Nested Array With Time-Delayers for Target Range and Angle Estimation

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Abstract—Since nested array with phased-array cannot directly estimate the range of targets due to range ambiguity, this paper proposes a nested array with diverse time-delayers for target range and angle estimation. The essence is to construct a new array structure by systematically nesting two uniform linear arrays through diverse time-delays which yields range-dependent receiving array beampattern. Using second-order statistics of the received data, it is capable of providing a dramatic increase of the system degrees-of-freedom which means more sources can be resolved. More importantly, target range and angle can be estimated by exploiting the range-dependent array beampattern. The performance improvements are evaluated by examining the Cramér-Rao lower bounds.

I. INTRODUCTION

The number of sources that can be resolved by a N -element uniform linear phased array is $N - 1$. To overcome this constraint, multiple techniques [1], [2] have been proposed for detecting more sources than physical sensors. Minimum redundancy array is suggested in [1], but there are no closed form expressions for the array geometry and achievable degrees-of-freedom (DOFs) for a given N elements. In [3], the fourth-order cumulants are applied to increase DOFs, but its applicability is restricted to non-Gaussian sources. Multiple-input multiple-output (MIMO) can significantly increase DOFs, but it is not applicable to passive arrays.

Nested array [4], [5], which nests two or more uniform linear arrays (ULAs), is a promising technique to significantly increase the DOFs. It provides $O(N^2)$ DOFs using N physical elements by exploiting the second-order statistics of the received data through difference co-array processing. Nested array has closed form expression for the array geometry and achievable DOFs for given N elements, but only phased arrays are employed in the literatures. This limits its performance in mitigating undesired range-dependent interferences and thus limits its ability to resolve targets from noise and interference by adaptive beamforming. Frequency diverse array (FDA) offers range-dependent transmit beampattern by employing a small frequency increment compared to the carrier frequency across the array elements [6]. This range-dependent beampattern provides a potential to estimate target range, but only active transmitting FDA is presented in existing literatures.

Inspired by active FDA using diverse frequency increments to achieve range-dependent transmit beampattern, this paper proposes a new nested array with diverse time-delayers to achieve range-dependent receiving array beampattern for target range and angle estimation. The essence is to systematically nest two ULAs through diverse time-delayers. The use of

time-delayers generates a receiving array beampattern that is a function of range, time, and angle. Moreover, the DOFs are increased by exploiting the second-order statistics of the received data through the difference co-array processing. In doing so, the targets can be localized in range and angle domains without sacrificing the advantage of nested array in increasing array DOFs.

This paper is organized as follows. Section II proposes the new nested array with diverse time-delayers. The signal model and corresponding difference co-array processing algorithm are presented. Section III is devoted to evaluate the performance of the designed nested array in target range and angle estimations. Finally, simulation results are provided in Section IV and concluding remarks are given in Section V.

II. NEW NESTED ARRAY SIGNAL AND DATA MODELS

Like the basic nested array using phased arrays, the new nested array also is basically a concatenation of two ULAs through diverse time-delayers whereas they are not employed in the basic nested array, namely, the inner and outer arrays consist of N_1 and N_2 elements with inter-element spacings d_1 and $d_2 = (N_1 + 1)d_1$, respectively. More precisely it is a linear array with element locations given by the union of $S_{\text{iner}} = \{n_1 d_1, n_1 = 1, 2, \dots, N_1\}$ and $S_{\text{outer}} = \{n_2(N_1 + 1)d_1, n_2 = 1, 2, \dots, N_2\}$. Figure 1 illustrates the proposed nested array based on diverse time-delayers. Since it provides not only phase-dependent beampattern but also range-dependent beampattern, there are rather different characteristics in beampattern, signal model and direction-of-arrival (DOA) estimation from basic nested array.

We derive the proposed nested array receiver pattern in a rather general way by assuming the signal coming to the array is a linearly frequency modulated (LFM) waveform

$$s(t) = \exp \left\{ j2\pi \left(f_0 t + \frac{1}{2} k_r t^2 \right) \right\} \quad (1)$$

where f_0 is the starting frequency (equal to the carrier frequency) and k_r is the chirp rate. The signal related with the direction θ and range r in far-field and received by the n th sensor can be modeled as

$$x_n(r, \theta, t) = \frac{\sqrt{\sigma}}{r_n} s(t - \tau_n) \exp \left\{ -j \frac{2\pi f_0 + k_r(t - \tau_n)}{c_0} r_n \right\} + v_n(t) \quad (2)$$

where r_n and τ_n denote the target range to the n th sensor and corresponding propagation time, respectively, σ is the target reflection coefficient, c_0 is the speed of light, and $v_n(t)$ is the

Fig. 1: Illustration of proposed nested array receiver based on diverse time-delayers.

noise. Here, $\tau_n = nT_e$ with T_e being a fixed inter-element time delay and r_n can be approximated as $r_n = r - nd \sin \theta$ with d being the inter-element spacing.

In amplitude sense, r_n can be replaced by r_0 . We then have

$$x_n(r, \theta, t) = \frac{\sqrt{\sigma}}{r_0} \exp \left\{ j \left(2\pi f_0 t + \frac{k_r t^2}{2} - \frac{2\pi f_0 r_0}{c_0} - \frac{k_r r_0 t}{c_0} \right) \right\} \\ \times \exp \left\{ j n \left(-2\pi f_0 T_e + \frac{k_r x_n T_e^2}{2} - k_r t T_e + \frac{2\pi f_0 d \sin \theta}{c_0} \right. \right. \\ \left. \left. + \frac{k_r r_0 T_e}{c_0} + \frac{k_r d \sin \theta t}{c_0} - \frac{k_r x_n d \sin \theta T_e}{c_0} \right) \right\} + v_n(t) \quad (3)$$

Since $r_0 \gg \frac{x_n c_0 T_e}{2}$, the constant phase term $\frac{k_r x_n T_e^2}{2}$ can be ignored. As $x_n \frac{T_e}{2}$ is a constant depending on the physical parameters of the array, ignoring $\frac{k_r x_n d \sin(\theta) T_e}{c_0}$ will cause deformation in the pulse shapes for larger observation point angles and a shift in the timings. But the deformation can be ignored in analytical derivations and thus it is ignored in the derivations. In this case, after matched filtering, the received array data can be expressed as

$$\mathbf{y}(t) = \frac{\sigma_s}{r} \mathbf{b}(r, \theta, t) + \mathbf{v}(t) \quad (4)$$

where $\mathbf{v}(t)$ is the noise vector and $\mathbf{b}(r, \theta, t)$ is the receive steering vector:

$$\mathbf{b}(r, \theta, t) = \begin{bmatrix} 1 & e^{j\varphi_0(r, \theta, t)} & \dots & e^{j(N-1)\varphi_0(r, \theta, t)} \end{bmatrix}^T \quad (5)$$

where

$$\varphi_0(r, \theta, t) = -2\pi f_0 T_e - k_r t T_e + \frac{2\pi f_0 d \sin \theta}{c_0} + \frac{k_r r_0 T_e}{c_0} + \frac{k_r d \sin \theta t}{c_0} \quad (6)$$

Suppose there are D sources, the covariance matrix is

$$\mathbf{R}_{yy} = E \{ \mathbf{y}(t) \mathbf{y}^H(t) \} \\ = \mathbf{B} \mathbf{R}_{ss} \mathbf{B}^H + \sigma_n^2 \mathbf{I}_N \quad (7)$$

where E represents the expectation operator, H is the conjugate transpose, $\mathbf{B} = \begin{bmatrix} \mathbf{b}(r_1, \theta_1, t) & \mathbf{b}(r_2, \theta_2, t) & \dots & \mathbf{b}(r_D, \theta_D, t) \end{bmatrix}$ is the array manifold matrix, $\mathbf{R}_{ss}$ is the source autocorrelation matrix, σ_n^2 is the noise variance and $\mathbf{I}_N$ is the $N \times N$ identity matrix. When the sources are temporally uncorrelated, $\mathbf{R}_{yy}$ can be vectorized as [7]

$$\mathbf{z} = \text{vec}(\mathbf{R}_{yy}) \\ = \sum_{i=1}^D [\mathbf{b}^*(r_i, \theta_i, t) \otimes \mathbf{b}(r_i, \theta_i, t)] \sigma_i^2 + \sigma_n^2 \text{vec}(\mathbf{I}_N). \quad (8)$$

The $N^2 \times 1$ vector $\mathbf{z}$ can be seen as the signal received at the difference co-array with the amplitudes of the source signal vector replaced by their corresponding powers. Correspondingly, the equivalent difference co-array steering vector is $\mathbf{u}(r, \theta, t) = \mathbf{b}^*(r, \theta, t) \otimes \mathbf{b}(r, \theta, t)$. The distinct elements of $\mathbf{u}(r, \theta, t)$ behave like the manifold of a (longer) array whose element positions are given by the distinct values in the array position set. Therefore, the maximum DOFs for a N -element nested array is $N(N-1) + 1$.

III. BEAMFORMING-BASED TARGET RANGE AND ANGLE ESTIMATION

Since the virtual data $\mathbf{z}$ of (8) can be seen as the signal received at the difference co-array of nested array receiver with the amplitudes of the source signal vector replaced by their corresponding powers, beamforming can be further applied in the receiver.

Though the incident jammers are originally assumed uncorrelated, after the difference co-array processing, they are represented by their powers and consequently the resulting covariance matrix will be of rank 1. In [5], this problem is tackled with the use of the forward spatial smoothing method at an expense of aperture length loss because it essentially halves the total DOFs offered by the difference co-array, meaning that less sources can be identified. To reduce the aperture length loss, we use a two-way smoothing method based on the forward-backward spatial smoothing technique [8].

We divide the virtual L -element co-array into K overlapping subarrays, each with P elements, namely, $L = P + K - 1$. That is, the number of smoothing times is K . Define two $P \times L$ matrices:

$$\mathbf{F}_k = [\mathbf{0}_{P \times (k-1)} \mid \mathbf{I}_P \mid \mathbf{0}_{P \times (K-k)}] \quad (9a)$$

$$\mathbf{G}_k = [\mathbf{0}_{P \times (k-1)} \mid \mathbf{J}_P \mid \mathbf{0}_{P \times (K-k)}] \quad (9b)$$

where k represents the k th smoothing subarray, $\mathbf{0}_{P \times (k-1)} \in \mathbb{R}^{P \times (k-1)}$ and $\mathbf{0}_{P \times (K-k)} \in \mathbb{R}^{P \times (K-k)}$ are zero matrices, and $\mathbf{J}_P$ is the $P \times P$ exchange matrix whose anti-diagonal elements are 1 and zero otherwise. Multiplying $\mathbf{F}_k$ and $\mathbf{G}_k$ by (8) and its conjugate, respectively, yields

$$\mathbf{z}_k^f = \mathbf{F}_k \mathbf{z} \quad (10a)$$

$$\mathbf{z}_k^b = \mathbf{G}_k \mathbf{z}^* \quad (10b)$$

where $*$ is the conjugate operator. Their covariance matrices are

$$\mathbf{R}_k^f = E \{ \mathbf{z}_k^f (\mathbf{z}_k^f)^H \} = \mathbf{F}_k E \{ \mathbf{z} \mathbf{z}^H \} \mathbf{F}_k^H = \mathbf{F}_k \mathbf{R}_{zz} \mathbf{F}_k^H \quad (11a)$$

$$\mathbf{R}_k^b = E \{ \mathbf{z}_k^b (\mathbf{z}_k^b)^H \} = \mathbf{G}_k E \{ \mathbf{z}^* \mathbf{z}^H \} \mathbf{G}_k^H = \mathbf{G}_k \mathbf{R}_{zz}^* \mathbf{G}_k^H \quad (11b)$$

The forward and backward spatial smoothing data covariance matrices are expressed, respectively, as

$$\mathbf{R}_{zz}^f = \frac{1}{K} \sum_{k=1}^K \mathbf{R}_k^f \quad (12a)$$

$$\mathbf{R}_{zz}^b = \frac{1}{K} \sum_{k=1}^K \mathbf{R}_k^b \quad (12b)$$

Correspondingly, the forward-backward spatial smoothing data covariance is

$$\begin{aligned} \mathbf{R}_{zz}^{fb} &= \frac{1}{2} (\mathbf{R}_{zz}^f + \mathbf{R}_{zz}^b) \\ &= \frac{1}{2K} \left(\sum_{k=1}^K \mathbf{F}_k \mathbf{R}_{zz} \mathbf{F}_k^H + \sum_{k=1}^K \mathbf{G}_k \mathbf{R}_{zz}^* \mathbf{G}_k^H \right). \end{aligned} \quad (13)$$

It has already been proved that when $P \geq D$ and $2K \geq D$, the forward-backward smoothing data covariance $\mathbf{R}_{zz}^{fb}$ will be of full-rank [8]. We then have

$$L = P + K - 1 \geq D + \frac{D}{2} - 1 \Rightarrow D \leq \frac{2(L+1)}{3}. \quad (14)$$

Therefore, after forward-backward spatial smoothing [9], [10], we can use the full-rank covariance matrix $\mathbf{R}_{zz}^{fb}$ for minimum variance distortionless response (MVDR) beamforming. By eigendecomposing $\mathbf{R}_{zz}^{fb}$ as $\mathbf{R}_{zz}^{fb} \triangleq \mathbf{E}_s \mathbf{\Lambda}_s \mathbf{E}_s^H + \mathbf{E}_n \mathbf{\Lambda}_n \mathbf{E}_n^H$, where the diagonal matrix $\mathbf{\Lambda}_s$ contains the P largest eigenvalues and the columns of $\mathbf{E}_s$ are the corresponding eigenvectors, and the diagonal matrix $\mathbf{\Lambda}_n$ contains the remaining $(L-P)$ eigenvalues and the columns of $\mathbf{E}_n$ are the corresponding eigenvectors, the range and angle of targets can be estimated from the L magnitude peaks of the multiple signal classification (MUSIC) estimator [11]:

$$\begin{aligned} \{\hat{r}, \hat{\theta}\} &= \arg \max_{r, \theta} |f(r, \theta)|^2 \\ &= \arg \max_{r, \theta} \left| \frac{\mathbf{b}_s^H(r, \theta, t) \mathbf{b}_s(r, \theta, t)}{\mathbf{b}_s^H(r, \theta, t) \mathbf{E}_n \mathbf{E}_n^H \mathbf{b}_s(r, \theta, t)} \right|^2 \end{aligned} \quad (15)$$

where $\mathbf{b}_s(r, \theta, t)$ is the sub-array steering vector of the form:

$$\mathbf{b}_s(r, \theta, t) = \begin{bmatrix} 1 & e^{j\varphi_0(r, \theta, t)} & \dots & e^{j(P-1)\varphi_0(r, \theta, t)} \end{bmatrix}^T. \quad (16)$$

The Cramér-Rao lower bound (CRLB) of target range estimation and angle estimation can be derived with the following data model [5]

$$\tilde{\mathbf{z}}_{li} = \mathbf{B}_{li}(r, \theta, t) \sigma_s^2 + \sigma_n^2 \mathbf{e}_i' \quad (17)$$

where $\mathbf{B}_{li}$ is a $\left(\frac{N^2}{4} + \frac{N}{2}\right) \times D$ matrix consists of the $\left(\frac{N^2}{4} + \frac{N}{2} + 1 - i\right)$ th to $\left(\frac{N^2-2}{2} + N + 1 - i\right)$ th rows of $\mathbf{B}_1$ which is obtained from $\mathbf{B}^* * \mathbf{B}$ by removing the repeated rows (after their first occurrence), and $\mathbf{e}_i'$ is a vector of all zeros except a 1 at the i th position. The CRLBs can be obtained as the inverse of the Fisher information matrix (FIM). Specifically, when the angle is known and range is unknown, the range FIM can be derived as

$$\begin{aligned} \mathbf{I}_{rr} &= 2\text{Re} \left\{ \frac{\partial \mathbf{B}_{li}^H(r, \theta, t)}{\partial r} (\mathbf{C}_n^{-1}) \frac{\partial \mathbf{B}_{li}(r, \theta, t)}{\partial r} \right\} \\ &= 4\text{SNR} \sum_{p=0}^{P-1} \frac{k_r^2 T_d^2 p^2}{c_0^2} \end{aligned} \quad (18)$$

Fig. 2: Illustration of nested array positions.

where SNR denote the signal-to-noise ratio (SNR), namely, $\text{SNR} = \sigma_s^2 / \sigma_n^2$. Similarly, when the range is known, the angle FIM can be derived as

$$\begin{aligned} \mathbf{I}_{\theta\theta} &= 2\text{Re} \left\{ \frac{\partial \mathbf{B}_{li}^H(r, \theta, t)}{\partial \theta} (\mathbf{C}_n^{-1}) \frac{\partial \mathbf{B}_{li}(r, \theta, t)}{\partial \theta} \right\} \\ &= 4d^2 \cos^2 \theta \cdot \text{SNR} \left(\frac{4\pi^2}{\lambda^2} + \frac{k_r 4\pi t}{\lambda c_0} + \frac{k_r^2 t^2}{c_0^2} \right) \sum_{p=0}^{P-1} p^2 \end{aligned} \quad (19)$$

IV. SIMULATION RESULTS

In the following simulations, we assume that the array with $N_1 = 3$ and $N_2 = 3$ operates at a carrier frequency $f_0 = 10$ GHz, $d = \lambda_0/2$ and $T_e = 5$ ns. The array elements are arranged as Figure 2. The additive noise is modeled as a complex Gaussian zero mean spatially and temporally distributed sequence. We compare the proposed nested array with the same array structure FDA denoted as SAS-FDA and the equal aperture length FDA denoted as EAL-FDA.

Suppose there are 5 sources located as the directions -50° , -20° , -5° , 10° and 45° , respectively. Figure 3(a) shows the MUSIC spectra for the three tested arrays. We can see that they all perform well in resolving 5 sources because 6 physical array sensors provide 5 DOFs and thus 5 sources can be well resolved. However, when there are 11 sources, Figure 3(b) shows that the nested array outperforms the EAL-FDA. The latter misses the sources located at -40° and 45° , and also has an obvious deviation at 50° . This superior performance is expected since the former enjoys the advantages of nested array in increasing DOFs. Note that, due to its limited DOFs, when there are 11 sources, the MUSIC-based algorithm is not suitable for the SAS-FDA any more and thus no corresponding comparisons are given in Figure 2(b).

Figure 4 compares the range and angle estimation CRLBs versus SNR. It can be seen that the proposed nested array exhibits better CRLBs in both range estimation and angle estimation than the SAS-FDA and EAL-FDA schemes. This is attributed to the reason that the proposed nested array exploits the second-order statistics of the received data with the difference co-array processing algorithm.

V. CONCLUSION

This paper proposed a new receiving array design scheme by nesting two ULAs with diverse time-delayers. It jointly exploits the advantages of nested array in increasing DOFs to resolve more sources and FDA in range-dependent array beampattern in detect range-dependent sources. The diverse time-delayers are used to yield range-dependent receiving array

(a) 5 sources.

(b) 11 sources.

Fig. 3: Comparisons of MUSIC spectra.

(a) Range CRLB when the angle is known.

(b) Angle CRLB when the range is known.

Fig. 4: Comparisons of CRLBs versus SNR.

beam pattern, which performs similar function as the frequency increments in active transmitting FDA. When the difference co-array processing algorithm is applied in the receiver, much more sources can be resolved in the MUSIC-based range and angle estimator and better (lower) CRLBs can be achieved. Since the proposed array falls into the area of FDA and conventional FDA can only be used in transmitter, this work opens a new avenue to develop new FDA techniques.

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