An *hp*/spectral element model for efficient long-time integration of Boussinesq-type equations

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Abstract

We present an hp /spectral element method for modelling one-dimensional non-linear dispersive water waves, described by a set of enhanced Boussinesq-type equations. The model uses nodal basis functions of arbitrary order in space and the third-order Adams-Bashforth scheme to advance in time. Numerical computations are used to show that the hp /spectral element model exhibits exponential convergence. The model is compared to two numerical methods frequently used for solving Boussinesq-type equations; a finite element model using linear basis functions and a finite difference model using a five-point stencil for estimating the first-order derivatives. Using numerical examples we show that the hp /spectral element model gives great savings in computational time, compared to the other models, if: (i) highly accurate results are requested, or, more importantly, (ii) results of “engineering accuracy” are called for in combination with long-time integration.

Keywords: Boussinesq-type equations, hp /spectral element method

1 Introduction

Over the last decade Boussinesq-type equations have become a popular choice in coastal engineering to simulate nonlinear dispersive wave propagation in shallow and intermediate water. This is mainly due to the improvements in the dispersive, but also in the nonlinear, characteristics of the equations. There is an ever growing body of studies concerned with deriving more and more sophisticated equations, e.g. Madsen and Schäffer (1998) and Gobbi *et al.* (2000). In this paper, however, we turn our attention towards the numerical solution of the one-dimensional enhanced low-order Boussinesq-type equations of Nwogu (1993):

$$\begin{aligned}\partial_t \eta + \partial_x ((h + \eta) u) + \partial_x (a_1 h^3 \partial_{xx} u + a_2 h^2 \partial_{xx} (hu)) &= 0, \\ \partial_t u + b_1 h^2 \partial_{xxt} u + b_2 h \partial_{xx} (h \partial_t u) + u \partial_x u + g \partial_x \eta &= 0,\end{aligned}$$

where $\eta(x, t)$ is the free surface elevation measured from the still water level, $u(x, t)$ is the horizontal velocity at an arbitrary depth z_α , $h(x)$ is the water depth measured from the still water level and g is the acceleration of gravity. The Nwogu equations are valid for weakly dispersive and nonlinear waves in shallow and intermediate waters.

The first numerical Boussinesq model, presented by Peregrine (1967), was a finite difference method (FDM). With the introduction of enhanced low-order Boussinesq-type equations (Madsen *et al.*, 1991; Nwogu, 1993; Beji and Nadaoka, 1996) FDM remained the numerical method of choice. Over the years FDM has been – and is still to this day – the predominant method for solving Boussinesq-type equations. One particularly popular version is due to Wei and Kirby (1995), who used it to solve the equations of Nwogu. In this method the first-order spatial derivatives are estimated to fourth-order by using a centred five-point stencil, while the second- and third-order spatial derivatives are approximated to second-order. The semi-discrete equations are then advanced in time using a predictor-corrector Adams-Bashforth-Moulton scheme, usually with iteration of the corrector step. The outline of this FDM has then been extended and used in studies

of higher-order Boussinesq-type equations (Madsen *et al.*, 1996; Gobbi and Kirby, 1999). The popularity of FDM is to be attributed to the ease in which higher-order derivatives are resolved, as well as giving rise to tri-diagonal systems which can be very efficiently solved using e.g. the Thomas algorithm. One drawback is that we must use structured spatial meshes in 2D, even though some negative aspects of this shortcoming can be avoided by using curvilinear coordinates (Shi *et al.*, 2001). Another problem with FDM is the treatment of the boundary conditions.

In order to be able to use unstructured spatial meshes in complex 2D geometries we have seen – over the last four years – a number of studies using finite element methods (FEM) to numerically solve enhanced Boussinesq-type equations (note that we with FEM here mean a Galerkin method using low order polynomials as shape functions, generally linear). The main disadvantage of FEM is that the third-order spatial derivatives present in the equations must be reduced, since a C^0 finite element approximation does not support derivatives higher than two. Walkley and Berzins (1999) reduced the third-order derivatives by introducing an auxiliary variable to the system of equations – resulting in an additional equation to be solved in 1D. A different approach is to resolve the third-order derivatives by estimating the first-order derivatives of the free surface elevation using central differences (Li *et al.*, 1999).

In the present paper we will investigate the use of an hp /spectral element method (hp /SEM), i.e. a Galerkin method that is of arbitrary order in space and has a spatial order of accuracy of at least $\mathcal{O}(\Delta x^P)$, in which Δx is the mesh size and P is the order of the interpolating polynomial. With an h -type Galerkin method it is understood that the polynomial expansion is of a fixed – usually low – order and convergence is achieved by refining the spatial mesh, known as h -type refinement (from the convention in applied mathematics of denoting the characteristic element size with h). Hence, the models described above are all h -type methods. In p -type Galerkin methods the reversed

strategy is used: the spatial mesh is kept fixed and convergence is obtained by increasing the order p of the polynomial expansion, i.e. p -type refinement. For a p -type method in which the computational domain is treated as one element we obtain a polynomial spectral method (Gottlieb and Orzag, 1977).

Consequently, with hp /SEM is recognised a method that combines the h -type and p -type approaches. The hp /spectral element methods provide us with a general framework for solving partial differential equations and hp /SEM have over the years been applied to partial differential equations of parabolic and elliptic, as well as hyperbolic types. In computational fluid dynamics hp /SEM is especially popular for solving the Navier-Stokes equations, see the textbooks of Karniadakis and Sherwin (1999) and Deville *et al.* (2002) and the numerous references therein. The advantage of hp /SEM is that it combines the high accuracy of the polynomial spectral methods with the generality and flexible treatment of complex geometries of FEM. High-order numerical methods are not frequently used in coastal and ocean engineering – some exceptions being Ma (1993) and Iskandarani *et al.* (1995) who solved the oceanic shallow water equations using spectral elements, and Sanders *et al.* (1998) who used a Fourier spectral method to simulate Peregrine’s classical Boussinesq equations. Some common statements against the use of high-order numerical methods are:

1. high-order methods are tedious to implement.
2. there is no need to bother with high-order methods since the same accuracy can always be obtained using a low-order method and simply refine the spatial mesh.
3. as we are usually only interested in obtaining results of “engineering accuracy” it would be an overkill to use high-order methods.

It is definitely true that high-order methods are more complicated/tedious to implement although, as we shall demonstrate, there are gains in terms of accuracy. The second

statement is, certainly, true. However, with regard to computational efficiency high-order methods can generally achieve a greater accuracy at a lower computational cost. As for the third comment, obtaining engineering accuracy can be a very difficult task if long-time integration is to be performed since phase inaccuracy due to spatial resolution amplifies with time. Consequently, even for results of engineering accuracy higher-order methods can be computationally more efficient than low-order methods in the limit of long-time integration (Kreiss and Olinger, 1973), as numerical diffusion and dispersion decreases with increasing order of the scheme.

As mentioned earlier, in the present study we will develop an *hp*/SEM model to solve the enhanced Boussinesq-type equations of Nwogu. More specifically, the numerical model is based on the standard spectral element method of Patera (1984), which is a special class of nodal *p*-type elements using Lagrange polynomials through the zeros of the Gauss-Legendre-Lobatto integration points. For smooth problems the spectral element method is known to demonstrate exponentially fast convergence in the case of *p*-type refinement (Patera, 1984; Maday and Patera, 1989). The *hp*/SEM model will be compared to a linear FEM model as well as a FDM model in terms of computational efficiency. Using numerical examples we will show that the *hp*/SEM model gives significant savings in computational time, compared to the other models, if: (i) highly accurate results are requested, or – more importantly – (ii) results of engineering accuracy are called for in combination with long-time integration.

The paper is organised as follows. In section 2 we very briefly outline the Boussinesq-type equations of Nwogu. In section 3 we introduce the numerical models, with emphasis put on the spatial discretization of the *hp*/SEM model. Truncation error analyses are carried out in section 4, while the influence of the polynomial order on the stability is discussed in section 5. Computational examples, with weight on the computational efficiency of the models, are presented in section 6. Finally, in section 7 we summarise

the findings.

2 Governing equations

The low-order enhanced Boussinesq-type equations derived by Nwogu (1993) are written, in one dimension, as

$$\partial_t \eta + \partial_x ((h + \eta) u) + \partial_x (a_1 h^3 \partial_{xx} u + a_2 h^2 \partial_{xx} (hu)) = 0, \quad (1a)$$

$$\partial_t u + b_1 h^2 \partial_{xxt} u + b_2 h \partial_{xx} (h \partial_t u) + u \partial_x u + g \partial_x \eta = 0, \quad (1b)$$

where $\eta(x, t)$ is the free surface elevation measured from the still water level, $u(x, t)$ is the horizontal velocity at an arbitrary depth z_α , $h(x)$ is the water depth measured from the still water level and g is the acceleration of gravity. The coefficients a_1 , a_2 , b_1 and b_2 are defined as

$$a_1 = \frac{\alpha^2}{2} - \frac{1}{6}, \quad a_2 = \alpha + \frac{1}{2}, \quad b_1 = \frac{\alpha^2}{2}, \quad b_2 = \alpha, \quad (2)$$

in which $\alpha \equiv z_\alpha/h = -0.531$. We introduce the dimensionless nonlinearity, ε , and dispersion, μ , parameters, defined as

$$\varepsilon \equiv \frac{a_*}{h_*}, \quad \mu \equiv \frac{h_*}{l_*}, \quad (3)$$

where a_* , h_* and l_* are the characteristic amplitude, water depth and wavelength, respectively. The equations are valid for weakly dispersive and nonlinear waves in shallow and intermediate waters ($\mu \leq 0.5$).

3 Numerical schemes

In this section we outline three numerical methods to solve eqs. (1a-b). First we introduce the hp /spectral element method, then we briefly outline the FEM and FDM models. Both hp /SEM and FEM are formulated using standard Galerkin projection, and indeed FEM is nothing but a special case of hp /SEM.

For simplicity the models will be developed for the case of constant depth, even though the methodologies are equally valid for the variable depth case. Under this assumption, eqs. (1a-1b) simplify to:

$$\partial_t \eta + \partial_x ((h + \eta) u) + ah^3 \partial_{xxx} u = 0, \quad (4a)$$

$$\partial_t u + bh^2 \partial_{xxt} u + u \partial_x u + g \partial_x \eta = 0, \quad (4b)$$

where $a = (a_1 + a_2)$ and $b = (b_1 + b_2)$.

3.1 hp /SEM

Since the technique typically enforces C^0 continuity the third-order spatial derivative in eq. (4a) is reduced by introducing an auxiliary variable, w , into the system of equations (Walkley and Berzins, 1999), yielding

$$\partial_t \eta + \partial_x ((h + \eta) u + w) = 0, \quad (5a)$$

$$\partial_t u + bh^2 \partial_{xxt} u + \partial_x \left(\frac{u^2}{2} + g\eta \right) = 0, \quad (5b)$$

$$w - ah^3 \partial_{xxx} u = 0. \quad (5c)$$

We multiply eqs. (5a-c) with an arbitrary smooth function $v(x)$ and integrate over the domain Ω :

$$\int_{\Omega} v \partial_t \eta dx + \int_{\Omega} v \partial_x ((h + \eta) u + w) dx = 0, \quad (6a)$$

$$\int_{\Omega} v \partial_t u dx + bh^2 \int_{\Omega} v \partial_{xxt} u dx + \int_{\Omega} v \partial_x \left(\frac{u^2}{2} + g\eta \right) dx = 0, \quad (6b)$$

$$\int_{\Omega} v w dx - ah^3 \int_{\Omega} v \partial_{xxx} u dx = 0. \quad (6c)$$

We integrate the second-order spatial derivatives by parts to obtain

$$\int_{\Omega} v \partial_t \eta dx + \int_{\Omega} v \partial_x ((h + \eta) u + w) dx = 0, \quad (7a)$$

$$\int_{\Omega} v \partial_t u dx - bh^2 \int_{\Omega} \partial_x v \partial_{xt} u dx + \int_{\Omega} v \partial_x \left(\frac{u^2}{2} + g\eta \right) dx + bh^2 \int_{\partial\Omega} v \partial_{xt} u ds = 0, \quad (7b)$$

$$\int_{\Omega} v w dx + ah^3 \int_{\Omega} \partial_x v \partial_{xx} u dx - ah^3 \int_{\partial\Omega} v \partial_{xx} u ds = 0. \quad (7c)$$

We now partition the computational domain Ω into N non overlapping elements, and denote the elements by $I_e = [x_e, x_{e+1}]$, and the size of the elements with $\Delta x_e = (x_{e+1} - x_e)$, for $e = 1, \dots, N$. We map the element I_e into a standard element $I_{st} = [-1, 1]$, using the linear stretching

$$\xi = \chi_e(x) = \frac{2(x - x_e)}{x_{e+1} - x_e} - 1, \quad x \in I_e, \quad (8)$$

where ξ is the local coordinate. Following the Galerkin projection we now replace the smooth function v with a test function $v_\delta \in \mathcal{V}_\delta$ as well as approximating the solutions (η, u, w) with $(\eta_\delta, u_\delta, w_\delta) \in \mathcal{V}_\delta$, where $\mathcal{V}_\delta$ is the discrete space defined as

$$\mathcal{V}_\delta = \{v_\delta | v_\delta \in H^1, v_\delta(\chi_e^{-1}(\xi)) \in \mathcal{P}_e^P(I_{st}), e = 1, \dots, N\}, \quad (9)$$

in which $\mathcal{P}^P$ denote the space of all polynomials of degree at most P in the standard element I_{st} .

Thus the *hp*/spectral element method for eqs. (5a-c) is given as: Find $\eta_\delta, u_\delta, w_\delta \in \mathcal{V}_\delta$ such that for all $v_\delta \in \mathcal{V}_\delta$,

$$\int_{\Omega} v_\delta \partial_t \eta_\delta dx + \int_{\Omega} v_\delta \partial_x ((h + \eta_\delta) u_\delta + w_\delta) dx = 0, \quad (10a)$$

$$\int_{\Omega} v_\delta \partial_t u_\delta dx - b h^2 \int_{\Omega} \partial_x v_\delta \partial_{xt} u_\delta dx + \int_{\Omega} v_\delta \partial_x \left(\frac{u_\delta^2}{2} + g \eta_\delta \right) dx = 0, \quad (10b)$$

$$\int_{\Omega} v_\delta w_\delta dx + a h^3 \int_{\Omega} \partial_x v_\delta \partial_x u_\delta dx = 0, \quad (10c)$$

where the boundary terms were disregarded as only Dirichlet and periodic boundary conditions are to be considered in the test cases. We write the approximate solutions η_δ, u_δ and w_δ as

$$\eta_\delta(x, t) = \sum_{e=1}^N \sum_{p=0}^P \phi_p(\xi) \hat{\eta}_p^e(t), \quad (11a)$$

$$u_\delta(x, t) = \sum_{e=1}^N \sum_{p=0}^P \phi_p(\xi) \hat{u}_p^e(t), \quad (11b)$$

$$w_\delta(x, t) = \sum_{e=1}^N \sum_{p=0}^P \phi_p(\xi) \hat{w}_p^e(t), \quad (11c)$$

in which $\hat{\eta}_p^e$, $\hat{u}_p^e$ and $\hat{w}_p^e$ are the local expansion coefficients and $\phi_p(\xi)$ are the local p -type basis functions. The basis functions are in this study chosen to be the nodal Legendre cardinal functions through the $(P + 1)$ Gauss-Legendre-Lobatto quadrature points, defined in the standard element as

$$\phi_p(\xi) = \frac{(\xi - 1)(\xi + 1)\partial_\xi L_P(\xi)}{P(P + 1)L_P(\xi_p)(\xi - \xi_p)}, \quad 0 \leq p \leq P, \quad (12)$$

where L_P is the Legendre polynomial of arbitrary order P . An example of the shapes of the basis functions and the associated nodal points are shown in Figure 1.

Figure 1

We start by considering the elemental construction. Following the standard Galerkin method we use the same Legendre cardinal functions as test functions v_δ , and define the $(P + 1) \times (P + 1)$ element matrices as

$$\mathbf{M}_{qp}^e = \int_{-1}^1 J^e \phi_q \phi_p d\xi, \quad 0 \leq q, p \leq P, \quad (13a)$$

$$\mathbf{D}_{qp}^e = \int_{-1}^1 \phi_q \partial_\xi \phi_p d\xi, \quad 0 \leq q, p \leq P, \quad (13b)$$

$$\mathbf{K}_{qp}^e = \int_{-1}^1 (J^e)^{-1} \partial_\xi \phi_q \partial_\xi \phi_p d\xi, \quad 0 \leq q, p \leq P, \quad (13c)$$

in which $J^e = \Delta x_e/2$ is the Jacobian of the mapping. We now write the semi-discrete equations for a generic element in matrix form as

$$\mathbf{M}_{qp}^e \partial_t \hat{\eta}_p^e = -\mathbf{D}_{qp}^e \left((h + \hat{\eta}_p^e) \hat{u}_p^e + \hat{w}_p^e \right), \quad (14a)$$

$$(\mathbf{M}_{qp}^e - bh^2 \mathbf{K}_{qp}^e) \partial_t \hat{u}_p^e = -\mathbf{D}_{qp}^e \left(\frac{(\hat{u}_p^e)^2}{2} + g \hat{\eta}_p^e \right), \quad (14b)$$

$$\mathbf{M}_{qp}^e \hat{w}_p^e = -ah^3 \mathbf{K}_{qp}^e \hat{u}_p^e. \quad (14c)$$

The integrals in eqs. (13a-c) are evaluated using the Gauss-Legendre-Lobatto quadrature rule with the quadrature points chosen at the $(P + 1)$ nodal locations. The under integration of the elemental mass matrix – we would need $(P + 2)$ quadrature points to evaluate the integral in eq. (13a) exactly – causes $\mathbf{M}^e$ to become diagonal due to the

Kronecker delta property (see Figure 1)

$$\mathbf{M}_{qp}^e \approx J^e \sum_{i=0}^P \rho_i \phi_q(\xi_i) \phi_p(\xi_i) = J^e \sum_{i=0}^P \rho_i \delta_{qi} \delta_{pi} = J^e \rho_q \delta_{qp}, \quad 0 \leq q, p \leq P, \quad (15)$$

in which ρ are the $(P + 1)$ Gauss-Legendre-Lobatto quadrature weights and δ denote the Kronecker delta. The diagonal local mass matrix is an important feature of the spectral element method from a computational point of view. The use of a reduced quadrature rule is equivalent to lumping (i.e. using the row sums as the diagonal entry) the mass matrix arising from exact integration. For finite elements mass lumping is a rather crude approximation, but for high-order spectral elements, however, lumping is a viable and attractive procedure. Lumping is acceptable for SEM as the error introduced by the lumped mass matrix is consistent with the approximation error of the expansion, i.e. the error introduced by the under integration is of the same order as the error of approximating a smooth function with a polynomial (Karniadakis and Sherwin, 1999) – giving that the error introduced by the reduced quadrature rule becomes insignificant for higher P . It should, nevertheless, be noted that for $P = 1$ the loss of accuracy caused by under integration is unacceptable in general and – as demonstrated in section 4 – for Boussinesq-type equations especially so. Therefore we will use $P \geq 2$ in the following test cases. The elemental weak derivative matrix $\mathbf{D}^e$ and the elemental Laplacian matrix $\mathbf{K}^e$ are both exactly integrated using the $(P + 1)$ quadrature points and are full.

The assembling procedure from local elemental matrices to a global matrix is straightforward and outlined in Figure 2, for details see Karniadakis and Sherwin (1999). The global mass matrix $\mathbf{M}$ is purely diagonal. The global weak derivative matrix $\mathbf{D}$ and the global Laplacian matrix $\mathbf{K}$ are block diagonal, resulting in a bandwidth of $(2P + 1)$.

Figure 2

We advance in time using the explicit third-order Adams-Bashforth (AB3) scheme,

giving:

$$\hat{\boldsymbol{\eta}}^{n+1} = \hat{\boldsymbol{\eta}}^n + \mathbf{M}^{-1} \frac{\Delta t}{12} (23\mathbf{E}^n - 16\mathbf{E}^{n-1} + 5\mathbf{E}^{n-2}) , \quad (16a)$$

$$\hat{\mathbf{u}}^{n+1} = \hat{\mathbf{u}}^n + (\mathbf{M} - bh^2\mathbf{K})^{-1} \frac{\Delta t}{12} (23\mathbf{F}^n - 16\mathbf{F}^{n-1} + 5\mathbf{F}^{n-2}) , \quad (16b)$$

where $\mathbf{E}$ and $\mathbf{F}$ are the right hand side of eqs. (14a) and (14b), respectively. Finally, after $\hat{\mathbf{u}}^{n+1}$ has been computed $\hat{\mathbf{w}}^{n+1}$ is obtained from eq. (14c) as:

$$\hat{\mathbf{w}}^{n+1} = -ah^3\mathbf{M}^{-1}\mathbf{K}\hat{\mathbf{u}}^{n+1} . \quad (16c)$$

3.2 FEM

As mentioned above, FEM can be regarded as a special case of hp /SEM. The derivation of the FEM model basically follows the derivation of the hp /spectral element method described in the previous section. A linear FEM model is given by eqs. (10a-c), with the major difference that the discrete space now is defined as

$$\mathcal{V}_\delta = \{v_\delta | v_\delta \in H^1, v_\delta(\chi_e^{-1}(\xi)) \in \mathcal{P}_e^1(I_{st}), e = 1, \dots, N\} , \quad (17)$$

i.e. the space consisting of at most linear polynomials ($P = 1$) in the standard element.

Thus, the order of the basis functions is kept fixed and given in I_{st} as

$$\phi_0(\xi) = \frac{1-\xi}{2}, \quad \phi_1(\xi) = \frac{1+\xi}{2} . \quad (18)$$

Furthermore, in order to avoid an unacceptable loss of accuracy, the elemental mass matrix (13a) is evaluated exactly for FEM. Hence, all elemental matrices are full and of size (2×2) . After the assembling procedure the resulting global matrices will be tri-diagonal. However, in order to speed up the computations we will use the consistent mass matrix only for eqs. (16a) and (16b). When we solve for the auxiliary variable $\hat{\mathbf{w}}$ in eq. (16c) we will resort to a lumped mass matrix, following Walkley and Berzins (1999). While previously stated that the use of a lumped mass matrix is unacceptable for linear elements, Walkley and Berzins (1999) showed that the use of a lumped mass

matrix for solving the auxiliary variable $\hat{\mathbf{w}}$ is acceptable, and it does not affect the formal second-order accuracy of the scheme. The time stepping is carried out using the AB3 scheme.

3.3 FDM

In this section we briefly describe the finite difference method of Wei and Kirby (1995), that is frequently used for Boussinesq-type equations. In this finite difference model the third-order spatial derivative is resolved directly and no auxiliary variable needs to be introduced. The first-order spatial derivatives are approximated to fourth-order by a centred five-point difference, while the higher-order derivatives are resolved using centred differences of second-order accuracy. Thus, if we divide the spatial domain into $N + 1$ nodes and let j denote the node of interest, the differences of an arbitrary variable f read:

$$\partial_x f_j \approx \frac{1}{12\Delta x} (f_{j-2} - 8f_{j-1} + 8f_{j+1} - f_{j+2}) , \quad (19a)$$

$$\partial_{xx} f_j \approx \frac{1}{\Delta x^2} (f_{j-1} - 2f_j + f_{j+1}) , \quad (19b)$$

$$\partial_{xxx} f_j \approx \frac{1}{2\Delta x^3} (-f_{j-2} + 2f_{j-1} - 2f_{j+1} + f_{j+2}) . \quad (19c)$$

Applying these differences to eqs. (4a-b) gives a scheme that is, as stated in Walkley and Berzins (1999), formally second-order accurate in space. Also here we employ the AB3 scheme to advance in time.

3.4 Remarks about the solution of the systems

In the case of Dirichlet boundary conditions, we solve the resulting matrix systems using LU factorization. For the tri-diagonal systems this technique reduces to the Thomas algorithm.

In the case of periodic boundary conditions for FEM and FDM we simply use the Sherman-Morrison formula and then use the Thomas algorithm on the resulting two

systems. For the hp /SEM model we use a technique known as static condensation to split the $(\mathbf{M} - bh^2\mathbf{K})$ matrix into components containing (elemental) boundary and interior basis functions, for details see Karniadakis and Sherwin (1999). The resulting system for the boundary components is tri-diagonal in 1D. Adjusting the matrices to account for the periodic boundary conditions, we again use the Sherman-Morrison formula in combination with the Thomas algorithm to solve the boundary nodes.

4 Truncation error analysis

It is clearly important that the numerical dispersion arising from the numerical discretization of the equations is kept considerably smaller than the physical dispersion. For example, a straightforward application of second-order finite differences to the first-order derivatives produces leading order truncation terms of the form of third-order derivatives, i.e. the same form as the physical dispersive terms present in the equations. Thus in order to keep the numerical dispersion insignificant compared to the physical dispersion, either a very dense mesh must be used or the leading errors identified and back-substituted. Wei and Kirby (1995) recognised that the back-substitution procedure could be avoided by applying fourth-order differences to the first-order derivatives. Applying the finite differences (19a-c) to eqs. (4a-b) we obtain the following local truncation errors (LTE) at a node j :

$$LTE_{\text{mass}} = -\frac{ah^3}{4}\Delta x^2\frac{\partial^5 u_j}{\partial x^5} + \mathcal{O}(\Delta x^4), \quad (20a)$$

$$LTE_{\text{momentum}} = -\frac{bh^2}{12}\Delta x^2\frac{\partial^5 u_j}{\partial x^4\partial t} + \mathcal{O}(\Delta x^4). \quad (20b)$$

We first note that the leading truncation term is of order $\mathcal{O}(\Delta x^2)$, demonstrates that the FDM scheme of Wei and Kirby (1995) is formally second-order accurate in space. Furthermore, the form of the leading truncation terms arise from the discretization of the dispersive terms and are of the form of fifth-order derivatives. As the physical

dispersion is of the form of third-order derivatives, the numerical dispersion does not significantly contaminate the physical dispersion for normal grid sizes.

As the linear FEM is second-order in space, we might suspect that it will introduce non-physical dispersion in line with a second-order finite difference approximation. However, as shown by Walkley and Berzins (1999), this is not the case. The equi-spaced FEM discretization has the following form (after substituting the mass lumped expression for w into the mass equation) at a node j :

$$\begin{aligned} \frac{1}{6}\partial_t(\eta_{j-1} + 4\eta_j + \eta_{j+1}) + \frac{1}{2\Delta x}(-r_{j-1} + r_{j+1}) \\ + \frac{1}{2\Delta x^3}(-u_{j-2} + 2u_{j-1} - 2u_{j+1} + u_{j+2}) = 0, \end{aligned} \quad (21a)$$

$$\begin{aligned} \frac{1}{6}\partial_t(u_{j-1} + 4u_j + u_{j+1}) + bh^2\frac{1}{\Delta x^2}\partial_t(u_{j-1} - 2u_j + u_{j+1}) \\ + \frac{1}{2\Delta x}(-s_{j-1} + s_{j+1}) = 0, \end{aligned} \quad (21b)$$

where $r = ((h + \eta)u)$ and $s = (0.5u^2 + g\eta)$. If we interpret eqs. (21a-b) as a “finite difference version” of the FEM model we can use Taylor series expansions in space to analyse the truncation errors. Note that although the Taylor series approach is useful for our present purpose of estimating numerical dispersion vs. physical dispersion, it is important to keep in mind that such an analysis does not provide accurate error estimates for a Galerkin method, see the discussion in Gresho and Sani (1998). Walkley and Berzins (1999) applied Taylor series expansions directly to eqs. (21a-b), while here we instead identify the finite difference discretization giving rise to eqs. (21a-b) and the truncation errors corresponding to that discretization. We find that eqs. (21a-b) can be obtained by resolving $\partial_x r$ and $\partial_x s$ using fourth-order compact differences, whereas the higher derivatives are approximated by centred as well as near-centred second-order differences. Thus, the corresponding truncation errors are:

$$LTE_{\text{mass}} = -\frac{ah^3}{12}\Delta x^2\frac{\partial^5 u_j}{\partial x^5} + \mathcal{O}(\Delta x^4), \quad (22a)$$

$$LTE_{\text{momentum}} = \frac{2bh^2}{3}\Delta x^2\frac{\partial^5 u_j}{\partial x^4\partial t} + \mathcal{O}(\Delta x^4). \quad (22b)$$

Here we again see that, as the leading truncation terms are fifth-order derivative terms, the leading error will not pollute the physical dispersion. It can also be seen that the use of a lumped mass matrix for solving the auxiliary variable w does not affect the formal second-order accuracy of the scheme.

For the hp /SEM model using $P = 1$ we obtain the following scheme (after substituting the mass lumped expression for w into the mass equation) at a node j

$$\partial_t \eta_j + \frac{1}{2\Delta x} (-r_{j-1} + r_{j+1}) + ah^3 \frac{1}{2\Delta x^3} (-u_{j-2} + 2u_{j-1} - 2u_{j+1} + u_{j+2}) = 0, \quad (23a)$$

$$\partial_t u_j + bh^2 \frac{1}{\Delta x^2} \partial_t (u_{j-1} - 2u_j + u_{j+1}) + \frac{1}{2\Delta x} (-s_{j-1} + s_{j+1}) = 0. \quad (23b)$$

From eqs. (23a-b) we see that for $P = 1$ the use of a reduced quadrature rule gives a scheme resembling a finite difference scheme using second-order centred differences for all spatial derivatives. Thus the truncation errors of (23a-b) read

$$LTE_{\text{mass}} = -\frac{1}{12}\Delta x^2 \left(2\frac{\partial^3 r_j}{\partial x^3} + ah^3 3\frac{\partial^5 u_j}{\partial x^5} \right) + \mathcal{O}(\Delta x^4), \quad (24a)$$

$$LTE_{\text{momentum}} = -\frac{1}{12}\Delta x^2 \left(2\frac{\partial^3 s_j}{\partial x^3} + bh^2 \frac{\partial^5 u_j}{\partial x^4 \partial t} \right) + \mathcal{O}(\Delta x^4). \quad (24b)$$

As the leading error terms contain third-order derivatives, the numerical dispersion will pollute the physical dispersion unless the model is run with a dense mesh. As already mentioned, the use of a under integrated hp /SEM model with linear basis functions is strongly discouraged. However examining the $P = 2$ scheme we get different expressions for the elemental boundary nodes and for the interior nodes (as evident from Figure 2). For $P = 2$ the Gauss-Legendre-Lobatto points are located at $\xi = (-1, 0, 1)$, giving a distance of $\Delta x/2$ between the nodes. The expressions at an elemental boundary node j

read:

$$\begin{aligned} \partial_t \eta_j + \frac{1}{2\Delta x} (r_{j-1} - 4r_{j-1/2} + 4r_{j+1/2} - r_{j+1}) \\ + \frac{1}{2\Delta x} (w_{j-1} - 4w_{j-1/2} + 4w_{j+1/2} - w_{j+1}) = 0, \end{aligned} \quad (25a)$$

$$\begin{aligned} \partial_t u_j - \frac{bh^2}{\Delta x^2} \partial_t (u_{j-1} - 8u_{j-1/2} + 14u_j - 8u_{j+1/2} + u_{j+1}) \\ + \frac{1}{2\Delta x} (s_{j-1} - 4s_{j-1/2} + 4s_{j+1/2} - s_{j+1}) = 0, \end{aligned} \quad (25b)$$

$$w_j + \frac{ah^3}{\Delta x^2} (u_{j-1} - 8u_{j-1/2} + 14u_j - 8u_{j+1/2} + u_{j+1}) = 0, \quad (25c)$$

and for an internal node $j + 1/2$:

$$\partial_t \eta_{j+1/2} + \frac{1}{\Delta x} (-r_j + r_{j+1}) + \frac{1}{\Delta x} (-w_j + w_{j+1}) = 0, \quad (26a)$$

$$\partial_t u_{j+1/2} - \frac{bh^2}{\Delta x^2} \partial_t (-4u_j + 8u_{j+1/2} - 4u_{j+1}) + \frac{1}{\Delta x} (-s_j + s_{j+1}) = 0, \quad (26b)$$

$$w_{j+1/2} + \frac{ah^3}{\Delta x^2} (-4u_j + 8u_{j+1/2} - 4u_{j+1}) = 0. \quad (26c)$$

The schemes at the interior nodes (26a-c) appear to give rise to truncation terms of the form of third-order derivatives, but – as the internal nodes are coupled to the elemental boundary nodes – the appearance is misleading. Adding the relations for nodes $j + 1/2$ and $j - 1/2$, given by eq. (26a), to the expression at node j (25a) we obtain – after rearranging and substitution of the w terms using eqs. (25c) and (26c):

$$\begin{aligned} \frac{1}{3} \partial_t (\eta_{j-1/2} + \eta_j + \eta_{j+1/2}) + \frac{1}{6\Delta x} (-r_{j-1} - 4r_{j-1/2} + 4r_{j+1/2} + r_{j+1}) \\ + \frac{ah^3}{6\Delta x^3} (u_{j-2} - 8u_{j-3/2} - 2u_{j-1} + 24u_{j-1/2} \\ - 24u_{j+1/2} + 2u_{j+1} + 8u_{j+3/2} - u_{j+2}) = 0. \end{aligned} \quad (27)$$

An analogous approach for the momentum equation gives the following expression

$$\begin{aligned} \frac{1}{3} \partial_t (u_{j-1/2} + u_j + u_{j+1/2}) - \frac{bh^2}{\Delta x^2} \partial_t (-u_{j-1} + 2u_j - u_{j+1}) \\ + \frac{1}{6\Delta x} (-s_{j-1} - 4s_{j-1/2} + 4s_{j+1/2} + s_{j+1}) = 0. \end{aligned} \quad (28)$$

Equations (27) and (28) can be obtained by first using a second-order compact difference to resolve one spatial derivative of the dispersive terms. Then the $\partial_x r$ and $\partial_x s$

terms are resolved using centred and near-centred fourth-order differences. The last spatial derivative of the $\partial_{xxt}u$ term is approximated by centred second-order differences, while the remaining two derivatives of the $\partial_{xxx}u$ term are resolved in sequence by (i) near-centred fourth-order approximation and (ii) centred second-order differences. The corresponding truncation errors read

$$LTE_{\text{mass}} = -\frac{ah^3}{12}\Delta x^2\frac{\partial^5 u_j}{\partial x^5} + \mathcal{O}(\Delta x^4), \quad (29a)$$

$$LTE_{\text{momentum}} = -\frac{bh^2}{12}\Delta x^2\frac{\partial^5 u_j}{\partial x^4\partial t} + \mathcal{O}(\Delta x^4). \quad (29b)$$

We see that for $P = 2$ the leading truncation errors are of order $\mathcal{O}(\Delta x^2)$ and are in the form of fifth-order derivatives. As the truncation error using higher P will be less than those stated in eqs. (29a-b), the numerical dispersion will be small compared to the physical dispersion for the hp /SEM model if $P \geq 2$.

Finally, as the leading term of the local truncation error of the third-order AB3 scheme contains a fourth-order derivative

$$f^{n+1} - f(t_{n+1}) = -\frac{27}{72}\Delta t^4\frac{\partial^4 f^n}{\partial t^4} + \mathcal{O}(\Delta t^5), \quad (30)$$

the AB3 scheme is of sufficiently high order for the Boussinesq-type equations and also the numerical dispersion caused by the temporal discretization will be small compared to the physical dispersion.

5 Stability considerations

The potential Achilles' heel of hp /SEM for equations involving second and higher spatial derivatives is that, using explicit time stepping schemes, the restriction on the time step is more severe than for low-order methods. Generally the time step needs to be of the order $\mathcal{O}(P^{-2\gamma})$, where γ is the order of the highest spatial derivative, in order to obtain a stable model. Thus, the third order derivative present in the Nwogu equations suggests

a time step of order $\mathcal{O}(P^{-6})$, which effectively would render the hp /SEM model useless for higher P .

The implicit treatment of the ∂_{xxt} term, resulting in an Helmholtz operator working on the time derivative in the momentum equations, forces the dominating time step restriction to arise from the first order derivation term. Considering the linearized eqs. (4a-b):

$$\partial_t \eta + h \partial_x u + ah^3 \partial_{xxx} u = 0, \quad (31a)$$

$$(1 + bh^2 \partial_{xx}) \partial_t u + g \partial_x \eta = 0, \quad (31b)$$

we write the semi-discrete equations as

$$\mathbf{A} \partial_t \hat{\mathbf{U}} + \mathbf{B} \hat{\mathbf{U}} = \mathbf{0}. \quad (32)$$

In which $\hat{\mathbf{U}} = [\hat{\boldsymbol{\eta}}, \hat{\mathbf{u}}]^T$ and, for the Galerkin models,

$$\mathbf{A} = \begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} - bh^2 \mathbf{K} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{0} & h\mathbf{D} - ah^3 \mathbf{D} (\mathbf{M}^{-1} \mathbf{K}) \\ g\mathbf{D} & \mathbf{0} \end{bmatrix}. \quad (33)$$

The eigenvalues of the semi-discrete operator $\mathbf{A}^{-1} \mathbf{B}$ are purely imaginary and in Figure 3 we plot the maximum eigenvalues $|\lambda|$ versus P for the hp /SEM model. We see that the growth of the maximum eigenvalue is $\mathcal{O}(P^2)$, and not $\mathcal{O}(P^6)$. Thus, the restriction on the time step is $\mathcal{O}(P^{-2})$, which is of the same order as the non-dispersive shallow water equations. It is noteworthy that for the classical Boussinesq equations of Peregrine, the implicit treatment of the ∂_{xxt} term makes the restriction on the time step independent of the polynomial order.

Figure 3

As the eigenvalues are purely imaginary, the AB3 scheme is stable if the following criterion is satisfied:

$$|\lambda| \Delta t \leq 0.723. \quad (34)$$

The eigenvalues are found from von Neumann analyses and read:

$$hp/\text{SEM: } \lambda = \pm \frac{\sqrt{gh}}{\Delta x} \sqrt{\frac{\sin^2 \theta (-1 + 2a\nu^2 - 2a\nu^2 \cos \theta)}{1 - 2b\nu^2 + 2b\nu^2 \cos \theta}} P^2, \quad (35a)$$

$$\text{FEM: } \lambda = \pm \frac{3\sqrt{gh}}{\Delta x} \sqrt{\frac{\sin^2 \theta (-1 + 2a\nu^2 - 2a\nu^2 \cos \theta)}{(2 + \cos \theta)(2 - 6b\nu^2 + \cos \theta + 6b\nu^2 \cos \theta)}}, \quad (35b)$$

$$\text{FDM: } \lambda = \pm \frac{\sqrt{gh}}{3\Delta x} \sqrt{\frac{\sin^2 \theta (-4 + \cos \theta)(4 - 6a\nu^2 - \cos \theta + 6a\nu^2 \cos \theta)}{1 - 2b\nu^2 + 2b\nu^2 \cos \theta}}, \quad (35c)$$

in which $\theta = k\Delta x$ and $\nu = h/\Delta x$.

6 Computational examples

6.1 Accuracy

We compute the linearized Nwogu equations for a constant depth, (31a-b), in $x \in [0, l]$ and $t \in [0, T]$, subject to periodic boundary conditions. T is the wave period. The time step is chosen sufficiently small so temporal error can be disregarded. The domain is uniformly discretized and the initial conditions are: $\eta(x, 0) = \eta_0 \sin(kx)$, $u(x, 0) = c\eta(x, 0)$ and $w(x, 0) = -ah^3k^2c\eta(x, 0)$, where η_0 is the amplitude, $c = \omega/(kh(1 - a(kh)^2))$ and ω is the frequency obtained from the linear dispersion relation (Nwogu, 1993):

$$\frac{\omega^2}{ghk^2} = \frac{1 - a(kh)^2}{1 - b(kh)^2}. \quad (36)$$

In order to obtain a μ of 0.5 we set $h = 0.56$, $l = 1.12$, $T = 0.84$ and $\eta_0 = 0.025$. The exact solution is simply $\eta(x, t) = \eta_0 \sin(kx - \omega t)$.

The performance of the models are measured in the L^∞ norm. The L^∞ error (normalized with the amplitude η_0), computed at the N_{dof} nodal points, is defined as

$$\|\eta - \hat{\eta}\|_{L^\infty} = \frac{\max_j |\eta_j - \hat{\eta}_j|}{\eta_0}, \quad \text{for } j = 1, \dots, N_{\text{dof}}. \quad (37)$$

Table 1 shows the L^∞ error and the associated numerical convergence when decreasing the mesh size Δx (i.e. h -type refinement). We see that the FEM and FDM models

exhibit a convergence of $\mathcal{O}(\Delta x^2)$, while the hp /SEM model shows a convergence of $\mathcal{O}(\Delta x^P)$. As the convergence of the hp /SEM model is a function of the polynomial order, we will obtain much better accuracy – for the same number of degrees of freedom – by increasing P instead of decreasing Δx (i.e. p -type refinement). In Figure 4 we plot the error versus the total degree of freedom on a lin-log scale and from the approximate straight line is demonstrated that the hp /SEM model exhibits exponential convergence if p -type refinement is performed (we use $N = 4$ with $P = 2, 4, 6, 8$ and 10).

Figure 4

6.2 CPU time versus integration time

We now compute the complete Nwogu equations for a constant depth, (4a-b), in $x \in [0, 10l]$, subject to periodic boundary conditions. The domain is uniformly discretized and we use the same initial conditions as above.

In Table 2 we present the CPU time required to advance one time step using a 2 GHz Pentium IV processor. We see that the computational effort for the models to advance one time step is: FDM $\sim C N$; FEM $\sim 2 C N$ and hp /SEM $\sim C N P^2$. However, as the models have different stability requirements as well as different convergence rates, it is more interesting to examine the computational cost of the methods for obtaining a given accuracy.

Table 2

We compute the CPU time for obtaining engineering accuracy, set to a fixed L^∞ error of 0.1 and 0.01, for different integration times of 10, 50 and 100 wave periods. As there does not exist any exact solution for this case, the L^∞ error is found by approximating the exact solutions with numerically obtained reference solutions. The reference solutions are given by the hp /SEM model run with significantly higher order and a very small time step.

From Figure 5 we see that the hp /SEM model is the most cost effective model and that the saving in CPU time increases with longer integration time and higher accuracy.

The FDM performs as good as the hp /SEM model for shorter integration times, while the FEM model is by far the most expensive of the models. Note that in constructing Figure 5 we have not tried to find the optimal hp -mesh for the hp /SEM model, but rather we illustrate the computational time for running the hp /SEM model using 10 elements with variable P and Δt .

Figure 5

6.3 CPU time versus accuracy

We consider solitary waves of $\varepsilon = 0.1$ and 0.3 propagating over a undisturbed depth of 1.0 m. The solitary waves are initially centered at $x = 50$ m and the spatial domain, $x \in [0, 300]$, is uniformly discretized. The domain is extended and so we can set all boundary conditions to zero without introducing any significant errors. The initial condition is given by the approximate analytical solitary wave solution of Wei and Kirby (1995). As the initial condition is non-exact, it sometimes creates numerical oscillations that causes instabilities to form at the boundaries. In order to avoid this we apply simple sponge layers (Larsen and Dancy, 1983) of 10 m width at the boundaries.

Figure 6

Figure 6 shows comparison between the simulated free surface elevation and the approximate analytical solutions after 50 s. The simulations were carried out using the hp /SEM model with $P = 5$, $N = 300$ and a time step of 1×10^{-3} . As discussed in Wei and Kirby (1995), the discrepancy between the numerical solution and the approximate analytical solution increases with increasing nonlinearity. We see in Figure 6b that the numerical solution gives a higher amplitude as well as a slower phase velocity than the analytical solution. Also illustrated in Figure 6 are the small amplitude wave trains – here called dispersive tails – developed behind the solitary waves. Note that the dispersive tails are not due to under resolution in time and/or space, but are a consequence of the non-exact initial conditions. If the simulation is run for a sufficiently long time, the dispersive tail is left behind and new a stable solitary wave shape will have evolved

(Wei and Kirby, 1995). The dispersive tails are hard to resolve properly and for the case of $\varepsilon = 0.3$ under resolution has triggered (very) small scale numerical oscillations, see Figure 6b.

As clearly seen from Figure 6 we can not use the analytical solution when computing the L^∞ error, due to the inconsistency between the analytical and numerical solution. Thus, the error is again computed by approximating the exact solutions with numerically obtained reference solutions.

In Figure 7 we have plotted the envelope curves of CPU time vs L^∞ error for the different methods using the solitary wave cases (for FDM and FEM we have used $N = 1\,200, 4\,000, 12\,000, 40\,000$ and $120\,000$, while for hp /SEM we have used $N = 300$ with variable P). We see that FDM is most cost effective when a relatively large error is permitted, while hp /SEM is preferable when high accuracy is required. The break even point, for these specific solitary wave cases, occurs around an L^∞ error of 1×10^{-4} . However, it should again be noted that the we have not attempted to obtain an optimal hp -mesh.

Figure 7

6.4 Comparison to experimental data

Finally, we compare simulations of wave transformation over a submerged bar with experimental data (Dingemans, 1994). This test illustrates the ability of the equations to predict generation of higher harmonics. The numerical wave tank is 45 m long with an undisturbed depth of 0.4 m upstream and downstream the submerged trapezoidal bar. The bar has an upward slope of 1:20 and a downward slope of 1:10, see Figure 8. Table 3 lists the location of the 8 wave gauges marked in Figure 8. We apply sponge layers at the boundaries in order to prevent reflection back into the domain while the waves are generated at $x = 0$ m using a source function method (Wei *et al.*, 1999).

Figure 8
Table 3

The generated waves are sinusoidal in shape with a wavelength of 3.73 m, a wave

period of 2.02 s and an amplitude of 0.01 m. We simulate up to $t = 40$ s using the hp /SEM model ($P = 5$) with a uniform element size of 0.2 m and time step of 0.01 s.

Figure 9 shows the simulated surface elevation compared to experimental data. The model is seen to perform satisfactorily before and on the top of the bar while the disagreement between computed and measured is quite pronounced after the bar. We stress that the disagreement is not caused by numerical errors but is due to limitations of the employed equations. The discrepancy is explained by the release of bound harmonics as free harmonics on the top and downward slope of the bar. As the Nwogu equations capture dispersion effects correctly up to $\mu \approx 0.5$, the second harmonic is correctly computed after the bar while the third and higher harmonics are not captured accurately. In order to improve the performance after the bar a more accurate set of Boussinesq-type equations is required (Madsen *et al.*, 1996; Gobbi and Kirby, 1999).

Figure 9

7 Summary and concluding remarks

It has been shown that the hp /spectral element method applied to the enhanced Boussinesq-type equations of Nwogu (1993) produces a numerical scheme that accurately and efficiently models the propagation and evolution of weakly nonlinear dispersive waves. The hp /SEM model was seen to exhibit exponential convergence and, thus, produced highly accurate results even on coarse meshes. The implicit treatment of the ∂_{xxt} terms gives a restriction of the time step of $\mathcal{O}(P^{-2})$, which is acceptable for higher P . It should be noted that even though the restriction on the time step appears to be more severe for the hp /SEM model than for the low-order models, in practice this effect is mitigated by the gain in accuracy with P , allowing us to use a coarser mesh.

Numerical computations of solitary waves showed that hp /SEM gives significant savings in CPU time, compared to the other models, if highly accurate results are requested, see Figure 7. For lower accuracy the FDM model was found to be most cost effective. A

more important result for practical engineering purposes is that we also see significant savings in CPU time when results of engineering accuracy are called for in combination with long-time integration. This was illustrated using numerical computations of regular dispersive waves (see Figure 5). Thus we can conclude that the *hp*/spectral element method is more computationally efficient for simulating the Boussinesq-type equations of Nwogu in the limit of high accuracy and long-time integration.

The diagonal mass matrix of the present *hp*/SEM model – due to the use of a reduced quadrature rule – plays a significant part in making the model computationally attractive. Nevertheless, it should be noted that the conclusion above that *hp*/SEM is more computationally efficient than the two low-order methods in the limit of high accuracy or long-time integration is not dependent on a diagonal mass matrix, but is to be attributed to the higher formal accuracy of the *hp*/SEM scheme when using high P . In the context of the present *hp*/SEM model an exactly evaluated mass matrix – resulting in a global mass matrix with a bandwidth of $(2P + 1)$ and a CPU per time step $\sim 3 CNP^2$ – of course would shift the break-even point in favour of the low-order methods, but the general conclusion still would be valid.

Finally, it should be mentioned that the savings in CPU time most likely are to be even more pronounced in 2D, as the *hp*/spectral element method usually becomes increasingly effective, compared to low-order methods, as the spatial dimension increases (Deville *et al.*, 2002). The development of an *hp*/SEM model for solving 2D Boussinesq-type equations on unstructured triangular meshes is ongoing work.

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Table 1: L^∞ error and order of convergence for the η component.

	$N = 10$		$N = 20$		$N = 40$	
Method	Error	Error	Order	Error	Order	
FDM	5.07E-02	8.47E-03	2.58	1.65E-03	2.36	
FEM	1.24E-01	2.98E-02	2.05	7.20E-03	2.05	
hp /SEM ($P = 2$)	1.75E-02	4.23E-03	2.05	1.04E-03	2.03	
hp /SEM ($P = 3$)	5.93E-04	4.86E-05	3.61	3.85E-06	3.66	
hp /SEM ($P = 4$)	2.24E-05	1.33E-06	4.07	8.20E-08	4.02	
hp /SEM ($P = 5$)	6.15E-07	1.06E-08	5.86	2.19E-10	5.60	

Table 2: CPU time to advance one time step ($\times 10^{-3}$ s) .

Method	$N = 600$	$N = 900$	$N = 1200$
FDM	0.26	0.38	0.51
FEM	0.56	0.81	1.15
hp /SEM ($P = 2$)	1.16	1.76	2.37
hp /SEM ($P = 3$)	2.21	3.34	4.50
hp /SEM ($P = 4$)	3.59	5.38	7.24
hp /SEM ($P = 5$)	5.23	7.89	10.61

Table 3: Location of wave gauges.

	Wave gauge no.							
	1	2	3	4	5	6	7	8
x -coordinate [m]	4.0	10.5	13.5	14.5	15.7	17.3	19.0	21.0

Figure 1: The Legendre cardinal functions of order $P = 5$ and the associated Gauss-Legendre-Lobatto nodal points.

Figure 2: Outline of the assembly of a global matrix. Three elements of order $P = 2$.

Figure 3: The growth of the maximum eigenvalue as a function of P .

Figure 4: Convergence in the L^∞ norm. For the hp /SEM model we use p -type refinement, while the FDM and FEM models are h -refined.

Figure 5: CPU time versus integration time for a fixed L^∞ error of (a) 0.1 and (b) 0.01.

Figure 6: Comparison between numerical (dots) and approximate analytical (solid line) solitary wave solutions after 50 s. (a) $\varepsilon = 0.1$; (b) $\varepsilon = 0.3$.

Figure 7: CPU time versus L^∞ error at $t = 50$ s. (a) $\varepsilon = 0.1$; (b) $\varepsilon = 0.3$.

Figure 8: Numerical wave tank.

Figure 9: Waves propagating over a submerged bar. Measured (solid) and computed (dashed) surface elevation.