Hierarchical hp Finite Elements in Hybrid Domains

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In this paper we introduce a new set of hp finite element expansions for use in hybrid domains using hexahedrons, prisms, pyramids and tetrahedrons. The expansions are explained in terms of a unified notation which incorporates the standard hexahedral hp finite element expansion. The new bases are motivated from a set of orthogonal polynomial expansions within each of the hybrid domains. The polynomial expansions are a *generalised* product of functions based upon a local co-ordinate system with independent limits. This construction leads to expansions with attractive computational properties and which exhibit a high level of orthogonality.

1 Introduction

Hybrid, or mixed, domain discretisation using a combination of structured and unstructured regions is a versatile way of dealing with complicated three-dimensional problems [1,2]. Such an approach combines the simplicity and convenience of structured domains with the geometric flexibility of an unstructured discretisation. In two-dimensions, hybrid discretisation simply implies the use of triangular and rectangular domains. In three-dimensions, however, the hybrid domain strategy is more complex requiring the use of hexahedrons, prisms, pyramids and tetrahedrons.

Hexahedral domains have been used quite extensively in the hp finite element field [3,4]. More recently an unstructured hp finite element approach, based upon theoretical work by Dubiner [5], has been developed for unsteady problems in fluid dynamics [6,7]. In this paper we shall show how these expansions can be constructed using a unified approach which incorporates all the hybrid domains.

This unified approach generates polynomial expansions which can be expressed in terms of a *generalised* product of the form $\phi_{pqr}(x, y, z) = \phi_p^a(x)\phi_{pq}^b(y)\phi_{pqr}^c(z)$.

Here we have used the Cartesian co-ordinates x, y and z but in general they can be any of set co-ordinates defining a specified region. The standard tensor product is simply a degenerate case of this product where the second and third functions are only dependent on one index. The primary motivation in developing an expansion of this form is computational efficiency. Finite element implementation frequently requires the evaluation of operations of the form:

$$I_{pqr} = \sum_i \sum_j \sum_k f_{pi}^a f_{pqj}^b f_{pqrk}^c u_{ijk}. \quad (1)$$

For example, if $f_{pi}^a = \phi_p^b(x_i)$, $f_{pqj}^b = \phi_{pq}^b(y_j)$, $f_{pqrk}^c = \phi_{pqr}^c(z_k)$ and $u_{ijk} = u(x_i, y_j, z_k)$ then equation (1) would represent the discrete inner product of $u(x, y, z)$ with respect to all the expansion modes $\phi_{pqr}(x, y, z)$. If we consider all indices as being $O(N)$ then the evaluation of equation (1) for every p, q and r index is at worst an operation of $O(N^6)$. By making use of the generalised product form, however, some terms can be factored out of the summation to give:

$$I_{pqr} = \sum_k f_{pqrk}^c \left\{ \sum_j f_{pqj}^b \left\{ \sum_i f_{pi}^a u_{ijk} \right\} \right\}.$$

This expression can be evaluated in an $O(N^4)$ operation by first performing the summation in the innermost brackets followed by that in the outer brackets and finally performing the summation over k . This technique is known as sum factorisation and can provide large computational savings.

A further feature of these new bases, which is beyond the scope of this paper, is the scaling of the discrete advection operator with polynomial order. This is important in relation to unsteady fluid problems as the scaling is related to the time step restriction, see [7] for further discussion.

The structure of this paper is as follows: In section 2 we introduce a local co-ordinate system for each hybrid domain from which we can construct the expansion bases. In section 3, we develop orthogonal polynomial expansions for all the domains which demonstrate the generalised product form. In section 4 we modify the orthogonal expansions to develop bases suitable for use as a C^0 continuous global finite element expansion. Finally, in section 5 we demonstrate the convergence properties of the expansions in prismatic and pyramidal domains.

2 Local Co-ordinate Systems

We start by defining a convenient set of local co-ordinates from which we can construct the expansions. Moving away from the use of barycentric, or volume, co-ordinates, which are typically applied to unstructured domains, we define a set of *collapsed Cartesian* co-ordinates in non-rectangular domains. These co-ordinates form the foundation of the polynomial expansions. The advantage of this system is that every the co-ordinate in all the hybrid domains vary between independent constant limits. Therefore, operations such as integration and differentiation can be performed using standard one-dimensional techniques.

Fig. 1. Triangle to rectangle transformation

The new co-ordinate systems are based upon the transformation of a triangular region to a rectangular domain (and vice versa) as shown in figure 1. The principal effect of the transformation is to map the vertical lines in the rectangular domain (i.e. lines of constant η_1) onto lines radiating out of the point $(\xi_1 = -1, \xi_2 = 1)$ in the triangular domain. The triangular region can now be described using the “ray” co-ordinate (η_1) and the standard horizontal co-ordinate ($\xi_2 = \eta_2$). The triangular domain is therefore defined by $(-1 \leq \eta_1, \eta_2 \leq 1)$ rather than the Cartesian description $(-1 \leq \xi_1, \xi_2; \xi_1 + \xi_2 \leq 0)$ where the upper bound couples the two co-ordinates. The “ray” co-ordinate (η_1) is multi-valued at $(\xi_1 = -1, \xi_2 = 1)$. Whilst such a singularity is not usually seen as favourable, we note that both cylindrical and spherical co-ordinate systems are singular.

Fig. 2. Hexahedron to tetrahedron transformation

As illustrated in figure 2, the same transformation can repeatedly be applied to generate new co-ordinate systems in three-dimensions. Here we start from the bi-unit hexahedral domain and apply the triangle to rectangle transformation in the vertical plane to generate a prismatic region. The transformation is then used in the second vertical plane to generate the pyramidic region. Finally the rectangle to triangle transformation is applied to every square cross section parallel to the base of the pyramidic region to arrive at the tetrahedral domain.

By determining the hexahedral co-ordinates (η_1, η_2, η_3) in terms of the Cartesian co-ordinates of the tetrahedral region (ξ_1, ξ_2, ξ_3) we can generate a new co-ordinate system for the tetrahedron. This new system and the planes described by fixing the local co-ordinates are shown in figure 3. Also shown are the new systems for the intermediate domains which are generated in the same fashion. Here we have assumed that the local Cartesian co-ordinates for every domain are (ξ_1, ξ_2, ξ_3) .

Fig. 3. Illustration of the local coordinate systems used in each of the hybrid domains and the planes described by fixing each local co-ordinate.

3 Orthogonal Expansions

For each of the hybrid domains we can develop a polynomial expansion based upon the local co-ordinate system derived in section 2. These expansions will be polynomials in terms of the local co-ordinates as well as the Cartesian co-ordinates (ξ_1, ξ_2, ξ_3) . This is a significant property as primary operations such as integration and differentiation can be performed with respect to the local co-ordinates but the expansion may still be considered as a polynomial expansion in terms of the Cartesian system.

We shall initially consider expansions which are orthogonal in the Legendre inner product. We define three principal functions $\phi_i^a(z)$, $\phi_{ij}^b(z)$ and $\phi_{ijk}^c(z)$, in terms of the Jacobi polynomial, $P_p^{\alpha, \beta}(z)$, as:

$$\phi_i^a(z) = P_i^{0,0}(z), \quad \phi_{ij}^b(z) = \left(\frac{1-z}{2}\right)^i P_j^{2i+1,0}(z),$$

$$\phi_{ijk}^c(z) = \left(\frac{1-z}{2}\right)^{i+j} P_k^{2i+2j+2,0}(z).$$

Using these functions we can construct the orthogonal polynomial expansions:

Hexahedral expansion:

Prismatic expansion:

$$\phi_{pqr}(\xi_1, \xi_2, \xi_3) = \phi_p^a(\xi_1) \phi_q^a(\xi_2) \phi_r^a(\xi_3) \quad \phi_{pqr}(\xi_1, \xi_2, \xi_3) = \phi_p^a(\xi_1) \phi_q^a(\eta_2) \phi_{qr}^b(\xi_3)$$

Pyramidal expansion:

Tetrahedral expansion:

$$\phi_{pqr}(\xi_1, \xi_2, \xi_3) = \phi_p^a(\overline{\eta}_1) \phi_q^a(\eta_2) \phi_{pqr}^c(\eta_3) \quad \phi_{pqr}(\xi_1, \xi_2, \xi_3) = \phi_p^a(\eta_1) \phi_{pq}^b(\eta_2) \phi_{pqr}^c(\eta_3)$$

where,

$$\eta_1 = \frac{2(1 + \xi_1)}{(-\xi_2 - \xi_3)} - 1, \quad \overline{\eta}_1 = \frac{2(1 + \xi_1)}{(1 - \xi_3)} - 1, \quad \eta_2 = \frac{2(1 + \xi_2)}{(1 - \xi_3)} - 1, \quad \eta_3 = \xi_3,$$

are the local co-ordinates illustrated in figure 3.

The hexahedral expansion is simply a standard tensor product of Legendre polynomials (since $P_p^{0,0}(z) = L_p(z)$). In the other expansions the introduction of the degenerate local co-ordinate systems is linked to the use of the more unusual functions $\phi_{ij}^b(z)$ and $\phi_{ijk}^c(z)$. These functions both contain factors of the form $\left(\frac{1-z}{2}\right)^n$ which is necessary to keep the expansion as a polynomial of the Cartesian co-ordinates (ξ_1, ξ_2, ξ_3) . For example, the co-ordinate η_2 in the prismatic expansion necessitates the use of the function $\phi_{qr}^b(\xi_3)$ which introduces a factor of $\left(\frac{1-\xi_3}{2}\right)^q$. The product of this factor with $\phi_q^a(\eta_2)$ is a polynomial function in ξ_2 and ξ_3 . Since the remaining part of the prismatic expansion, $\phi_p^a(\xi_1)$, is already in terms of a Cartesian co-ordinate the whole expansion is a polynomial in terms of the Cartesian system.

The polynomial space, in Cartesian co-ordinates, for each expansion is:

$$\mathcal{P} = \text{Span}\{\xi_1^p \xi_2^q \xi_3^r\} \quad (2)$$

where pqr for each domain is

$$\begin{array}{llll} \text{Hexahedron} & 0 \leq p \leq P_1 & 0 \leq q \leq P_2 & 0 \leq r \leq P_3 \\ \text{Prism} & 0 \leq p \leq P_1 & 0 \leq q \leq P_2 & 0 \leq q + r \leq P_3 \\ \text{Pyramidal} & 0 \leq p \leq P_1 & 0 \leq q \leq P_2 & 0 \leq p + q + r \leq P_3 \\ \text{Tetrahedron} & 0 \leq p \leq P_1 & 0 \leq p + q \leq P_2 & 0 \leq p + q + r \leq P_3. \end{array} \quad (3)$$

The range of the p, q and r indices indicate how the expansions can be ex-

panded to generate a complete polynomial space. We note that if $P_1 = P_2 = P_3$ then the tetrahedral and pyramidic expansions span the same space and are in a subspace of the prismatic expansion which is in turn a subspace of the hexahedral expansion.

3.1 Orthogonality of the pyramidic expansion

To gain an insight into the use of the Jacobi polynomial $P^{\alpha,0}(z)$ in the functions $\phi_{pq}^b(z)$ and $\phi_{pqr}^c(z)$ we consider the orthogonality of the pyramidic expansion in the Legendre inner product, i.e.

$$I = \int \phi_{pqr} \phi_{uvw} d(\xi_1, \xi_2, \xi_3) = \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \phi_p^a \phi_q^a \phi_{pqr}^c \cdot \phi_u^a \phi_v^a \phi_{uvw}^c J d\overline{\eta_1} d\eta_2 d\eta_3$$

where

$$J = \frac{\partial(\xi_1, \xi_2, \xi_3)}{\partial(\overline{\eta_1}, \eta_2, \eta_3)} = \left(\frac{1 - \eta_3}{2} \right)^2.$$

As the expansion is a product of functions based on the local co-ordinates and the Jacobian is only a function of η_3 , the integral can be expressed in terms of three one-dimensional integrals:

$$I = \int_{-1}^1 \phi_p^a \phi_u^a d\overline{\eta_1} \int_{-1}^1 \phi_q^a \phi_v^a d\eta_2 \int_{-1}^1 \left(\frac{1 - \eta_3}{2} \right)^2 \phi_{pqr}^c \phi_{uvw}^c d\eta_3. \quad (4)$$

The first two products contain the standard Legendre polynomials ($\phi_p^a(z) = P_p^{0,0}(z) = L_p(z)$) which are orthogonal in the interval $[-1, 1]$. The last integral can be written in full as

$$\int_{-1}^1 \left(\frac{1 - \eta_3}{2} \right)^2 \left(\frac{1 - \eta_3}{2} \right)^{p+q} P_r^{2p+2q+2,0}(\eta_3) \cdot \left(\frac{1 - \eta_3}{2} \right)^{u+v} P_w^{2u+2v+2,0}(\eta_3) d\eta_3.$$

The Jacobi polynomial $P_r^{2p+2q+2,0}(\eta_3)$ is orthogonal in this interval with respect to the weight function $\left(\frac{1 - \eta_3}{2} \right)^{(2p+2q+2)}$ and therefore if $p = u$ and $q = v$ the third integral is orthogonal. However if $p \neq u$ or $q \neq v$ the first or second integral in equation (4) is necessarily zero and therefore the expansion ϕ_{pqr} is orthogonal to ϕ_{uvw} .

4 C^0 Continuous Modified Expansions

The orthogonality of the previous expansions is attractive, however, we also need a way to impose the boundary conditions and tessellate the domains whilst maintaining some continuity in the global expansion.

To enforce C^0 continuity the orthogonal expansion is modified by decomposing the expansion into an interior and boundary contribution. The interior modes (or shape functions) are defined to be zero on the boundary of the local domain. The completeness of the expansion is then ensured by adding boundary modes which consist of vertex, edge and face contributions. The vertex modes have unit value at one vertex and decay to zero at all other vertices, edge modes have local support along one edge and are zero on all other edges and vertices and face modes have local support on one face and are zero on all other faces, edges and vertices. Using this decomposition C^0 continuity between elements can then be enforced by matching similar shaped boundary modes, see [7].

As for the orthogonal case we define three principal functions $\phi_i^a(z)$, $\phi_{ij}^b(z)$ and $\phi_{ijk}^c(z)$ ($0 \leq i \leq I$, $0 \leq j \leq J$, $0 \leq k \leq K$):

$$\phi_i^a(z) = \begin{cases} \left(\frac{1-z}{2}\right) & i = 0 \\ \left(\frac{1-z}{2}\right) \left(\frac{1+z}{2}\right) P_{i-1}^{1,1}(z) & 1 \leq i \leq I-1 \\ \left(\frac{1+z}{2}\right) & i = I \end{cases}$$

$$\phi_{ij}^b(z) = \begin{cases} \phi_j^a(z) & i = 0, \quad 0 \leq j \leq J \\ \left(\frac{1-z}{2}\right)^{i+1} & 1 \leq i \leq I-1, j = 0 \\ \left(\frac{1-z}{2}\right)^{i+1} \left(\frac{1+z}{2}\right) P_{j-1}^{2i+1,1}(z) & 1 \leq i \leq I-1, 1 \leq j \leq J-1 \\ \phi_j^a(z) & i = I, \quad 0 \leq j \leq J \end{cases},$$

$$\phi_{ijk}^c(z) = \begin{cases} \phi_{jk}^b(z) & i = 0, \quad 0 \leq j \leq J, \quad 0 \leq k \leq K \\ \phi_{ik}^b(z) & 0 \leq i \leq I, \quad j = 0, \quad 0 \leq k \leq K \\ \left(\frac{1-z}{2}\right)^{i+j+1} & 1 \leq i \leq I-1, 1 \leq j \leq J-1, k = 0 \\ \left(\frac{1-z}{2}\right)^{i+j+1} \left(\frac{1+z}{2}\right) P_{k-1}^{2i+2j+1,1}(z) & 1 \leq i \leq I-1, 1 \leq j \leq J-1, 1 \leq k \leq K-1 \\ \phi_{ik}^b(z) & 0 \leq i \leq I, \quad j = J, \quad 0 \leq k \leq K \\ \phi_{jk}^b(z) & i = I, \quad 0 \leq j \leq J, \quad 0 \leq k \leq K \end{cases}.$$

The function $\phi_i^a(z)$ has now been decomposed into two linearly varying components and an interior function. The linearly varying components generate the

vertex modes which are identical to the standard linear finite element expansion. The function $\phi_i^a(z)$, is the fundamental basis of the standard rectangular and hexahedral hp expansions [3,4] (note that the interior function can be defined in terms of the integral of the Legendre polynomial). The interior contribution of all the base functions (i.e. $1 \leq i \leq I-1$, $1 \leq j \leq J-1$, $1 \leq k \leq K-1$) is very similar to the orthogonal base functions defined in section 3 save that they are now premultiplied by a function of the form $\left(\frac{1-z}{2}\right)\left(\frac{1+z}{2}\right)$. This ensures that these modes are zero on the boundaries of the domain. We note that the symmetry of the boundary modes is ensured by the introduction of $\phi_i^a(z)$ into $\phi_{ij}^b(z)$ and the introduction of $\phi_{ij}^b(z)$ into $\phi_{ijk}^c(z)$.

Similar to the orthogonal expansions, the modified expansions for all the hybrid domains are

Hexahedral expansion:

Prismatic expansion:

$$\phi_{pqr}(\xi_1, \xi_2, \xi_3) = \phi_p^a(\xi_1)\phi_q^a(\xi_2)\phi_r^a(\xi_3), \quad \phi_{pqr}(\xi_1, \xi_2, \xi_3) = \phi_p^a(\xi_1)\phi_q^a(\eta_2)\phi_{qr}^b(\xi_3),$$

Pyramidic expansion:

Tetrahedral expansion:

$$\phi_{pqr}(\xi_1, \xi_2, \xi_3) = \phi_p^a(\overline{\eta_1})\phi_q^a(\eta_2)\phi_{pqr}^c(\eta_3), \quad \phi_{pqr}(\xi_1, \xi_2, \xi_3) = \phi_p^a(\eta_1)\phi_{pq}^b(\eta_2)\phi_{pqr}^c(\eta_3).$$

However, with the exception of the hexahedral expansion, we cannot consecutively loop over all the indices because the introduction of the boundary/interior decomposition destroys the dense packing of the functions $\phi_{ij}^b(z)$ and $\phi_{ijk}^c(z)$. Nevertheless, the construction of the expansion modes can be explained in an intuitive fashion.

Fig. 4. Degeneration from a hexahedral region into a tetrahedral domain.

As illustrated in figure 4, for the hexahedral domain the indices p, q, r correspond directly to a three dimensional array where all indices start from zero at the bottom left hand corner. Therefore, the vertex mode labelled A is described by $\phi_{(000)} = \phi_0^a(\xi_1)\phi_0^a(\xi_2)\phi_0^a(\xi_3)$, similarly the vertex mode labelled H is described by $\phi_{(P_1, P_2, P_3)}$ and the edge modes between C and G correspond to $\phi_{0, P_2, r}$ ($1 \leq r \leq P_3-1$).

When considering the prismatic domain we use the *equivalent* hexahedral indices. Accordingly, vertex A is now described by $\phi_{(000)} = \phi_0^a(\xi_1)\phi_0^a(\eta_2)\phi_{00}^b(\xi_3)$. In generating the new co-ordinate system, vertex G was mapped to vertex E and therefore the vertex mode, labelled EG in the prismatic domain, is

described by $\phi_{(0,0,P_3)} + \phi_{(0,P_2,P_3)}$ (i.e. adding the two vertices from the hexahedral domain which form the new vertex in the prismatic domain). A similar addition process is necessary for the prismatic edge $EG - FH$ which is constructed by adding the edge modes EF (i.e. $\phi_{(p,0,P_3)}$) to the edge modes GH (i.e. $\phi_{(p,P_2,P_3)}$). In degenerating from the hexahedral domain to the prismatic region, edge EG is removed and, therefore, does not contribute to the prismatic expansion.

This process can be extended to construct the expansion for the pyramidal and tetrahedral domains. For both these cases the top vertex is constructed by summing the contribution of E, F, G and H . In the tetrahedral domain edges CG and DH are also added.

The present construction has assumed that P_1, P_2 and P_3 have been fixed within the local expansion. In practice, it is possible to have different polynomial orders along each edge, face and in the interior of the domain. When connecting multiple domain the order along a connecting edge or face has to be the same for the global expansion to be continuous. There is, therefore, great flexibility in the choice of the polynomial order of the local expansion which may be adapted to comply with the solution in any specified region.

To obtain a complete polynomial space the interior functions of $\phi_{ij}^b(z)$ need only to be assembled up to the limit $i + j < I (= J)$ and similarly the interior functions of $\phi_{ijk}^c(z)$ need only to be assembled up to the limit $i + j + k < I (= J = K)$. Therefore we expect the pentahedral and tetrahedral expansions to contain less modes than the hexahedral expansion. The addition and degeneration of vertex and edges modes also reduces number of modes in each domain. If we fix the value of P_1, P_2 and P_3 to P then the number of modes required to generate a complete polynomial expansion as given by equation (2) are:

Hexahedral expansion	Prismatic expansion
$(P + 1)^3$	$\frac{1}{2}(P + 1)^2(P + 2)$
Pyramidal expansion	Tetrahedral expansion
$\frac{1}{6}(P + 1)(P + 2)(P + 3) + (P - 1)(P + 1) + 1$	$\frac{1}{6}(P + 1)(P + 2)(P + 3).$

The polynomial space of these expansions is the same as the orthogonal expansions (see equation (2)) and with the exception of the pyramidal expansion the number of modes is equal to the size of the polynomial space. The pyramidal expansion requires extra modes to perform the interior/boundary decomposition as compared to the orthogonal expansion. Unfortunately, the introduction of these extra modes does not increase the polynomial space of the expansion.

Although, the modified expansion allows the hybrid domains to be tessellated this process does destroy some of the orthogonality of the expansion.

Nevertheless, the Jacobi polynomials used in the interior of the functions $\phi_i^a(z)$, $\phi_{ij}^b(z)$ and $\phi_{ijk}^c(z)$ were chosen to maintain as much orthogonality in the expansions as possible. This point is illustrated in figure 5 where we see the non-zero entries of the mass matrix for the prismatic and pyramidic expansion within a single domain using an expansion of polynomial order $P = 10$. A high degree of sparsity is evident particularly in the prismatic region although it should be noted that for a fixed order of $P = 10$ the prismatic expansion has 726 modes whereas the pyramidic expansion only has 386 modes.

Fig. 5. Mass matrices for the prismatic (left) and pyramidic (right) domains at a fixed polynomial order of $P = 10$. Both expansions were ordered so that the vertex modes were first followed by the edge, face and finally the interior modes.

Finally, we note that to consistently support the internal modes of the rectangular face in the pyramidic expansion we should use the factor $(\frac{1-z}{2})^{i+j+2}$ in the function $\phi_{ijk}^c(z)$ instead of $(\frac{1-z}{2})^{i+j+1}$. However, this extra factor of $(\frac{1-z}{2})$ does not increase the polynomial space of the expansion and it is, therefore, preferable to leave $\phi_{ijk}^c(z)$ unaltered since it will then have the same form for both the pyramidic and tetrahedral expansions.

5 Convergence Tests

As mentioned, the hexahedral and tetrahedral expansions have previously been documented and examples using these expansions can be found in [3,4,6,7]. We therefore consider here the prismatic and pyramidic expansions. Figure 6 shows the L_2 projection of the function $u(\xi_1, \xi_2, \xi_3) = \sin(\xi_1) \sin(\xi_2) \sin(\xi_3)$ within these domains. Also shown in figure 6 is the convergence of the approximation in the L_∞ norm as a function of polynomial order P . As is typical in p -type refinement with smooth solutions, the approximation converges exponentially. Although the prismatic domain covers a larger region than the pyramidic domain its convergence is faster. This may well be explained by

Fig. 6. L_2 projection of $u(\xi_1, \xi_2, \xi_3) = \sin(\xi_1) \sin(\xi_2) \sin(\xi_3)$ in the prismatic and pyramidal domains. Convergence of the error in the L_∞ norm as a function of polynomial order.

the larger polynomial space of the prismatic expansion and the fact that the solution is aligned to the Cartesian directions which favours an expansion that is partly aligned to these co-ordinates.

6 Summary

In this paper, we have introduced new hp finite element bases for hybrid domains. The bases have been constructed in a *unified* approach which encompassed the standard h -type and structured hp -type finite elements. Therefore, it provides a clear framework within which the finite element of different polynomial orders and within different shaped regions can be assembled.

The new expansions are based upon orthogonal expansion within each hybrid region and although the modified C^0 continuous expansions are not strictly orthogonal a high level of orthogonality is preserved. Finally, the expansion in all the hybrid regions maintains computational efficiency because operations such as differentiation and integration can be performed using standard one-dimensional techniques.

Although not addressed in this paper the local co-ordinate systems do impose some restrictions on the orientation in which triangular faces may connect. It has, however, been shown in [6] that a C^0 tetrahedral expansion can be constructed for any tetrahedral mesh. It is believed that a similar strategy could be applied to a hybrid discretisation.

Finally we note that the bases are all hierarchical, which means that increasing the polynomial order of any expansion simply adds extra modes to the existing basis. Hierarchical expansions naturally lend themselves to p type adaptivity where the polynomial order of the expansion can differ within each elemental domain. This is an attractive property as it permits the polynomial order of the expansion to be altered in order to capture the spatial characteristics of

the solution.

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