Flexural-torsional buckling assessment of steel beam-columns through a stiffness reduction method

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Abstract

In this paper, a stiffness reduction method for the flexural-torsional buckling assessment of steel beam-columns subjected to major axis bending and axial compression is presented. The proposed method is applied by reducing the Young's E and shear G moduli through the developed stiffness reduction functions and performing Linear Buckling Analysis. To account for second-order forces induced prior to buckling, the in-plane (in the plane of bending) and out-of-plane analyses of a member are separated and stiffness reduction for the out-of-plane instability assessment is applied on the basis of member forces determined from the in-plane analysis. Since the developed stiffness reduction functions fully take into account the detrimental influence of imperfections and spread of plasticity, the proposed method does not require the use of member design equations, thus leading to practical design. For the purpose of verifying this approach, the strength predictions determined through the proposed stiffness reduction method are compared against those obtained from nonlinear finite element modelling for a large number of regular, irregular, single and multi-span beam-columns.

Keywords: Stiffness reduction; flexural-torsional buckling; steel beam-columns; inelastic buckling

1. Introduction

The elastic flexural-torsional buckling capacities of steel beam-columns are eroded by the effects of imperfections, residual stresses and the onset and spread of plasticity. In current steel design specifications [1–3], this is traditionally taken into account by reducing cross-sectional resistances through buckling reduction factors, separating the individual components of loading and resistance (i.e. flexural buckling under axial load and lateral-torsional buckling under major axis bending) and considering their interdependency through

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interaction equations. Another design strategy is based on the concept of stiffness reduction [4–7], where the inelastic flexural-torsional buckling load of a beam-column is determined by considering reduced stiffness calculated on the basis of the withstood forces. Though this approach represents the actual physical response more realistically, it is less suited to hand calculations and thus has not been widely applied in practice. Nevertheless, nowadays, structural analysis software, capable of providing elastic buckling loads of steel members through Linear Buckling Analysis (LBA), is widely available. Thus, the use of stiffness reduction approaches in conjunction with LBA may now provide a practical and accurate means of design. Instead of performing flexural and lateral-torsional buckling assessments independently and considering their interdependency through interaction equations, this approach takes into account compound buckling modes in a single step and thus leads to a more direct assessment of instability. Moreover, moment gradient effects, restraint type and position, the height of transverse loading and interactions between laterally unrestrained spans during buckling can directly be accounted for through LBA. Such an approach has been previously proposed by [8–12] for the determination of the inelastic flexural-torsional buckling resistances of steel beam-columns. Stiffness reduction functions have been put forward by Wongkaew [9] and Wongkaew and Chen [10] based on the AISC LRFD [13] member buckling equations, and by Trahair and Hancock [12] based on the AS 4100 [3] member instability assessment equations. However, in these studies, the considered cases were limited and the accuracy of the proposals were not verified against results obtained from non-linear finite element modelling.

With the aim of extending these previous studies, a stiffness reduction method applied performing LBA is presented in this paper. In the proposed method, a steel beam-column is assumed to be perfectly straight and the deleterious influence of the geometrical imperfections, residual stresses and the spread of plasticity is taken into account by reducing the Young’s modulus E and shear modulus G through the developed stiffness reduction functions. The assumption of a beam-column as perfectly straight leads to the response illustrated in Fig. 1, where the out-of-plane failure is a bifurcation form of buckling as the out-of-plane deformations are zero before instability occurs. Since the out-of-plane deformations prior to buckling are zero, the in-plane response of a member can be assessed separately from the out-of-plane response and the out-of-plane instability assessment can be performed considering member forces obtained from the in-plane analysis [8, 14, 15]. Based on this principle, in the design framework proposed in this paper, the in-plane analysis of a steel beam-column is initially carried out by performing Geometrically Nonlinear Analysis with reduced stiffness (GNA-SR) through the stiffness reduction functions developed in Kucukler et al. [16]. Provided the member does not fail in-plane, the out-of-plane analysis is then implemented by performing Linear Buckling Analysis with reduced stiffness (LBA-SR) based on the member forces determined through the in-plane GNA-SR. It should be noted that this type of separation of in-plane and out-of-plane analyses is also recommended in AS 4100 [3] and AISC 360-10 [2] for the traditional design of beam-columns subjected to in-plane loading.

The following sections first describe the finite element modelling approach adopted in this study, and then illustrate the development, application and practicality of the proposed

Figure 1: Response of a perfectly straight steel beam-column subjected to in-plane loading

stiffness reduction method for regular, irregular, single and multi-span beam-columns. The accuracy of the proposed method is verified by comparing its results against those obtained through Geometrically and Materially Nonlinear Analysis with Imperfections (GMNIA) using finite element modelling. Moreover, the proposed approach is also compared against the traditional beam-column design methods given in EN 1993-1-1 [1], and its qualities in comparison to these methods are shown. In this paper, the application of the proposed stiffness reduction method is investigated for hot-rolled steel members with Class 1 and 2 cross-sections [1].

2. Finite element modelling

This section addresses the development of finite element models and their validation against experimental results from the literature. To verify the proposed stiffness reduction method, the results of the finite element models considering material and geometric nonlinearities and involving geometrical imperfections and residual stresses are used in the next

sections.

2.1. Development of finite element models

The finite element models were created with shell elements using the finite element analysis software Abaqus [17]. In all numerical simulations, the four-noded reduced integration shell element, referred to as S4R in the Abaqus [17] element library and accounting for transverse shear deformations and finite membrane strains, was used to model the beam-columns. 16 elements were employed for each plate (i.e. the flanges and web) of the modelled I sections to ensure that the spread of plasticity through the depth of the cross-sections could be accurately captured. To avoid the overlapping of the flange and web plates, the web plate was offset by half the thickness of each of the flanges. For the beam-columns whose length to depth ratios were smaller than 20, 100 elements were used in the longitudinal direction, while 200 elements were used in the beam-column models with length to depths ratios larger than 20. For the purpose of satisfying the section properties given in steel section tables, the fillets were represented in the finite element models through additional beam elements placed at the centroids of the flanges similar to [18]. The Poisson's ratio was taken as 0.3 in the elastic range and 0.5 in the plastic range by defining the effective Poisson's ratio as 0.5 to allow for the change of cross-sectional area under load. The tri-linear stress-strain relationship shown in Fig. 2 was employed for all models, where E is the Young's modulus, E_{sh} is the strain hardening modulus, f_y and ϵ_y are the yield stress and strain respectively and ϵ_{sh} is the strain value at which the strain hardening commences. The parameters f_u and ϵ_u correspond to the ultimate stress and strain values respectively. E_{sh} was assumed to be 2 % of E and ϵ_{sh} was taken as $10\epsilon_y$, conforming to the ECCS recommendations [19]. The Simpson integration method was selected using five integration points through the thickness of the shell elements. The von Mises yield criterion with the associated flow rule and isotropic strain hardening was assumed in the models. The engineering stress-strain relationship shown in Fig. 2 was transformed to the true stress-strain relationship as the constitutive formulations of Abaqus [17] are based on the Cauchy (true) stress-strain assumption. In all simulations, grade S235 steel was used. The modified Riks method [20, 21] and the default convergence criteria recommended by Abaqus [17] were used to determine the load-displacement response of the finite element models.

The ECCS residual stress patterns [19], shown in Fig. 3, were employed in the finite element models. Unless otherwise indicated, the shape of the initial geometrical imperfection was assumed as the lowest global buckling mode of a member including twist, whose magnitude was taken as 1/1000 of the largest laterally unrestrained length of a member [22]. Adoption of the lowest global buckling mode, which is scaled as 1/1000 of the member length, as the geometrical imperfection of a beam-column subjected to axial compression plus uniform bending moment is illustrated in Fig. 4, where U is the translation. As the great majority of the cross-sections of beam-columns considered in this study were not slender according to criteria given in EN 1993-1-1 [1], local imperfections were not incorporated into the models. Fork-end support conditions allowing for warping deformations and rotations, but restraining translations and twist, were used in the finite element models. Kinematic coupling constraints were employed to satisfy the fork-end support conditions,

Figure 2: Material model used in finite element simulations

Figure 3: Residual stress patterns applied to finite element models (+ve=tension and -ve=compression)

whereby localised failure at the end supports was also prevented. The nodes within the end sections of the members were constrained to a reference point located at the centroid of the cross-section where the boundary conditions were applied. The accuracy of this modelling technique was verified by comparing the elastic buckling moments obtained through the finite element models subjected to uniform bending with those obtained through the analytical formulae provided by Trahair [8] for fork-end supported members.

2.2. Validation of finite element models

This section addresses the validation of the finite element approach adopted in this study against the steel beam-column experiments of Van Kuren and Galambos [23]. Additional results from the experiments were also provided in Galambos and Lay [24]. To observe the influence of out-of-plane instability effects on the response, the specimens subjected to major axis bending plus axial compression were not laterally restrained between the supports. The test setup was such that the specimens were pin-ended in the in-plane direction, but they were elastically restrained against end rotation in the out-of-plane direction. The degree of the rotational restraint corresponded to an effective length L_{eff} approximately equal to 0.6

Figure 4: Adoption of the lowest global buckling mode as the geometrical imperfection for a fork-end supported beam-column under axial compression plus uniform bending

times of the actual length L [23]. In the finite element models, these restraint conditions were replicated using elastic rotational springs at the supports whose stiffness was specified to provide this same effective length value. Since the shapes and magnitudes of the geometrical imperfections of the specimens were not provided by Van Kuren and Galambos [23], they were assumed as a half-sine wave in shape and $L/1000$ in magnitude in both the in-plane and out-of-plane directions. The patterns and magnitudes of the residual stresses of the cross-sections used in the experiments were investigated in separate studies [25, 26], whose recommendations were used herein. The loading sequence, whereby the axial compression was applied first and kept constant while the bending moment was increased up to the collapse, and material properties reported by Van Kuren and Galambos [23] were adopted in the finite element models. The normalised moment-rotation paths of the specimens T23, T26, T31 and T32 observed in the experiments and those determined through the finite element models are provided in Fig. 5. The geometrical properties and loading conditions of the specimens, which had $W 100 \times 100 \times 19.3$ steel cross-sections, are also shown in the figure, where $M_{y,Ed}$ is the applied major axis bending moment, θ is the end rotation, $N_{pl} = Af_y$ is the yield load, in which f_y is the yield stress and A is the cross-sectional area, and $M_{y,pl} = W_{pl,y}f_y$ is the major axis plastic bending moment resistance, in which $W_{pl,y}$ is the major axis plastic section modulus. Fig. 5 shows that the agreement between the normalised moment-rotation paths obtained through the GMNIA of the finite element models and those of the experiments is good, which indicates that the finite element models are able to replicate the physical response of steel beam-columns influenced by out-of-plane instability effects. Table 1 also shows the comparison between the ultimate strength values predicted

Figure 5: Comparison between the normalised moment end-deformation paths of the FE models and those from experiments

by the shell finite element models against those obtained from the experiments for eleven specimens, where λ_y is the slenderness of the specimen determined by dividing its length L to the radius of gyration of its cross-section about the major axis i_y , (i.e. $\lambda = L/i_y$), N_{Ed} is the applied axial load, $M_{ult,exp}$ and $M_{ult,FE}$ are the ultimate bending moment resistances obtained from the experiment and the finite element model respectively, and ζ is the ratio of the ultimate bending moment resistance obtained from the finite element model to that observed in the experiment, (i.e. $\zeta = M_{ult,FE}/M_{ult,exp}$). ζ_{av} and ζ_{cov} are the average and coefficient of variation of ζ values, and ζ_{max} and ζ_{min} are the maximum and minimum of ζ values respectively. The three types of loading conditions of the specimens (i.e. a, b and c) are illustrated in Fig. 6. It should be noted that the specimens of Van Kuren and Galambos [23] not influenced by instability effects (i.e. those exhibiting significant amounts of strain hardening) and those subjected to minor axis bending or pure compression were not considered herein. As can be seen from Table 1, the agreement between the ultimate bending moment strengths determined through the finite element models and those observed in the experiments is generally good. The discrepancies in the maximum strength values may result from the differences between the actual shapes and magnitudes of the geometrical imperfections of the specimens which were not reported by Van Kuren and Galambos [23] and those assumed in the finite element models. Additional validation studies of the shell finite element modelling approach adopted in this paper against different experiments from the literature can be found in Kucukler et al. [27] and Kucukler [28].

Table 1: Comparison of the ultimate strength values determined through the shell finite element models against those obtained from the beam-column experiments of Van Kuren and Galambos [23]

Specimen	Section	Loading	λ_y	N_{Ed}/N_{pl}	$M_{ult,exp}/M_{y,pl}$	$M_{ult,FE}/M_{y,pl}$	ζ
T12	W200×46.1	b	55.2	0.122	0.762	0.736	0.97
T13	W200×46.1	c	54.9	0.122	1.016	0.989	0.97
T14	W200×46.1	a	55.2	0.230	0.900	0.953	1.06
T16	W200×46.1	b	41.3	0.123	0.752	0.811	1.08
T19	W200×46.1	b	27.5	0.121	0.776	0.880	1.13
T20	W100×19.3	b	56.1	0.117	0.768	0.819	1.07
T23	W100×19.3	c	83.2	0.114	0.932	0.978	1.05
T26	W100×19.3	b	83.7	0.122	0.723	0.692	0.96
T30	W100×19.3	a	111.6	0.122	0.972	0.997	1.03
T31	W100×19.3	c	111.6	0.122	0.835	0.876	1.05
T32	W100×19.3	b	111.6	0.122	0.642	0.554	0.86
ζ_{av}							1.02
ζ_{cov}							0.073
ζ_{max}							1.13
ζ_{min}							0.86

Figure 6: Considered loading cases in Van Kuren and Galambos [23] beam-column experiments

3. Stiffness reduction functions for flexural and lateral-torsional buckling

This section briefly describes the stiffness reduction functions for the flexural buckling assessment of columns and lateral-torsional buckling (LTB) assessment of beams proposed

Table 2: Imperfection factors α_z for minor axis flexural buckling of hot-rolled I section columns from EN 1993-1-1[1] and the imperfection factors α_{LT} for LTB proposed by Kucukler et al. [27]

Aspect ratio	α_z	α_{LT}
$h/b \leq 1.2$	0.49	$0.22 \sqrt{\frac{W_{el,y}}{W_{el,z}}}$
$h/b > 1.2$	0.34	$0.17 \sqrt{\frac{W_{el,y}}{W_{el,z}}}$

by Kucukler et al. [16, 27]. In the following section, these functions will be utilised in the derivation of a stiffness reduction function for the flexural-torsional buckling assessment of beam-columns.

3.1. Stiffness reduction function for flexural buckling of columns

Using the European column buckling curves [1], a stiffness reduction function for the flexural buckling of columns τ_{Nz} has been proposed by Kucukler et al. [16]. The function is given by eq. (1), where N_{Ed} is the applied axial load, N_{pl} is the yield load and α_z is the imperfection factor for flexural buckling about the minor axis. The values of α_z for the minor axis flexural buckling of hot-rolled I section members, given in EN 1993-1-1 [1], are provided in Table 2, in which h and b are the overall cross-section height and width. The proposed stiffness reduction function is employed to reduce the Young's modulus E in the implementation of LBA to account for the deleterious effects of imperfections and spread of plasticity on the flexural buckling strengths of columns.

$$\tau_{Nz} = \frac{4\psi^2}{\alpha_z^2 N_{Ed}/N_{pl} \left[1 + \sqrt{1 - 4\psi \frac{N_{Ed}/N_{pl} - 1}{\alpha_z^2 N_{Ed}/N_{pl}}} \right]^2} \quad \text{but} \quad \tau_N \leq 1$$

$$\text{where} \quad \psi = 1 + 0.2\alpha_z \frac{N_{Ed}}{N_{pl}} - \frac{N_{Ed}}{N_{pl}} \quad (1)$$

It is worth noting that the proposed stiffness reduction function yields the same flexural buckling strengths as those determined through the EN 1993-1-1 [1] column buckling curves and fully accounts for the development of plasticity and the effects of residual stresses and geometrical imperfections on the flexural buckling strength.

3.2. Stiffness reduction function for lateral-torsional buckling of beams

In Kucukler et al. [27], a stiffness reduction function for the LTB assessment of steel beams has been proposed and verified against a large number of GMNIA simulations. The proposed stiffness reduction function τ_{LT} is provided by eq. (2) where α_{LT} is the imperfection factor for LTB and κ is an imperfection modification factor. The imperfection modification factor κ , used for the consideration of the susceptibility of the member cross-section to LTB, is determined through eq. (3) in which G is the shear modulus, I_z , I_t and I_w are

the second moment of area about the minor axis, torsion constant and warping constant respectively. The recommended values of the imperfection factor α_{LT} are provided in Table 2, where $W_{el,y}$ and $W_{el,z}$ are the elastic section moduli about the major and minor axes. The proposed stiffness reduction expression is used to reduce the Young's E and shear G moduli for application in a LBA, thereby taking into account fully the detrimental influence of geometrical imperfections, residual stresses and the spread of plasticity on the LTB strengths of steel beams.

$$\tau_{LT} = \frac{4\psi_{LT}^2}{\kappa^2 \alpha_{LT}^2 M_{y,Ed}/M_{y,pl} \left[1 + \sqrt{1 - 3.2\psi_{LT} \frac{M_{y,Ed}/M_{y,pl} - 1}{\kappa^2 \alpha_{LT}^2 M_{y,Ed}/M_{y,pl}}} \right]^2} \quad \text{but} \quad \tau_{LT} \leq 1$$

$$\text{where} \quad \psi_{LT} = 1 + 0.2\kappa\alpha_{LT} \frac{M_{y,Ed}}{M_{y,pl}} - \frac{M_{y,Ed}}{M_{y,pl}} \quad (2)$$

$$\kappa = \frac{W_{pl,y}/A}{\frac{GI_t}{8M_{y,pl}} + \sqrt{\left(\frac{GI_t}{8M_{y,pl}}\right)^2 + I_w/I_z}} \quad (3)$$

4. Stiffness reduction for flexural-torsional buckling of beam-columns

In this section, a stiffness reduction expression for the flexural-torsional buckling assessment of beam-columns is proposed. Consideration of the influence of moment gradient on the development of plasticity is described, and the accuracy of the proposed design approach is verified by comparisons with GMNIA results for a large number of beam-columns.

4.1. Ultimate cross-section strength interaction equation

Before the development of the stiffness reduction function, an ultimate cross-section strength interaction equation is first presented. In view of its accuracy and simple form, the continuous cross-section interaction equation proposed by Duan and Chen [29], which is given by eq. (4), where $\alpha_{ult,c}$ is the cross-section utilisation factor under combined axial load and major axis bending, is adopted for the determination of the ultimate cross-section resistance of beam-columns subjected to major axis bending and axial load.

$$1/\alpha_{ult,c} = \left(\frac{N_{Ed}}{N_{pl}} \right)^{1.3} + \frac{M_{y,Ed}}{M_{y,pl}} \quad (4)$$

4.2. Derivation of stiffness reduction function for flexural-torsional buckling of beam-columns

The inelastic flexural-torsional buckling moment of a fork-end supported beam-column subjected to uniform major axis bending plus axial compression $M_{cr,N,i}$ may be determined by reducing the minor axis EI_z , torsional GI_t and warping EI_w stiffnesses through the single stiffness reduction factor $\tau_{N,LT}$ as given by eq. (5), where M_{cr} is the elastic buckling

moment, $N_{cr,T}$ is the elastic torsional buckling load and $N_{cr,z}$ is the elastic flexural buckling load about the minor axis, given by eq. (6), eq. (7) and eq. (8) respectively. Note that L is the length of the beam-column and I_y is the second moment of area about the major axis. Since the same stiffness reduction rate is used for EI_z , GI_t and EI_w , $\tau_{N,LT}$ may conveniently be applied directly to the Young's modulus E and shear modulus G , leading to the same strength predictions.

$$M_{cr,N,i} = \tau_{N,LT} M_{cr} \sqrt{\left(1 - \frac{N_{Ed}}{\tau_{N,LT} N_{cr,z}}\right) \left(1 - \frac{N_{Ed}}{\tau_{N,LT} N_{cr,T}}\right)} \quad (5)$$

$$M_{cr} = \sqrt{\frac{\pi^2 EI_z}{L^2} \left[GI_t + \frac{\pi^2 EI_w}{L^2} \right]} \quad (6)$$

$$N_{cr,T} = \frac{A}{I_y + I_z} \left(GI_t + \frac{\pi^2 EI_w}{L^2} \right) \quad (7)$$

$$N_{cr,z} = \frac{\pi^2 EI_z}{L^2} \quad (8)$$

For a beam-column to be deemed adequate, the out-of-plane buckling load factor $\alpha_{cr,op}$ must be greater than or equal to 1.0 as shown in eq. (9). Note that after reducing the Young's E and shear G moduli through $\tau_{N,LT}$, Linear Buckling Analysis (LBA-SR) directly provides $\alpha_{cr,op}$ values in conjunction with the corresponding buckling modes, where the buckling mode with the lowest $\alpha_{cr,op}$ controls the design.

$$\alpha_{cr,op} = \frac{M_{cr,N,i}}{M_{y,Ed}} \geq 1.0$$

$$\alpha_{cr,op} = \frac{\tau_{N,LT} N_{cr,z}}{N_{Ed}} \geq 1.0 ; \quad \alpha_{cr,op} = \frac{\tau_{N,LT} N_{cr,T}}{N_{Ed}} \geq 1.0 \quad (9)$$

In the stiffness reduction functions for flexural buckling τ_N given by eq. (1) and for LTB τ_{LT} by eq. (2), members subjected to pure compression alone or major axis bending moment alone are considered. Thus, the full cross-section capacities N_{pl} and $M_{y,pl}$ are used in the functions. However, for members subjected to combined major axis bending and axial compression, the combined effect of the actions on the degree of stiffness reduction should be considered, as recommended by Trahair and Hancock [12]. Considering combined loading, the reduced bending moment $M_{y,pl,r}$ and axial load $N_{pl,r}$ resistances can be determined using eq. (10) and eq. (11) respectively, which were determined by re-arrangement of the ultimate cross-section interaction equation given in eq. (4). It should be noted that though the

ultimate cross-section resistance equation proposed by Duan and Chen [29] is used herein, alternative equations such as that given in EN 1993-1-1 [1] can be employed to determine the reduced cross-section resistances.

$$N_{pl,r} = N_{pl} \left(1 - \frac{M_{y,Ed}}{M_{y,pl}} \right)^{1/1.3} \quad (10)$$

$$M_{y,pl,r} = M_{y,pl} \left[1 - \left(\frac{N_{Ed}}{N_{pl}} \right)^{1.3} \right] \quad (11)$$

The ratios between the applied loads and corresponding reduced section capacities are used to modify the stiffness reduction functions for flexural buckling τ_{Nz} and LTB τ_{LT} as shown in eq. (12) and eq. (13). The only difference between the stiffness reduction functions given in eq. (1) and eq. (2) and those presented in eq. (12) and eq. (13) is the use of the reduced section capacities. It is worth noting that the stiffness reduction functions given by eq. (1) and eq. (2) were originally derived from the Perry-Robertson equations that take into account the elastic and first yield mechanical flexural and lateral-torsional buckling response of geometrically imperfect steel members but involve imperfection factors α_z and α_{LT} calibrated to the GMNIA results for the consideration of residual stresses and the spread of plasticity [16, 27].

$$\tau_{Nz,R} = \frac{4\psi^2}{\alpha_z^2 N_{Ed}/N_{pl,r} \left[1 + \sqrt{1 - 4\psi \frac{N_{Ed}/N_{pl,r} - 1}{\alpha_z^2 N_{Ed}/N_{pl,r}}} \right]^2} \quad \text{but} \quad \tau_{Nz,R} \leq 1$$

$$\psi = 1 + 0.2\alpha_z \frac{N_{Ed}}{N_{pl,r}} - \frac{N_{Ed}}{N_{pl,r}} \quad (12)$$

$$\tau_{LT,R} = \frac{4\psi_{LT}^2}{\kappa^2 \alpha_{LT}^2 M_{y,Ed}/M_{y,pl,r} \left[1 + \sqrt{1 - 3.2\psi_{LT} \frac{M_{y,Ed}/M_{y,pl,r} - 1}{\kappa^2 \alpha_{LT}^2 M_{y,Ed}/M_{y,pl,r}}} \right]^2} \quad \text{but} \quad \tau_{LT,R} \leq 1.0$$

$$\text{where} \quad \psi_{LT} = 1 + 0.2\kappa \alpha_{LT} \frac{M_{y,Ed}}{M_{y,pl,r}} - \frac{M_{y,Ed}}{M_{y,pl,r}} \quad (13)$$

It is proposed herein that the expression for the stiffness reduction function for the flexural-torsional buckling assessment of beam-columns $\tau_{N,LT}$ is given by the multiplication of the modified stiffness reduction functions $\tau_{Nz,R}$ and $\tau_{LT,R}$. The proposed expression for $\tau_{N,LT}$ is therefore as given by eq. (14). Note that $\tau_{N,LT}$ degenerates into the stiffness reduction functions developed for pure loading cases: when the bending moment is zero $\tau_{N,LT} = \tau_{Nz}$ and when the axial force is zero $\tau_{N,LT} = \tau_{LT}$.

$$\tau_{N,LT} = \tau_{Nz,R} \tau_{LT,R} \quad (14)$$

4.3. Consideration of pre-buckling effects and influence of moment gradient

The stiffness reduction method proposed in this paper assumes the beam-column under consideration to be perfectly straight, which leads to the response illustrated in Fig. 1. Owing to zero deformations in the out-of-plane direction prior to the buckling, the in-plane and out-of-plane effects can be isolated and the in-plane assessment can be carried out separately from the out-of-plane instability assessment. In this paper, it is recommended that the in-plane assessment of beam-columns is implemented by means of Geometrically Nonlinear Analysis (GNA-SR) with the stiffness reduction scheme set out in Kucukler et al. [16]. The application of the GNA-SR approach, where the section forces at the most heavily loaded cross-section are checked against the ultimate cross-section resistance, is explained and applied to the in-plane design of beam-columns in [16, 30], thus it will not be described further herein. Owing to in-plane deformations, which may increase with the development of plasticity, significant second-order bending moments can be induced prior to out-of-plane buckling. In this paper, these pre-buckling effects are taken into account by considering the second-order forces obtained from the in-plane GNA-SR to reduce the nominal Young's E and shear G moduli through the stiffness reduction function for flexural-torsional buckling $\tau_{N,LT}$ given by eq. (14). Thus, the bending moment values $M_{y,Ed}$ used in eq. (10) and eq. (13) should be the second-order moments obtained from GNA-SR. The application of the described design framework, involving the in-plane assessment (GNA-SR) and out-of-plane assessment (LBA-SR) stages, is illustrated by Fig. 7.

The in-plane deformations occurring prior to the out-of-plane buckling results in second-order moments varying along the length of a beam-column. Thus, it can be said that a steel beam-column is always subjected to varying major axis bending along its length before it buckles in the out-of-plane direction. The moment gradient may also result from applied transverse loading or unequal end-moments. To make accurate flexural-torsional buckling capacity predictions, the influence of varying bending moment on the development of plasticity should be accounted for, since reducing stiffness considering the maximum bending moment value $M_{y,Ed}$ along the length in eq. (10) and eq. (13) may lead to uneconomic design. In Kucukler et al. [27], two different approaches were proposed for the consideration of moment gradient effects on plasticity. The first one is based on factoring the largest bending moment along the length by an equivalent uniform moment factor and reducing the Young's E and shear moduli G at the same rate along the whole member length. The second one is based on the division of the member into portions and reducing the Young's E and shear moduli G of each portion considering the corresponding forces. Both approaches have qualities and limitations, as the first approach is easier to implement, while the second approach can be applied to members with any shape of bending moment. In the following two subsections, the adoption of these two different approaches for the flexural-torsional buckling assessment of beam-columns is explained.

4.3.1. Incorporation of moment gradient factors $C_{m,LT}$ into the stiffness reduction functions

This subsection describes the incorporation of moment gradient factors $C_{m,LT}$ into the stiffness reduction functions for the consideration of the influence of moment gradient on

Stage 1: In-plane analysis GNA-SR

Stage 2: Out of-plane analysis LBA-SR

Figure 7: Example application of the stiffness reduction method to steel beam-columns subjected to major axis bending and axial compression

the development of plasticity. The moment gradient factors are incorporated into the components of the stiffness reduction function $\tau_{N,LT}$ associated with bending as shown in eq. (15) and eq. (16), where $M_{y,Ed}$ is the absolute maximum value of the bending moment (i.e. first + second-order) along the laterally unrestrained length of the beam-column. Similar to Kucukler et al. [27], this study also recommends the determination of different $C_{m,LT}$ and $\tau_{N,LT}$ for each laterally unrestrained segment of the beam-column to accurately locate the failing segment.

$$N_{pl,r} = N_{pl} \left(1 - \frac{C_{m,LT} M_{y,Ed}}{M_{y,pl}} \right)^{1/1.3} \quad (15)$$

$$\tau_{LT,R} = \frac{4\psi_{LT}^2}{\kappa^2 \alpha_{LT}^2 C_{m,LT} M_{y,Ed} / M_{y,pl,r} \left[1 + \sqrt{1 - 3.2\psi_{LT} \frac{C_{m,LT} M_{y,Ed} / M_{y,pl,r} - 1}{\kappa^2 \alpha_{LT}^2 C_{m,LT} M_{y,Ed} / M_{y,pl,r}}} \right]^2} \quad (16)$$

where $\psi_{LT} = 1 + 0.2\kappa\alpha_{LT} \frac{C_{m,LT} M_{y,Ed}}{M_{y,pl,r}} - \frac{C_{m,LT} M_{y,Ed}}{M_{y,pl,r}}$

In this study, a quarter point formula given by eq. (17), which was obtained by calibration against GMNIA results, is proposed for the determination of the moment gradient factor

$C_{m,LT}$, where M_A , M_B and M_C are the absolute values of bending moments at the quarter, centre and three-quarter points of the laterally unrestrained segment. It should be noted that eq. (17) should be used only when a beam-column is not subjected to transverse loading between lateral restraints. For beam-columns subjected to transverse loading between lateral restraints, expressions developed for $C_{m,LT}$ on the basis of a large number of GMNIA results, as described by Kucukler et al. [27], should be used. The determination of the $C_{m,LT}$ factors and the application of the proposed approach to laterally restrained members are described in detail in [27], and the same principles apply to the approach proposed herein for beam-columns.

$$C_{m,LT} = \frac{\sqrt{7M_{y,Ed}^2 + 5M_A^2 + 8M_B^2 + 5M_C^2}}{5M_{y,Ed}} \geq 0.7 \quad (17)$$

It should be emphasised that the moment gradient factors $C_{m,LT}$ incorporated into the stiffness reduction functions only take into account the influence of the bending moment shapes on the development of plasticity in this study. This approach is different to the usual adoption of the moment gradient factors in traditional design, where they are generally used for the determination of the most heavily loaded cross-section along the length of a member.

4.3.2. Tapering approach

In this subsection, an approach based on the division of a member along its length into portions and the application of stiffness reduction to each portion considering the corresponding section forces to take into account moment gradient effect on plasticity is described. The application of this approach is illustrated in Fig. 8, where it can be seen that the bending moment value at the middle of each portion is used within the components of $\tau_{N,LT}$ associated with bending, which are given by eq. (10) and eq. (13). Due to the gradual reduction of stiffness, this approach is referred to as the tapering approach in [27] and this definition will also be adopted in this paper. It was shown in [27] that transverse loading between lateral restraints generates second-order torsion, which leads to additional plasticity. To allow for this effect, it was therefore proposed to increase the rate of stiffness reduction by using a larger imperfection factor; the same approach is adopted herein, with the increased imperfection factor is denoted $\alpha_{LT,F}$ and used in eq. (13), where 'F' signifies the transverse loading. It is proposed to take $\alpha_{LT,F}$ as 1.4 times the α_{LT} values given in Table 2, i.e. $\alpha_{LT,F} = 1.4\alpha_{LT}$, when a beam-column is subjected to transverse loading between lateral restraints.

Owing to the use of bending moment values at the middle of each portion, the tapering approach may lead to unconservative results if a member is not divided into a sufficient number of portions. This sufficient number is dependent on the shape of the bending moment diagram, and can only be determined through a convergence study. Similar to [27], the division of a beam-column into 40 portions is found to be sufficient for all the cases considered in this study, though fewer portions provided accurate strength predictions in many cases.

Figure 8: Application of the stiffness reduction considering varying bending moments - tapering approach

4.4. Application of the stiffness reduction function to beam-columns subjected to uniform bending and axial compression

The accuracy of the proposed stiffness reduction method for the determination of the flexural-torsional buckling strengths of beam-columns subjected to uniform bending plus axial compression is assessed in this section. As described previously, in-plane analysis is initially performed through GNA-SR, and then Linear Buckling Analysis (LBA) is carried out reducing the Young's and shear moduli through $\tau_{N,LT}$ given by eq. (14) on the basis of section forces obtained from GNA-SR. To account for the influence of moment gradient on the development of plasticity, both the moment gradient factor approach, which will henceforth be referred to as the LBA-SR $C_{m,LT}$ approach, and the tapering approach, which will henceforth be referred to as the LBA-SR tapering approach, are used. Since members considered in this section are not subjected to transverse loading, eq. (17) is used for the determination of the $C_{m,LT}$ factors in the LBA-SR $C_{m,LT}$ approach.

The strengths obtained through LBA-SR are compared against those determined through GMNIA in Fig. 9 for beam-columns with IPE 240 and HEA 240 cross-sections. It can be seen that there is a good agreement between the results for beam-columns with both cross-sections, as well as different member slendernesses and ratios of bending moment to axial load. Moreover, the figure shows that the strength predictions obtained through the LBA-SR $C_{m,LT}$ and LBA-SR tapering approaches are quite close.

The accuracy of the proposed stiffness reduction method was also assessed for 30 IPE and HE European cross-sections. The properties of the considered sections are provided in Table 3, showing that a wide range of cross-section shapes were considered. The number of analysed beam-columns for different slenderness values and different aspect ratios (corresponding to different magnitudes of residual stresses as illustrated in Fig. 3) is shown in Table 4, where N is the number of beam-columns investigated for a particular group. Three non-dimensional slenderness values $\bar{\lambda}_z = 0.4, 1.0, 1.5$ were considered to verify the accuracy

Figure 9: Comparison of the flexural-torsional buckling strengths determined through the proposed stiffness reduction method (LBA-SR) with those obtained through GMNIA for fork-end-supported beam-columns subjected to uniform bending plus axial compression

of the proposed method for beam-columns with small, intermediate and large slendernesses. In total, the strength predictions obtained through the proposed method were compared against those obtained from GMNIA of the shell element models of 780 fork-end supported beam-columns. The accuracy of the LBA-SR approach was assessed through the ratio ϵ , defined by eq. (18), where R_{LBA-SR} and R_{GMNIA} are the radial distances measured from the origin to the interaction curves determined through LBA-SR and GMNIA respectively. Values of ϵ greater than 1.0 indicate unconservative strength predictions. The comparison of the results are presented in Table 5, where ϵ_{av} and ϵ_{cov} are the average and coefficient of variation of ϵ values and ϵ_{min} and ϵ_{max} are the minimum and maximum values of ϵ values respectively. Table 5 shows that both the LBA-SR $C_{m,LT}$ approach and the LBA-SR tapering approach provide accurate results, though the predictions are somewhat conservative for cross-sections with $h/b \leq 1.2$. For all considered sections, LBA-SR results in unconservative errors of no greater than 4% (i.e. $\epsilon \leq 1.04$), indicating the reliability of the proposed stiffness reduction method.

$$\epsilon = \frac{R_{LBA-SR}}{R_{GMNIA}} \quad (18)$$

To assess the importance of the consideration of pre-buckling effects, LBA-SR was also performed reducing the stiffness on the basis of the applied bending moments (first-order bending moments) and axial compression. The comparison of the strengths obtained through LBA-SR where pre-buckling effects are neglected against those of GMNIA are given in Table

Table 3: Range of European cross-section dimensions considered to assess the accuracy of the proposed stiffness reduction method

	$h/b \leq 1.2$ - 10 sections		$h/b > 1.2$ - 20 sections	
	Max	Min	Max	Min
h/b	1.2	0.91	3.34	1.22
b/t_f	25	5.3	22.22	7.73
h/t_w	37.7	10	60.63	17.95

Table 4: Number of beam-column cases examined to assess the accuracy of the proposed stiffness reduction method

	$h/b \leq 1.2$	$h/b > 1.2$
$\bar{\lambda}_z$	N	N
0.4	110	220
1.0	90	180
1.5	60	120

5. It is seen that up to non-dimensional slenderness $\bar{\lambda}_z$ values of 1.0, this approach, which does not require the second-order moments obtained from GNA-SR, also provides accurate strength predictions. Nevertheless, with increasing slenderness, the second-order moments induced by pre-buckling deformations become significant, thus this approach provides quite unconservative results particularly for slender beam-columns with small aspect ratios where the non-dimensional slenderness for in-plane flexural buckling $\bar{\lambda}_y = \sqrt{N_{pl}/N_{cr,y}}$ and that for out-of-plane flexural buckling $\bar{\lambda}_z = \sqrt{N_{pl}/N_{cr,z}}$ are similar. Note that $N_{cr,y}$ is the elastic flexural buckling load about the major axis. The pre-buckling effects may also be of importance for beam-columns laterally restrained along the length. An example, which was also investigated by Greiner and Lindner [31], where the pre-buckling effects are of great significance is shown in Fig. 10. In the example, a beam-column with an IPE 500 cross-section is subjected to uniform major axis bending plus axial compression, and has intermediate lateral restraints in the out-of-plane direction, resulting in non-dimensional slendernesses $\bar{\lambda}_y = 1.26$ and $\bar{\lambda}_z = 1.0$. As can be seen from the figure, the LBA-SR approach leads to very unconservative strength predictions when the pre-buckling effects are neglected. For axial loads larger than $0.16N_{pl}$ (i.e. $N_{Ed} > 0.16N_{pl}$), the in-plane GNA-SR analysis controls the design and provides smaller strength predictions, but these are still quite unconservative. In contrast, both the LBA-SR $C_{m,LT}$ and LBA-SR tapering approaches, where stiffness is reduced on the basis of forces obtained from GNA-SR, provide accurate strength predictions. It is worthwhile noting that the strengths determined through beam-column design methods provided in Annexes A and B of EN 1993-1-1 [1] are also in a good agreement with those of GMNIA.

Table 5: Comparison of the results obtained through LBA-SR (i.e. the proposed stiffness reduction method) and Eurocode 3 [1] Annexes A and B with those of GMNIA for fork-end beam-columns subjected to uniform major axis bending plus axial compression

	$\bar{\lambda}_z$	$h/b \leq 1.2$				$h/b > 1.2$			
		ϵ_{av}	ϵ_{cov}	ϵ_{max}	ϵ_{min}	ϵ_{av}	ϵ_{cov}	ϵ_{max}	ϵ_{min}
Proposed stiffness reduction method - $C_{m,LT}$ approach	0.4	0.95	0.024	1.02	0.92	0.97	0.017	1.01	0.94
	1.0	0.94	0.042	1.02	0.86	0.99	0.021	1.04	0.94
	1.5	0.96	0.026	1.03	0.92	0.99	0.018	1.03	0.96
Proposed stiffness reduction method - tapering approach	0.4	0.96	0.022	1.02	0.93	0.97	0.017	1.01	0.94
	1.0	0.95	0.037	1.02	0.88	0.99	0.021	1.04	0.94
	1.5	0.97	0.023	1.03	0.93	0.99	0.018	1.03	0.96
Proposed stiffness reduction method - pre-buckling effects neglected	0.4	0.98	0.015	1.02	0.95	0.98	0.016	1.01	0.94
	1.0	0.99	0.018	1.02	0.96	1.01	0.020	1.06	0.96
	1.5	0.99	0.041	1.09	0.94	1.00	0.026	1.10	0.96
Eurocode 3 Annex A	0.4	1.00	0.019	1.05	0.96	1.00	0.021	1.07	0.96
	1.0	0.94	0.053	1.08	0.86	0.97	0.048	1.11	0.87
	1.5	0.94	0.049	1.08	0.87	0.95	0.045	1.10	0.87
Eurocode 3 Annex B	0.4	0.97	0.033	1.05	0.92	0.98	0.028	1.07	0.94
	1.0	0.94	0.060	1.08	0.84	0.98	0.051	1.11	0.88
	1.5	0.92	0.058	1.08	0.84	0.94	0.059	1.10	0.85

For the purpose of illustrating the accuracy of the stiffness reduction method in comparison to Eurocode 3 [1], the results obtained through the beam-column design methods provided in Annexes A and B of Eurocode 3 [1] are also compared with those of GMNIA in Table 5. The ϵ values were determined through the ratio of the radial distances measured from the origin to the interaction curves determined from Annex A or Annex B and GMNIA respectively. Note that the LTB formula given in the Clause 6.3.2.3. of Eurocode 3 [1] was used in all the calculations in this paper. As seen from the table, the proposed stiffness reduction method is more accurate than Annex B in almost all considered cases and is more accurate than Annex A for beam-columns with aspect ratios larger than 1.2.

4.5. Application of the stiffness reduction method to beam-columns subjected to unequal end moments or transverse loading

In this section, the accuracy of the proposed LBA-SR approach is assessed for fork-end-supported beam-columns subjected to unequal end moments or transverse loading. The strength predictions obtained through the LBA-SR $C_{m,LT}$ approach and LBA-SR tapering approach are compared against those determined through GMNIA and Eurocode 3 Annexes A and B in Figs. 11 (a) and (b) for beam-columns with IPE 240 and HEA 240 cross-sections and subjected to unequal end moments plus axial compression. Since all the beam-columns analysed in this subsection have fork-end support conditions, the expressions provided in

Figure 10: Comparison of the flexural-torsional buckling strengths determined through the proposed stiffness reduction method (LBA-SR) with those obtained through GMNIA for fork-end-supported beam-columns subjected to uniform bending plus axial compression

Eurocode 3 [1] for the determination of the moment gradient factors are directly used in the calculations to obtain strength predictions according to Eurocode 3 Annexes A and B. In the LBA-SR $C_{m,LT}$ approach, the quarter point formula given by eq. (17) is employed to determine the moment gradient factors $C_{m,LT}$. As can be seen from the figure, the LBA-SR $C_{m,LT}$ and LBA-SR tapering approaches lead to very accurate strength predictions for the case of single curvature bending given in Fig. 11 (a) and double curvature bending shown in Fig. 11 (b). Though the Eurocode 3 [1] design methods are also in a good agreement with the GMNIA results, the proposed stiffness reduction approaches provide more accurate strength predictions.

Figs 11 (c) and (d) show comparisons of the strength predictions determined through the LBA-SR $C_{m,LT}$ and LBA-SR tapering approaches against those calculated through GMNIA and Eurocode 3 [1] for beam-columns subjected to transverse loading and axial compression. Note that transverse loading was applied at the shear centre of the beam-column cross-sections in both cases. In the LBA-SR $C_{m,LT}$ approach, the moment gradient factors $C_{m,LT}$ were calculated using the expressions provided by Kucukler et al. [27]. Since the beam-columns are subjected to transverse loading, the increased imperfection factor $\alpha_{LT,F}$ equal to $\alpha_{LT,F} = 1.4\alpha_{LT}$ was used for the determination of $\tau_{LT,R}$ given by eq. (13) in the LBA-SR tapering approach. As can be seen from Fig. 11 (c) and (d), the LBA-SR $C_{m,LT}$ and LBA-SR tapering approaches provide results that are in a good agreement with those obtained through GMNIA and are more accurate than Eurocode 3 [1], though the results are rather conservative for beam-columns subjected to a point load at the mid-span and axial compression. It should be noted that for the case of beam-columns subjected to uniformly

Figure 11: Comparison of the results obtained through Linear Buckling Analysis with stiffness reduction (LBA-SR) with those obtained through GMNIA and Eurocode 3 (2005) for fork-end-supported beam-columns subjected to different bending moment shapes and axial compression

distributed load, end moments and axial compression, shear forces become very significant in some of the low slenderness ($\bar{\lambda}_z = 0.4$) members, leading to shear failure, which limited the maximum bending capacity. In the proposed LBA-SR approach, the shear capacity checks were performed using the design rules given in Clause 6.2.6 of Eurocode 3 [1], which lead to safe results for members failing due to shear as can be seen from Fig. 11 (c).

5. Application of the stiffness reduction method to irregular and multi-span beam-columns

In this section, the accuracy of the proposed stiffness reduction method for the design of irregular and multi-span beam-columns is investigated. In accordance with previous sections, in-plane analysis GNA-SR is initially implemented as described in [16], and LBA-SR is subsequently performed with reduced Young's and shear moduli through $\tau_{N,LT}$ given by eq. (14), considering section forces from GNA-SR. In all the considered cases, the results obtained through the proposed approach are compared against those determined through GMNIA using shell finite element models. Additionally, the results of the proposed stiffness reduction method are also compared against those obtained from the beam-column design methods of Eurocode 3 [1] so as to demonstrate the accuracy of the proposed approach in comparison to traditional design. In traditional design, irregularities are accounted for by calculating elastic buckling loads and moments and using these values in the associated formulae such as those used for the determination of non-dimensional slendernesses. This principle, which is referred to as the design by buckling analysis in AS 4100 [3] and implicitly allowed in Eurocode 3 [1], is adopted herein so as to establish a fair comparison between traditional design and the proposed approach.

In the implementation of the design by buckling analysis (DBA) according to Eurocode 3 [1] for the assessment of the irregular and multi-span beam-columns performed herein, Linear Buckling Analyses of the finite element models of the investigated members are initially carried out considering only the corresponding component of loading that causes pure compression or pure bending respectively. These Linear Buckling Analyses furnish the elastic flexural buckling loads $N_{cr,y}$, $N_{cr,z}$, the elastic torsional buckling load $N_{cr,T}$ and the elastic critical moment M_{cr} of the investigated member, from which non-dimensional slendernesses for flexural buckling $\bar{\lambda}_y$, $\bar{\lambda}_z$ and that for lateral-torsional buckling $\bar{\lambda}_{LT}$ for the most heavily stressed cross-section are determined. These slenderness values and elastic buckling loads and moments are then used within the beam-column interaction equations of Eurocode 3 [1], thereby assessing the utilisation rate of the most heavily stressed cross-section. The imperfection factors used for the calculation of buckling reduction factors and constants within the interaction equations that are functions of cross-section dimensions are determined considering the cross-section properties of the most heavily stressed cross-section. Following the recommendations of Goncalves and Camotim [32] and Boissonnade et al. [33], exact equivalent moment factors are determined performing Geometrically Nonlinear Analyses of the irregular and multi-span members investigated in this section and used within the beam-column design equations of Eurocode 3 [1]. It should be noted that the same principles followed in this paper were also adopted by Trahair et al. [34], Simoes da Silva et al. [35] and Trahair [36] for the design of irregular and multi-span members within steel frames according to the Eurocode 3 [1] and AS 4100 [3] provisions. Additional information for the implementation of the DBA in conjunction with code provisions for irregular and multi-span members can be found in these studies [34–36].

5.1. Stepped beam-column

In this subsection, the accuracy of the proposed stiffness reduction method for the design of fork-end supported beam-columns strengthened with additional plates, which are referred to as stepped beam-columns, is investigated. As shown in Fig. 12, a stepped beam-column with an HEB 400 cross-section and subjected to uniform bending plus axial compression is strengthened in the central half by attaching additional plates ($t = 21.6$ mm) to the flanges, such that the second-moment area about the major axis is two times of that of the original section. In the implementation of the stiffness reduction method, stiffness reduction was applied to the original and strengthened portions considering corresponding cross-section properties within the stiffness reduction functions. Note that warping moments and deformations are fully transferred between the strengthened and original segments in the shell finite element models of the stepped beam-columns. The flexural-torsional buckling loads obtained through the proposed stiffness reduction method are compared against those obtained from GMNIA in Fig. 12 for different slenderness values, where the non-dimensional slendernesses $\bar{\lambda}_{z,0}$ for the minor axis buckling are determined considering the original beam-columns without the additional plates. Fig. 12 shows that both the LBA-SR $C_{m,LT}$ and LBA-SR tapering approaches provide very accurate results owing to the consideration of the development of different rates of plasticity within the original and strengthened portions. In addition to LBA-SR, the beam-column design methods of Eurocode 3 are also employed considering the increased values of elastic buckling loads and moments. Fig. 12 illustrates that the proposed stiffness reduction method brings about significant improvements in accuracy in comparison to traditional design, and that the strength predictions obtained through Eurocode 3 [1] are often rather conservative.

5.2. Beam-column with an intermediate compressive force

This subsection addresses the application of the stiffness reduction method to a fork-end supported beam-column with an HEB 400 cross-section, and subjected to uniform major axis bending and compressive forces applied at the mid-height and at one end - see Fig. 13. In the application of the stiffness reduction method, the change of the axial force along the beam-column length is considered within the stiffness reduction functions associated with the axial loading. The capacity predictions made through the LBA-SR $C_{m,LT}$ and LBA-SR tapering approaches are compared against those obtained through GMNIA in Fig. 13, where the non-dimensional slendernesses $\bar{\lambda}_{z,0}$ are calculated assuming that the beam-columns are subjected to uniform axial loading. It is seen from the figure that both the LBA-SR $C_{m,LT}$ and LBA-SR tapering approaches provide capacity predictions in a very close agreement with those obtained from GMNIA. Fig. 13 also shows capacity predictions calculated through Eurocode 3 [1]. It is seen that the proposed approach leads to considerably more accurate capacity predictions in comparison to Eurocode 3 [1].

5.3. Beam-column with an intermediate elastic restraint

In some instances, the stiffness of lateral restraints may not be sufficient to assume them to be rigid; the actual level of restraint stiffness should therefore be considered in design calculations. In this subsection, application of the proposed stiffness reduction method

Figure 12: Comparison of the flexural-torsional buckling strengths determined through the proposed stiffness reduction method (LBA-SR) with those obtained through GMNIA for a stepped beam-column subjected to uniform bending plus axial compression

Figure 13: Comparison of the flexural-torsional buckling strengths determined through the proposed stiffness reduction method (LBA-SR) with those obtained through GMNIA for beam-column subjected to uniform bending plus intermediate axial compression

to a fork-end supported beam-column with a discrete elastic lateral restraint at the mid-height is investigated. The geometrical properties and loading conditions of the beam-column with an HEB 400 cross-section and subjected to uniform major axis bending and axial compression are illustrated in Fig. 14. It is seen that the elastic lateral restraint is attached to the critical flange subjected to compressive stresses due to both bending and axial load. To assess the accuracy of the design approaches for the consideration of the influence of the lateral restraint, the stiffness of the elastic lateral restraint K was varied and the strengths of beam-columns were determined through GMNIA, the LBA-SR $C_{m,LT}$ and LBA-SR tapering approaches and the beam-column design methods of Eurocode 3 Annexes A and B. In the GMNIA simulations, two different shapes of geometrical imperfections were considered: one-half sine wave and two-half sine waves, which correspond to the first and second buckling modes. Moreover, non-proportional loading was applied in the GMNIA simulations, where the axial loading was first applied up to $0.5N_{pl}$, and then the bending moment was increased up to the collapse while the axial load was kept constant. The results are compared in Fig. 14, where K_L is the elastic threshold stiffness of the restraint leading to the elastic flexural-torsional buckling of the beam-column in the second mode. Fig. 14 shows that up to a specific slenderness value, GMNIA with an imperfection in the form of the first buckling mode shape leads to lower strength predictions than those obtained with the imperfection in the form of the second buckling mode shape, while the latter provides lower strength predictions after this value is exceeded. This specific stiffness can be referred to as the inelastic threshold stiffness $K_{L,inelastic}$ leading to inelastic buckling of the beam-column in the second-mode. With the development of plasticity in the beam-column, the effectiveness of the lateral restraint increases as the ratio of the restraint stiffness to the beam-column stiffness becomes larger [16]. Thus, the inelastic threshold stiffness $K_{L,inelastic}$ leading to inelastic buckling in the second mode is considerably smaller than that for elastic buckling K_L . Fig. 14 shows that both the LBA-SR $C_{m,LT}$ approach and LBA-SR tapering approach are able to capture the described response and the increased effectiveness of the elastic lateral restraint. Despite slight conservatism, both the LBA-SR $C_{m,LT}$ and LBA-SR tapering approaches also provide accurate strength predictions. A very important advantage of the LBA-SR $C_{m,LT}$ and LBA-SR tapering approaches is that they are able to capture the transition between the first and second inelastic buckling modes directly without the need for explicitly modelling the geometrical imperfections. Since the increased effectiveness of the elastic restraint with the development of plasticity within the beam-column is not considered, the beam-column design methods provided in Eurocode Annexes A and B cannot capture the described response and lead to overly conservative results.

5.4. Continuous beam-column

In this subsection, application of the proposed stiffness reduction method to a three-span continuous beam-column with an HEB 400 cross-section is investigated. The geometrical properties and loading conditions of the continuous beam-column are shown in Fig. 15, where it is shown that the beam-column is subjected to a uniformly distributed load applied to the top flange and equal-magnitude point loads at the locations of the intermediate and end-supports. Note that lateral deflection and twist are prevented at the supports. The

Figure 14: Comparison of the flexural-torsional buckling strengths determined through the proposed stiffness reduction method (LBA-SR) with those obtained through GMNIA for a beam-column with an elastic restraint at the mid-height

flexural-torsional buckling strengths of the beam-column were determined for different slenderness values $\bar{\lambda}_{z,0}$ and different ratios of transverse loading to axial force through GMNIA, the LBA-SR $C_{m,LT}$ approach and LBA-SR tapering approach and the beam-column design formulae given in Annexes A and B of Eurocode 3 [1]. In the GMNIA simulations, the eigenmode corresponding to the lowest buckling load was adopted as an imperfection shape, which was scaled by $L/1000$, i.e. $1/1000$ of the length of the laterally unrestrained span. The in-plane GNA-SR of the continuous beam-column was performed by reducing the stiffness of each span separately through the stiffness reduction expressions and quarter point moment gradient formula given in [16] on the basis of forces obtained through first-order elastic analysis. After performing GNA-SR, the corresponding section forces were employed to reduce the Young's and shear moduli through $\tau_{N,LT}$, and then LBA was performed in the application of the LBA-SR $C_{m,LT}$ and LBA-SR tapering approaches. In the LBA-SR $C_{m,LT}$ approach, the $C_{m,LT}$ factors for each span were determined using the expressions developed in [27]. Since the beam-column is subjected to transverse loading between lateral restraints, the increased imperfection factor $\alpha_{LT,F} = 1.4\alpha_{LT}$ was employed within $\tau_{LT,R}$ given by eq. (13) in the application of the LBA-SR tapering approach. A comparison of the results is displayed in Fig. 15, where the non-dimensional slendernesses for the flexural buckling $\bar{\lambda}_{z,0}$ are determined assuming each laterally unrestrained segment as simply supported. Fig. 15 shows that the strength predictions obtained through both the LBA-SR $C_{m,LT}$ approach and

the LBA-SR tapering approach are in a very good agreement with those determined through GMNIA. For the studied continuous beam-column, the lowest segment of the member is subjected to the largest forces and therefore experiences the greatest extent of plasticity. Since the middle segment remains relatively elastic, the support afforded by this segment to the critical lower segment effectively increases. Owing to the consideration of this behaviour through the stiffness reduction functions, the LBA-SR $C_{m,LT}$ and LBA-SR tapering approaches result in considerably more accurate strength predictions in comparison to those obtained through Eurocode 3 [1] as can be seen in Fig. 15. It should be noted that Lim and Lu [6] also investigated the response of continuous beam-columns, observing results similar to those described herein. For the beam-columns with $\bar{\lambda}_{z,0} = 0.4$, the maximum bending $M_{y,Ed}$ capacity is limited by shear failure similar to the case illustrated in Fig. 11 (c). It is seen from Fig. 15 that the use of the equations given in the Clause 6.2.6 of Eurocode 3 [1] within the LBA-SR approaches to perform the shear capacity checks leads to safe results.

Figure 15: Comparison of the flexural-torsional buckling strengths determined through the proposed stiffness reduction method (LBA-SR) with those obtained through GMNIA for a continuous beam-column

6. Conclusions

This study presented a stiffness reduction method for the flexural-torsional buckling assessment of steel beam-columns. According to the proposed approach, the in-plane assessment of a beam-column is initially carried out by performing Geometrically Nonlinear Analysis with stiffness reduction (GNA-SR) where the section forces at the most heavily loaded section are checked against the ultimate cross-section resistance as described in detail in Kucukler et al. [16]. Provided the member does not fail in the in-plane analysis,

the out-of-plane assessment (flexural-torsional buckling assessment) is carried out by performing Linear Buckling Analysis with stiffness reduction (LBA-SR). Stiffness reduction in the out-of-plane assessment is applied on the basis of the section forces obtained from the in-plane analysis (GNA-SR) through the stiffness reduction functions derived in this study. For the consideration of the influence of moment gradient on the development of plasticity, two alternative approaches may be employed: the LBA-SR $C_{m,LT}$ approach and LBA-SR tapering approach. The first is based on the incorporation of moment gradient factors into the stiffness reduction functions, while the latter involves the division of a member into portions along the length, with the reduction of stiffness for each portion based on the corresponding section forces. Shell finite element models of steel beam-columns were developed and validated against experimental results from the literature. Geometrically and Materially Nonlinear Analysis with Imperfections (GMNIA) of these shell finite element models furnished benchmark results used for the verification of the proposed stiffness reduction method. The proposed method was verified against GMNIA results for 780 fork-end supported beam-columns subjected to uniform bending plus axial load covering 30 different European IPE and HE shapes and different member slendernesses. The accuracy of the proposed method for beam-columns subjected to unequal end moments and transverse loading was also illustrated. In addition to regular and single span beam-columns, the proposed stiffness reduction method was also assessed for irregular and multi-span beam-columns, where the proposed method provided capacity predictions in very good agreement with those obtained through GMNIA. It was also shown that since the influence of the development of plasticity on the response of steel members is considered, the proposed stiffness reduction method leads to considerably more accurate results in comparison to traditional design based on the beam-column design methods of Eurocode 3 [1] for irregular and multi-span beam-columns.

The proposed stiffness reduction method obviates the need of using member buckling equations in design, considers compound buckling modes and assesses the out-of-plane instability in a single step, so offering a realistic and practical way of designing steel members. An important advantage of the proposed method is that it can be readily applied through any conventional structural analysis software capable of providing elastic flexural-torsional buckling loads for beam-columns. Future research will be directed towards the application of the method to steel beam-columns with welded or monosymmetric cross-sections, those susceptible to local buckling effects and to steel frames.

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