drawn in Section V.

Notation: For matrices A and B , A' denotes the transpose of A , $A \succ 0$ ($A \succeq 0$) indicates that A is positive (semi)definite and $A \otimes B$ is the Kronecker product of A and B . The standard Euclidean or 2-norm is denoted $\|\cdot\|_2$, with associated induced matrix norm $\|\cdot\|_2$. The weighted 2-norm of a vector x is denoted $\|x\|_P = \sqrt{x'Px}$. I_N is the $N \times N$ identity matrix. Vertical stacking of vectors x and y is denoted $(x; y) := \begin{bmatrix} x \\ y \end{bmatrix}$.

II. FINITE HORIZON ℓ_2 GAIN MINIMIZATION

Consider the discrete-time linear time-invariant system

$$x^+ = Ax + Bu + Gw \quad (1a)$$

$$z = Cx + Du \quad (1b)$$

where $x \in \mathbb{R}^n$ is the system state at the current time instant, x^+ is the state at the next time instant, $u \in \mathbb{R}^m$ is the control input/manipulated variable, $w \in \mathbb{R}^l$ is a disturbance and $z \in \mathbb{R}^p$ is the costed/controlled variable. It is assumed that at each sample instant a measurement of the state is available. The current and future values of the disturbance are unknown and may change from one time instant to the next, but are contained in a convex and compact set W , assumed known, containing the origin.

The system is subject to convex constraints on the state and input, so that it must satisfy $(x, u) \in Z$, where $Z \in \mathbb{R}^n \times \mathbb{R}^m$ is a closed and convex set containing the origin in its interior. A design goal is to guarantee that the state and input of the closed-loop system remain in Z for all disturbance sequences generated from W . In addition to Z , a target/terminal constraint set $X_f \subset \mathbb{R}^n$ is given, which is closed and

convex and contains the origin in its interior. The set X_f can be used to design an RHC law with guaranteed infinite-horizon ℓ_2 gain properties, as will be shown in Section III.

Predictions of the system's evolution over a finite control/planning horizon of length N will be used to define a suitable control policy with a bounded finite-horizon ℓ_2 gain. Define stacked versions of the predicted input, state and disturbance vectors $\mathbf{u} \in \mathbb{R}^{mN}$, $\mathbf{x} \in \mathbb{R}^{n(N+1)}$ and $\mathbf{w} \in \mathbb{R}^{lN}$, respectively, as $\mathbf{x} := (x_0; \dots; x_N)$, $\mathbf{u} := (u_0; \dots; u_{N-1})$ and $\mathbf{w} := (w_0; \dots; w_{N-1})$, where $x_0 := x$ denotes the value of the initial state and $x_{i+1} := Ax_i + Bu_i + Gw_i$ denotes the state after i time instants. Let the set $\mathcal{W} := W^N := W \times \dots \times W$, so that $\mathbf{w} \in \mathcal{W}$. Finally, define a closed and convex set $\mathcal{Z}$, appropriately constructed from Z and X_f , such that the constraints to be satisfied are equivalent to requiring that $(\mathbf{x}, \mathbf{u}) \in \mathcal{Z}$, i.e.

$$\mathcal{Z} := \{(\mathbf{x}, \mathbf{u}) | (x_i, u_i) \in Z, i = 0, 1, \dots, N-1, x_N \in X_f\}.$$

We are interested in finding a *feedback policy* π of the form

$$\pi := \{\kappa_0(\cdot), \kappa_1(\cdot), \dots, \kappa_{N-1}(\cdot)\} \quad (2)$$

such that the control inputs $u_i = \kappa_i(x_0, \dots, x_i)$ will satisfy the constraints $\mathcal{Z}$ for all possible disturbance sequences. Given a positive scalar γ and an initial state x , we wish to determine whether there exists a constraint admissible feedback policy π and a non-negative scalar $\beta(x)$ such that the following finite-horizon ℓ_2 gain property [17]–[19] holds for all allowable disturbance sequences $\mathbf{w} \in \mathcal{W}$

$$\|x_N\|_P^2 + \sum_{i=0}^{N-1} \|z_i\|_2^2 \leq \beta(x) + \gamma^2 \sum_{i=0}^{N-1} \|w_i\|_2^2 \quad (3)$$

where $P \succeq 0$ is given. The main aim of this section is to demonstrate that by restricting the class of policies π considered, convex optimization methods can be used to solve the above finite-horizon ℓ_2 gain problem efficiently. In Section III, it will be shown that such policies can be used in the synthesis of a computationally tractable receding horizon control law with an infinite-horizon ℓ_2 gain guarantee.

A. An Affine Policy Parameterization

Finding an *arbitrary* control policy π that satisfies the constraints $\mathcal{Z}$ for all admissible disturbance sequences while guaranteeing satisfaction of the ℓ_2 gain property (3) is extremely difficult in general, and current proposals for defining such policies generally require solution via dynamic programming [11]. As a result, we find it convenient to restrict the class of control policies to those of the form

$$u_i = v_i + \sum_{j=0}^{i-1} M_{i,j} G w_j, \quad \forall i \in \{0, 1, \dots, N-1\} \quad (4)$$

where each $M_{i,j} \in \mathbb{R}^{m \times n}$ and each $v_i \in \mathbb{R}^m$. Note that since full state feedback is assumed, the uncertain terms Gw_j represent the difference between the predicted and actual states at each step, i.e. $Gw_j = x_{j+1} - Ax_j - Bu_j$. For notational convenience, define the variable $\mathbf{v} \in \mathbb{R}^{mN}$ and the block lower triangular matrix $\mathbf{M} \in \mathbb{R}^{mN \times nN}$ such that

$$\mathbf{M} := \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ M_{1,0} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ M_{N-1,0} & \cdots & M_{N-1,N-2} & 0 \end{pmatrix}, \quad \mathbf{v} := \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-1} \end{pmatrix} \quad (5)$$

so that the control input and state sequences can be written as

$$\mathbf{u} = \mathbf{v} + \mathbf{M}\mathbf{G}\mathbf{w}, \quad (6a)$$

$$\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{v} + (\mathbf{B}\mathbf{M}\mathbf{G} + \mathbf{G})\mathbf{w} \quad (6b)$$

where all other matrices are defined in the Appendix. The set of admissible $(\mathbf{M}, \mathbf{v})$, for which the constraints $\mathcal{Z}$ are satisfied for all allowable disturbance sequences of length N , is defined as

$$\Pi_N(x) := \left\{ (\mathbf{M}, \mathbf{v}) \left| \begin{array}{l} (\mathbf{M}, \mathbf{v}) \text{ satisfies (5)} \\ \mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{G}\mathbf{w} \\ \mathbf{u} = \mathbf{v} + \mathbf{M}\mathbf{G}\mathbf{w} \\ (\mathbf{x}, \mathbf{u}) \in \mathcal{Z}, \forall \mathbf{w} \in \mathcal{W} \end{array} \right. \right\} \quad (7)$$

and the set of initial states x for which an admissible control policy of the form (4) exists as

$$X_N := \{x \in \mathbb{R}^n | \Pi_N(x) \neq \emptyset\}. \quad (8)$$

Remark 1: The control parameterization (4) was proposed in [14], [20] and is of particular interest because the set of feasible policy parameters $\Pi_N(x)$ for each state x is convex. Importantly, this class of policies is equivalent to one defined as an affine function of current and prior states, where $u_i = g_i + \sum_{j=0}^{i-1} K_{i,j}x_j$ [13].

B. A Finite Horizon Cost

For a given initial state x , we wish to find a control policy $(\mathbf{M}, \mathbf{v}) \in \Pi_N(x)$ to satisfy both the system constraints $\mathcal{Z}$ and the ℓ_2 gain property (3). We thus define the finite-horizon quadratic cost function

$$J_N(x, \gamma, \mathbf{M}, \mathbf{v}, \mathbf{w}) := \|x_N\|_P^2 + \sum_{i=0}^{N-1} (\|z_i\|_2^2 - \gamma^2 \|w_i\|_2^2)$$

where $x_0 := x$ and the sequence $\{u_i\}_{i=0}^{N-1}$ is given by (4), and will consider a zero-sum game of the form:

$$\min_{(\mathbf{M}, \mathbf{v}) \in \Pi_N(x)} \max_{\mathbf{w} \in \mathcal{W}} J_N(x, \gamma, \mathbf{M}, \mathbf{v}, \mathbf{w}). \quad (9)$$

One can readily obtain matrices $\mathcal{H}_x$, $\mathcal{H}_u$ and $\mathcal{H}_w$ (given in the Appendix) such that

$$J_N(x, \gamma, \mathbf{M}, \mathbf{v}, \mathbf{w}) = \|\mathcal{H}_x x + \mathcal{H}_u \mathbf{v} + (\mathcal{H}_u \mathbf{M}\mathbf{G} + \mathcal{H}_w) \mathbf{w}\|_2^2 - \gamma^2 \|\mathbf{w}\|_2^2. \quad (10)$$

In this section, we will provide constraints on the feedback policy $(\mathbf{M}, \mathbf{v})$ and the gain γ such that this zero-sum game can be guaranteed to be convex-concave. This will ensure the existence of a saddle point solution in pure policies [15, Sect. 2.1] and that (9) is solvable via convex optimization techniques. We first define the function $J_N^*(\cdot)$ as

$$J_N^*(x, \gamma, \mathbf{M}, \mathbf{v}) := \max_{\mathbf{w} \in \mathcal{W}} J_N(x, \gamma, \mathbf{M}, \mathbf{v}, \mathbf{w}) \quad (11)$$

so that the following condition, guaranteeing that the minimization problem in (9) is *convex*, is easily established:

Proposition 1: The function $(\mathbf{M}, \mathbf{v}) \mapsto J_N^*(x, \gamma, \mathbf{M}, \mathbf{v})$ is convex and lower bounded by zero.

Proof: For any fixed $\mathbf{w} \in \mathcal{W}$, the function $(x, \mathbf{M}, \mathbf{v}) \mapsto J_N(x, \gamma, \mathbf{M}, \mathbf{v}, \mathbf{w})$ is convex, and the pointwise supremum of convex functions is convex. The lower bound follows from the assumption that $0 \in \mathcal{W}$. ■

We next impose a condition on γ such that the maximization problem in (11) is *concave*, so that our eventual min-max control policy optimization problem will be convex-concave:

Proposition 2: For any given $\mathbf{M}$, there exists a $\gamma > 0$ such that the following quadratic matrix inequality (QMI) holds:

$$H_N := \gamma^2 I - (\mathcal{H}_u \mathbf{M}\mathbf{G} + \mathcal{H}_w)' (\mathcal{H}_u \mathbf{M}\mathbf{G} + \mathcal{H}_w) \succeq 0. \quad (12)$$

Furthermore, if (12) is satisfied then the function $\mathbf{w} \mapsto J_N(x, \gamma, \mathbf{M}, \mathbf{v}, \mathbf{w})$ is concave.

Proof: For fixed $\mathbf{M}$, the inequality (12) is satisfied for any $\gamma \geq \|\mathcal{H}_u \mathbf{M}\mathbf{G} + \mathcal{H}_w\|_2$. Concavity is obvious from inspection of (10). ■

Multiplying (12) by $-1/\gamma$ and applying the Schur complement inequality [21, Sec. A.5.5], the above QMI is equivalent to the following linear matrix inequality (LMI) in γ and $\mathbf{M}$:

$$\begin{pmatrix} -\gamma I & (\mathcal{H}_u \mathbf{M}\mathbf{G} + \mathcal{H}_w)' \\ (\mathcal{H}_u \mathbf{M}\mathbf{G} + \mathcal{H}_w) & -\gamma I \end{pmatrix} \preceq 0. \quad (13)$$

As a result, we will consider a policy of the form (4) to be feasible (with respect to a given γ) if, given an initial state x , it is both constraint admissible and satisfies the LMI condition (13). For a given state x and positive scalar γ , we therefore define the set of policies satisfying these conditions as

$$\tilde{\Pi}_N(x, \gamma) := \{(\mathbf{M}, \mathbf{v}) \in \Pi_N(x) | (\gamma, \mathbf{M}) \text{ satisfies (13)}\}$$

and the set of states for which a policy of this form exists as $\tilde{X}_N(\gamma) := \{x | \tilde{\Pi}_N(x, \gamma) \neq \emptyset\}$.

Inclusion of these LMI constraints imposes tighter requirements on admissibility for the sets $\tilde{\Pi}_N(x, \gamma)$ and $\tilde{X}_N(\gamma)$ than for the related sets $\Pi_N(x)$ and X_N ; i.e. for any state x and positive scalar γ , the set inclusions $\tilde{X}_N(\gamma) \subseteq X_N$ and $\tilde{\Pi}_N(x, \gamma) \subseteq \Pi_N(x)$ hold. It is also easily shown that $\tilde{\Pi}_N(x, \gamma_1) \subseteq \tilde{\Pi}_N(x, \gamma_2)$ and $\tilde{X}_N(\gamma_1) \subseteq \tilde{X}_N(\gamma_2)$ for any $\gamma_1 \leq \gamma_2$.

C. Finite Horizon ℓ_2 Gain

For a given initial state $x \in X_N$, it is of interest to compute the smallest positive value of γ for which one can ensure the existence of an admissible disturbance feedback policy $(\mathbf{M}, \mathbf{v})$ such that the finite-horizon ℓ_2 gain property (3) holds. For this purpose, define the function $\gamma_N^* : X_N \rightarrow \mathbb{R}_+$ as

$$\gamma_N^*(x) := \inf \left\{ \gamma | \tilde{\Pi}_N(x, \gamma) \neq \emptyset \right\}. \quad (14)$$

We are interested in the following optimization problem for a given $\gamma \geq \gamma_N^*(x)$:

$$V_N^*(x, \gamma) := \min_{(\mathbf{M}, \mathbf{v})} J_N^*(x, \gamma, \mathbf{M}, \mathbf{v}) \text{ s.t. } (\mathbf{M}, \mathbf{v}) \in \tilde{\Pi}_N(x, \gamma). \quad (15)$$

We denote a minimizer¹ of (15) as $(\mathbf{M}^*(x, \gamma), \mathbf{v}^*(x, \gamma))$, and assume that it can be partitioned in a manner identical to that in (5), so that the k th element of $\mathbf{v}^*(x, \gamma)$ is denoted $v_k^*(x, \gamma)$, and the (i, j) th submatrix of $\mathbf{M}^*(x, \gamma)$ is denoted $M_{i,j}^*(x, \gamma)$.

The main result of this section follows immediately from the discussion thus far:

Theorem 1: For a given initial state $x(0) \in X_N$ and $\gamma \geq \gamma_N^*(x(0))$, consider implementing the following time-varying control policy on the system (1):

$$u(k) = v_k^*(x(0), \gamma) + \sum_{j=0}^{k-1} M_{k,j}^*(x(0), \gamma) (x(j+1) - Ax(j) - Bu(j))$$

¹We assume that $\tilde{\Pi}_N(\mathbf{x}, \gamma) \neq \emptyset$ for all $\gamma \geq \gamma_N^*(\mathbf{x})$ and that a minimizer for (15) exists for all $\mathbf{x} \in \tilde{X}_N(\gamma)$ and all $\gamma \geq \gamma_N^*(\mathbf{x})$. These assumptions are guaranteed to hold if $\mathbf{D}$ is full column rank, $\mathbf{0} \in \text{interior}(\mathcal{W})$ and there exists a bounded set $\mathcal{B}$ such that $\mathcal{Z} \subseteq \mathbb{R}^n \times \mathcal{B}$ [22].

for all $k \in \{0, 1, \dots, N-1\}$. For all disturbance sequences $\{w(k)\}_{k=0}^{N-1}$ drawing values from W , we have that $(x(k), u(k)) \in Z$ for $k \in \{0, 1, \dots, N-1\}$, $x(N) \in X_f$ and the following ℓ_2 gain property holds:

$$\|x(N)\|_P^2 + \sum_{k=0}^{N-1} \|z(k)\|_2^2 \leq V_N^*(x(0), \gamma) + \sum_{k=0}^{N-1} \gamma^2 \|w(k)\|_2^2.$$

III. INFINITE HORIZON ℓ_2 GAIN MINIMIZATION

We next consider the *infinite*-horizon case, and seek a control law that satisfies

$$\sum_{k=0}^{\infty} \|z(k)\|_2^2 \leq \beta(x(0)) + \gamma^2 \sum_{k=0}^{\infty} \|w(k)\|_2^2 \quad (16)$$

for all disturbance sequences satisfying $\sum_{k=0}^{\infty} \|w(k)\|_2^2 < \infty$ and $w(k) \in W$ for all $k \in \{0, 1, \dots\}$. We construct our control law by exploiting the parameterization introduced in (4) and implementing the control in a receding horizon fashion. We therefore define the receding horizon control law $\mu_N : X_N \times \mathbb{R}_+ \rightarrow \mathbb{R}^m$ as

$$u = \mu_N(x, \gamma) := v_0^*(x, \gamma) \quad (17)$$

which is *nonlinear* in general. The closed-loop system dynamics become

$$x(k+1) = Ax(k) + B\mu_N(x(k), \gamma) + Gw(k) \quad (18a)$$

$$z(k) = Cx(k) + D\mu_N(x(k), \gamma). \quad (18b)$$

In order to show that this closed-loop system satisfies the ℓ_2 gain condition (16) while simultaneously guaranteeing that $(x(k), \mu_N(x(k), \gamma)) \in Z$ for all time, we require the following assumption on the terminal set X_f and cost P :

Terminal Cost and Constraint: (A, B) is stabilizable, (C, A) is detectable and (C, A, B) has no zeros on the unit circle. The control feedback matrix K and terminal cost P satisfy the solution to an *unconstrained* $\mathcal{H}_\infty$ state feedback control problem [15], [16] with ℓ_2 gain γ_f

$$P := C'C + A'\bar{P}A - (A'\bar{P}B + C'D) \times (D'D + B'\bar{P}B)^{-1}(B'\bar{P}A + D'C) \quad (19a)$$

$$K := -(D'D + B'\bar{P}B)^{-1}(B'\bar{P}A + D'C) \quad (19b)$$

where $\bar{P} := P + PG(\gamma_f^2 I - G'PG)^{-1}G'P$ and $\gamma_f^2 I - G'PG \succ 0$. We further assume that the terminal constraint set X_f is chosen to be both constraint admissible and robust positively invariant under the control $u = Kx$, i.e.

$$X_f \subseteq \{x | (x, Kx) \in Z\} \quad (20a)$$

$$(A + BK)x + Gw \in X_f, \quad \forall x \in X_f, \forall w \in W. \quad (20b)$$

A variety of methods are available for the computation of a set X_f satisfying the conditions (20)—the reader is referred to [23] and the references therein for details.

A. Invariance Properties

We first consider whether, for a given fixed $\gamma \geq \gamma_f$, the optimization problem (15) required to implement the receding horizon control law $\mu_N(\cdot, \gamma)$ can be solved for all time for the closed-loop system (18). In order to provide conditions under which the set $\tilde{X}_N(\gamma)$

is robust positively invariant for the closed-loop system (18), we first establish the following nesting condition (see the Appendix for proofs):

Proposition 3: If **A1** holds with $\gamma \geq \gamma_f$, then the following set inclusion property holds:

$$X_f \subseteq \tilde{X}_1(\gamma) \subseteq \dots \subseteq \tilde{X}_{N-1}(\gamma) \subseteq \tilde{X}_N(\gamma) \subseteq \tilde{X}_{N+1}(\gamma) \dots$$

A similar nesting condition holds with respect to the minimum achievable gain:

Corollary 1: If **A1** holds, then $\gamma_{N+1}^*(x) \leq \max\{\gamma_f, \gamma_N^*(x)\}$ for all $x \in X_N$.

The significance of Corollary 1 is that, unlike existing $\mathcal{H}_\infty$ receding horizon control schemes such as [10, Chap. 9], it will allow us to guarantee that the achievable infinite-horizon ℓ_2 gain for the closed-loop system (18) is non-increasing with increasing horizon length. Using the above results we can also demonstrate that for any $\gamma \geq \gamma_f$, each of the sets $\tilde{X}_N(\gamma)$ is robust positively invariant for the closed-loop system (18).

Lemma 1: If **A1** holds with $\gamma \geq \gamma_f$ then, for each $N \in \{1, 2, \dots\}$, the set $\tilde{X}_N(\gamma)$ is robust positively invariant for the closed-loop system (18), i.e. if $x \in \tilde{X}_N(\gamma)$ then $x^+ = Ax + B\mu_N(x, \gamma) + Gw \in \tilde{X}_N(\gamma)$ for all $w \in W$.

Corollary 2: If one employs instead the minimum-gain control law $u = \mu_N(x, \max\{\gamma_f, \gamma_N^*(x)\})$, then the sequence $\{\max\{\gamma_f, \gamma_N^*(x(k))\}\}_{k=0}^\infty$ is non-increasing for the closed-loop system, provided that the disturbance sequence $w(\cdot)$ remains in W .

B. Finite ℓ_2 Gain in Receding Horizon Control

We can now state the main result of this section, which allows us to place an upper bound on the ℓ_2 gain of the closed-loop system (18) under the proposed RHC law $\mu_N(\cdot, \gamma)$:

Theorem 2: If **A1** holds and $\gamma \geq \max\{\gamma_f, \gamma_N^*(x(0))\}$ then, for the closed-loop system (18), the ℓ_2 gain from the disturbance $w(\cdot)$ to the cost/controlled variable $z(\cdot)$ is upper bounded by γ . Furthermore, the constraints are satisfied for all time if the initial state $x(0) \in \tilde{X}_N(\gamma)$.

Proof: For any function $\varphi(\cdot)$, define

$$\Delta\varphi(x, u, w) := \varphi(Ax + Bu + Gw) - \varphi(x)$$

and functions $V_f(x) := x'Px$, $\ell(x, u, w) := (\|z\|_2^2 - \gamma^2\|w\|_2^2)$ and $\ell_f(x, u, w) := (\|z\|_2^2 - \gamma_f^2\|w\|_2^2)$, where $z = Cx + Du$. If **A1** holds, then a standard result in $\mathcal{H}_\infty$ control is that

$$\max_{w \in \mathbb{R}^p} [\Delta V_f + \ell_f](x, Kx, w) = 0.$$

Since $\gamma_f \leq \gamma$ implies $\ell(x, u, w) \leq \ell_f(x, u, w)$ for all (x, u, w)

$$[\Delta V_f + \ell](x, u, w) \leq [\Delta V_f + \ell_f](x, u, w), \quad \forall (x, u, w)$$

so that

$$\max_{w \in W} [\Delta V_f + \ell](x, Kx, w) \leq 0. \quad (21)$$

Given a state $x^+ = Ax + B\mu_N(x, \gamma) + Gw$, Lemma 1 ensures the existence of a feasible control policy $(\mathbf{M}, \hat{\mathbf{v}}) \in \tilde{\Pi}_N(x^+, \gamma)$ such that the terminal control input is $u_N = Kx_N$. Application of (21) guarantees that

$$[\Delta V_\gamma + \ell](x, \mu_N(x, \gamma), w) \leq 0, \quad \forall x \in \tilde{X}_N(\gamma), \forall w \in W \quad (22)$$

TABLE I
MINIMUM GAIN $\gamma_N^*(0)$ AND REGION OF ATTRACTION $|X_N|$

N	$\gamma_N^*(0)$			$ X_N $		
	OL	PS	(M, v)	OL	PS	(M, v)
1	1.41	1.41	1.41	0.83	0.83	0.83
2	3.29 [†]	1.91	1.41	—	0.93	0.99
5	28.2 [†]	2.65	1.41	—	0.98	1.13
10	903.1 [†]	3.09	1.41	—	0.98	1.15

[†] Infeasible; result is for system (27) without constraints.

[†] Infeasible; result is for system (27) without constraints.

where $V_\gamma(\cdot) := V_N^*(\cdot, \gamma)$. For any integer q and disturbance sequence $\{w(k)\}_{k=0}^{q-1}$ taking values in W , it then follows from (22) that

$$V_N^*(x(q), \gamma) \leq V_N^*(x(0), \gamma) - \sum_{k=0}^{q-1} \ell(x(k), \mu_N(x(k), \gamma), w(k)) \quad (23)$$

where $x(k+1) = Ax(k) + B\mu_N(x(k), \gamma) + Gw(k)$ for all $k \in \{0, \dots, q-1\}$. It follows from Proposition 1 that $V_N^*(x, \gamma) \geq 0$, so that for any integer $q \geq 0$, (23) implies

$$\sum_{k=0}^{q-1} \|z(k)\|_2^2 \leq V_N^*(x(0), \gamma) + \gamma^2 \sum_{k=0}^{q-1} \|w(k)\|_2^2 \quad (24)$$

and the claim about finite gain follows. Constraint satisfaction is guaranteed by Lemma 1. ■

Note that for any initial state $x(0)$, Corollary 1 guarantees that the minimum achievable closed-loop gain in Theorem 2 is a non-increasing function of the horizon length N . Further, given the same assumptions as Theorem 2 and using standard results [1], one can show that the origin of the undisturbed closed-loop system $x^+ = Ax + B\mu_N(x, \gamma)$ is exponentially stable with region of attraction $\tilde{X}_N(\gamma)$. Finally, if $x(0) \in X_f$ and $\gamma = \gamma_f$, then $V_N^*(x(0), \gamma_f) = \|x(0)\|_P^2$ in (24) and $\mu_N(x(k), \gamma_f) = Kx(k)$ for all $k \in \{0, 1, \dots\}$, matching the well-known $\mathcal{H}_\infty$ optimal control law for unconstrained linear systems [16].

IV. COMPUTATIONAL ISSUES

When the state and input constraint set Z is polyhedral, it is often possible to eliminate the universal quantifier in (7) and characterize $\Pi_N(x)$ using a set of convex constraints [13]. For example, if the set W is polyhedral or 2-norm bounded, then (7) can be rewritten using only linear or second-order cone constraints, respectively. We show here how the min-max optimization problem (15) can be solved numerically in such a case.

We assume for the purposes of this section that the disturbance set W can be characterized using a single affine conic inequality in the form

$$W := \{w | Sw \preceq_K h\}$$

for some convex cone K with dual cone K^* , where we denote $Sw \preceq_K h \Leftrightarrow h - Sw \in K$; note that both polyhedral and norm-bounded disturbance sets can be characterized in this fashion [21]. In such a case it is possible to eliminate the maximization over w implicit in (15) using duality arguments:

Proposition 4 ([22]): If the QMI (12) holds, then the optimization problem

$$\max_{w \in W} (\|(\mathcal{H}_x x + \mathcal{H}_u v) + (\mathcal{H}_u M G + \mathcal{H}_w)w\|_2^2 - \gamma^2 \|w\|_2^2)$$

is equivalent to (i.e. achieves the same optimal value as) the minimization problem

$$\begin{aligned} & \min_{(\delta, y)} (2h'y + \delta) \\ & \text{subject to } y \succeq_{K^*} 0 \text{ and} \\ & \begin{pmatrix} \delta & y'S & (\mathcal{H}_x x + \mathcal{H}_u v)' \\ S'y & \gamma^2 I & (\mathcal{H}_u M G + \mathcal{H}_w)' \\ (\mathcal{H}_x x + \mathcal{H}_u v) & (\mathcal{H}_u M G + \mathcal{H}_w) & I \end{pmatrix} \succeq 0. \end{aligned} \quad (25)$$

Given this result, it is possible to solve the min-max optimization problem (12) via the solution of a single convex optimization problem, i.e.

$$V_N^*(x, \gamma) = \min_{(\delta, M, v, y)} 2h'y + \delta \quad \text{s.t. } (M, v) \in \Pi_N(x), y \succeq_{K^*} 0 \text{ and (25)}. \quad (26)$$

In cases where the the state and input constraints are polyhedral and the disturbance set is either polytopic or 2-norm bounded, the constraint $(M, v) \in \Pi_N(x)$ can be reformulated as a set of linear or second-order-cone constraints respectively using the methods in [13]—such constraints are readily converted to LMIs using standard techniques [22], [24]. The optimization problem (26) is therefore a semidefinite program (SDP) in a tractable number of variables in these cases. Note that in posing such a problem it is not necessary to explicitly include the LMI condition (13), since its satisfaction is implied by satisfaction of (25).

A. Numerical Example

Consider the system

$$x^+ = 2x + 2u + w \quad (27a)$$

$$z = [x \ u]'. \quad (27b)$$

subject to the constraints $|u| \leq 0.7$ and $|w| \leq 0.25$. We calculated the minimum gain $\gamma_N^*(0)$ defined in (14) given terminal conditions $X_f = \{x | |x| \leq 0.5\}$ and $P = 2$ for varying horizon lengths N . We also consider the minimum achievable gain for an *open-loop* (OL) control policy (equivalent to setting $M = 0$) defined in the sense of [10, Chap. 9], as well as for a control policy based on a *pre-stabilizing* (PS) control sequence $u_i = v_i + Kx_i$ in a manner similar to [5], [6] with $K = -1.25$. Note that the latter case is equivalent to specifying a fixed feedback structure $M = (I - KB)^{-1}KG$, where $K := I \otimes K$. The minimum guaranteed gains at the origin $\gamma_N^*(0)$, as well as the region of attraction $|X_N| = \max\{|x| | x \in X_N\}$, are shown in Table I.

The interesting point to note is that, for the feedback approach adopted in this technical note, $\gamma_N^*(0)$ does not change with horizon length N . Since the aim is to minimize the closed-loop ℓ_2 gain, this is a much more favorable result than can be guaranteed by the open-loop or pre-stabilizing methods, for which the minimal gain increases with N . The feedback approach allows one to shape the matrix $(\mathcal{H}_u M G + \mathcal{H}_w)'(\mathcal{H}_u M G + \mathcal{H}_w)$ on-line via M , while still ensuring the optimization problem is convex-concave (see Proposition 2), whereas this matrix is fixed for the open-loop and pre-stabilizing methods. Furthermore, not only does the feedback approach guarantee better disturbance rejection, but it also guarantees a larger region of attraction X_N than the pre-stabilized or open-loop methods, where the latter is infeasible for horizons larger than one.

V. CONCLUSION

We have shown that, by imposing additional convex constraints on the class of robust control policies proposed in [13] for control of con-

strained linear systems with bounded additive disturbances, one can formulate a robust RHC law with guaranteed bounds on the ℓ_2 gain of the closed-loop system, while simultaneously guaranteeing feasibility and constraint satisfaction for all time. The proposed control law requires the solution, at each time instant, of a convex-concave min-max problem in a number of variables that grows polynomially in both the number of states and inputs and the length of the planning horizon, making it a suitable candidate for online implementation.

APPENDIX

Define $\mathbf{A} \in \mathbb{R}^{n(N+1) \times n}$ and $\mathbf{E} \in \mathbb{R}^{n(N+1) \times nN}$ as

$$\mathbf{A} := \begin{pmatrix} I_n \\ A \\ A^2 \\ \vdots \\ A^N \end{pmatrix}, \quad \mathbf{E} := \begin{pmatrix} 0 & 0 & \cdots & 0 \\ I_n & 0 & \cdots & 0 \\ A & I_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1} & A^{N-2} & \cdots & I_n \end{pmatrix}.$$

Define $\mathbf{G} := \mathbf{E}(I_N \otimes G)$, $\mathbf{B} := \mathbf{E}(I_N \otimes B)$, $\mathcal{G} := I_N \otimes G$, $\tilde{B} := (A^{N-1}B \cdots AB B)$, $\tilde{G} := (A^{N-1}G \cdots AG G)$, $\mathbf{C} := (I_N \otimes C, 0)$, and $\mathbf{D} := I_N \otimes D$. Finally, define

$$\mathcal{H}_x := \begin{pmatrix} \mathbf{C}\mathbf{A} \\ P^{\frac{1}{2}}A^N \end{pmatrix}, \quad \mathcal{H}_u := \begin{pmatrix} \mathbf{C}\mathbf{B} + \mathbf{D} \\ P^{\frac{1}{2}}\tilde{B} \end{pmatrix}, \quad \text{and } \mathcal{H}_w := \begin{pmatrix} \mathbf{C}\mathbf{G} \\ P^{\frac{1}{2}}\tilde{G} \end{pmatrix}.$$

Proof of Proposition 3: The proof is by induction. For any $x \in \tilde{X}_N(\gamma)$, there exists a policy $(\bar{\mathbf{M}}, \bar{\mathbf{v}}) \in \bar{\Pi}_N(x, \gamma)$ by definition. If **A1** holds, then it is possible to construct a pair $(\bar{\mathbf{M}}, \bar{\mathbf{v}}) \in \bar{\Pi}_{N+1}(x)$ such that the final input is $u_N := Kx_N$ [13, Prop. 11], where $\bar{\mathbf{M}} := \begin{bmatrix} \bar{\mathbf{M}} & 0 \\ \bar{\mathbf{M}} & 0 \end{bmatrix}$, $\bar{\mathbf{v}} := [\bar{\mathbf{v}}]$, $\bar{M} := K(\tilde{G} + \tilde{B}\bar{\mathbf{M}}\mathcal{G})$, $\bar{v} := K(A^N x + \tilde{B}\bar{\mathbf{v}})$, and $\tilde{B}$ and $\tilde{G}$ are defined above. We need only show that this choice of $\bar{\mathbf{M}}$ also satisfies a QMI condition of the form (12).

After considerable, though straightforward, algebraic manipulation (see [22] for details), one can show that this is equivalent to requiring that, for horizon length $N + 1$

$$H_{N+1} = \begin{pmatrix} X & -Y'A'_K P G \\ -G'P A_K Y & \gamma^2 I - G'P G \end{pmatrix} \succeq 0 \quad (28)$$

where $A_K := (A + BK)$, $Y := (\tilde{G} + \tilde{B}\bar{\mathbf{M}}\mathcal{G})$ and

$$X := H_N + Y'(P - A'_K P A_K - (C + DK)'(C + DK))Y.$$

Noting that $(\gamma^2 I - G'P G) \succ 0$, which is always true if $\gamma \geq \gamma_f$, and computing the Schur complement, the condition (28) is satisfied iff

$$H_N + Y'(P - A'_K \bar{P}_\gamma A_K - (C + DK)'(C + DK))Y \succeq 0 \quad (29)$$

where $\bar{P}_\gamma := P + PG(\gamma^2 I - G'P G)^{-1}G'P$ and $H_N \succeq 0$ by assumption. Using the definitions (19) to verify the matrix identity $(P - (C + DK)'(C + DK)) = A'_K \bar{P}_\gamma A_K$, one can show that (29) is satisfied if $\bar{P} \succeq \bar{P}_\gamma$, which is always true for $\gamma_f \leq \gamma$. The proof is completed by verifying, in a similar manner, that $X_f \subseteq \tilde{X}_1(\gamma)$. ■

Proof of Corollary 1: For any $\gamma > \max\{\gamma_f, \gamma_N^*(x)\}$ and any $x \in \tilde{X}_N(\gamma)$, the proof of Proposition 3 ensures the existence of a policy $\bar{\Pi}_{N+1}(x, \gamma)$. From the definition (14), this implies that $\gamma_{N+1}^*(x) \leq \gamma$ for all $\gamma > \max\{\gamma_f, \gamma_N^*(x)\}$, so that $\gamma_{N+1}^*(x) \leq \max\{\gamma_f, \gamma_N^*(x)\}$. ■

Proof of Lemma 1: If **A1** holds then, from Proposition 3, there exists a policy pair $(\bar{\mathbf{M}}, \bar{\mathbf{v}})$ satisfying $(\bar{\mathbf{M}}, \bar{\mathbf{v}}) \in \bar{\Pi}_{N+1}(x, \gamma)$ with

$u_N = Kx_N$. From [13, Prop. 13], one can construct a pair $(\bar{\mathbf{M}}, \bar{\mathbf{v}}) \in \bar{\Pi}_N(x^+)$ via appropriate definition of $\bar{\mathbf{v}}$ and by partitioning $\bar{\mathbf{M}} = \begin{bmatrix} 0 & 0 \\ \bar{\mathbf{M}} & \mathbf{M} \end{bmatrix}$. We need only consider whether this pair also satisfies the condition (12). Given this partitioning for $\bar{\mathbf{M}}$, one can show that H_{N+1} can be partitioned as

$$H_{N+1} = \begin{bmatrix} U & V \\ V' & H_N \end{bmatrix} \succeq 0$$

for appropriately defined matrices U and V , where H_N is as defined as in (12). Since H_N is a principal sub-matrix of H_{N+1} , it follows that $H_N \succeq 0$. ■

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On Delay-Independent Diagonal Stability of Max-Min Congestion Control

Yueping Zhang, *Member, IEEE*, and
Dmitri Loguinov, *Senior Member, IEEE*

Abstract—Network feedback in a congestion-control system is subject to delay, which can significantly affect stability and performance of the entire system. While most existing stability conditions explicitly depend on delay D_i of individual flow i , a recent study [1] shows that the combination of a symmetric Jacobian A and condition $\rho(A) < 1$ guarantees local stability of the system regardless of D_i . However, the requirement of symmetry is very conservative and no further results have been obtained beyond this point. In this technical note, we proceed in this direction and gain a better understanding of conditions under which congestion-control systems can achieve delay-independent stability. Towards this end, we first prove that if Jacobian matrix A satisfies $\|A\| < 1$ for any *monotonic* induced matrix norm $\|\cdot\|$, the system is locally stable under arbitrary diagonal delay D_i . We then derive a more generic result and prove that delay-independent stability is guaranteed as long as A is Schur diagonally stable [2], which is also observed to be a necessary condition in simulations. Utilizing these results, we identify several classes of well-known matrices that are stable under diagonal delays if $\rho(A) < 1$ and prove stability of MKC [1] with arbitrary parameters α_i and β_i .

Index Terms—Delay-independent stability, diagonal delay, max-min congestion control.

I. INTRODUCTION

Several max-min congestion control algorithms (e.g., XCP [3], RCP [4], VCP [5], MKC [1], and JetMax [6]) have been recently proposed. These protocols receive feedback from the most-congested router in their path and exhibit appealing performance from both theoretical and practical perspectives. Thus, stability of these systems, especially when delay is present in the network feedback, has recently received a fair amount of attention [1], [7]–[15]. However, most existing stability conditions (e.g., [3], [8], [10], [12]) require that parameters of the control equation be adaptively tuned according to feedback delay D_i of user i , making them undesirable in practice due to the resulting unfairness between flows with different RTTs and oscillations when delays are not

properly estimated by end-users. Although this limitation is partially resolved by the recent work [1], which proves that max-min systems with a symmetric Jacobian matrix are locally stable regardless of delay, the requirement of symmetry is very restrictive in practice and understanding whether the same result holds for a wider class of matrices remains open.

In this technical note, we gain a deeper insight into stability of max-min congestion control systems under diagonal delays. Most max-min systems (e.g., MKC [1], RCP [4], and XCP [3]) can be linearized to the following shape (more on this in Section II):

$$x_i(n) = \sum_{j=1}^N a_{ij} x_j(n - D_i) \quad (1)$$

where a_{ij} are some constants, N is the number of flows, n is the discrete time variable, and $x_i(n)$ and D_i are, respectively, the sending rate and round-trip time of user i . Using this model, we first present an alternative proof of Theorem 1 in [1] under time-invariant delay D_i , based on which we derive a sufficient stability condition of (1) to be $\|A\|_2 < 1$, where $A = (a_{ij})$ is the coefficient matrix of the system and matrix norm $\|\cdot\|_2$ is induced by the L^2 vector norm. Clearly, this condition is more generic than the one obtained in [1], which required A to be real and symmetric.

Subsequently, we prove that this result actually extends to any matrix norm induced by a *monotonic* vector norm (which subsumes all standard vector norms, such as $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty$, and $\|\cdot\|_\infty^w$). Moreover, we prove that a special norm $\|A\|_s = \inf_{W \in \mathcal{P}^*} \|WAW^{-1}\|_2$ (where $\mathcal{P}^*$ is the set of all positive diagonal matrices) is a monotonic induced norm and further generalize the sufficient stability condition of system (1) to $\|A\|_s < 1$, whose necessity is also indicated by simulations. Armed with these results, we identify several classes of systems that are stable under diagonal delays if and only if they are stable under undelayed feedback. This finding allows us to prove stability of Max-min Kelly Control (MKC) [1] with arbitrary parameters α_i and β_i . We also discuss and verify obtained results using Matlab simulations.

The rest of the technical note is organized as follows. In Section II, we describe modeling assumptions of this technical note and review existing work in the area of delay-independent stability. In Section III, we present our main results of the technical note and verify them via Matlab simulations. In Section IV, we conclude the technical note and suggest directions for future work.

II. BACKGROUND

A. Modeling Single-Link Congestion Control

Assume a generic model of max-min congestion control that is composed of M links and N users. Each flow i solicits feedback p_i from a particular link, which we call the *bottleneck* link of user i , in its path. Assuming that bottleneck assignments of the network are fixed [11], [15], it is proven in [6] that the entire system can be decomposed into subsystems of users constrained by individual bottlenecks, which is stable independent of delays if and only if individual subsystems are. Thus, in what follows we restrict our focus to congestion control in single-link networks and note that the obtained results are applicable to all multi-link max-min systems with time-invariant bottleneck assignments.

Consider a network with N users accessing a single bottleneck link. Feedback delays arise from transmission, propagation, and queuing

¹Throughout the technical note, we use index i to denote both flows and senders (sources).

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D. Loguinov is with Texas A&M University, College Station, TX 77843 USA (e-mail: dmitri@cs.tamu.edu).

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