Stochastic Exit Time Problems Arising in Process Control

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Abstract

This paper concerns the problem of controlling a stochastic system, with small noise parameter, to prevent it leaving a safe region of the state space. Such problems arise in flow control and other areas. We consider a formulation of the problem, in which a control is sought to maximize a cost which is related to the expected exit time, but modified to reduce the probability of an early exit, according to a specified level of risk aversion ('risk sensitive' stochastic control). Formally letting the noise parameter tend to zero, the optimal control strategy for this problem coincides with the optimal feedback control strategy for a differential game. We identify a class of differential games arising in this way, so called 'decomposable differential games', for which the optimal control strategy can be easily obtained and illustrate the proposed solution technique by applying it to a flow control problem arising in process systems engineering.

Keywords: Stochastic Control, Exit Problems, Risk Sensitive Control, Differential Games

1 Introduction

We consider the problem of controlling a stochastic system with noise intensity parameter ϵ , modelled as a stochastic differential equation with control term, to prevent the state leaving a safe region. Such problems arise in the control of dams and surge tanks to avoid overflow, control of telecommunications systems to avoid the failing of a communications link, control of queues to avoid excessive waiting times, and other areas. Maximizing the expected value of the exit time is unsatisfactory for many of these applications, because such an approach may give control strategies for which the probability of an early exit is unacceptably high. These concerns have given rise to a substantial literature, risk sensitive stochastic optimal control, in which the cost is modified to reduce the probability of an early exit, according to the level of risk 'aversion' implicit in the problem formulation. In certain circumstances, the limiting control strategy as $\epsilon \rightarrow 0$ can be interpreted as the solution to a differential game. The monographs by Whittle [11], Fleming and Soner [6] and the paper by Dupuis and McEneaney [5] provide detailed coverage of this body of research, and extensive references.

In this paper we identify a class of differential games arising in this way, so called 'decomposable differential games', for which the optimal control strategy can be easily obtained. We then illustrate the proposed solution technique, by applying it to a flow control problem arising in process systems engineering.

For decomposable differential games, the safe set in the state space can be partitioned into a finite number of regions, in each of which the optimal control takes a fixed value. The regions

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are determined by constructing value functions for a number of simple, unconstrained dynamic optimization problems, each associated with a fixed control value, and examining their level sets. The conditions under which this is possible require the control system to have a certain monotonicity property and the collection of dynamic optimization problems to have ‘non-overlapping’ extremals.

For the example considered, which has state space dimension 2, it is possible to obtain formulae for the value functions of the dynamic optimization problems involved. However the proposed methodology would appear to have a role, even for differential games problems involving higher dimensional state spaces, for which such formulae are not available. In this broader context, the methodology can still be used in a modified form, in which several optimization problems are solved on-line and their costs compared, to determine the optimal control. These research directions are currently being explored.

2 The Stochastic Exit Problem

The state x of a stochastic control system is related to the control u by the stochastic differential equation on $[0, \infty)$

$$\begin{cases} dx = f(x, u)dt + \epsilon^{\frac{1}{2}}\sigma(x)dB \\ x(0) = x_0 \\ u(t) \in U \end{cases} \quad (2.1)$$

in which ϵ is a positive number, $f : R^n \times R^{m_1} \rightarrow R^n$ and $\sigma : R^n \rightarrow R^{n \times m_2}$ are given functions, $x_0 \in R^n$ is a given point and $U \subset R^{m_1}$ is a given subset. B is an R^{m_2} -valued Brownian motion adapted to a filtered probability space $(\Omega, \mathcal{F} = (\mathcal{F}_t), P)$.

To be more precise, control functions u are $\mathcal{F}$ -progressively measurable processes on $[0, \infty)$ taking values in $U \subset R^{m_1}$. Under the hypotheses to be imposed, the stochastic differential equation (2.1) has a unique strong-sense solution for a given control u ; this solution is the state x corresponding to the control u .

We specify an open set $A \subset R^n$ in which the initial state x_0 is located. A is interpreted as the ‘safe’ region of the state space.

The control problem considered is that of choosing a control which, in some probabilistic sense, maximizes the time when the state first exits from A . Interest will focus on the case when the noise intensity scaling factor ϵ is small. It is assumed that

(H1): A has a C^1 boundary

(H2): U is compact

(H3): $f(x, u)$ is continuous and Lipschitz continuous with respect to x , with Lipschitz constant independent of $u \in U$. $\sigma(x)$ is bounded and of class C^2 .

Given a control $u(\cdot)$, let $x(\cdot)$ be the corresponding state. Denote by τ^ϵ the first exit time of $x(\cdot)$ from A . We consider the cost

$$E_{x_0} \exp \{-\theta \tau^\epsilon / \epsilon\} .$$

(The subscript x_0 has been appended to the expectation operator, to emphasize the specified initial state.) The positive number θ is regarded as a design parameter.

The positive number ϵ in the cost is the same scaling factor as that appearing in the stochastic differential equation (2.1). It is introduced in such a manner to permit an asymptotic analysis of maximizing controls, in the limit as $\epsilon \downarrow 0$.

Cost functions of this kind are referred to as ‘risk averse’ costs for exit time, and can be thought of as refinements of the expected exit time criterion. Risk averse costs have received considerable attention in the literature. Suffice it to mention here that, for small θ , the effect of minimizing the cost approximates that of maximizing the expected exit time. Increasing θ produces a control for which the probability of an early exit is lessened, at the price of a reduction in the average exit time.

We formalize the control design problem, to choose a control that confines the state to the safe set A , as

$$(P_{x_0}^\epsilon) \begin{cases} \text{Minimize } E_{x_0} \exp \{-\theta \tau^\epsilon / \epsilon\} \\ \text{over controls } u \text{ such that} \\ dx = f(x, u)dt + \epsilon^{\frac{1}{2}} \sigma(x)dB \\ x(0) = x_0 \\ u(t) \in U \end{cases}$$

where, as before, τ^ϵ is the first exit time of the state x from A .

3 Asymptotic Properties For Vanishingly Small Noise

Fix $\epsilon > 0$ and consider problem (P_x^ϵ) of the previous section, when a generic point $x \in A$ replaces the initial state x_0 . Define the value function $\tilde{W}^\epsilon : A \rightarrow R \cup \{\infty\}$ for (P_x^ϵ) to be the infimum cost parameterized by the initial state. That is

$$\tilde{W}^\epsilon(x) = \inf(P_x^\epsilon) \quad \text{for } x \in A.$$

By convention, we set $\tilde{W}^\epsilon(x) = 0$ if, for some control, the values of the state $x(\cdot)$ are confined to A for all time with probability one. It is known [5] that, if we assume

(H4): $\bar{A}$ is compact, $m_2 = n$ and there exists $\gamma > 0$ such that, for all $x \in A$

$$\xi^T a(x) \xi \geq \gamma |\xi|^2 \quad \text{for all } \xi \in R^n,$$

where

$$a(x) := \sigma(x) \sigma^T(x),$$

then $\tilde{W}^\epsilon(x)$ is a finite valued, continuous function, C^2 on the interior of A , and the function together with its first and second derivatives are extendible to all of $\bar{A}$ as continuous functions. Furthermore $\tilde{W}^\epsilon(x)$ satisfies the Hamilton Jacobi equation

$$\begin{cases} (\epsilon/2) \text{tr}\{\tilde{W}_{xx}^\epsilon(x) a(x)\} + \min_{u \in U} \tilde{W}_x^\epsilon f(x, u) - (\theta/\epsilon) \tilde{W}^\epsilon(x) = 0 & \text{for } x \in A \\ \tilde{W}^\epsilon(x) = 1 & \text{for } x \in \partial A. \end{cases} \quad (3.1)$$

While studying the limits of solutions to this equation as $\epsilon \downarrow 0$ is apparently unrevealing, an asymptotic analysis can be carried on the log transformed function

$$W^\epsilon := -\epsilon \ln \tilde{W}^\epsilon.$$

Suppressing the superscript ϵ we have

$$W_x = -\epsilon \tilde{W}^{-1} \tilde{W}_x, \quad W_{xx} = \epsilon \tilde{W}^{-2} \tilde{W}_x^T \tilde{W}_x - \epsilon \tilde{W}^{-1} \tilde{W}_{xx},$$

in which expressions gradients are taken to be row vectors. We deduce that, for all $x \in A$

$$\begin{aligned} & (\epsilon/2) \operatorname{tr} \{ \tilde{W}_{xx}^\epsilon a \} + \tilde{W}_x^\epsilon f - (\theta/\epsilon) \tilde{W}^\epsilon \\ & = -\epsilon^{-1} \tilde{W}^\epsilon \left((\epsilon/2) \operatorname{tr} \{ W_{xx}^\epsilon a \} + W_x f - (1/2) \tilde{W}_x a W_x^T + \theta \right) . \end{aligned}$$

Since $\tilde{W}^\epsilon$ is positive on A , we deduce that

$$(\epsilon/2) \operatorname{tr} \{ W_{xx}^\epsilon a \} + \max_u W_x^\epsilon f(x, u) - (1/2) W_x^\epsilon a W_x^{\epsilon T} + \theta = 0 .$$

This equation can be written

$$(\epsilon/2) \operatorname{tr} \{ W_{xx}^\epsilon a \} + \max_u \min_v \{ W_x^\epsilon (f(x, u) + \sigma v) + (1/2) |v|^2 \} + \theta = 0 .$$

It is known that, as $\epsilon \downarrow 0$, $W^\epsilon(x)$ converges pointwise on A to $W(x)$, the unique continuous viscosity solution [1] to

$$\max_{u \in U} \min_{v \in R^{m_2}} \{ W_x(x) (f(x, u) + \sigma(x)v) + (1/2) |v|^2 \} + \theta = 0 , \quad (3.2)$$

vanishing on ∂A . See [5], [6].

4 A Related Differential Game

As in [5], we now link the stochastic problem $(P_{x_0}^\epsilon)$, for vanishingly small ϵ , with the differential game, labelled (P_{x_0}) : find a closed loop u -player control $\phi \in \Phi$ to maximize

$$\inf_{v \in \mathcal{V}} J(\phi(v), v; x_0) \quad (4.1)$$

in which the ‘pay-off’ function $J : \mathcal{U} \times \mathcal{V} \rightarrow R \cup \{+\infty\}$, for a given initial state x_0 , is

$$J(u, v; x_0) = \int_0^\tau \left(\frac{1}{2} |v|^2 + \theta \right) dt . \quad (4.2)$$

Here τ is the first exit time from A for

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) + \sigma(x(t))v(t) \quad \text{a.e.} \\ x(0) &= x_0 . \end{aligned}$$

The data A , f , σ and x_0 (and the set U in the definition of the set $\mathcal{U}$ below) are the same as for $(P_{x_0}^\epsilon)$, and are assumed to satisfy hypotheses $(H1) - (H4)$. The class of closed loop u -controls Φ and the class of open loop v -controls $\mathcal{V}$ are defined presently.

The infimum is assigned the value $+\infty$ if, for all v -controls, the state trajectory remains in A for all time.

The control spaces are defined as follows. The spaces of open loop controls $\mathcal{U}$ and $\mathcal{V}$ for the u - and v -players are, respectively,

$$\begin{aligned} \mathcal{U} &:= \{ \text{measurable functions } u : [0, \infty) \rightarrow R^{m_1} \text{ s.t. } u(t) \in U \text{ a.e.} \} \\ \mathcal{V} &:= L_{\text{loc}}^2([0, \infty); R^{m_2}) . \end{aligned}$$

(It is convenient to take as time domain the set $[0, \infty)$, even though values of control functions after the first exit time the corresponding state trajectory are irrelevant.)

We adopt the Elliott-Kalton ‘non-anticipative’ framework for defining closed loop u -controls, according to which the spaces of closed loop controls for the u -player and the v -player are taken to be

$$\begin{aligned}\Phi &:= \{\text{non-anticipative mappings } \phi : \mathcal{V} \rightarrow \mathcal{U}\}, \\ \Psi &:= \{\text{non-anticipative mappings } \psi : \mathcal{U} \rightarrow \mathcal{V}\},\end{aligned}$$

in which a ‘non-anticipative mapping’ $\phi : \mathcal{V} \rightarrow \mathcal{U}$ is a mapping such that, for each pair of open loop controls $v, v' \in \mathcal{V}$ and each $T \geq 0$, we have, writing $u = \phi(v)$ and $u' = \phi(v')$:

$$'v(t) = v'(t) \text{ a.e. } t \in [0, T]' \quad \text{implies} \quad 'u(t) = u'(t) \text{ a.e. } t \in [0, T]'$$

Defining closed loop controls in terms of non-anticipative mappings is a well-established procedure for avoiding the technical difficulties that arise when a closed loop control for the u -player is specified, in a classical sense, as a state feedback map: namely, that the state equation may fail to have a unique solution for a given open loop $v \in \mathcal{V}$, (possibly discontinuous) state feedback for the u -player and initial state, thereby leaving ambiguous what is meant by the corresponding ‘state trajectory’.

Within the Elliott-Kalton framework, on the other hand, the state trajectory for $\phi \in \Phi$ and $v \in \mathcal{V}$ is clear. It is simply the state trajectory corresponding to the *open loop* controls $u \in \mathcal{U}$ and $v \in \mathcal{V}$, where $u = \phi(v)$.

Following [5], we define the upper value function $W : A \rightarrow R \cup \{\infty\}$ of the game (P_{x_0}) to be

$$W(x) = \sup_{\phi \in \Phi} \inf_{v \in \mathcal{V}} J(\phi(v), v; x) \quad \text{for } x \in A.$$

Under hypotheses (H1)-(H4) $W(\cdot)$ is the unique continuous viscosity solution to the Isaacs equation

$$\max_{u \in U} \min_{v \in R^{m_2}} \{W_x(f(x, u) + \sigma(x)v) + (1/2)|v|^2\} + \theta = 0,$$

vanishing on ∂A . See [5]. We recognize this equation as that satisfied by the scaled, log transformed value function of the stochastic exit time problem, in the limit as $\epsilon \downarrow 0$. (See (3.2)).

We mention that W is referred to as the upper value, to distinguish it from the lower value of the game, namely

$$W^-(x) = \inf_{v \in \Psi} \sup_{u \in \mathcal{U}} J(u, \psi(u); x) \quad \text{for } x \in A.$$

A central issue elsewhere in the differential games literature is the existence of the *value* of the game, that is the common value of $W(x)$ and $W^-(x)$ parameterized by the initial state x , when they coincide. (See, e.g., [1].) For the problems considered in this paper the value may fail to exist. But in such circumstances it still makes sense to formulate (P_{x_0}) , which may be thought of as the ‘upper’ game because it is associated with the upper value. It is the closed loop control strategy solving the upper game that is linked to the stochastic exit problem.

Compatibility of Non-anticipative and State-Feedback Controls. The non-anticipativity approach is advantageous for formulating unrestrictive conditions for existence of optimal closed loop controls and for establishing that the value function satisfies the Isaacs equation, as compared with the classical approach in which closed loop controls are defined via a state feedback, a general description of which is provided by:

$$u \in \chi(x) \text{ for all } x \in A, \tag{4.3}$$

in which $\chi(\cdot) : A \rightsquigarrow U$ is a set valued function.

It is often the case in specific applications, however, that a candidate for the optimizing closed loop control presents itself as a classical state feedback control. In order to verify optimality in these circumstances it is necessary as a first step to establish that the candidate can be interpreted as a non-anticipative control. The following definition of compatibility and conditions under which state feedback controls for the control system

$$\dot{x}(t) = f(x(t), u(t)) + \sigma(x(t))v(t), \quad x(0) = x_0 \quad (4.4)$$

can be so interpreted are useful in this regard. See [3],[2].

Definition 4.1 *A closed loop control $\phi \in \Phi$ is said to be compatible with the feedback relation (4.3) if, for any $v \in L^1_{loc}([0, \infty))$, the solution x to (4.4), in which $u = \phi(v)$, satisfies*

$$u(t) \in \chi(x(t)) \text{ for a.e. } t \in [0, \infty).$$

Proposition 4.2 *Consider the control system (4.4) and feedback relation (4.3). Assume that (H1)-(H3) are satisfied. Assume furthermore that*

(C): $\chi(\cdot)$ has bounded values and has closed graph.

Then there exists a closed loop control compatible with (4.3).

The connection with the stochastic exit problem is now evident: the upper value function for the differential games problem coincides with the log-transformed value function of the stochastic exit problem, in the pointwise limit as the noise intensity parameter ϵ vanishes.

The log transformation is monotone, so minimizers for the stochastic control problem are unaffected when the original cost is replaced by its log transformation. It is therefore a plausible conjecture that an optimizing closed loop strategy for the differential games problem will provide a sub-optimal control, which will approximate the optimal control for the stochastic control problem with increasing accuracy, as ϵ decreases to zero.

5 Decomposable Control Systems

Henceforth we concentrate on the task of solving the differential game (P_{x_0}) , bearing in mind throughout its relation with the stochastic control problem for vanishingly small values of the noise intensity parameter ϵ , as described above.

We first prepare the ground for describing a class of differential games problems, for which the optimal closed loop strategy has a simple structure, which greatly facilitates computation. The problems considered have the special feature that, given any $\bar{u} \in U$, the optimal control and the value function of the (one player) optimal control problem that arises when we freeze $u(\cdot) = \bar{u}$, may be calculated. The defining properties of the class are expressed then in terms of properties of optimal controls and value functions of the associated optimal control problems.

Take a collection of open sets $\{A_j\}_{j=1}^N$ in R^n , each covering A , and u -player control values $\{u_j\}_{j=1}^N$ in U , such that

$$\cap_{j=1}^N A_j = A \quad \text{and} \quad \cap_{j=1}^N \partial A_j = \emptyset.$$

(In applications, the set A is typically a region of the state space comprising points satisfying several functional inequality constraints. For each j , A_j is the larger region, resulting from imposition of only the j 'th constraint. The associated u_j is a control value that maximizes the generalized minimum energy of the v -control for exiting A_j . In the example considered below, A is the region defined by the requirements that the level of water in a tank lies between -1 and $+1$. Here, it is natural to take A_1 to be the region for which the level is less than $+1$ and u_1 to be the permissible control value which maximizes the fluid extraction rate, and to take A_2 to be the region for which the level is greater than -1 and u_2 to be the permissible control value which minimizes the fluid extraction rate.)

For every j and $x \in A_j$ consider the optimal control problem

$$(P_x^j) \begin{cases} \text{Minimize } \int_0^\tau \left(\frac{1}{2}|v|^2 + \theta \right) dt \\ \text{over } v \in \mathcal{V} \text{ satisfying} \\ \dot{x} = f(x, u_j) + \sigma(x)v \quad \text{a.e. on } [0, \infty) \\ x(0) = x . \end{cases}$$

Now τ denotes the first exit time from A_j . Notice that each (P_x^j) is a (one player) optimal control problem, because the u -player control value is fixed at u_j . For each j let $W^j : A_j \rightarrow R$ be the value function for (P_x^j)

$$W^j(x) = \inf_{v \in \mathcal{V}} \int_0^\tau \left(\frac{1}{2}|v|^2 + \theta \right) dt \quad \text{for } x \in A_j .$$

(We set $W(x) = +\infty$ if no $v \in \mathcal{V}$ exists such that the corresponding state trajectory leaves A_j in finite time.) Define

$$D_j = \{x \in A \mid W^j(x) \leq W^{j'}(x) \text{ for all } j' \neq j\} .$$

We shall assume that

(H5): For each j , $W^j(\cdot)$ is locally uniformly Lipschitz continuous and continuously differentiable on $\{x \in A_j \mid W^j(x) > 0\}$.

Definition 5.1 Problem (P_{x_0}) is said to be decomposable w.r.t. $\{A_j, u_j\}$ if (H1)-(H3) and (H5), and also the following conditions, are satisfied: for each j and $x \in A_j$, (P_x^j) has a minimizer $\bar{v}$, with corresponding state trajectory $\bar{x}$ and first exit time $\bar{\tau}$, such that

(a): (monotonicity) For any measurable $u : [0, \bar{\tau}] \rightarrow U$ the solution $x(\cdot)$ to $\dot{x}(t) = f(x(t), u(t)) + \sigma(x(t))\bar{v}(t)$ and $x(0) = x$ satisfies

$$x(\tau') \notin A_j \quad \text{for some } \tau' \in [0, \bar{\tau}] .$$

(b): (non-intersecting extremals)

(i): $\bar{x}(t) \in \text{int } D_j$ for all $t \in (0, \bar{\tau})$

(ii): if $W^j(x) > 0$ and $j \in \arg \min_{j'} \{W^{j'}(x)\}$, then $\bar{v}(t)$ is continuous at $t = 0$ and

$$\nabla(W^j - W^{j'})(x)(f(x, u_{j'}) + \sigma(x)\bar{v}(0)) < 0 .$$

for every $j' \in \arg \min_{j'} \{W^{j'}(x)\}$, $j' \neq j$.

Remarks: Some comments on the nature of the above conditions will be helpful.

- (i): For differential games that are decomposable, the ‘safe’ set A can be expressed as a union of N sets with non-intersecting interiors (the D_j sets) and the optimal closed-loop control will assign the fixed control value u_j when the state is in the j ’th open set. This requires that, if an optimal state originates in D_j , it will evolve in the interior of D_j up till the exit time. This is the ‘non-intersecting extremals’ condition.
- (ii): For the differential game of interest here, the boundary of A can be partitioned into N sub-boundaries, each associated with one of the subsets D_j of A . The optimal state feedback in D_j (the constant control u^j) will tend to direct the state as much as possible away from the associated sub-boundary. The monotonicity condition is the requirement that if the v -player continues to apply the same control, but the u -control generated by the optimal feedback is substituted by any open loop control, then the state will enter the relevant subboundary earlier.

6 Optimal Strategies for Decomposable Problems

The following theorem provides a complete description of the optimal closed loop control for the differential game, in the case that it is decomposable.

Theorem 6.1 *Suppose there exists a collection of open subsets of R^n and points in U*

$$\{(A_j, u_j) \mid j = 1, \dots, N\}$$

such that (P_{x_0}) is decomposable w.r.t. $\{(A_j, u_j)\}$. Let $W^j(\cdot)$, $j = 1, \dots, N$ be the value functions for the optimal control problems introduced in Def. 5.1.

Then the value function for (P_{x_0}) is the lower envelope of the family of optimal control value functions $\{W^j(\cdot)\}_{j=1}^N$:

$$W(x) = \min_j W^j(x) \quad \text{for all } x \in A.$$

Furthermore, any non-anticipative control $\bar{u}$ compatible with the set-valued feedback

$$\chi(x) = \bar{co}\{u_j \mid j \in \arg \min_j W^j(x)\} \quad \text{for } x \in A \quad (6.1)$$

(and one such non-anticipative control exists) is an optimal closed loop control for the u -player.

We provide a sketch of the proof, full details of which will be reported elsewhere.

Outline of Proof: We notice right away that there exists a non-anticipative control compatible with the set-valued feedback function (4.3). This follows from Prop. 4.2, since χ is bounded and since, in consequence of the continuity of the W^j ’s, χ has a closed graph. Write, once again, $J(\phi, v; x_0)$ for the payoff of the game (P_{x_0}) , for given initial state x_0 , $\phi \in \Phi$ and $v \in \mathcal{V}$. It suffices to show, for an arbitrary initial state x_0 in A , that:

(A): For any $\phi \in \Phi$

$$\min_{v \in \mathcal{V}} J(\phi(v), v; x_0) \leq W(x_0). \quad (6.2)$$

(B): Take any non-anticipative control $\bar{\phi}$ compatible with the set-valued state feedback $u = \chi(x)$ and any $v \in \mathcal{V}$. Take any $j \in \arg \min_j W^j(x_0)$ and let $\bar{v}$ be a minimizer for $(P_{x_0}^j)$, with state trajectory $\bar{x}$ and first exit time $\bar{\tau}$. Then

$$J(\bar{\phi}(v), v; x_0) \geq W(x_0) \quad (6.3)$$

and

$$J(\bar{\phi}(\bar{v}), \bar{v}; x_0) = W(x_0) . \quad (6.4)$$

Consider (A). Select any $\phi \in \Phi$ and take any $j \in \arg \min_{j'} \{W^{j'}(x_0)\}$. By definition of ‘decomposable problem’ the optimal control problem $(P_{x_0}^j)$ has a minimizer $\bar{v}$ with state trajectory $\bar{x}$ and first exit time $\bar{\tau}$. Since $\bar{\tau}$ is the first exit time from A , $\bar{x}(\bar{\tau}) \notin A$. Set $u = \phi(\bar{v})$. Write also x for the state trajectory obtained by solving $\dot{x}(t) = f(x(t), u(t)) + \sigma(x)\bar{v}(t)$ with initial value x_0 and write τ for the first exit time of x . From the conditions for decomposability we have $x(\tau) \notin A$, from which it follows that $0 \leq \tau \leq \bar{\tau}$. But then

$$\begin{aligned} \inf_{v \in \mathcal{V}} J(\phi(v), v; x_0) &\leq J(u, \bar{v}; x_0) = \frac{1}{2} \int_0^\tau |\bar{v}|^2 dt + \theta \tau \\ &\leq \frac{1}{2} \int_0^{\bar{\tau}} |\bar{v}|^2 dt + \theta \bar{\tau} = W^j(x_0) = W(x_0) . \end{aligned}$$

Relation (6.2) in (A) is confirmed.

The function $W(\cdot)$ is not continuously differentiable, but it is locally uniformly Lipschitz continuous as a lower envelope of locally uniformly Lipschitz continuous functions. Relation (6.3) in (B), for arbitrary v , is proved by employing $W(\cdot)$ as a nonsmooth verification function, as in Chapter 12 of [10]. Proof of relation (6.4) in (B), for the open loop v -player control $\bar{v}$, makes key use of the ‘non-intersecting extremals’ condition.

In Section 2 we posed a small-noise optimal stochastic control problem. Ideally, we would like to know what the control strategy for this problem is, in the limit as the noise parameter vanishes. It should be clear that Thm. 6.1 falls short of providing such information in full. The theorem merely characterizes the solution to the associated differential games problem. Under hypotheses including (H4), which are rather restrictive from an engineering perspective, the differential games problem is linked to the stochastic control problem, to the extent that the value function of the differential games problem can be interpreted as the pointwise limit of (a transformed, scaled version of) the value function of the stochastic control problem, as the noise parameter vanishes. Even when (H4) is satisfied, a rigorous justification of the convergence of the optimal control policy for the stochastic control problem to that of the differential games problem, as the noise vanishes, is lacking, since convergence of value functions does not guarantee convergence of control strategies.

Nonetheless, it is reasonable to expect that solving the differential games problem will provide a good sub-optimal approximation to the small noise stochastic control problem for examples of interest arising in an engineering context, such as that considered in Section 8, even when (H4) (requiring a bounded ‘safe’ set A and noise to enter all the states) is not satisfied. Showing this would appear however to be technically challenging.

7 An Application to Flow Control

We illustrate the practical implications of the preceding theory, by considering a problem of surge tank control. Surge tanks are buffer devices used in chemical processing, to prevent excessive flow rate fluctuations, as fluids flow between reactors.

Denote by y the height of the fluid in a surge tank of uniform cross-section. Then y is related to the rate of change of inflow v and the rate of change of outflow u according to the following

differential equation.

$$d^2y/dt^2 = -u + v.$$

Here v and u are interpreted as a random disturbance and control signal, respectively.

We represent the constraints that the tank neither empties nor overflows as

$$-1 \leq y \leq +1. \quad (7.1)$$

There is also a constraint on the maximum rate of change of outflow. This is expressed as

$$-1 \leq u \leq +1. \quad (7.2)$$

The design problem is to find a state feedback control

$$u = \chi(y, \dot{y}),$$

the control signal for which depends on the current height and rate of change of height, to prevent violation of the state constraint (7.1) with a high probability. Control signals are required to satisfy the control constraint (7.2). In this idealized description of constraints and surge tank dynamics, we have taken unit values for all constants involved.

Let us suppose the disturbance v is modelled as ‘white noise’, with intensity parameter $\epsilon^{1/2}$. For cases when the noise intensity is very small, a possible control strategy is one which minimizes the ‘risk sensitive’ modification of the expectation of exit time τ^ϵ :

$$E_{x_0} \exp \{-\theta \tau^\epsilon / \epsilon\},$$

where x_0 is the initial value of $(y, \dot{y})$, in the limit as $\epsilon \downarrow 0$. θ is a specified risk aversion parameter. This leads to consideration of the following game, in which the state vector is the 2-vector $x = [y, \dot{y}]^T$:

$$\begin{aligned} & \text{Maximize } \inf_{v \in \mathcal{V}} \int_0^\tau \left(\frac{1}{2} |v|^2 + \theta \right) dt \\ & \text{over } \phi \in \Phi, \end{aligned}$$

in which the infimum is taken over $v \in \mathcal{V}$ and τ is the first exit time from A for the state trajectory x satisfying

$$\begin{cases} \dot{x} = Fx + b(-\phi(v) + v) & \text{a.e. } t \in [0, \tau] \\ x(0) = x_0. \end{cases}$$

The matrix F and vector b are

$$F = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and the set A is

$$A = \{x = (x_1, x_2) \in R^2 \mid -1 < x_1 < +1\}.$$

For purposes of applying the preceding theory, we take $N = 2$ and identify

$$A_1 = \{(x_1, x_2) \mid x_1 < 1\}, \quad A_2 = \{(x_1, x_2) \mid x_1 > -1\}$$

The boundaries of these sets are, (respectively)

$$\partial A_1 = \{+1\} \times R, \quad \text{and} \quad \partial A_2 = \{-1\} \times R.$$

The fixed u -control values associated with A_1 and A_2 are

$$u_1 = +1 \text{ and } u_2 = -1 .$$

We claim that the problem is decomposable. To justify this assertion, we must show that the conditions of Def. 5.1 are satisfied. The two optimal control problems (P_1) and (P_2) are, in the present context,

$$(P_{x_0}^1) \left\{ \begin{array}{l} \text{Minimize } \int_0^\tau (\frac{1}{2}|v|^2 + \theta) dt \\ \text{over } \tau \geq 0 \text{ and } v \in L^2[0, \tau]; R) \text{ satisfying} \\ \dot{x} = Fx + b(-1 + v) \quad \text{a.e. } t \in [0, \tau] \\ x(0) = x_0, x(\tau) \in \{+1\} \times R. \end{array} \right.$$

and

$$(P_{x_0}^2) \left\{ \begin{array}{l} \text{Minimize } \int_0^\tau (\frac{1}{2}|v|^2 + \theta) dt \\ \text{over } \tau \geq 0 \text{ and } v \in L^2[0, \tau]; R) \text{ satisfying} \\ \dot{x} = Fx + b(+1 + v) \quad \text{a.e. } t \in [0, \tau] \\ x(0) = x_0, x(\tau) \in \{-1\} \times R . \end{array} \right.$$

(a): (*the monotonicity condition*) We consider only (A_1, u_1) . ((A_2, u_2) is treated analogously.) Take then $x_0 \in A_1$ and consider a minimizer $(\bar{v}, \bar{\tau})$ for $(P_{x_0}^1)$, with state trajectory $\bar{x}$. Then the first component of $\bar{x} = (\bar{x}_1, \bar{x}_2)$ satisfies

$$\left\{ \begin{array}{l} \ddot{\bar{x}}_1(t) = -1 + \bar{v}(t) \quad \text{a.e. } t \in [0, \bar{\tau}] \\ (\bar{x}_1(0), \dot{\bar{x}}_1(0)) = x_0 \end{array} \right.$$

Now take any u -control $u : [0, \bar{\tau}] \rightarrow [-1, +1]$. Then the first component x_1 of the state $x = (x_1, x_2)$ corresponding to u and $\bar{v}$ satisfies

$$\left\{ \begin{array}{l} \ddot{x}_1(t) = -u(t) + \bar{v}(t) \quad \text{a.e. } t \in [0, \bar{\tau}] \\ (\bar{x}_1(0), \dot{\bar{x}}_1(0)) = x_0 \end{array} \right.$$

Since $u(t) \geq -1$ a.e.

$$\left\{ \begin{array}{l} (\ddot{x}_1 - \ddot{\bar{x}}_1)(t) \geq 0 \quad \text{a.e. } t \in [0, \bar{\tau}] \\ (x_1(0) - \bar{x}_1(0)) = (\dot{x}_1(0) - \dot{\bar{x}}_1(0)) = 0 . \end{array} \right.$$

It follows that $x_1(\bar{\tau}) \geq \bar{x}_1(\bar{\tau})$. But $\bar{x}_1(\bar{\tau}) = +1$. Since $x_1(0) < +1$ and $x_1(\cdot)$ is continuous, we can conclude that, for some $\tau \in [0, \bar{\tau}]$, $x_1(\tau) = +1$, i.e. $x(\tau) \in \partial A_1$. The monotonicity condition is verified.

(b): (*Non-intersecting extremals*) To verify satisfaction of this condition, we need to examine, in some detail, solutions to two optimal control problems. It is helpful to consider the two cases $\theta > 0$ and $\theta = 0$ separately.

Case (i): $\theta > 0$. Define the mappings $\eta^1 : (0, \infty) \times (0, \infty) \rightarrow R^2$ and $\eta^2 : (0, \infty) \times (0, \infty) \rightarrow R^2$ to be:

$$\begin{aligned} \eta^1(q, \tau) &= (1 - \frac{1}{2}\tau^2 + \frac{1}{6}q\tau^3 - \frac{1}{2}\theta\tau q, \tau - \frac{1}{2}q\tau^2 + \frac{1}{2}\theta q) , \\ \eta^2(r, \sigma) &= (-1 + \frac{1}{2}\sigma^2 - \frac{1}{6}r\sigma^3 + \frac{1}{2}\theta\sigma r, -\sigma + \frac{1}{2}r\sigma^2 - \frac{1}{2}\theta r) . \end{aligned}$$

It can be shown that there exists open sets $\mathcal{O}^j$, $j = 1, 2$, such that, for each j , η^j is one-to-one on $\mathcal{O}^j$ and

$$\eta^j(\mathcal{O}^j) = A_j .$$

It can be shown furthermore that, for any initial state $x \in A_1$, the value function, first exit time and optimal control for problem (P_x^1) is

$$\begin{cases} V^1(x) = \frac{1}{6}q^2\tau^3 + \theta\tau \\ \text{first exit time} = \tau \\ v(t) = q(\tau - t), \quad \text{for a.e. } t \in [0, \tau], \end{cases}$$

in which $(q, \tau) = (\eta^1)^{-1}(x)$.

Also that, for any initial state $x \in A_2$, the value function, first exit time and optimal control for problem (P_x^2) are

$$\begin{cases} V^2(x) = \frac{1}{6}r^2\sigma^3 + \theta\sigma \\ \text{first exit time} = \sigma \\ v(t) = r(\sigma - t), \quad \text{for a.e. } t \in [0, \sigma], \end{cases}$$

in which $(r, \sigma) = (\eta^2)^{-1}(x)$.

Notice that optimal controls are continuous, as required for checking the ‘non-intersecting extremals’ condition. The set A decomposes into disjoint regions

$$A = \{x \in A \mid V^2(x) < V^1(x)\} \cup \Sigma \cup \{x \in A \mid V^1(x) < V^2(x)\};$$

The switching set Σ , a smooth submanifold, is

$$\begin{aligned} \Sigma = \{x \in A \mid \frac{1}{6}q^2\tau^3 + \theta\tau = \frac{1}{6}r^2\sigma^3 + \theta\sigma, \\ \text{in which } (q, \tau) = (\eta^1)^{-1}(x) \text{ and } (r, \sigma) = (\eta^2)^{-1}(x)\}. \end{aligned}$$

It can be shown that optimal state trajectories for (P_x^1) issuing from $\{x \in A \mid V^1(x) \leq V^2(x)\}$ evolve in $\{x \in A \mid V^1(x) < V^2(x)\}$ for $t \in (0, \tau)$; also that, at points $x \in A$ for which $V^1(x) = V^2(x)$, we have

$$\nabla(V^1 - V^2)(x)(f(x, u_2) + \sigma(x)\bar{v}_1(0)) < 0,$$

where v_1 is the solution to (P_x^1) , and

$$\nabla(V^2 - V^1)(x)(f(x, u_1) + \sigma(x)\bar{v}_2(0)) < 0,$$

where v_2 is the solution to (P_x^2) . Analogous statements can be made about optimal state trajectories for These are the ‘non-intersecting extremals’ conditions (P_x^2) .

Case(ii): $\theta=0$. The reason this case has to be treated a little differently is that, now, V^1 and V^2 are not continuously differentiable on the sets $A_1 = \{x = (x_1, x_2) \mid x_1 < 1\}$ and $A_2 = \{x = (x_1, x_2) \mid x_1 > -1\}$, respectively. But checking the conditions for decomposable problems merely required that these functions are continuously differentiable on their support sets (sets on which they have strictly positive values), as indeed they are. These two support sets are:

$$\begin{aligned} A_{\text{support}}^1 &= \{x \in A_1 \mid x_1 < 1 + \frac{1}{2}x_2^2\} \\ A_{\text{support}}^2 &= \{x \in A_2 \mid x_1 > -1 + \frac{1}{2}x_2^2\} \end{aligned}$$

(The significance of these sets is that if, for either $j = 1$ or 2 , the initial condition x lies in $A \setminus A_{\text{support}}^j$, then the related state trajectory with zero v control exits from A , whatever the

u -player control happens to be, and therefore incurs zero cost.)

Take the functions η^1 and η^2 as before (with $\theta = 0$). We can choose $\mathcal{O}^1$ and $\mathcal{O}^2$ such that η^j is one-to-one on $\mathcal{O}^j$ for $j = 1, 2$ and

$$\begin{aligned}\eta(\mathcal{O}^1) &= \{(x_1, x_2) \mid x_1 < 1\} \cap A_{\text{support}}^1, \\ \eta(\mathcal{O}^2) &= \{(x_1, x_2) \mid x_1 > -1\} \cap A_{\text{support}}^2.\end{aligned}$$

It can be shown that, for any initial state $x \in A_{\text{support}}^1$, the value function, first exit time and optimal control for problem (P_x^1) are

$$\begin{cases} V^1(x) = \frac{1}{6}q^2\tau^3 + \theta\tau \\ \text{first exit time} = \tau \\ v(t) = q(\tau - t), \quad \text{for a.e. } t \in [0, \tau], \end{cases}$$

in which $(q, \tau) = (\eta^1)^{-1}(x)$. Also, for any initial state $x \in A_{\text{support}}^2$, the value function, first exit time and optimal control for problem (P_x^2) are

$$\begin{cases} V^2(x) = \frac{1}{6}r^2\sigma^3 + \theta\sigma \\ \text{first exit time} = \sigma \\ v(t) = r(\sigma - t), \quad \text{for a.e. } t \in [0, \sigma], \end{cases}$$

in which $(r, \sigma) = (\eta^2)^{-1}(x)$. These semi-explicit formulae for the case $\theta = 0$ can be used to check the conditions confirming decomposibility of the differential games problem.

In the case $\theta = 0$ the switching surface is generated by the points

$$\begin{aligned}x_1(\alpha) &= \frac{(1 + 3\alpha - 2\alpha^2 + 2\alpha^3 - 3\alpha^4 - \alpha^5)}{(1 + 3\alpha - 2\alpha^2 - 2\alpha^3 + 3\alpha^4 + \alpha^5)} \\ x_2(\alpha) &= \frac{\sqrt{12}\alpha(\alpha - 1)}{(1 + \alpha)^{1/2}(1 + 3\alpha - 2\alpha^2 - 2\alpha^3 + 3\alpha^4 + \alpha^5)},\end{aligned}$$

as α ranges over the interval

$$\frac{4}{3 + \sqrt{17}} \leq \alpha \leq \frac{3 + \sqrt{17}}{4}.$$

(We remark that the denominators in these formulae are positive for all non-negative α 's.)

Fig. 1 provides plots of the switching sets for a number of values of constant θ . Fig. 2 illustrates the ‘non-intersecting’ properties of extremals. Indeed, optimal state trajectories for the two optimal control problems remain in regions where the u -control has constant value, and they depart from the switching surface at a positive angle.

We point out that the Maximum Rate of Change of Outflow (MROC) is an important performance indicator in surge tank control, since it is the *rate of change* of outflow that has adverse effects on downstream process units (disturbed sediments, turbulence, etc), not the flow itself [9], [7]. Our formulation of the control problem (proposed in [4]) is to maximize in some sense the elapsed time before the surge tank fills or empties, while observing a constraint on the maximum rate of change of outflow. Previous treatments have limited attention to deterministic disturbances with a particular profile (step changes in inflow rate). The formulation of this paper, by contrast, allows for random inputs.

Figure 1: Switching curves

Figure 2: Non-intersecting Optimal Trajectories

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