A high order splitting scheme for the Navier Stokes equations with variable viscosity.

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Abstract

The objective of this paper is to extend the splitting scheme of Karniadakis *et al.* (1991) to temporally and spatially varying viscosity, while retaining the decoupling of the viscous term. The derivation of the algorithm and a simplified von Neumann stability analysis for the one-dimensional diffusion equation is presented, demonstrating that for a linear diffusion equation, the scheme is unconditionally stable if the implicit part of the viscosity is larger than the explicit part.

Key words: Splitting scheme, Variable Viscosity, Navier-Stokes

1 Introduction.

An algorithm for incompressible flows, appropriate to large eddy simulation is proposed, that includes modelling of the incompressible Navier-Stokes equations with a viscosity which may vary spatially and temporally, i.e.

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} = -\nabla p + \nabla \cdot [\nu (\nabla \mathbf{V} + \nabla^T \mathbf{V})], \quad \mathbf{Q} = \nabla \cdot \mathbf{V} = 0. \quad (1)$$

Explicit advancement of the viscous term in time would lead to stability restrictions. Semi-implicit (Kim & Moin 1982) and fully implicit schemes (Choi & Moin 1994) have already been suggested, though they lead to coupled systems of equations which may be computationally expensive. To circumvent this problem, we propose rearranging the viscous term in the splitting scheme of Karniadakis *et al.* (1991), in order to retain the decoupling of velocities. The viscosity is then split into two components, one treated explicitly while the other implicitly. A full derivation of the scheme is presented, together with

a von Neumann stability analysis for a similarly split one dimensional diffusion equation. The paper concludes with numerical examples validating the scheme.

2 Derivation of a splitting scheme with variable viscosity.

2.1 Definition of the three step solution.

The non linear and linear operators for the Navier Stokes equation, given by equation (1), are respectively defined as $N(\mathbf{V}) = (\mathbf{V} \cdot \nabla)\mathbf{V}$ and $L(\mathbf{V}) = \nabla \cdot [\nu (\nabla \mathbf{V} + \nabla^T \mathbf{V})]$. Considering a time dependent spatially varying viscosity $\nu(x, t)$ one can decompose it into two components $\nu = \bar{\nu} + \nu_s$ where $\bar{\nu}(x)$ is an implicitly advanced, spatially varying component, and $\nu_s(x, t)$ is an instantaneous, explicitly advanced, spatially and temporally varying component. The form of $\bar{\nu}(x)$ can be a spatial average, as suggested by Moin & Kim (1982) or a time average, as suggested by Leslie & Gao (1988). In order to decouple the velocity components, the linear term is split into $L(\mathbf{V}) = L^+(\mathbf{V}) + N^+(\mathbf{V})$ where $L^+(\mathbf{V}) = \nabla \cdot [\bar{\nu} \nabla \mathbf{V}]$, $N^+(\mathbf{V}) = \nabla \cdot [\bar{\nu} \nabla^T \mathbf{V}] + \nabla \cdot [\nu_s \cdot (\nabla \mathbf{V} + \nabla^T \mathbf{V})]$. $N^+(\mathbf{V})$ is treated explicitly, in a similar fashion to the non-linear terms, while $L^+(\mathbf{V})$ is treated implicitly, similarly to the linear term. Integrating over a time step Δt equation (1) may be approximated as

$$\mathbf{V}^{n+1} - \mathbf{V}^n = \Delta t \nabla p^{n+1} + \Delta t \sum_{q=0}^{J_i-1} \gamma_q L^+(\mathbf{V}) + \Delta t \sum_{q=0}^{J_e-1} \beta_q [N^+(\mathbf{V}) + N(\mathbf{V})], (2)$$

where β_q are integration weights of order J_e , such as the explicit Adams-Bashforth scheme and γ_q correspond to weights for an implicit scheme of order J_i . Following the procedure from Karniadakis *et al.* (1991), a high order splitting scheme with spatially varying viscosity can then be constructed, resulting into the three step solution process

$$\frac{\hat{\mathbf{V}} - \mathbf{V}^n}{\Delta t} = \sum_{q=0}^{J_e-1} \beta_q [N(\mathbf{V}^{n-q}) + N^+(\mathbf{V}^{n-q})] \quad (3)$$

$$\frac{\hat{\hat{\mathbf{V}}} - \hat{\mathbf{V}}}{\Delta t} = -\nabla \bar{p}^{n+1} \quad (4)$$

$$\frac{\mathbf{V}^{n+1} - \hat{\hat{\mathbf{V}}}}{\Delta t} = \sum_{q=0}^{J_i-1} \gamma_q L^+(\mathbf{V}^{n+1-q}) \quad (5)$$

with $\hat{\mathbf{V}}, \hat{\hat{\mathbf{V}}}$ being the intermediate velocity fields.

2.2 Derivation of the Poisson type equation and its boundary condition for the Pressure.

At time step $n + 1$, the divergence is denoted as $Q^{n+1} = \nabla \cdot \mathbf{V}^{n+1}$. Taking the divergence of equation (5) and combining it with equation (4) leads to $\nabla^2 p^{n+1} = \nabla \cdot \left(\frac{\dot{\mathbf{V}}}{\Delta t} \right) + \nabla \cdot \left[\sum_{q=0}^{J_i-1} \gamma_q L^+(\mathbf{V})^{n+1-q} \right]$. Setting the divergence as zero at time steps $n, n + 1$, leads to

$$\frac{\partial^2 p^{n+1}}{\partial x_k^2} = \frac{\partial}{\partial x_k} \left(\frac{\hat{u}_k}{\Delta t} \right) + \sum_{q=0}^{J_i-1} \left[\frac{\partial^2 \bar{\nu}}{\partial x_j \partial x_k} \frac{\partial u_k}{\partial x_j} + \frac{\partial \bar{\nu}}{\partial x_k} \frac{\partial^2 u_k}{\partial x_j^2} \right]^{n+1-q}. \quad (6)$$

It is noted that at the right hand side, terms exist at time level $(n + 1)$, leading to a coupled system. This term is decomposed by extrapolating the value at $n + 1$ from the previous time levels. It can be shown that imposition of this equation results in the time accuracy of the global solution being directly dependent on the boundary values of the divergence and therefore on the treatment of the pressure boundary conditions. Taking the dot product of equation (2) and assuming time invariant velocity boundary conditions, leads to

$$\frac{\partial p^{n+1}}{\partial \mathbf{n}} = \mathbf{n} \cdot \left[\sum_{q=0}^{J_e-1} \beta_q \left[N(\mathbf{V})^{n-q} + N^+(\mathbf{V})^{n-q} \right] + \sum_{q=0}^{J_i-1} \gamma_q L^+(\mathbf{V})^{n+1-q} \right]. \quad (7)$$

A stable scheme can be constructed by rewriting the viscous linear terms of a solenoidal part, which is approximated by an explicit scheme, and an irrotational part, ∇Q , which is zero. Equation (7) can be rewritten as

$$\begin{aligned} \frac{\partial p^{n+1}}{\partial \mathbf{n}} = & n \cdot \left[\sum_{q=0}^{J_e-1} \beta_q \left[N(\mathbf{V})^{n-q} + N^+(\mathbf{V})^{n-q} \right] + \sum_{q=0}^{J_i-1} \gamma_q \bar{\nu} \nabla \cdot \nabla \mathbf{Q}^{n+1-q} \right] + \\ & n \cdot \left[\sum_{q=0}^{J_i-1} \gamma_q \left(\nabla \bar{\nu} \cdot \nabla \mathbf{V}^{n+1-q} - \bar{\nu} \nabla \times (\nabla \times \mathbf{V})^{n+1-q} \right) \right]. \end{aligned} \quad (8)$$

The term $\gamma_0 \nabla Q^{n+1}$ is dropped since $Q^{n+1} = 0$. It is again observed that there are terms at $(n + 1)$, which are extrapolated from the previous time levels. Karniadakis *et al.* (1991) demonstrated the accumulation of divergence flux at the boundary at every time step and therefore the possibility for instability if the Laplacian form is used for the second derivative of velocity. If the rotational form is used, the magnitude of the *boundary divergence flux* is controlled directly by the time step. The boundary divergence errors may be further reduced by using a multi-step approximation. Therefore, the pressure boundary condition can be approximated as

$$\begin{aligned} \frac{\partial p^{n+1}}{\partial \mathbf{n}} = & n \cdot \left[\sum_{q=0}^{J_e-1} \beta_q \left[N(\mathbf{V})^{n-q} + N^+(\mathbf{V})^{n-q} \right] + \right. \\ & \left. n \cdot \left[\sum_{q=0}^{J_e-1} \beta_q \left(\nabla \bar{\nu} \cdot \nabla \mathbf{V}^{n-q} - \bar{\nu} \nabla \times (\nabla \times \mathbf{V})^{n-q} \right) \right] \right] \end{aligned} \quad (9)$$

Solution of the Poisson equation for the pressure p^{n+1} with only Neumann boundary conditions requires that the compatibility condition hold. The solvability condition may be shown to require

$$\sum_{q=0}^{J_e-1} \beta_q \int_{\Omega} [\nabla \cdot \mathbf{N} + \nabla \cdot [\nabla \bar{\nu} \nabla \mathbf{V}]]^{n-q} dv = \int_{\Omega} \nabla \cdot \left(\frac{\hat{\mathbf{V}}}{\Delta t} \right) + \nabla \cdot \left[\sum_{q=0}^{J_e-1} \beta_q L^+(\mathbf{V})^{n-q} \right]$$

which is valid under the formulation used.

2.3 Summary and implementation of the high order splitting scheme.

The final step is to use a Helmholtz type equation to find the velocities at a time step $n+1$. For a θ scheme, equation (5) can be written as

$$(1 - \theta)L^+(\mathbf{V})^{n+1} - \frac{1}{\Delta t(1 - \theta)}\mathbf{V}^{n+1} = -\frac{1}{\Delta t}\hat{\mathbf{V}} - \theta L^+(\mathbf{V})^n$$

with $\hat{\mathbf{V}} = \hat{\mathbf{V}} - \Delta t \nabla p^{n+1}$. In order to eliminate both an implicit and explicit evaluation of the Laplacian, the averaged variable $\mathbf{V}^*$ is used, defined as $\mathbf{V}^* = (1 - \theta)\mathbf{V}^{n+1} + \theta\mathbf{V}^n$. The new linear and non linear components $L^+(\mathbf{V})$, $N^+(\mathbf{V})$ can be expanded by considering the property $Q^{n+1} = 0$, in order to improve the stability conditions of the scheme $L^+(\mathbf{V}) = \frac{\partial \bar{\nu}}{\partial x_j} \frac{\partial u_i}{\partial x_j} + \bar{\nu} \frac{\partial^2 u_i}{\partial x_j^2}$, $N^+(\mathbf{V}) = \frac{\partial \bar{\nu}}{\partial x_j} \frac{\partial u_i}{\partial x_j} + \frac{\partial}{\partial x_j} \left[\nu_s \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right]$. The scheme may therefore be summarised as

$$\frac{\hat{\mathbf{V}} - \mathbf{V}^n}{\Delta t} = \sum_{q=0}^{J_e-1} \beta_q \left(-(\mathbf{V} \cdot \nabla) \mathbf{V} + (\nabla \bar{\nu}) (\nabla^T \mathbf{V}) + \nabla \cdot \nu_s (\nabla \mathbf{V} + \nabla^T \mathbf{V}) \right)^{n-q} \quad (10)$$

$$\nabla^2 p^{n+1} = \nabla \cdot \left(\frac{\hat{\mathbf{V}}}{\Delta t} \right) + \nabla \cdot \sum_{q=0}^{J_e-1} \beta_q [\nabla \cdot \bar{\nu} \nabla \mathbf{V}]^{n+1-q} \quad (11)$$

$$\nabla \cdot \bar{\nu} \nabla \mathbf{V}^* - \frac{1}{\Delta t(1 - \theta)} \mathbf{V}^* = -\frac{\hat{\mathbf{V}} + \frac{\theta}{1-\theta} \mathbf{V}^n}{\Delta t} \quad (12)$$

where $\hat{\mathbf{V}} = \hat{\mathbf{V}} + \Delta t \nabla p^{n+1}$ and the pressure equation is supplemented with the boundary condition

$$\begin{aligned} \frac{\partial p^{n+1}}{\partial \mathbf{n}} = & n \cdot \left[- \sum_{q=0}^{J_e-1} \beta_q \left[(\mathbf{V} \cdot \nabla) \mathbf{V} - (\nabla \bar{v}) \nabla^T \mathbf{V} - \nabla \cdot \nu_s (\nabla \mathbf{V} + \nabla^T \mathbf{V}) \right]^{n-q} \right] + \\ & n \cdot \left[\sum_{q=0}^{J_e-1} \beta_q \left[\nabla \bar{v} \nabla \mathbf{V} - \bar{v} \nabla \times (\nabla \times \mathbf{V}) \right]^{n-q} \right] \end{aligned} \quad (13)$$

3 Stability Analysis.

To obtain an indication of the stability of the viscous splitting approach, the diffusion equation in one dimension is considered using the second order, i.e. Adams-Bashforth, Crank-Nicholson scheme,

$$\begin{aligned} \frac{u^{n+1} - u^n}{\Delta t} = & \frac{3}{2} \left[(\nu_s)_x \frac{\partial u}{\partial x} + \nu_s \frac{\partial^2 u}{\partial x^2} + \bar{\nu}_x \frac{\partial u}{\partial x} \right]^n - \frac{1}{2} \left[(\nu_s)_x \frac{\partial u}{\partial x} + \nu_s \frac{\partial^2 u}{\partial x^2} + \bar{\nu}_x \frac{\partial u}{\partial x} \right]^{n-1} \\ & + \frac{1}{2} \left[\bar{\nu}_x \frac{\partial u}{\partial x} + \bar{\nu} \frac{\partial^2 u}{\partial x^2} \right]^{n+1} + \frac{1}{2} \left[\bar{\nu}_x \frac{\partial u}{\partial x} + \bar{\nu} \frac{\partial^2 u}{\partial x^2} \right]^n. \end{aligned} \quad (14)$$

Under the standard von Neumann techniques, $u_{xx}^n = -2(1 - \cos\theta) \frac{u^n}{\Delta x^2}$, $u_x^n = i \frac{\sin\theta}{\Delta x} u^n$, $(\nu_s)_x^n = i \frac{\sin\theta}{\Delta x} \nu_s^n$, $\bar{\nu}_x^n = i \frac{\sin\theta}{\Delta x} \bar{\nu}^n$, where $\theta = k\Delta x$, k being the wavenumber. Setting $u^{n+1} = \lambda u^n$ and assuming the viscosity does not vary in time, equation (14) may be written in the form $a\lambda^2 - b\lambda - c = 0$, where $a = \frac{1}{\Delta t} - \frac{1}{2} \left[-\frac{\bar{\nu}}{\Delta x^2} \sin^2\theta - \frac{2\bar{\nu}}{\Delta x^2} (1 - \cos\theta) \right]$ and $b = \frac{1}{\Delta t} - \frac{\sin^2\theta}{\Delta x^2} \left[\frac{3}{2} \nu_s + 2\bar{\nu} \right] - \frac{1 - \cos\theta}{\Delta x^2} [3\nu_s + \bar{\nu}]$, $c = \nu \frac{1 - \cos\theta}{\Delta x^2} + \frac{\sin^2\theta}{2\Delta x^2} (\nu + \bar{\nu})$. For stability, $|\lambda| < 1$, which translates into (Miller 1971) $|c/a| < 1$, $|b/a| + c/a < 1$, and therefore

- $(2\nu_s + 4\bar{\nu})\sin^2\theta + (1 - \cos\theta)(4\nu_s + 4\bar{\nu}) > 0$ which is always true if $\nu_s + \bar{\nu} > 0$ and $\bar{\nu} > 0$
- $\bar{\nu} > \frac{5}{4}\nu_s$, if ν_s and $\bar{\nu} \neq 0$,
- $\nu_s < \frac{\Delta x^2}{4\Delta t}$ if $\bar{\nu} = 0$,
- $-2\bar{\nu}(1 - \cos\theta) < \frac{2\Delta x^2}{\Delta t}$, which is always true if $\nu_s = 0$.

It is interesting to note, that the analysis of Leslie & Gao (1988) which did not include the spatial derivative of viscosity obtained $\nu_s < \frac{\Delta x^2}{4\Delta t}$, while unconditional stability was achieved if $\bar{\nu} > \nu_s$.

Fig. 1. Kovasnay flow mesh and spatial convergence for Kovasnay flow.

4 Validation.

4.1 Kovasnay and Stokes Flows.

The Kovasnay flow is a laminar flow behind a two-dimensional grid, the exact solution of which is given by Kovasnay (1948). It is of the form $u = 1 - e^{\lambda x} \cos(2\pi y)$, $v = \frac{1}{2\pi} e^{\lambda x} \sin(2\pi y)$, where $\lambda = Re^2/2 - (Re^2/4 + e\pi^2)^{0.5}$, Re being the Reynolds number. All boundary conditions are defined by the above exact solution. The mesh shown in figure 1 is used, with viscosities of the form $\bar{\nu} = 0.024 + \frac{(y-0.5)(x-0.25)}{0.5625} 10^{-3}$ and $\nu_s = 0.001 - \bar{\nu}$. It is noted that although the implicit and explicit viscosities are varying in space, the total viscosity is constant, so that the kovasnay solution is valid. Figure 1 shows the exponential spatial convergence achieved, as a function of polynomial order.

Fig. 2. Mesh and temporal convergence for Stokes flow.

The Stokes channel flow is a time dependent problem of a flow between parallel plates, for a viscosity equal to 1. Although it is linear, it embodies all the essential features of the Navier Stokes equations and can also serve as a model to assess the high order splitting method derived in this paper. The initial conditions from Karniadakis *et al.* (1991) were applied and the viscosity was split into two components in a manner similar to the Kovasnay flow. The implicit viscosity is of the form $\bar{\nu} = 0.99 + 10^{-2} \frac{x^2 y^2}{\pi}$, while the explicit

$\nu_s = 0.01 - \bar{\nu}$, with the total viscosity equal to 1.

The important variable is the decay rate $\tilde{\sigma} = -\frac{1}{\nu T} \ln \frac{\nu(y=0, t+T)}{\nu(y=0, t)}$, where the time period T is taken to be $T=0.3$. After time T , the energy of the initial field has been reduced by almost five orders of magnitude. A high-resolution spectral element mesh is employed to eliminate any residual spatial discretisation errors. Simulations were carried out for various time steps Δt with a third order Adams- Bashforth used for the explicit terms, while an Euler-forward and a Crank-Nicholson scheme was used for the viscous terms. Figure 2 demonstrates the first and second order accuracy obtained with the two methods respectively.

4.2 Variable Viscosity flows.

Fig. 3. Variable viscosity channel flow and spatial convergence.

The above test cases, were for mild variations in viscosity. By assuming a computational domain as shown in figure 3, the viscosity varies $\bar{\nu} = 1 - \alpha y^2$, with $\nu_s = 0$, and with pressure gradients as $\frac{dp}{dx} = 2\alpha$, $\frac{dp}{dy} = 0$. The velocity profile takes the form $u = \ln \frac{1-\alpha y^2}{1-\alpha}$. and spatial convergence was achieved for $\alpha = 0.9$, as shown in figure 3.

Temporal accuracy and convergence were validated with a decaying Couette flow. Assuming a Couette flow with the same variation in viscosity as the previous example, and an adverse pressure gradient of $\frac{dp}{dx} = -2\alpha$, $\frac{dp}{dy} = 0$, the pressure gradient is set to zero and the flow tends from the solid line profile to the dotted line profile, as shown in figure 4. The initial velocity is of the form $u = \left[-1 + \frac{y\sqrt{\alpha}}{2} \ln \frac{1+\sqrt{\alpha}y}{1-\sqrt{\alpha}y} + 0.725129784y \right] e^{-2\alpha t}$ and first and second order accuracy for the Euler backward and Crank Nicholson schemes is achieved as shown in figure 4, quantified by $\alpha = \frac{1}{2T} \ln \frac{u(t+T)}{u(t)}$.

5 Conclusion.

The high order splitting scheme by Karniadakis *et al.* (1991) has been modified to take into account spatially variable viscosity by splitting the viscosity into an implicit and explicit term. The explicit term is treated using the Adams-Bashforth scheme together with the non-linear terms, while the implicit term is treated using a Crank Nicholson scheme. To understand the stability restrictions, a simplified Von-Neumann stability analysis is performed on a one dimensional diffusion equation indicating that the scheme is unconditionally stable if the implicit viscosity $\bar{\nu} > \frac{5}{4}\nu_s$. Numerical examples are given, with differing variances in viscosity, showing first and second order temporal accuracy, even with a viscosity variation of 90%.

Fig. 4. Variable viscosity Couette flow and temporal Convergence.

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