Unsteady Navier-Stokes Solvers Using Hybrid Spectral/ hp Element Methods

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Abstract

A domain decomposition unsteady Stokes solver has been implemented in a hybrid spectral/ hp element solver. To extend the algorithm to solve the Navier-Stokes equations the non-linear terms are explicitly advanced in time and treated as a forcing function to the Stokes solver. This scheme is compared with the computationally efficient high order splitting method and it is found that for polynomial orders typically used in practice that the Stokes scheme is approximately 4/3 times more expensive in two-dimensions. A preliminary result for the Oseen problem using an analytic solution due to Kovasznay is also presented.

Key words: Splitting scheme, Stokes Solver, Oseen problem.

1 Introduction

Two very popular approaches to numerically solving the discrete incompressible Navier-Stokes equations are to either directly solve the matrix problem arising from the discretisation of the Stokes problem or to use a splitting/projection method where the velocity matrix system and pressure systems are typically decoupled.

The direct solution of the Stokes system introduces the problem of appropriate spaces for the velocity and pressure systems to satisfy the inf-sup condition [1] and requires the solution of the full velocity-pressure system. However, if a discontinuous pressure space is used then all but the constant mode of the pressure system can be decoupled from the velocity. Furthermore, when implementing this approach with a spectral/ hp element discretisation, the remaining velocity system may then be statically condensed to decouple the so called interior elemental degrees of freedom, reducing the Stokes problem to a

smaller system expressed on the elemental boundaries [2,3]. The direct solution of the Stokes problem provides a very natural setting for the solution of the pressure system which is not so easily dealt with in a splitting scheme. Further, the solution of the full coupled velocity systems means that the introduction of a spatially varying viscosity, which arise for example in non-Newtonian flows, is only a minor modification.

Splitting schemes are typically favoured for their numerical efficiency since for a Newtonian fluid the velocity and pressure systems are handled independently requiring the solution of three (in two-dimensions) elliptic systems of rank N as opposed to a single system of rank $3N$ solved in the Stokes problem. However, a drawback of this approach is the splitting error which is introduced when decoupling the pressure and velocity systems, although this can be made consistent with the overall temporal accuracy of the scheme by appropriate discretisation of the pressure boundary conditions [4].

Both of these approaches have been implemented within the framework of a Galerkin spectral/ hp element method and in this paper we outline the formulation of the schemes and then compare the force characteristics and computational cost of unsteady flow past a circular cylinder at Reynolds number of $Re = 100$. Preliminary results of the extension of the Stokes solver to the Oseen problem are also presented.

2 Algorithms

2.1 Splitting Scheme

Following the implementation in [4] the splitting scheme scheme discretises the incompressible Navier Stokes equations,

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla p + \nu \nabla^2 \mathbf{v} + \mathbf{f}, \quad \nabla \cdot \mathbf{v} = 0,$$

in the following three steps,

$$\begin{aligned} \frac{\tilde{\mathbf{v}} - \mathbf{v}^n}{\Delta t} &= \sum_{q=0}^{J_e-1} \beta_q [(\mathbf{v} \cdot \nabla) \mathbf{v}]^{n-q} + \mathbf{f}^{n+1}, \\ \frac{\tilde{\tilde{\mathbf{v}}} - \tilde{\mathbf{v}}}{\Delta t} &= -\nabla \bar{p}^{n+1}, \\ \frac{\mathbf{v}^{n+1} - \tilde{\tilde{\mathbf{v}}}}{\Delta t} &= \nu \sum_{q=0}^{J_i-1} \gamma_q \nabla^2 \mathbf{v}^{n+1-q}, \end{aligned}$$

where $\nabla \bar{p}^{n+1} = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \nabla p \, dt$, and J_e, J_i represent the order of an explicit and implicit time integration scheme related to the coefficients β_q and γ_q . In the first step the non-linear advection terms are propagated using explicit time stepping scheme. Taking the divergence of the second step produces a Poisson equation for pressure which is augmented with the boundary conditions

$$\frac{\partial \bar{p}^{n+1}}{\partial n} = \mathbf{n} \cdot \left[\sum_{q=0}^{J_e-1} \beta_q [(\mathbf{v} \cdot \nabla) \mathbf{v}]^{n-q} + \nu \sum_{q=0}^{J_e-1} \beta_q (-\nabla \times (\nabla \times \mathbf{v}))^{n-q} \right].$$

Finally the third step can be reformulated into a Helmholtz equation which accounts for diffusion. This approach has a boundary divergence error which scales as $\sqrt{\nu}(\Delta t)^{J_e}$ which is consistent with the overall temporal discretisation.

2.2 Stokes Algorithm

Following the construction in [2,3] we consider the weak form of the Stokes problem for the velocity field $\mathbf{v} = [u, v]^T$ and pressure field p :

$$\begin{aligned} (\nabla \phi, \nu \nabla \mathbf{v}) - (\nabla \cdot \phi, p) &= (\phi, \mathbf{f}) \\ (q, \nabla \cdot \mathbf{v}) &= 0 \end{aligned} \tag{1}$$

where $\mathbf{v}, \phi \in \mathbf{V}$, $p, q \in W$ and $\mathbf{V}, W$ are appropriate spaces for the velocity and pressure system to satisfy the inf-sup condition. In our computations we have adopted a $Q_{p+1} - Q_{p-1}$ space for quadrilateral sub-domains and a $P_{p+1} - P_{p-1}$ space triangular domains. Using matrix notation the discrete system may be written as:

$$\begin{bmatrix} A & D_b^T & B \\ D_b & 0 & D_i^T \\ B^T & D_i & C \end{bmatrix} \begin{bmatrix} \mathbf{v}_b \\ p \\ \mathbf{v}_i \end{bmatrix} = \begin{bmatrix} \mathbf{f}_b \\ 0 \\ \mathbf{f}_i \end{bmatrix} \tag{2}$$

where the components of A, B and C are $(\nabla \phi_b, \nu \nabla \mathbf{v}_b)$, $(\nabla \phi_b, \nu \nabla \mathbf{v}_i)$ and $(\nabla \phi_i, \nu \nabla \mathbf{v}_i)$ and the components of D_b and D_i are $(q, \nabla \cdot \mathbf{v}_b)$ and $(q, \nabla \cdot \mathbf{v}_i)$. The indices b and i refer to the degrees of freedom on the elemental boundary and interior, respectively. In constructing the system we have lumped the contributions from each component of the velocity field into the matrices A, B and C . However we note that for a Newtonian fluid the contributions from each field are decoupled. Since the interior degrees of freedom of the velocity field do not overlap, the matrix C is block diagonal and to take advantage

of this structure we can statically condense out the C matrix to obtain the system:

$$\begin{bmatrix} A - BC^{-1}B^T & D_b^T - BC^{-1}D_i & 0 \\ D_b - D_i^T C^{-1}B^T & -D_i^T C^{-1}D_i & 0 \\ B^T & D_i & C \end{bmatrix} \begin{bmatrix} \mathbf{v}_b \\ p \\ \mathbf{v}_i \end{bmatrix} = \begin{bmatrix} \mathbf{f}_b - BC^{-1}\mathbf{f}_i \\ -D_i^T C^{-1}\mathbf{f}_i \\ \mathbf{f}_i \end{bmatrix}. \quad (3)$$

In this formulation the pressure space is also piecewise continuous with no overlap at the elemental boundaries and so the matrix $-D_i^T C^{-1}D_i$ is also block diagonal. It is therefore possible to repeat the static condensation process and decouple the pressure degrees of freedom. However in this second condensation the constant pressure degrees of freedom must be kept within the inner boundary system in order to constrain the constant pressure modes which provide the only coupling between the pressure fields of each element. As a final point we note that this static condensation process may be constructed at an elemental level allowing the inner boundary system to be solved using either a direct or an iterative technique as discussed in [3]. Furthermore it has been shown in [3] that solving the boundary system iteratively, using a block diagonal preconditioner corresponding to the edges, then the condition number grows at most logarithmically in the polynomial degrees p .

To extend the above Stokes solver to an unsteady Navier-Stokes solver we first introduce the unsteady term, $\mathbf{v}_t$, into the Stokes problem given in equation (1). This has the principal effect of modifying the weak Laplacian operator $(\nabla\phi, \nu\nabla\mathbf{v})$ into a weak Helmholtz operator $(\nabla\phi, \nu\nabla\mathbf{v}) - \lambda(\phi, \mathbf{v})$ where λ depends on the time integration scheme. The second modification requires the explicit discretisation of the non-linear terms in a similar manner to the splitting scheme described in 2.1 and this term is then introduced as the forcing term $\mathbf{f}$.

3 Comparison of Schemes

To compare the computational cost and memory requirements of the two scheme we consider the unsteady Stokes problem and determined the number of floating point operations per time step. We presume that a large number of time steps are being performed in any given computation and so we assume that the computational cost required to assemble the matrix systems is negligible.

Considering a rectangular mesh of N_{el} quadrilateral elements at a polynomial order of p the computational cost of both solution methods involves a bound-

ary, interior and coupling matrix operations. The coupling system being used twice. For each elliptic solve in the the splitting scheme where there are $2N_{el} p$ boundary degrees of freedom and $N_{el} (p - 1)^2$ interior degrees of freedom the operation count is

Matrix System	Operation Count
Boundary	$[2N_{el}p]^2 + O(N_{el} p)$
Coupling (x2)	$4N_{el}p(p - 1)^2$
Interior	$N_{el}(p - 1)^4 + O(N_{el} p^2)$.

Therefore, the salient cost of the splitting scheme which uses three elliptic solves is: $C_{split} \approx 12N_{el}^2 p^2 + 3N_{el} p^4 + O(N_{el} p^3)$.

For the Stokes solver assuming a Newtonian fluid and using a $Q_{p+1} - Q_{p-1}$ expansion space we have a velocity/constant pressure boundary system of size $([4N_{el} (p + 1) + N_{el}]^2)$ and an interior system of size $2p^2 + (p - 1)^2$ and so the operation count of the scheme can be broken down into:

Matrix System	Operation Count
Boundary	$[N_{el}(4p + 5)]^2 + O(N_{el} p)$
Coupling (x2)	$8N_{el}(p + 1)(p^2 + (p - 1)^2) + 2p^2(p - 1)^2$
Interior	$N_{el}[(p - 1)^4 + 2p^4] + O(N_{el} p^2)$.

Therefore the salient cost of the Stokes solve is: $C_{Stokes} \approx 16N_{el}^2 p^2 + 7N_{el} p^4 + O(N_{el} p^3, N_{el}^2 p)$.

These approximations also reflect the memory requirement for each scheme although it should be noted that a further memory saving is possible in the splitting scheme since only one Helmholtz system is required for both velocity solves. From these approximations we see that when N_{el} is large compared with p the cost of the Stokes solver is approximately 4/3 times more expensive than the splitting scheme. However in the case when p is large compared with N_{el} , the Stokes solve is about twice as expensive as the splitting scheme. In practice the former case is more relevant since p is typically of order 10 and N_{el} is typically larger than 100.)

To provide a direct comparison of the schemes we have considered the two computational problems shown in figure 1. For both tests the CPU cost per time-step was measured on an R10000 195MHz Silicon graphics. In the first problem we have considered the solution of a Stokes problem on a regular grid with an increasing number of element at a polynomial order of $p = 10$. We see that the ratio of the CPU cost of the Stokes solver to the CPU cost of

Fig. 1. Comparison of CPU time per time step

the Splitting scheme is approximately $4/3$ and N_{el} increases. In the second case we have solved the full Navier-Stokes equations and so the timings of the non-linear term evaluations have been included. In this test the hybrid mesh with triangular and quadrilateral elements contained 780 sub-domains and polynomial orders ranging from $p = 4$ to $p = 10$ were used. For lower polynomial orders we see a very negligible difference in CPU time although at $p = 10$ the Stokes solver was 1.3 times more expensive than the splitting scheme.

4 Unsteady flow simulation

To determine whether we are achieving a comparable resolution from the two schemes we have simulated unsteady flow past a circular cylinder at a Reynolds number of $Re = 100$ using the 780 element hybrid mesh shown in figure 2. At this Reynolds number the flow is unsteady but completely two-dimensional. This test case has been widely studied both experimentally and numerically and is of interest since there have been noticeable discrepancies in the force characteristics, particularly rms values, when using different numerical schemes [5].

The boundary conditions on the simulation were free stream values of $\mathbf{v} = [1, 0]^T$ at the inflow and the far field boundaries and no-slip conditions at the cylinder wall. In the splitting scheme a zero value was imposed on the pressure and natural boundary conditions enforcing zero normal derivatives were applied to the velocity. For the Stokes solver the natural traction boundary condition was applied at outflow thereby fixing the constant pressure value. For a range of polynomial orders the flow was time marched using a first order time scheme (thereby using the same time integration in both methods) with a time step of $\Delta t = 0.003$ until a time periodic state was reached. The following

Fig. 2. Hybrid Mesh around a circular cylinder and instantaneous vorticity contours at $Re=100$

force measurements were calculated using a time period of four lift cycles and as can be appreciated the forces characteristic compare very well achieving an agreement to 2 significant figures by $p = 6$.

Poly. order	Splitting Scheme			Stokes Solver		
	C_{Lrms}	$\overline{C}_D$	C_{Drms}	C_{Lrms}	$\overline{C}_D$	C_{Drms}
2	0.206	1.314	22.25×10^{-3}	0.122	1.220	3.89×10^{-3}
4	0.236	1.351	5.96×10^{-3}	0.232	1.344	6.40×10^{-3}
6	0.235	1.347	6.73×10^{-3}	0.235	1.347	6.72×10^{-3}
8	0.235	1.348	6.73×10^{-3}	0.235	1.348	6.72×10^{-3}
10	0.235	1.348	6.72×10^{-3}	0.235	1.348	6.72×10^{-3}

5 Oseen Flow

An interesting extension which is possible with the Stokes solver but has not been developed for the splitting scheme is the solution to the Oseen problem where the momentum equation contains a linear advection term of the form:

$$-\nu \nabla^2 \mathbf{v} + (\mathbf{u} \cdot \nabla) \mathbf{v} + \nabla p = \mathbf{f},$$

where $\mathbf{u}$ is a known velocity. As noted by [6] this system can be used as a basis for a steady-state Navier-Stokes solver or to improve the convective time step restriction associated with the advection term of the Navier-Stokes equations.

The coupled velocity pressure solver can be formulated in an similar fashion to that described in section 2.2 although the matrix system are now

Fig. 3. Solution to the Kovasznay problem using a Oseen solver.

non-symmetric and so the transpose matrices must now be treated as separate quantities. This scheme has been implemented as a direct solver and as validation we have considered the Kovasznay problem simulating flow past a two-dimensional grid. The exact solution is of the form $u(x, y) = 1 - e^{\lambda x} \cos(2\pi y)$, $v(x, y) = \lambda/(2\pi) e^{\lambda x} \sin(2\pi y)$ where $\lambda = 1/(2\nu) - \sqrt{(1/(4\nu^2) + 4\pi^2)}$. In the calculation shown in figure 3 the value of $\mathbf{u}$ was prescribed to be the exact solution.

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