A Unified Scheme to Achieve the Degrees-of-Freedom Region of the MIMO Interference Channel with Delayed Channel State Information

Mohsen Rezaee[†], Peter J. Schreier[†], Maxime Guillaud[‡], Bruno Clerckx^{*}

[†]Signal and System Theory Group, University of Paderborn,

Germany – e-mail: {mohsen.rezaee, peter.schreier}@sst.upb.de

[‡]Mathematical and Algorithmic Sciences Lab, Huawei Technologies, France Research Center,

Paris, France – e-mail: maxime.guillaud@huawei.com

^{*}Imperial College London, London SW7 2AZ, United Kingdom – e-mail:

b.clerckx@imperial.ac.uk

^{*}School of Electrical Engineering, Korea University, Seoul 136-701, Korea

Abstract

In interference channels (ICs), channel state information (CSI) can be utilized to design transmit signals that are optimally adapted to the state of the channels. This requires feeding CSI back to the transmitters. The CSI available at the transmitters (CSIT) is usually degraded, due to the limited capacity of the feedback link and the delays involved in the channel estimation and feedback. We discuss the scenario of fast fading and delayed CSIT, which is highly relevant in mobile environments with short channel coherence times. We consider the two-user Multiple-Input-Multiple-Output (MIMO) IC where the transmitters are provided with delayed CSIT. The DoF region for this channel was characterized by Vaze and Varanasi. We devise a simple and intuitive achievable scheme, which has a unified structure for different antenna configurations. We show that the proposed scheme also achieves the DoF region of the two-user MIMO broadcast channel (BC). In the IC, our scheme does not require the knowledge

of direct channels at the transmitters. Moreover, we show that the amount of feedback can be further reduced by exploiting the invariances of the problem. Our approach can be helpful when analyzing more complicated networks.

I. INTRODUCTION

The rapid growth in demand for wireless connectivity requires that new systems make optimal use of available resources. Employing multiple antennas at the transceivers is known to increase the capacity and reliability of transmission compared to single-antenna systems. However, due to practical limitations, transmission devices can only support a limited number of antennas. Thus, it is important to develop efficient and (if possible) close to optimal transmission methods to exploit the available antennas at the communicating nodes. As the communication channel between the nodes varies over time/frequency/space, channel state information (CSI) at the nodes can potentially be useful to design efficient transmission schemes. The benefit of CSI is particularly important in multi-user systems where transmitters create interference for non-intended receivers. CSI at the receiver side can be utilized to separate the desired signal from interference, while CSI at the transmitter side (CSIT) can potentially prevent (or reduce) interference for non-intended receivers by properly designing the transmission signals. In order to optimize the transmission scheme for any instance of the channel, it is important that every transmitter have access to the instantaneous CSIT. However, providing instantaneous CSIT is not always feasible in practice, especially in fast-fading channels. It was shown in [2] that for the MIMO broadcast channel (BC) even outdated CSIT (which may be independent of the current CSIT) may allow performance improvements over the case where no CSIT is available. The performance improvement is in the form of higher achievable Degrees-of-Freedom (DoF). As the DoF represent a capacity approximation at high Signal-to-Noise-Ratio (SNR), this translates into an asymptotic capacity increase.

The DoF gains in [2] are realized by an interference alignment scheme. This scheme is based on the idea that the interference seen at each of the receivers in the previous time slot (which is a linear combination of the data symbols intended for the other users) can be reconstructed by the transmitters at the current time instant using delayed CSIT, and subsequently forwarded to the receivers. The receivers can exploit this interference to decode their own signal. This interference alignment scheme achieves DoF

Part of the results of this paper are presented in [1].

gains and is sum-DoF optimal for the class of MISO BCs, where the transmitter has at least as many antennas as the number of users K and the receivers have independent and identically distributed (i.i.d.) channels. These results were then extended to the case of the MIMO BC in [3], which also derived an outer bound for the DoF region in the K -user case.

In the interference channel and the X-channel, the authors of [4] showed that using ideas similar to those of [2], higher DoF can be achieved compared to the no-CSIT case. The results of [4] were improved upon in [5], [6]. Several other works studied the effect of delayed CSIT in different interference networks [7]–[10]. The benefit of combining delayed CSIT and instantaneous CSIT in correlated channels was studied in [11], [12]. Other interesting scenarios such as mixed CSIT, alternating CSIT, hybrid CSIT and local CSIT were considered in [13]–[18].

The DoF region of the two-user MIMO IC was fully characterized in [10] for different possible antenna configurations. The proposed achievable scheme in [10] is based on transmitting on a stream-per-antenna basis, which makes it difficult to implement. Furthermore, the scheme requires (delayed) knowledge of the direct channels at the transmitters. Another technique was proposed in [12] for the same channel with temporally correlated CSI. While this includes our setting, the scheme in [12] has several disadvantages. An infinite number of channel extensions is required to achieve the full DoF. This results in an infinite delay due to the required backward decoding. Furthermore, while [12] proved the existence of parameters for the power allocation that are DoF-achieving, it did not provide an algorithm to determine these parameters.

In our paper, we propose a unified DoF-achieving scheme for the two-user MIMO IC (and also MIMO BC) with outdated CSIT based on linear matrix precoding, which covers all possible antenna configurations, avoids the many case distinctions required by [10], and overcomes the shortcomings of [12]. The fundamental idea behind our technique is that each user should reduce the dimension of its interference for the other user when the other user cannot accommodate all the incoming streams. Transmitting over multiple time slots and exploiting the CSI of previous time slots creates a channel structure that allows reducing the dimension of the interference without reducing the number of transmit streams.

In the following, we highlight the differences and benefits of our scheme compared to [10], [12]:

- Unlike [10], our proposed scheme is a unified scheme; not only does it unify different configurations

of the MIMO IC, the same scheme also works for the MIMO BC. Most of the literature on delayed CSIT is based on using heuristic methods, which design deterministic precoders that achieve the DoF in specific scenarios. Our approach employs randomness as a novel design element for precoders that work in a variety of different scenarios.

- Because our method has an intuitive interpretation, it is easier to extend than the schemes in [10] and [12]. For instance, a possible extension is the case of channels correlated over successive time slots.
- Unlike [10], we do not require knowledge of the direct channels, which conforms with the idea of interference alignment and is important in practice: It reduces the complexity of the network and avoids interference created by imperfect feedback. Moreover, the feedback links can use their limited capacity to feedback the necessary information more accurately.
- We require strictly less feedback (in terms of the number of real parameters that need to be known to achieve the DoF) for the interfering channels compared to [10] and [12].
- Since our scheme is based on precoding over a finite number of channel extensions (on the order of the number of antennas), it reduces the complexity of the transceivers and also imposes very little decoding delay at the receiver, unlike the method of [12]. Furthermore, since our method uses simple matrix precoding and decoding, it can easily be integrated into existing systems. The other methods presented in [10], [12] require more sophisticated transceivers.

Notation: Non-bold letters represent scalar quantities, boldface lowercase and boldface uppercase letters indicate vectors and matrices, respectively. $\mathbf{I}_N$ is the $N \times N$ identity matrix. A block diagonal matrix is denoted by $\text{Bdiag}(\cdot)$, where the argument contains the blocks on its diagonal. $\mathcal{CN}(0, 1)$ denotes the complex Gaussian circularly symmetric distribution with zero mean and unit variance. The i th block of a matrix $\mathbf{X}$ (where $\mathbf{X}$ consists of a row or column of blocks, or diagonal blocks in case of a block-diagonal matrix) is represented by $\mathbf{X}^{(i)}$. For a matrix $\mathbf{X}$, we use $\bar{\mathbf{X}}$ to denote the submatrix that is constructed by removing some of the columns/rows of $\mathbf{X}$. Similarly we use $\tilde{\mathbf{X}}$ when removing columns/rows from $\bar{\mathbf{X}}$. If we use two different submatrices of $\mathbf{X}$ simultaneously, we denote them by $\bar{\mathbf{X}}$ and $\tilde{\mathbf{X}}$ to avoid confusion. When $\mathbf{X}_i = \mathbf{A}\mathbf{B}$, we use the notation $\mathbf{X}_i = \mathbf{X}_{i,1}\mathbf{X}_{i,2} + \mathbf{Y}$ to write a decomposition for $\mathbf{X}_i$ in which $\mathbf{X}_{i,1}$ contains some (or all) of the columns of $\mathbf{A}$ and $\mathbf{X}_{i,2}$ contains the corresponding rows of $\mathbf{B}$, and $\mathbf{Y}$ is the product of the remaining columns of $\mathbf{A}$ and the corresponding rows of $\mathbf{B}$. Finally, $\mathcal{R}(\cdot)$ denotes

the row space spanned by the rows of the matrix in the argument.

Our program for this paper is as follows. In Section II, the system model is described. In Section III, our unified DoF-achieving scheme is introduced. The proof of the achievability of the proposed scheme is detailed in Section IV. The extension of the proposed scheme to the two-user MIMO BC is presented in Section V. Finally, an efficient feedback method is presented in Section VI, followed by simulation results in Section VII.

II. SYSTEM MODEL

Consider a MIMO two-user interference channel. For user $i \in \{1, 2\}$, transmitter i and receiver i have M_i and N_i antennas, respectively. The channel between transmitter j and receiver i at time slot t is $\mathbf{H}_{ij}^{(t)} \in \mathbb{C}^{N_i \times M_j}$ for $i, j \in \{1, 2\}$. The channel matrices have random entries drawn from a continuous distribution, which are independent and identically distributed (i.i.d.) over time and space. The received signal at time instant t at receiver $i \neq j$ reads

$$\mathbf{y}_i^{(t)} = \mathbf{H}_{ii}^{(t)} \mathbf{V}_i^{(t)} \mathbf{x}_i + \mathbf{H}_{ij}^{(t)} \mathbf{V}_j^{(t)} \mathbf{x}_j + \mathbf{n}_i^{(t)}, \text{ for } t = 1, \dots, T$$

where $\mathbf{x}_j \in \mathbb{C}^{T d_j \times 1}$ is the symbol vector of transmitter j which is transmitted over T time slots (hence up to d_j DoF are achievable for transmitter j), $\mathbf{V}_j^{(t)} \in \mathbb{C}^{M_j \times T d_j}$ is the precoder of transmitter j at time slot t , and $\mathbf{n}_i^{(t)}$ is the additive white Gaussian noise vector at time slot t at receiver i whose entries have zero mean and unit variance.

The transmitters have access to delayed CSIT, i.e., at a given time instant t , the transmitters have perfect information about all the channels corresponding to the previous time slots, i.e., $\{\mathbf{H}_{ij}^{(t-1)}, \mathbf{H}_{ij}^{(t-2)}, \dots\}$ for $i, j \in \{1, 2\}$. By the assumption of temporal independence of the channels, the available delayed CSIT is completely outdated. In other words, the delayed CSIT does not provide any information about the current CSIT. The transmitters wish to convey their symbols to their respective receivers exploiting the outdated CSIT. DoF-achieving schemes are presented in [10] for different antenna configurations.

III. GENERAL ACHIEVABLE SCHEME FOR DIFFERENT ANTENNA CONFIGURATIONS

We provide a general achievable scheme, which encompasses all possible antenna configurations. If we stack the channel outputs and noise vectors (for all T time slots) as $\mathbf{y}_i = \begin{bmatrix} \mathbf{y}_i^{(1)} \\ \vdots \\ \mathbf{y}_i^{(T)} \end{bmatrix}$ and $\mathbf{n}_i = \begin{bmatrix} \mathbf{n}_i^{(1)} \\ \vdots \\ \mathbf{n}_i^{(T)} \end{bmatrix}$, then for $i, j \in \{1, 2\}$, $i \neq j$

$$\mathbf{y}_i = \mathbf{H}_{ii} \mathbf{V}_i \mathbf{x}_i + \mathbf{H}_{ij} \mathbf{V}_j \mathbf{x}_j + \mathbf{n}_i$$

where $\mathbf{V}_j = \begin{bmatrix} \mathbf{V}_j^{(1)} \\ \vdots \\ \mathbf{V}_j^{(T)} \end{bmatrix}$ and $\mathbf{H}_{ij} = \text{Bdiag}(\mathbf{H}_{ij}^{(1)}, \dots, \mathbf{H}_{ij}^{(T)})$. The proposed scheme has two phases (successive in time) where each phase is composed of a number of time slots. The scheme is based on transmitting with random precoders in phase 1 and transmitting with a designed precoder in phase 2. We assume that the duration of phase 1 for transmitter j is $T - q_i$ with $j \neq i \in \{1, 2\}$ where q_i will be determined in a way that satisfies the DoF requirement. At transmitter j , we denote the precoders of phase 1 and phase 2 by $\mathbf{V}_j^{\text{ph1}} = \begin{bmatrix} \mathbf{V}_j^{(1)} \\ \vdots \\ \mathbf{V}_j^{(T-q_i)} \end{bmatrix}$ and $\mathbf{V}_j^{\text{ph2}} = \begin{bmatrix} \mathbf{V}_j^{(T-q_i+1)} \\ \vdots \\ \mathbf{V}_j^{(T)} \end{bmatrix}$ respectively. Therefore transmitter j uses the last q_i time slots to facilitate the decoding at both receivers using a precoder in phase 2 designed to exploit the CSIT from phase 1.

Let us now consider the received signal space (signal and interference) at receiver i with $i, j \in \{1, 2\}$ and $i \neq j$. We have $\mathbf{R}_i = [\mathbf{H}_{ii} \mathbf{V}_i \quad \mathbf{H}_{ij} \mathbf{V}_j]$ which can be further written as:

$$\mathbf{R}_i = \begin{bmatrix} \mathbf{H}_{ii}^{(1)} \mathbf{V}_i^{(1)} & \mathbf{H}_{ij}^{(1)} \mathbf{V}_j^{(1)} \\ \mathbf{H}_{ii}^{(2)} \mathbf{V}_i^{(2)} & \mathbf{H}_{ij}^{(2)} \mathbf{V}_j^{(2)} \\ \vdots & \vdots \\ \mathbf{H}_{ii}^{(T)} \mathbf{V}_i^{(T)} & \mathbf{H}_{ij}^{(T)} \mathbf{V}_j^{(T)} \end{bmatrix} = \begin{bmatrix} \mathbf{H}_{ii}^{\text{ph1}} \mathbf{V}_i^{\text{ph1}} & \mathbf{H}_{ij}^{\text{ph1}} \mathbf{V}_j^{\text{ph1}} \\ \mathbf{H}_{ii}^{\text{ph2}} \mathbf{V}_i^{\text{ph2}} & \mathbf{H}_{ij}^{\text{ph2}} \mathbf{V}_j^{\text{ph2}} \end{bmatrix}.$$

where $\mathbf{H}_{ij}^{\text{ph1}} = \text{Bdiag}(\mathbf{H}_{ij}^{(1)}, \dots, \mathbf{H}_{ij}^{(T-q_i)})$ and $\mathbf{H}_{ij}^{\text{ph2}} = \text{Bdiag}(\mathbf{H}_{ij}^{(T-q_i+1)}, \dots, \mathbf{H}_{ij}^{(T)})$.

We define $d_1^* = Td_1$ and $d_2^* = Td_2$. Furthermore, q_1 and q_2 are chosen as follows. For $i, j \in \{1, 2\}$

Group	Name in [10]	Condition
G_2	A.I.3	$M_1 > N_1 \geq N_2$ and $M_2 > N_2$ and $M_2 > N_1$
G_1	A.II.2	$M_1 > N_1 > M_2 \geq N_2$
	B.I	$N_1 \geq M_1 > N_2 > M_2$
	B.II.1	$M_1 > N_1 > N_2 > M_2$ and $M_2 \leq m$
	B.II.2	$M_1 > N_1 > N_2 > M_2$ and $M_2 > m$ and $M_1 = M'_1$
	B.III.1	$M_1 > M'_1 > N_1 > N_2 > M_2 > m$ and $M_1 \geq N_1 + N_2 - m$
	B.III.2	$M_1 > M'_1 > N_1 > N_2 > M_2 > m$ and $M_1 < N_1 + N_2 - m$

TABLE I. Antenna configurations with DoF larger for delayed CSIT than without CSIT. We define $m = N_2 \frac{M'_1 - N_1}{M'_1 - N_2}$ and $M'_1 = \min(M_1, N_1 + N_2 - M_2)$ with the assumption of $N_1 \geq N_2$.

and $j \neq i$ we set

$$q_i = \begin{cases} \frac{d_j^*}{N_j} & d_1 + d_2 \leq N_i \\ \frac{d_i^*}{N_i} & d_1 + d_2 > N_i. \end{cases} \quad (1)$$

We will elaborate on the intuition behind this particular choice of q_1 and q_2 later on. We choose the number of time slots T such that d_1^*, d_2^* and q_1, q_2 are integers.

First we present our simple achievable scheme, which is based on retrospective interference alignment.

Proposition 1. *The following scheme almost surely achieves all the corner points (d_1, d_2) of the optimal DoF region for all antenna configurations where delayed CSIT is beneficial compared to no-CSIT.*

For $i, j \in \{1, 2\}$ and $j \neq i$, the precoder of transmitter j is designed as $\mathbf{V}_j = \begin{bmatrix} \mathbf{V}_j^{\text{ph1}} \\ \mathbf{V}_j^{\text{ph2}} \end{bmatrix}$ with $\mathbf{V}_j^{\text{ph1}} \in \mathbb{C}^{(T-q_i)M_j \times d_j^*}$ and $\mathbf{V}_j^{\text{ph2}} \in \mathbb{C}^{q_i M_j \times d_j^*}$.

- If $d_1 + d_2 \leq N_i$, randomly generate $\mathbf{V}_j^{\text{ph1}}$ and $\mathbf{V}_j^{\text{ph2}}$ (with i.i.d. Gaussian entries).
- If $d_1 + d_2 > N_i$, randomly generate $\mathbf{V}_j^{\text{ph1}}$ (with i.i.d. Gaussian entries). Set $\mathbf{V}_j^{\text{ph2}} = \mathbf{V}_{j,1}^{\text{ph2}} \mathbf{H}_{ij}^{\text{ph1}} \mathbf{V}_j^{\text{ph1}}$, where $\mathbf{V}_{j,1}^{\text{ph2}} \in \mathbb{C}^{q_i M_j \times (T-q_i)N_i}$ is generated randomly (with i.i.d. Gaussian entries).

Without loss of generality we assume that $N_1 \geq N_2$. According to [10] we list in Table I the configurations for which the delayed CSIT DoF region is strictly larger than the no-CSIT case.

Remark 1. *For the case A.I.3 we have $d_1 + d_2 > N_1 \geq N_2$ and for all other cases we have $N_2 < d_1 + d_2 \leq N_1$. This can be easily shown from the achievable points and the relations between number of antennas for each case.*

Interference alignment is needed when the rank of the received space is smaller than the total number of streams. As a consequence of Remark 1, for case A.I.3 we need to align interference at both receivers in order to achieve the DoF region, while for the other cases it is sufficient to align interference only at one receiver (receiver 2 in our setting). Therefore we divide all cases into the following two groups and prove the achievability for each group separately:

- Group G_1 : This group includes the cases A.II.2, B.I, B.II.1, B.II.2, B.III.1, B.III.2 (which satisfy $N_2 < d_1 + d_2 \leq N_1$)
- Group G_2 : This group includes only the case A.I.3 (which satisfies $d_1 + d_2 > N_1$).

Intuitively speaking, by choosing $\mathbf{V}_j^{\text{ph2}} = \mathbf{V}_{j,1}^{\text{ph2}} \mathbf{H}_{ij}^{\text{ph1}} \mathbf{V}_j^{\text{ph1}}$ for receiver i with $d_1 + d_2 > N_i$, the interference in phase 2 is aligned in the interference subspace of phase 1. In addition, an important aspect of our scheme is that we incorporate a random matrix $\mathbf{V}_{j,1}^{\text{ph2}}$ which is pre-multiplied by $\mathbf{H}_{ij}^{\text{ph1}} \mathbf{V}_j^{\text{ph1}}$ (which represents the interference subspace of phase 1). This is in fact not necessary for interference alignment, but ensures that the received signal space will maintain its maximum possible dimension after the alignment procedure. In other words, when we align the interference in a reduced-dimensional subspace at receiver i , it is possible that the entire received space (including desired signal and interference) also has a reduced dimension (less than TN_i), which means a reduced DoF for user i . We will show that the extra precoding in our scheme guarantees the maximum use of the available signal space without changing the dimension of the interference subspace.

Remark 2. In phase 2, transmitter 1 requires $\mathbf{H}_{21}^{\text{ph1}} \mathbf{V}_1^{\text{ph1}}$ and transmitter 2 requires $\mathbf{H}_{12}^{\text{ph1}} \mathbf{V}_2^{\text{ph1}}$. This means that, even though the phases are not synchronized (q_1 and q_2 might be different), the transmitters have access to the necessary information after transmitting in their corresponding first phase.

Let us now explain the intuition behind the choice of q_1 and q_2 according to (1). If alignment is required at both receivers ($d_1 + d_2 > N_1$) then the interference at receiver $i \in \{1, 2\}$ should be aligned in a subspace of dimension $(T - q_i)N_i$ (which is the rank of the interference space in phase 1 according to Lemma 1). This leaves $q_i N_i$ dimensions free for the desired signal, i.e., $d_i^* = q_i N_i$ for $i \in \{1, 2\}$. If alignment is required only at receiver 2 ($d_1 + d_2 \leq N_1$) then we only need $d_2^* = q_2 N_2$. For simplicity of analysis in this case we also choose $q_1 = q_2$. Putting all these in compact form gives (1).

In the next section we prove the achievability of our scheme. Readers not interested in this proof may

skip Section IV and proceed directly to Section V.

IV. PROOF OF THE ACHIEVABILITY OF THE PROPOSED SCHEME

The overall procedure to prove the achievability of our scheme consists of the following key steps:

- First we establish some properties of operations involving random matrices (in Lemmas 1 and 2). These properties will be helpful throughout the proofs.
- The corner points of the DoF region depend on the different cases in each of the groups G_1 and G_2 . We show (in Lemmas 3 and 4) that for all the corner points corresponding to one of the groups, a certain set of conditions is always satisfied. These conditions are equalities and inequalities that are derived based on the antenna configurations in each group.
- There are separate proofs of achievability for group G_1 (see Section IV-A) and G_2 (see Section IV-B). Instead of proving the achievability for every single corner point in every case, we provide a single proof which covers all the points that belong to one of the groups. Throughout the proof we use the conditions derived earlier for each group. The proof for a given group proceeds as follows:
 - First we show that the interference subspace has a maximum rank of r_i at receiver i .
 - After that, we prove theorems that at each receiver i , $\text{rank}(\mathbf{R}_i) = d_i^* + r_i$. Since the interference has a maximum rank of r_i , the desired messages can be successfully decoded.

Lemma 1. *Assume that $\mathbf{H}$ is a block-diagonal matrix with L blocks on its diagonal and each block is an $a \times b$ matrix with random i.i.d. elements from a continuous probability distribution and $\mathbf{V}$ is an $Lb \times s$ matrix with random i.i.d. elements also from a continuous distribution. Then, almost surely we have*

$$\text{rank}(\mathbf{H}\mathbf{V}) = \min(\text{rank}(\mathbf{H}), \text{rank}(\mathbf{V})) = \min(La, Lb, s).$$

Proof. See [1]. □

Lemma 2. *Consider the matrix $\mathbf{Z} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{0} \end{bmatrix}$ where $\mathbf{A} \in \mathbb{C}^{a \times l}$, $\mathbf{B} \in \mathbb{C}^{a \times (a+b-l)}$ (with $l \leq a+b$) and $\mathbf{C} \in \mathbb{C}^{b \times l}$ are all matrices with random i.i.d. elements. The matrix $\mathbf{Z}$ has full rank almost surely when $l \geq b$.*

Proof. See Appendix A. □

Lemma 3. *The following conditions hold for the corner points of the DoF region for all cases in group G_1 in Table I (in this group we define $q \triangleq q_1 = q_2$):*

$$\text{C.I : } d_i \leq \min(M_i, N_i), \ i \in \{1, 2\} \text{ and } N_2 < d_1 + d_2 \leq N_1$$

$$\text{C.II : } T(d_1 + d_2) \leq (T - q)N_1 + l$$

$$\text{C.III : } Td_1 \leq (T - q)M_1$$

$$\text{with } l = \min \left(qN_1, \min(qM_1, qN_1, (T - q)N_2) + \min(qM_2, qN_2) \right).$$

Proof. See Appendix B. □

Lemma 4. *The following conditions hold for the corner points of the DoF region for the case in group G_2 in Table I:*

$$\bar{\text{C.I : }} d_i \leq \min(M_i, N_i), \ i \in \{1, 2\} \text{ and } d_1 + d_2 > N_1 \geq N_2$$

$$\bar{\text{C.II : }} Td_j = (T - q_i)M_j''$$

$$\text{with } M_j'' = \min(M_j, N_1 + N_2), \ i, j \in \{1, 2\}, \ i \neq j.$$

Proof. See Appendix C. □

Remark 3. *In order to show the achievability of a point (d_1, d_2) it is sufficient to show that*

$$\text{rank}(\mathbf{R}_i) = d_i^* + r_i, \text{ for } i \in \{1, 2\}$$

where $\text{rank}(\mathbf{H}_{ij}\mathbf{V}_j) \leq r_i$ for $j \neq i$.

Remark 3 says that if the received space is large enough to accommodate the desired signal and interference (which is assumed to have a rank not greater than r_i), then the desired signal can be decoded.

We now present the achievability proofs for groups G_1 and G_2 .

A. Transmission Scheme for Group G_1

In this group we have $N_2 < d_1 + d_2 \leq N_1$. Therefore according to (1) we have $q_1 = q_2 = \frac{d_2^*}{N_2}$. For simplicity we define $q \triangleq q_1 = q_2 = \frac{d_2^*}{N_2}$. According to Proposition 1, $\mathbf{V}_2^{\text{ph1}} \in \mathbb{C}^{(T-q)M_2 \times d_2^*}$ and

$\mathbf{V}_2^{\text{ph2}} \in \mathbb{C}^{qM_2 \times d_2^*}$ are generated randomly. Furthermore $\mathbf{V}_1^{\text{ph1}} \in \mathbb{C}^{(T-q)M_1 \times d_1^*}$ is generated randomly and $\mathbf{V}_1^{\text{ph2}} \in \mathbb{C}^{qM_1 \times d_1^*}$ is constructed as $\mathbf{V}_1^{\text{ph2}} = \mathbf{V}_{1,1}^{\text{ph2}} \mathbf{H}_{21}^{\text{ph1}} \mathbf{V}_1^{\text{ph1}}$ where $\mathbf{V}_{1,1}^{\text{ph2}} \in \mathbb{C}^{qM_1 \times (T-q)N_2}$ is generated randomly. For $i, j \in \{1, 2\}$, the channels of phase 1 and 2 are $\mathbf{H}_{ij}^{\text{ph1}} \in \mathbb{C}^{(T-q)N_i \times (T-q)M_j}$ and $\mathbf{H}_{ij}^{\text{ph2}} \in \mathbb{C}^{qN_i \times qM_j}$ respectively.

If we consider the matrix that spans the interference subspace at each receiver, i.e., $\mathbf{H}_{ij} \mathbf{V}_j$, we have

$$\text{rank}(\mathbf{H}_{12} \mathbf{V}_2) \leq d_2^*,$$

$$\text{rank}(\mathbf{H}_{21} \mathbf{V}_1) = \text{rank} \left(\begin{bmatrix} \mathbf{H}_{21}^{\text{ph1}} \mathbf{V}_1^{\text{ph1}} \\ \mathbf{H}_{21}^{\text{ph2}} \mathbf{V}_1^{\text{ph2}} \end{bmatrix} \right) = \text{rank} \left(\begin{bmatrix} \mathbf{I} \\ \mathbf{H}_{21}^{\text{ph2}} \mathbf{V}_{1,1}^{\text{ph2}} \end{bmatrix} \mathbf{H}_{21}^{\text{ph1}} \mathbf{V}_1^{\text{ph1}} \right) \leq (T-q)N_2.$$

In other words, at receiver 2, the interference subspace of phase 2 is aligned within the interference subspace of phase 1 since $\mathbf{H}_{21}^{\text{ph2}} \mathbf{V}_1^{\text{ph2}} = \mathbf{H}_{21}^{\text{ph2}} \mathbf{V}_{1,1}^{\text{ph2}} \mathbf{H}_{21}^{\text{ph1}} \mathbf{V}_1^{\text{ph1}} \in \mathcal{R}(\mathbf{H}_{21}^{\text{ph1}} \mathbf{V}_1^{\text{ph1}})$. Therefore according to Remark 3 we can set $r_1 = d_2^*$ and $r_2 = (T-q)N_2$.

Theorem 1. For group G_1 , the scheme in Proposition 1 gives

$$\text{rank}(\mathbf{R}_1) = d_1^* + r_1 = d_1^* + d_2^*$$

Proof. $\mathbf{R}_1$ can be written as

$$\mathbf{R}_1 = \begin{bmatrix} \mathbf{I}_{(T-q)N_1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{11}^{\text{ph2}} & \mathbf{H}_{12}^{\text{ph2}} \end{bmatrix} \begin{bmatrix} \mathbf{H}_{11}^{\text{ph1}} \mathbf{V}_1^{\text{ph1}} & \mathbf{H}_{12}^{\text{ph1}} \mathbf{V}_2^{\text{ph1}} \\ \mathbf{V}_{1,1}^{\text{ph2}} \mathbf{H}_{21}^{\text{ph1}} \mathbf{V}_1^{\text{ph1}} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_2^{\text{ph2}} \end{bmatrix}.$$

Defining $l = \min(qN_1, \min(qM_1, qN_1, (T-q)N_2) + \min(qM_2, qN_2))$, we construct $\bar{\mathbf{H}}_{12}^{\text{ph2}} \in \mathbb{C}^{qN_1 \times \min(qM_2, qN_2)}$ and $\bar{\mathbf{V}}_2^{\text{ph2}} \in \mathbb{C}^{\min(qM_2, qN_2) \times d_2^*}$ by removing $qM_2 - \min(qM_2, qN_2)$ columns from $\mathbf{H}_{12}^{\text{ph2}}$ and the corresponding rows from $\mathbf{V}_2^{\text{ph2}}$ such that every block on the diagonal of $\bar{\mathbf{H}}_{12}^{\text{ph2}}$ has $\min(M_2, N_2)$ columns. Furthermore, we construct $\bar{\mathbf{H}}_{11}^{\text{ph2}} \in \mathbb{C}^{qN_1 \times [l - \min(qM_2, qN_2)]}$ and $\bar{\mathbf{V}}_{1,1}^{\text{ph2}} \in \mathbb{C}^{[l - \min(qM_2, qN_2)] \times (T-q)N_2}$ by removing columns and rows from $\mathbf{H}_{11}^{\text{ph2}}$ and $\mathbf{V}_{1,1}^{\text{ph2}}$ respectively such that every block on the diagonal of $\bar{\mathbf{H}}_{11}^{\text{ph2}}$ has at most $N_1 - \min(M_2, N_2)$ columns. The choice of the columns for removal is arbitrary as long as the conditions regarding the number of remaining columns on each block are respected. Therefore

$$\mathbf{R}_1 = \mathbf{R}_{1,1} \mathbf{R}_{1,2} + \mathbf{Q}$$

with

$$\mathbf{R}_{1,1} = \begin{bmatrix} \mathbf{I}_{(T-q)N_1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{H}}_{11}^{\text{ph2}} & \tilde{\mathbf{H}}_{12}^{\text{ph2}} \end{bmatrix}, \quad \mathbf{R}_{1,2} = \begin{bmatrix} \mathbf{H}_{11}^{\text{ph1}} \mathbf{V}_1^{\text{ph1}} & \mathbf{H}_{12}^{\text{ph1}} \mathbf{V}_2^{\text{ph1}} \\ \tilde{\mathbf{V}}_{1,1}^{\text{ph2}} \mathbf{H}_{21}^{\text{ph1}} \mathbf{V}_1^{\text{ph1}} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{V}}_2^{\text{ph2}} \end{bmatrix},$$

$$\mathbf{Q} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{H}}_{11}^{\text{ph2}} & \tilde{\mathbf{H}}_{12}^{\text{ph2}} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \tilde{\mathbf{V}}_{1,1}^{\text{ph2}} \mathbf{H}_{21}^{\text{ph1}} \mathbf{V}_1^{\text{ph1}} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{V}}_2^{\text{ph2}} \end{bmatrix}$$

where $\tilde{\mathbf{H}}_{11}^{\text{ph2}} \in \mathbb{C}^{qN_1 \times [qM_1 - l + \min(qM_2, qN_2)]}$ and $\tilde{\mathbf{H}}_{12}^{\text{ph2}} \in \mathbb{C}^{qN_1 \times [qM_2 - \min(qM_2, qN_2)]}$ are composed of the excluded columns from $\mathbf{H}_{11}^{\text{ph2}}$ and $\mathbf{H}_{12}^{\text{ph2}}$ respectively. Similarly, $\tilde{\mathbf{V}}_{1,1}^{\text{ph2}} \in \mathbb{C}^{[qM_1 - l + \min(qM_2, qN_2)] \times (T-q)N_2}$ and $\tilde{\mathbf{V}}_2^{\text{ph2}} \in \mathbb{C}^{[qM_2 - \min(qM_2, qN_2)] \times d_2^*}$ are composed of the excluded rows from $\mathbf{V}_{1,1}^{\text{ph2}}$ and $\mathbf{V}_2^{\text{ph2}}$ respectively. The entries of $\mathbf{Q}$ are either zero or random values independent of the entries of $\mathbf{R}_{1,1}\mathbf{R}_{1,2}$. Although $\mathbf{V}_1^{\text{ph1}}$ appears in both $\mathbf{Q}$ and $\mathbf{R}_{1,1}\mathbf{R}_{1,2}$, it can be simply shown that the entries of $\mathbf{Q}$ are independent from the entries in $\mathbf{R}_{1,1}\mathbf{R}_{1,2}$ (using the fact that all the precoders have Gaussian i.i.d. elements). Therefore we can conclude that adding $\mathbf{Q}$ to $\mathbf{R}_{1,1}\mathbf{R}_{1,2}$ will almost surely not decrease the rank. Hence

$$\text{rank}(\mathbf{R}_1) \geq \text{rank}(\mathbf{R}_{1,1}\mathbf{R}_{1,2}). \quad (2)$$

For the matrix $\mathbf{R}_{1,1}$ we have

$$\text{rank}(\mathbf{R}_{1,1}) = (T-q)N_1 + \text{rank} \left(\begin{bmatrix} \tilde{\mathbf{H}}_{11}^{\text{ph2}} & \tilde{\mathbf{H}}_{12}^{\text{ph2}} \end{bmatrix} \right)$$

where $\begin{bmatrix} \tilde{\mathbf{H}}_{11}^{\text{ph2}} & \tilde{\mathbf{H}}_{12}^{\text{ph2}} \end{bmatrix}$ is of size $qN_1 \times (l - \min(qM_2, qN_2))$. Since $\begin{bmatrix} \tilde{\mathbf{H}}_{11}^{\text{ph2}} & \tilde{\mathbf{H}}_{12}^{\text{ph2}} \end{bmatrix}$ contains two block-diagonal matrices, by changing the position of columns we can turn $\begin{bmatrix} \tilde{\mathbf{H}}_{11}^{\text{ph2}} & \tilde{\mathbf{H}}_{12}^{\text{ph2}} \end{bmatrix}$ into a block-diagonal matrix with q blocks, where each block has N_1 rows and at most N_1 columns (according to the construction). Note that the rank will not change by changing the order of columns in the matrix. Therefore each block of the resulting matrix will have full column rank, which is less than or equal to N_1 . We conclude that $\begin{bmatrix} \tilde{\mathbf{H}}_{11}^{\text{ph2}} & \tilde{\mathbf{H}}_{12}^{\text{ph2}} \end{bmatrix}$ has full column rank, which is less than or equal to qN_1 . This implies that $\mathbf{R}_{1,1}$ has full column rank. This yields

$$\text{rank}(\mathbf{R}_{1,1}\mathbf{R}_{1,2}) = \text{rank}(\mathbf{R}_{1,2}). \quad (3)$$

Now we consider $\mathbf{R}_{1,2}$. The matrix $\mathbf{R}_{1,2}$ can be written as

$$\mathbf{R}_{1,2} = \begin{bmatrix} \mathbf{I}_{(T-q)N_1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{V}}_{1,1}^{\text{ph2}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{\min(qM_2, qN_2)} \end{bmatrix} \begin{bmatrix} \mathbf{H}_{11}^{\text{ph1}} \mathbf{V}_1^{\text{ph1}} & \mathbf{H}_{12}^{\text{ph1}} \mathbf{V}_2^{\text{ph1}} \\ \mathbf{H}_{21}^{\text{ph1}} \mathbf{V}_1^{\text{ph1}} & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{V}}_2^{\text{ph2}} \end{bmatrix}.$$

By eliminating rows and columns we have

$$\text{rank}(\mathbf{R}_{1,2}) \geq \text{rank}(\mathbf{R}_{1,2,1} \mathbf{R}_{1,2,2}) \quad (4)$$

where

$$\mathbf{R}_{1,2,1} = \begin{bmatrix} \mathbf{I}_{(T-q)N_1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \bar{\bar{\mathbf{V}}}_{1,1}^{\text{ph2}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{\min(qM_2, qN_2)} \end{bmatrix}, \quad \mathbf{R}_{1,2,2} = \begin{bmatrix} \mathbf{H}_{11}^{\text{ph1}} \mathbf{V}_1^{\text{ph1}} & \mathbf{H}_{12}^{\text{ph1}} \mathbf{V}_2^{\text{ph1}} \\ \bar{\mathbf{H}}_{21}^{\text{ph1}} \mathbf{V}_1^{\text{ph1}} & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{V}}_2^{\text{ph2}} \end{bmatrix}$$

where we construct $\bar{\mathbf{H}}_{21}^{\text{ph1}} \in \mathbb{C}^{(l-\min(qM_2, qN_2)) \times (T-q)M_1}$ and $\bar{\bar{\mathbf{V}}}_{1,1}^{\text{ph2}} \in \mathbb{C}^{(l-\min(qM_2, qN_2)) \times (l-\min(qM_2, qN_2))}$ by removing rows from $\mathbf{H}_{21}^{\text{ph1}}$ and the corresponding columns from $\bar{\mathbf{V}}_{1,1}^{\text{ph2}}$ as follows. From $\mathbf{H}_{21}^{\text{ph1}}$ we remove one row from the first block, then one row from the second block, and so on, until we reach the final block. Then we start over by removing a second row from the first block, then the second block, etc. This procedure is aborted as soon as we reach the desired number of rows. In this way the i th block of $\bar{\mathbf{H}}_{21}^{\text{ph1}}$ will have $s_i = h_1$ or $s_i = h_1 - 1$ rows for some $h_1 \leq N_2$ and we have $\sum_{i=1}^{T-q} s_i = l - \min(qM_2, qN_2)$. Since $\mathbf{R}_{1,2,1}$ is square and has full rank,

$$\text{rank}(\mathbf{R}_{1,2,1} \mathbf{R}_{1,2,2}) = \text{rank}(\mathbf{R}_{1,2,2}). \quad (5)$$

Thus we need to focus on $\mathbf{R}_{1,2,2}$.

It can be shown that up to a permutation of rows (which does not change the rank), $\mathbf{R}_{1,2,2}$ can be written as

$$\mathbf{R}'_{1,2,2} = \mathbf{H}_e \mathbf{V}_e \quad (6)$$

where

$$\text{rank}(\mathbf{R}'_{1,2,2}) = \text{rank}(\mathbf{R}_{1,2,2}), \quad (7)$$

$$\mathbf{H}_e = \text{Bdiag} \left(\begin{bmatrix} \mathbf{H}_{11}^{(1)} & \mathbf{H}_{12}^{(1)} \\ \bar{\mathbf{H}}_{21}^{(1)} & \mathbf{0} \end{bmatrix}, \dots, \begin{bmatrix} \mathbf{H}_{11}^{(T-q)} & \mathbf{H}_{12}^{(T-q)} \\ \bar{\mathbf{H}}_{21}^{(T-q)} & \mathbf{0} \end{bmatrix}, \mathbf{I}_{q \min(M_2, N_2)} \right), \quad \mathbf{V}_e = \begin{bmatrix} \mathbf{V}_1^{(1)} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_2^{(1)} \\ \mathbf{V}_1^{(2)} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_2^{(2)} \\ \vdots & \vdots \\ \mathbf{V}_1^{(T-q)} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_2^{(T-q)} \\ \mathbf{0} & \bar{\mathbf{V}}_2^{\text{ph2}} \end{bmatrix}.$$

Further we have

$$\text{rank}(\mathbf{H}_e \mathbf{V}_e) \geq \text{rank}(\bar{\mathbf{H}}_e \bar{\mathbf{V}}_e) \quad (8)$$

where

$$\bar{\mathbf{H}}_e = \text{Bdiag} \left(\begin{bmatrix} \bar{\mathbf{H}}_{11}^{(1)} & \bar{\mathbf{H}}_{12}^{(1)} \\ \bar{\bar{\mathbf{H}}}_{21}^{(1)} & \mathbf{0} \end{bmatrix}, \dots, \begin{bmatrix} \bar{\mathbf{H}}_{11}^{(T-q)} & \bar{\mathbf{H}}_{12}^{(T-q)} \\ \bar{\bar{\mathbf{H}}}_{21}^{(T-q)} & \mathbf{0} \end{bmatrix}, \mathbf{I}_{q \min(M_2, N_2)} \right), \quad \bar{\mathbf{V}}_e = \begin{bmatrix} \bar{\mathbf{V}}_1^{(1)} & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{V}}_2^{(1)} \\ \bar{\mathbf{V}}_1^{(2)} & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{V}}_2^{(2)} \\ \vdots & \vdots \\ \bar{\mathbf{V}}_1^{(T-q)} & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{V}}_2^{(T-q)} \\ \mathbf{0} & \bar{\mathbf{V}}_2^{\text{ph2}} \end{bmatrix}.$$

Here, $\bar{\mathbf{H}}_{11}^{(i)}$ and $\bar{\bar{\mathbf{H}}}_{21}^{(i)}$ consist of a_i (arbitrary) columns from $\mathbf{H}_{11}^{(i)}$ and $\bar{\mathbf{H}}_{21}^{(i)}$ respectively and $\bar{\mathbf{H}}_{12}^{(i)}$ consists of b_i (arbitrary) columns from $\mathbf{H}_{12}^{(i)}$. Accordingly $\bar{\mathbf{V}}_1^{(i)}$ and $\bar{\mathbf{V}}_2^{(i)}$ have a_i and b_i rows respectively. The values a_i and b_i are defined as $a_i = s_i + \alpha_i$ where s_i is the number of rows in $\bar{\mathbf{H}}_{21}^{(i)}$ and the α_i values are non-negative integers that must be chosen such that $\text{rank}(\bar{\mathbf{V}}_e) = d_1^* + d_2^*$ and $a_i \leq M_1, \forall i$. We will momentarily elaborate on how to do that. Furthermore, $b_i = \min(N_1 - \alpha_i, M_2)$. Therefore every block $\begin{bmatrix} \bar{\mathbf{H}}_{11}^{(i)} & \bar{\mathbf{H}}_{12}^{(i)} \\ \bar{\bar{\mathbf{H}}}_{21}^{(i)} & \mathbf{0} \end{bmatrix}$ will have full column rank of $\min(N_1 + s_i, \alpha_i + s_i + M_2)$ according to Lemma 2. This

means that $\bar{\mathbf{H}}_e$ has full column rank and thus

$$\text{rank}(\bar{\mathbf{H}}_e \bar{\mathbf{V}}_e) = \text{rank}(\bar{\mathbf{V}}_e). \quad (9)$$

Now we show how to choose the α_i values such that $\text{rank}(\bar{\mathbf{V}}_e) = d_1^* + d_2^*$ and $a_i \leq M_1, \forall i$: We know that $\bar{\mathbf{H}}_{21}^{(i)}$ has h_1 or $h_1 - 1$ rows, which means that for every i , $s_i = h_1$ or $s_i = h_1 - 1$. We initialize all α_i values with zeros and increase one α_i by 1 at a time (starting from those i indices where $s_i = h_1 - 1$). Then we move to another α_i and repeat this process until $\sum_{i=1}^{T-q} \alpha_i = d_1^* - l + \min(qM_2, qN_2)$. This procedure leads to the following conditions:

$$1) a_i \leq M_1, \forall i$$

$$2) \text{ Either } N_1 - \alpha_i \leq M_2, \forall i \text{ or } N_1 - \alpha_i \geq M_2, \forall i.$$

Using C.III, condition 1) holds because $\sum_{i=1}^{T-q} a_i = \sum_{i=1}^{T-q} s_i + \sum_{i=1}^{T-q} \alpha_i = d_1^* \leq (T - q)M_1$ by construction, so that the a_i values cannot differ by more than 1 (since we start incrementing α_i from those i indices that satisfy $s_i = h_1 - 1$). Therefore for each i we have $a_i \leq M_1$ in order to satisfy $\sum_{i=1}^{T-q} a_i \leq (T - q)M_1$. Condition 2) is based on a similar argument because the α_i values in our construction cannot differ by more than 1. Consequently only one of the inequalities in 2) holds for all values of i .

Now we have

$$\text{rank}(\bar{\mathbf{V}}_e) = \text{rank} \left(\begin{bmatrix} \bar{\mathbf{V}}_1^{(1)} & \mathbf{0} \\ \bar{\mathbf{V}}_1^{(2)} & \mathbf{0} \\ \vdots & \vdots \\ \bar{\mathbf{V}}_1^{(T-q)} & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{V}}_2^{(1)} \\ \mathbf{0} & \bar{\mathbf{V}}_2^{(2)} \\ \vdots & \vdots \\ \mathbf{0} & \bar{\mathbf{V}}_2^{(T-q)} \\ \mathbf{0} & \bar{\mathbf{V}}_2^{\text{ph2}} \end{bmatrix} \right) = \min \left(d_1^*, \sum_{i=1}^{T-q} a_i \right) + \min \left(d_2^*, \sum_{i=1}^{T-q} b_i + \min(qM_2, qN_2) \right).$$

We know that $d_1^* = \sum_{i=1}^{T-q} a_i$. Using condition 2) we have

$$\begin{aligned} \sum_{i=1}^{T-q} b_i &= \sum_{i=1}^{T-q} \min(N_1 - \alpha_i, M_2) = \min \left((T-q)N_1 - \sum_{i=1}^{T-q} \alpha_i, (T-q)M_2 \right) \\ &= \min \left((T-q)N_1 - d_1^* + l - \min(qM_2, qN_2), (T-q)M_2 \right). \end{aligned}$$

Next we discuss all possible cases: If $N_2 \leq M_2$ then we immediately have $\text{rank}(\bar{\mathbf{V}}_e) = d_1^* + d_2^*$ because $d_2^* = qN_2$. If $N_2 > M_2$ then there are two possibilities: If $(T-q)M_2 \leq (T-q)N_1 - d_1^* + l - \min(qM_2, qN_2)$ then

$$\text{rank}(\bar{\mathbf{V}}_e) = d_1^* + \min \left(d_2^*, (T-q)M_2 + qM_2 \right) = d_1^* + \min(d_2^*, TM_2) = d_1^* + d_2^*.$$

On the other hand, if $(T-q)M_2 > (T-q)N_1 - d_1^* + l - \min(qM_2, qN_2)$, then

$$\text{rank}(\bar{\mathbf{V}}_e) = d_1^* + \min \left(d_2^*, (T-q)N_1 - d_1^* + l - \min(qM_2, qN_2) + qM_2 \right) = d_1^* + d_2^*$$

where we have used condition C.II. Therefore in all cases $\text{rank}(\bar{\mathbf{V}}_e) = d_1^* + d_2^*$. Hence from (2), (3), (5), (4), (6), (7), (8) and (9) we get

$$\begin{aligned} \text{rank}(\mathbf{R}_1) &\geq \text{rank}(\mathbf{R}_{1,1}\mathbf{R}_{1,2}) = \text{rank}(\mathbf{R}_{1,2}) \geq \text{rank}(\mathbf{R}_{1,2,1}\mathbf{R}_{1,2,2}) = \text{rank}(\mathbf{R}_{1,2,2}) \\ &= \text{rank}(\mathbf{R}'_{1,2,2}) = \text{rank}(\mathbf{H}_e\mathbf{V}_e) \geq \text{rank}(\bar{\mathbf{H}}_e\bar{\mathbf{V}}_e) = \text{rank}(\bar{\mathbf{V}}_e) = d_1^* + d_2^* \end{aligned}$$

which means $\text{rank}(\mathbf{R}_1) = d_1^* + d_2^*$. □

Theorem 2. For group G_1 , the scheme in Proposition 1 gives

$$\text{rank}(\mathbf{R}_2) = d_2^* + r_2 = TN_2. \quad (10)$$

Proof. $\mathbf{R}_2$ can be written as

$$\mathbf{R}_2 = \begin{bmatrix} \mathbf{I}_{(T-q)N_2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{22}^{\text{ph2}} & \mathbf{H}_{21}^{\text{ph2}} \end{bmatrix} \begin{bmatrix} \mathbf{H}_{22}^{\text{ph1}}\mathbf{V}_2^{\text{ph1}} & \mathbf{H}_{21}^{\text{ph1}}\mathbf{V}_1^{\text{ph1}} \\ \mathbf{V}_2^{\text{ph2}} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_{1,1}^{\text{ph2}}\mathbf{H}_{21}^{\text{ph1}}\mathbf{V}_1^{\text{ph1}} \end{bmatrix}.$$

We construct $\bar{\mathbf{H}}_{22}^{\text{ph2}}$ and $\bar{\mathbf{V}}_2^{\text{ph2}}$ by removing $qM_2 - \min(qM_2, qN_2)$ columns from $\mathbf{H}_{22}^{\text{ph2}}$ and the corresponding rows from $\mathbf{V}_2^{\text{ph2}}$ such that every block on the diagonal of $\bar{\mathbf{H}}_{22}^{\text{ph2}}$ has $\min(M_2, N_2)$ columns.

Furthermore, we construct $\bar{\mathbf{H}}_{21}^{\text{ph2}} \in \mathbb{C}^{qN_2 \times [qN_2 - \min(qM_2, qN_2)]}$ and $\bar{\mathbf{V}}_{1,1}^{\text{ph2}} \in \mathbb{C}^{[qN_2 - \min(qM_2, qN_2)] \times (T-q)N_2}$ by removing columns and rows from $\mathbf{H}_{21}^{\text{ph2}}$ and $\mathbf{V}_{1,1}^{\text{ph2}}$ respectively such that every block on the diagonal of $\bar{\mathbf{H}}_{21}^{\text{ph2}}$ has $N_2 - \min(M_2, N_2)$ columns. Therefore we have

$$\text{rank}(\mathbf{R}_2) \geq \text{rank}(\mathbf{R}_{2,1}\mathbf{R}_{2,2}) \quad (11)$$

where

$$\mathbf{R}_{2,1} = \begin{bmatrix} \mathbf{I}_{(T-q)N_2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{H}}_{22}^{\text{ph2}} & \bar{\mathbf{H}}_{21}^{\text{ph2}} \end{bmatrix}, \quad \mathbf{R}_{2,2} = \begin{bmatrix} \mathbf{H}_{22}^{\text{ph1}} \mathbf{V}_2^{\text{ph1}} & \mathbf{H}_{21}^{\text{ph1}} \mathbf{V}_1^{\text{ph1}} \\ \bar{\mathbf{V}}_2^{\text{ph2}} & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{V}}_{1,1}^{\text{ph2}} \mathbf{H}_{21}^{\text{ph1}} \mathbf{V}_1^{\text{ph1}} \end{bmatrix}$$

where we have used again the argument that the rank will not be reduced by addition of an independent matrix with zero or random elements.

Similar to $\mathbf{R}_{1,1}$ in the previous section, it can be shown that $\mathbf{R}_{2,1}$ has full column rank. As a result,

$$\text{rank}(\mathbf{R}_{2,1}\mathbf{R}_{2,2}) = \text{rank}(\mathbf{R}_{2,2}).$$

The matrix $\mathbf{R}_{2,2}$ can be written as

$$\mathbf{R}_{2,2} = \begin{bmatrix} \mathbf{I}_{(T-q)N_2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{\min(qM_2, qN_2)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{V}}_{1,1}^{\text{ph2}} \end{bmatrix} \begin{bmatrix} \mathbf{H}_{22}^{\text{ph1}} \mathbf{V}_2^{\text{ph1}} & \mathbf{H}_{21}^{\text{ph1}} \mathbf{V}_1^{\text{ph1}} \\ \bar{\mathbf{V}}_2^{\text{ph2}} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{21}^{\text{ph1}} \mathbf{V}_1^{\text{ph1}} \end{bmatrix}$$

We can further write

$$\text{rank}(\mathbf{R}_{2,2}) \geq \text{rank}(\mathbf{R}_{2,2,1}\mathbf{R}_{2,2,2}) \quad (12)$$

in which

$$\mathbf{R}_{2,2,1} = \begin{bmatrix} \mathbf{I}_{(T-q)N_2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{\min(qM_2, qN_2)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \bar{\bar{\mathbf{V}}}_{1,1}^{\text{ph2}} \end{bmatrix}, \quad \mathbf{R}_{2,2,2} = \begin{bmatrix} \mathbf{H}_{22}^{\text{ph1}} \mathbf{V}_2^{\text{ph1}} & \mathbf{H}_{21}^{\text{ph1}} \mathbf{V}_1^{\text{ph1}} \\ \bar{\mathbf{V}}_2^{\text{ph2}} & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{H}}_{21}^{\text{ph1}} \mathbf{V}_1^{\text{ph1}} \end{bmatrix}$$

where we construct $\bar{\bar{\mathbf{V}}}_{1,1}^{\text{ph2}} \in \mathbb{C}^{(qN_2 - \min(qM_2, qN_2)) \times (qN_2 - \min(qM_2, qN_2))}$ and $\bar{\mathbf{H}}_{21}^{\text{ph1}} \in \mathbb{C}^{(qN_2 - \min(qM_2, qN_2)) \times (T-q)M_1}$ by removing columns and rows from $\bar{\mathbf{V}}_{1,1}^{\text{ph2}}$ and $\mathbf{H}_{21}^{\text{ph1}}$ respectively. The process of removing the rows

from $\mathbf{H}_{21}^{\text{ph1}}$ is such that we start by removing one row from the first block, then the second block and so on (if necessary going back to the first block and removing further rows) until we reach the desired number of rows. Eventually the number of rows in each block will be less than or equal to M_2 since the desired total number of rows is $qN_2 - \min(qM_2, qN_2)$ which is less than or equal to $(T - q)M_2$.

The southeast block in $\mathbf{R}_{2,2,2}$ can be made equal to $\mathbf{0}$ by subtracting the corresponding rows from the northern blocks, which does not change the rank. This is possible since every row in $\bar{\mathbf{H}}_{21}^{\text{ph1}} \mathbf{V}_1^{\text{ph1}}$ is contained in $\mathbf{H}_{21}^{\text{ph1}} \mathbf{V}_1^{\text{ph1}}$. After this operation the resulting matrix is

$$\mathbf{R}'_{2,2,2} = \begin{bmatrix} \mathbf{H}_{22}^{\text{ph1}} \mathbf{V}_2^{\text{ph1}} & \mathbf{H}_{21}^{\text{ph1}} \mathbf{V}_1^{\text{ph1}} \\ \bar{\mathbf{V}}_2^{\text{ph2}} & \mathbf{0} \\ -\bar{\mathbf{H}}_{22}^{\text{ph1}} \mathbf{V}_2^{\text{ph1}} & \mathbf{0} \end{bmatrix}$$

where $\bar{\mathbf{H}}_{22}^{\text{ph1}} \in \mathbb{C}^{(qN_2 - \min(qM_2, qN_2)) \times (T-q)M_2}$ is constructed from $\mathbf{H}_{22}^{\text{ph1}}$ by removing the rows that correspond to the rows that were removed from $\mathbf{H}_{21}^{\text{ph1}}$ (to build $\bar{\mathbf{H}}_{21}^{\text{ph1}}$). From Lemma 1 we have $\text{rank}(\mathbf{H}_{21}^{\text{ph1}} \mathbf{V}_1^{\text{ph1}}) = (T - q)N_2$. This yields

$$\begin{aligned} \text{rank}(\mathbf{R}'_{2,2,2}) &\geq \text{rank} \left(\begin{bmatrix} \bar{\mathbf{V}}_2^{\text{ph2}} \\ -\bar{\mathbf{H}}_{22}^{\text{ph1}} \mathbf{V}_2^{\text{ph1}} \end{bmatrix} \right) + \text{rank}(\mathbf{H}_{21}^{\text{ph1}} \mathbf{V}_1^{\text{ph1}}) \\ &= \text{rank} \left(\begin{bmatrix} \mathbf{I}_{\min(qM_2, qN_2)} & \mathbf{0} \\ \mathbf{0} & -\bar{\mathbf{H}}_{22}^{\text{ph1}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{V}}_2^{\text{ph2}} \\ \mathbf{V}_2^{\text{ph1}} \end{bmatrix} \right) + (T - q)N_2 \\ &\geq \text{rank} \left(\begin{bmatrix} \mathbf{I}_{\min(qM_2, qN_2)} & \mathbf{0} \\ \mathbf{0} & -\bar{\mathbf{H}}_{22}^{\text{ph1}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{V}}_2^{\text{ph2}} \\ \bar{\mathbf{V}}_2^{\text{ph1}} \end{bmatrix} \right) + (T - q)N_2 \\ &= \text{rank} \left(\begin{bmatrix} \bar{\mathbf{V}}_2^{\text{ph2}} \\ \bar{\mathbf{V}}_2^{\text{ph1}} \end{bmatrix} \right) + (T - q)N_2 \\ &= d_2^* + (T - q)N_2 \\ &= TN_2 \end{aligned}$$

where we construct $\bar{\bar{\mathbf{H}}}_{22}^{\text{ph1}} \in \mathbb{C}^{(qN_2 - \min(qM_2, qN_2)) \times (qN_2 - \min(qM_2, qN_2))}$ and $\bar{\mathbf{V}}_2^{\text{ph1}} \in \mathbb{C}^{(qN_2 - \min(qM_2, qN_2)) \times d_2^*}$ by removing columns and rows from $\bar{\mathbf{H}}_{22}^{\text{ph1}}$ and $\mathbf{V}_2^{\text{ph1}}$ as follows. We know that similar to $\bar{\mathbf{H}}_{21}^{\text{ph1}}$, the number of rows on every block of $\bar{\bar{\mathbf{H}}}_{22}^{\text{ph1}}$ is less than or equal to M_2 with the total number of rows being

$qN_2 - \min(qN_2, qM_2)$. Since every block of $\bar{\mathbf{H}}_{22}^{\text{ph1}}$ has M_2 columns, we can remove columns from each block in a way that every block becomes a square matrix, which achieves the desired total number of columns $qN_2 - \min(qN_2, qM_2)$. With this elimination method, $\bar{\bar{\mathbf{H}}}_{22}^{\text{ph1}}$ will be a square (and full rank) matrix. Thus we can write

$$\text{rank}(\mathbf{R}_{2,2}) \geq \text{rank}(\mathbf{R}_{2,2,1}\mathbf{R}_{2,2,2}) = \text{rank}(\mathbf{R}_{2,2,2}) = \text{rank}(\mathbf{R}'_{2,2,2}) \geq TN_2 \quad (13)$$

where we have also used the fact that $\mathbf{R}_{2,2,1}$ is square and therefore full column rank. Since $\text{rank}(\mathbf{R}_{2,2}) \leq TN_2$, we have $\text{rank}(\mathbf{R}_{2,2}) = TN_2$. Therefore $\text{rank}(\mathbf{R}_{2,1}\mathbf{R}_{2,2}) = \text{rank}(\mathbf{R}_{2,2}) = TN_2$. Hence from (11) we get $\text{rank}(\mathbf{R}_2) \geq TN_2$ which gives $\text{rank}(\mathbf{R}_2) = TN_2$. $\square$

B. Transmission Scheme for Group G_2

In this group we have $d_1 + d_2 > N_1 \geq N_2$. Therefore according to (1) we select $q_1 = \frac{d_1^*}{N_1}$ and $q_2 = \frac{d_2^*}{N_2}$. The matrices $\mathbf{V}_1^{\text{ph1}} \in \mathbb{C}^{(T-q_2)M_1 \times d_1^*}$ and $\mathbf{V}_2^{\text{ph1}} \in \mathbb{C}^{(T-q_1)M_2 \times d_2^*}$ are generated randomly. Furthermore, for $j \neq i$, $i, j \in \{1, 2\}$, $\mathbf{V}_j^{\text{ph2}} \in \mathbb{C}^{q_i M_j \times d_j^*}$ is constructed as $\mathbf{V}_j^{\text{ph2}} = \mathbf{V}_{j,1}^{\text{ph2}} \mathbf{H}_{ij}^{\text{ph1}} \mathbf{V}_j^{\text{ph1}}$ where $\mathbf{V}_{j,1}^{\text{ph2}} \in \mathbb{C}^{q_i M_j \times (T-q_i)N_i}$ is generated randomly. For $i, j \in \{1, 2\}$, $\mathbf{H}_{ij}^{\text{ph1}} \in \mathbb{C}^{(T-q_i)N_i \times (T-q_i)M_j}$ and $\mathbf{H}_{ij}^{\text{ph2}} \in \mathbb{C}^{q_i N_i \times q_i M_j}$ are the channel matrices in phase 1 and 2 respectively.

For the interference subspace at receiver $i \in \{1, 2\}$, $i \neq j$ we have

$$\text{rank}(\mathbf{H}_{ij}\mathbf{V}_j) = \text{rank} \left(\begin{bmatrix} \mathbf{H}_{ij}^{\text{ph1}} \mathbf{V}_j^{\text{ph1}} \\ \mathbf{H}_{ij}^{\text{ph2}} \mathbf{V}_j^{\text{ph2}} \end{bmatrix} \right) = \text{rank} \left(\begin{bmatrix} \mathbf{I} \\ \mathbf{H}_{ij}^{\text{ph2}} \mathbf{V}_{j,1}^{\text{ph2}} \end{bmatrix} \mathbf{H}_{ij}^{\text{ph1}} \mathbf{V}_j^{\text{ph1}} \right) \leq (T - q_i)N_i.$$

With this particular choice of precoders, at each receiver we align the interference subspace of phase 2 within the interference subspace of phase 1. Then, at receiver $i \in \{1, 2\}$, the whole interference lies in a subspace of dimension $(T - q_i)N_i = TN_i - d_i^*$, which leaves d_i^* dimensions for the desired signal. Hence we can set $r_i = (T - q_i)N_i$, for $i \in \{1, 2\}$.

Theorem 3. For group G_2 , for $i \in \{1, 2\}$, the scheme in Proposition 1 gives

$$\text{rank}(\mathbf{R}_i) = d_i^* + r_i = TN_i$$

Proof. See Appendix D. $\square$

From Theorems 1, 2, 3, we can conclude that the scheme provided by Proposition 1 achieves all the

corner points in the DoF region for the studied cases.

V. EXTENSION OF THE PROPOSED SCHEME TO THE TWO-USER MIMO BC

We will now utilize the same ideas and intuitions as before to find an achievable scheme for the MIMO BC. Indeed, we will show that the same scheme used in Proposition 1 for the MIMO IC is also an achievable scheme for the MIMO BC (except that both precoders are designed at one transmitter). Let us consider the 2-user MIMO BC with M antennas at the transmitter and N_1 and N_2 antennas at receivers 1 and 2 respectively. The received signal at time instant t at receiver $i \neq j$ reads

$$\mathbf{y}_i^{(t)} = \mathbf{H}_i^{(t)} \mathbf{V}_i^{(t)} \mathbf{x}_i + \mathbf{H}_i^{(t)} \mathbf{V}_j^{(t)} \mathbf{x}_j + \mathbf{n}_i^{(t)}, \text{ for } t = 1, \dots, T$$

where $\mathbf{x}_j \in \mathbb{C}^{T d_j \times 1}$ is the symbol vector that is desired by receiver j , $\mathbf{V}_j^{(t)} \in \mathbb{C}^{M \times T d_j}$ is the precoder that is used for $\mathbf{x}_j$ at time slot t , and $\mathbf{n}_i^{(t)}$ is the additive white Gaussian noise vector at time slot t at receiver i whose entries have zero mean and unit variance. The channel between the transmitter and receiver i at time slot t is $\mathbf{H}_i^{(t)} \in \mathbb{C}^{N_i \times M}$ and we assume the same channel model as in the previous

problem. If we stack the vectors as $\mathbf{y}_i = \begin{bmatrix} \mathbf{y}_i^{(1)} \\ \vdots \\ \mathbf{y}_i^{(T)} \end{bmatrix}$ and $\mathbf{n}_i = \begin{bmatrix} \mathbf{n}_i^{(1)} \\ \vdots \\ \mathbf{n}_i^{(T)} \end{bmatrix}$, then for $i, j \in \{1, 2\}$, $i \neq j$

$$\mathbf{y}_i = \mathbf{H}_i \mathbf{V}_i \mathbf{x}_i + \mathbf{H}_i \mathbf{V}_j \mathbf{x}_j + \mathbf{n}_i$$

with $\mathbf{V}_j = \begin{bmatrix} \mathbf{V}_j^{(1)} \\ \vdots \\ \mathbf{V}_j^{(T)} \end{bmatrix}$ and $\mathbf{H}_i = \text{Bdiag}(\mathbf{H}_i^{(1)}, \dots, \mathbf{H}_i^{(T)})$.

Proposition 2. *Using the scheme of Proposition 1, the corner points of the DoF region of the 2-user MIMO BC can be achieved (almost surely) when delayed CSIT is beneficial compared to no CSIT.*

Proof. Note that the two precoders that are designed in Proposition 1 will be used at the one transmitter and $\mathbf{H}_i$ instead of $\mathbf{H}_{ij}$ is used for designing the precoders. In order to apply the same ideas as before,

we look at the received signal spaces for $i \in \{1, 2\}$, $j \neq i$

$$\mathbf{R}_i = [\mathbf{H}_i \mathbf{V}_i \quad \mathbf{H}_i \mathbf{V}_j] = \begin{bmatrix} \mathbf{H}_i^{\text{ph1}} \mathbf{V}_i^{\text{ph1}} & \mathbf{H}_i^{\text{ph1}} \mathbf{V}_j^{\text{ph1}} \\ \mathbf{H}_i^{\text{ph2}} \mathbf{V}_i^{\text{ph2}} & \mathbf{H}_i^{\text{ph2}} \mathbf{V}_j^{\text{ph2}} \end{bmatrix}.$$

At this point we decide based on the dimension of the received signal and interference subspaces, whether we need interference alignment or not. If alignment is required, we align the interference of the second phase into the subspace corresponding to the interference of the first phase, i.e., we choose $\mathbf{V}_j^{\text{ph2}}$ such that $\mathbf{H}_i^{\text{ph2}} \mathbf{V}_j^{\text{ph2}} \in \mathcal{R}(\mathbf{H}_i^{\text{ph1}} \mathbf{V}_j^{\text{ph1}})$. According to [3], the corner points for which delayed CSIT is beneficial compared to no CSIT correspond to the configurations with $M > N_1 \geq N_2$. The corresponding DoF corner point (obtained by unifying the points in Sections IV.B and IV.C in [3]) is $d_1 = \frac{N_1 M''(M'' - N_2)}{N_1(M'' - N_2) + M''(M'' - N_1)}$ and $d_2 = \frac{N_2 M''(M'' - N_1)}{N_1(M'' - N_2) + M''(M'' - N_1)}$ with $M'' = \min(M, N_1 + N_2)$. Note that for this set of configurations, this point is exactly the same as the corner point of the MIMO IC with (M, M, N_1, N_2) which belongs to group G_2 . Therefore, alignment is required at both receivers since we have $d_1 + d_2 > N_1 \geq N_2$ in group G_2 . By choosing the precoders according to Proposition 1, we ensure that the interference has a reduced rank and can be separated from the desired signal. The proof would follow a line of argument similar to the proof for group G_2 in Section IV-B, except that the channels are now $\mathbf{H}_i$ instead of $\mathbf{H}_{ij}$. This, however, does not affect the idea of the proof. $\square$

VI. FEEDBACK REQUIREMENT

Our technique does not require the knowledge of direct channels at the transmitters. In this section we show that our scheme can be slightly modified to further relax the feedback requirement for the interfering channels. First of all, according to Proposition 1, for $i, j \in \{1, 2\}$, $j \neq i$, no CSI is required at transmitter j if $d_1 + d_2 \leq N_i$. If $d_1 + d_2 > N_i$, transmitter j needs to have access to $\mathbf{H}_{ij}^{\text{ph1}}$ in order to construct $\mathbf{V}_j^{\text{ph2}}$ since $\mathbf{V}_j^{\text{ph2}} = \mathbf{V}_{j,1}^{\text{ph2}} \mathbf{H}_{ij}^{\text{ph1}} \mathbf{V}_j^{\text{ph1}}$, where $\mathbf{V}_{j,1}^{\text{ph2}} \in \mathbb{C}^{q_i M_j \times (T - q_i) N_i}$ is generated randomly (with i.i.d. Gaussian entries).

However, we show that it is possible to reduce the feedback requirement because knowledge of the row span of $\mathbf{H}_{ij}^{\text{ph1}}$ is sufficient to achieve the desired DoF. The matrix $\mathbf{H}_{ij}^{\text{ph1}}$ is block-diagonal and therefore it is sufficient to feedback the row span of each of the diagonal blocks $(\mathbf{H}_{ij}^{(t)}, t = 1, \dots, T - q_i)$. The row span of each matrix $\mathbf{H}_{ij}^{(t)}$ can be represented by a point on the Grassmann manifold $\mathcal{G}_{M_j, N_i}$, which is the set of all N_i -dimensional subspaces in the M_j -dimensional vector space. A point on the Grassmann

manifold can be represented by any matrix $\mathbf{X}$ whose rows span the subspace defined by $\mathbf{X}$, i.e., $\text{span}(\mathbf{X})$. The real dimension of $\mathcal{G}_{M_j, N_i}$ is $2N_i(M_j - N_i)$ which is less than the real dimension of the original matrices, which is $2N_i M_j$. Therefore feeding back the points on the Grassmann manifold has the potential to significantly reduce the amount of feedback.

Lemma 5. *In order to achieve the DoF region of the MIMO IC with delayed CSIT for the configurations listed in Table I, it is sufficient to feedback the row span of the channel matrices $\mathbf{H}_{ij}^{(t)}$, $t = 1, \dots, T - q_i$.*

Proof. Every point on the Grassmann manifold can be represented by a truncated unitary matrix. Suppose that $\mathbf{F}_{ij}^{(t)}$ is such a matrix that spans the same row space as $\mathbf{H}_{ij}^{(t)}$ and therefore the row span of $\mathbf{F}_{ij}^{(t)}$ is fed back to transmitter. Hence we can write $\mathbf{H}_{ij}^{(t)} = \mathbf{C}_{ij}^{(t)} \mathbf{F}_{ij}^{(t)}$ where $\mathbf{C}_{ij}^{(t)}$ is an invertible matrix. As the row span of $\mathbf{F}_{ij}^{(t)}$ is sent to the transmitter, the feedback as received by the transmitters takes the form of a truncated unitary matrix which is in the row space of $\mathbf{F}_{ij}^{(t)}$ and can be written as $\mathbf{G}_{ij}^{(t)} = \mathbf{O}_{ij}^{(t)} \mathbf{F}_{ij}^{(t)}$ for some unitary matrix $\mathbf{O}_{ij}^{(t)}$. Next we show that the transmitter can use $\mathbf{G}_{ij}^{(t)}$ to design the precoder of phase 2. By designing $\mathbf{V}_j^{\text{ph2}}$ using $\mathbf{G}_{ij}^{\text{ph1}} = \text{Bdiag}(\mathbf{G}_{ij}^{(1)}, \dots, \mathbf{G}_{ij}^{(T-q_i)})$ instead of $\mathbf{H}_{ij}^{\text{ph1}}$ we have

$$\begin{aligned} \mathbf{V}_j^{\text{ph2}} &= \mathbf{V}_{j,1}^{\text{ph2}} \mathbf{G}_{ij}^{\text{ph1}} \mathbf{V}_j^{\text{ph1}} \\ &= \mathbf{V}_{j,1}^{\text{ph2}} \text{Bdiag}(\mathbf{G}_{ij}^{(1)}, \dots, \mathbf{G}_{ij}^{(T-q_i)}) \mathbf{V}_j^{\text{ph1}} \\ &= \mathbf{V}_{j,1}^{\text{ph2}} \text{Bdiag}(\mathbf{O}_{ij}^{(1)} (\mathbf{C}_{ij}^{(1)})^{-1}, \dots, \mathbf{O}_{ij}^{(T-q_i)} (\mathbf{C}_{ij}^{(T-q_i)})^{-1}) \text{Bdiag}(\mathbf{H}_{ij}^{(1)}, \dots, \mathbf{H}_{ij}^{(T-q_i)}) \mathbf{V}_j^{\text{ph1}} \\ &= \dot{\mathbf{V}}_{j,1}^{\text{ph2}} \mathbf{H}_{ij}^{\text{ph1}} \mathbf{V}_j^{\text{ph1}} \end{aligned}$$

with $\dot{\mathbf{V}}_{j,1}^{\text{ph2}} = \mathbf{V}_{j,1}^{\text{ph2}} \text{Bdiag}(\mathbf{O}_{ij}^{(1)} (\mathbf{C}_{ij}^{(1)})^{-1}, \dots, \mathbf{O}_{ij}^{(T-q_i)} (\mathbf{C}_{ij}^{(T-q_i)})^{-1})$. Clearly $\dot{\mathbf{V}}_{j,1}^{\text{ph2}}$ is a full rank random matrix. We have thus effectively designed $\mathbf{V}_j^{\text{ph2}}$ according to the original construction (with $\mathbf{H}_{ij}^{\text{ph1}}$) using a different random precoder $\dot{\mathbf{V}}_{j,1}^{\text{ph2}}$ instead of $\mathbf{V}_{j,1}^{\text{ph2}}$. $\square$

Quantized CSI feedback

Sending back the row space of the channel matrices still requires perfect feedback of the matrix elements, which are continuous values. In practice, the accuracy of the feedback is limited by the capacity of the feedback channel. As a result, the precoders that are designed based on imperfect feedback will deviate from the ideal solutions and therefore create interference at the receivers. This interference can

lead to a reduction in the DoF if the accuracy of feedback is not properly scaled with the SNR. To analyze this situation, in the following we assume that each receiver can feedback a limited number of bits B to the transmitter. According to the proposed scheme, the receiver quantizes the subspace spanned by the rows of $\mathbf{F}_{ij}^{(t)}$ using B bits and feeds the index of the quantized codeword back to the transmitter. We further assume that the receivers and the transmitters share a predefined codebook $\mathcal{S} = \{\mathbf{S}_1, \dots, \mathbf{S}_{2^B}\}$ which is composed of 2^B truncated unitary matrices of size $N_i \times M_j$ and is designed via Grassmannian subspace packing [19]. The quantized codeword at the receiver is the point in $\mathcal{S}$ closest to $\mathbf{F}_{ij}^{(t)}$, i.e.,

$$\hat{\mathbf{F}}_{ij}^{(t)} = \arg \min_{\mathbf{S} \in \mathcal{S}} d_c(\mathbf{S}, \mathbf{F}_{ij}^{(t)}) \quad (14)$$

where $d_c(\mathbf{X}, \mathbf{Y}) = \frac{1}{\sqrt{2}} \|\mathbf{X}\mathbf{X}^H - \mathbf{Y}\mathbf{Y}^H\|_F$ is the chordal distance between $\mathbf{X}$ and $\mathbf{Y}$ in $\mathcal{G}_{M_j, N_i}$. The transmitters use $\hat{\mathbf{F}}_{ij}^{\text{ph1}} = \text{Bdiag}(\hat{\mathbf{F}}_{ij}^{(1)}, \dots, \hat{\mathbf{F}}_{ij}^{(T-q_i)})$ instead of $\mathbf{H}_{ij}^{\text{ph1}}$ to design the precoders.

VII. SIMULATION RESULTS

We have verified the achievability of the proposed scheme by the following procedure: 1) generating random i.i.d. channel matrices, 2) finding the precoders according to Proposition 1 (and finding the values of T, q_1, q_2), 3) determining the signal and interference subspaces at each receiver, i.e., $\mathbf{H}_{ii}\mathbf{V}_i$ and $\mathbf{H}_{ij}\mathbf{V}_j$, 4) verifying that $\text{rank}(\mathbf{H}_{ii}\mathbf{V}_i) = d_i^*$ and $\text{rank}(\mathbf{H}_{ii}\mathbf{V}_i) + \text{rank}(\mathbf{H}_{ij}\mathbf{V}_j) = \text{rank}([\mathbf{H}_{ii}\mathbf{V}_i \ \mathbf{H}_{ij}\mathbf{V}_j])$. The same procedure can be followed to verify the achievability of the scheme with Grassmannian feedback using $\mathbf{G}_{ij}^{\text{ph1}}$ instead of $\mathbf{H}_{ij}^{\text{ph1}}$.

In this section, we provide simulation results on our quantized feedback method to show the sum-rate performance compared to a conventional feedback method. We consider the two-user MIMO IC with the antenna configurations according to groups G_1 and G_2 . The sum-rate for different assumptions on the quality of delayed CSIT is plotted for two configurations in Figures 1 and 2. Figure 1 corresponds to the configuration of $M_1 = 3, M_2 = 1, N_1 = 4, N_2 = 2$ which belongs to group G_1 . Figure 2 corresponds to the configuration of $M_1 = 3, M_2 = 3, N_1 = 2, N_2 = 2$ which belongs to group G_2 . The delayed CSIT is either perfect (no quantization, represented by R_∞), quantized with our proposed method (R_p), or quantized using the conventional method (R_c). In the conventional quantization method we vectorize the channel matrices and quantize the normalized version of the vectors. In both quantization methods we use random vector quantization (RVQ) to generate the codebooks. The number of quantization bits is chosen as $B = 15$ in both methods. Clearly the proposed scheme provides a sum-rate improvement over

Fig. 1. Sum-rate comparison between the proposed scheme and the conventional method for $K = 2, M_1 = 3, M_2 = 1, N_1 = 4, N_2 = 2$

Fig. 2. Sum-rate comparison between the proposed scheme and the conventional method for $K = 2, M_1 = 3, M_2 = 3, N_1 = 2, N_2 = 2$

the conventional method using the same number of quantization bits. The performance improvement is larger at higher SNR values as interference becomes the dominant impairment at high SNR. Note that in the conventional method, we assume that the norm of the vectorized channels is known perfectly at the transmitters. Since this is idealistic, in practice the performance gain of the proposed method will be higher than what is shown in the figures. From the slope of the curves corresponding to perfect (and delayed) CSIT at high SNR, it is possible to calculate the achievable DoF and verify that the optimal (sum) DoF is achieved. In Figure 1, the configuration corresponds to $d_1 = d_2 = \frac{6}{5}$ and Figure 2 corresponds to $d_1 = \frac{3}{2}$ and $d_2 = 1$.

VIII. CONCLUSION AND FUTURE WORK

We proposed a DoF-achieving scheme for the two-user MIMO IC where the transmitters are provided with delayed CSI. The DoF region for this channel was characterized in [10]. We developed an intuitive achievable scheme based on linear projection at the transceivers which is simpler than the one in [10] and has a unified structure for different antenna configurations. We showed that the scheme is also DoF-achieving for the two-user MIMO BC. Furthermore we showed that our scheme requires less information feedback compared to the existing methods. While the simple and unified structure of our method is obviously beneficial for practical implementation, it is also important to note that the proposed method can be a starting point for future extensions to other interference networks or other scenarios such as

temporally correlated channels.

APPENDIX A

PROOF OF LEMMA 2

Since $l \geq b$, we start by splitting $\mathbf{Z}$ into the following two matrices

$$\mathbf{Z} = \begin{bmatrix} \tilde{\mathbf{A}} & \bar{\mathbf{A}} & \mathbf{B} \\ \tilde{\mathbf{C}} & \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{C}} & \mathbf{0} \end{bmatrix} \quad (15)$$

where $\tilde{\mathbf{X}}$ and $\bar{\mathbf{X}}$ denote the first b columns and the last $l - b$ columns of the matrix $\mathbf{X}$. Since $\bar{\mathbf{C}}$ has random elements independent of the other matrices, addition of the second term in (15) does not reduce the rank of the first term. Therefore

$$\text{rank}(\mathbf{Z}) \geq \text{rank} \left(\begin{bmatrix} \tilde{\mathbf{A}} & \bar{\mathbf{A}} & \mathbf{B} \\ \tilde{\mathbf{C}} & \mathbf{0} & \mathbf{0} \end{bmatrix} \right) = \text{rank} \left(\begin{bmatrix} \mathbf{B} & \bar{\mathbf{A}} & \tilde{\mathbf{A}} \\ \mathbf{0} & \mathbf{0} & \tilde{\mathbf{C}} \end{bmatrix} \right) \geq \text{rank} \left(\begin{bmatrix} \mathbf{B} & \bar{\mathbf{A}} \end{bmatrix} \right) + \text{rank} \left(\begin{bmatrix} \tilde{\mathbf{C}} \end{bmatrix} \right) = a + b$$

where we have exchanged the columns inside the matrix (which does not change the rank) and we have used the fact that the rank of a block triangular matrix is greater or equal to the sum of the ranks of the diagonal blocks. Here $\begin{bmatrix} \mathbf{B} & \bar{\mathbf{A}} \end{bmatrix}$ and $\begin{bmatrix} \tilde{\mathbf{C}} \end{bmatrix}$ are matrices with random i.i.d. elements of size $a \times a$ and $b \times b$ respectively. Since $\text{rank}(\mathbf{Z}) \leq a + b$ we have $\text{rank}(\mathbf{Z}) = a + b$.

APPENDIX B

PROOF OF LEMMA 3

We verify the conditions for two cases in the group G_1 . The proofs for the other cases follow using similar techniques and are excluded due to space constraints.

- **Case A.II.2:** $M_1 > N_1 > M_2 \geq N_2$:

In this case we have $d_1 = \frac{M_1''(N_1 - N_2)}{(M_1'' - N_2)}$ and $d_2 = \frac{N_2(M_1'' - N_1)}{(M_1'' - N_2)}$ with $M_1'' = \min(M_1, N_1 + N_2)$. We set $T = (M_1'' - N_2)$ which gives $q = \frac{d_2^*}{N_2} = (M_1'' - N_1)$. The first condition is clearly satisfied. The second condition can be simplified as:

$$T(d_1 + d_2) \leq (T - q)N_1 + \min \left(qN_1, qN_2 + \min(qN_1, (T - q)N_2) \right).$$

Since we have $M_1'' \leq N_1 + N_2$,

$$M_1''(N_1 - N_2) \leq (N_1 + N_2)(N_1 - N_2) = N_1^2 - N_2^2$$

Thus $(M_1'' - N_1)N_1 \leq (M_1'' - N_2)N_2$ which means $qN_1 \leq TN_2$. If $qN_1 > (T - q)N_2$ then

$$qN_1 \leq TN_2 = qN_2 + (T - q)N_2 = qN_2 + \min(qN_1, (T - q)N_2)$$

Hence the condition simplifies to $T(d_1 + d_2) \leq TN_1$ which is satisfied in group G_1 . If we have $qN_1 \leq (T - q)N_2$ then again the condition simplifies to $T(d_1 + d_2) \leq TN_1$ which is satisfied in group G_1 .

The third condition is also satisfied since we have $d_1^* = M_1''(N_1 - N_2) \leq M_1(N_1 - N_2) = (T - q)M_1$.

• **Case B.I:** $N_1 \geq M_1 > N_2 > M_2$:

In this case we have $d_1 = \frac{M_1(N_2 - M_2)}{N_2}$ and $d_2 = M_2$. We choose $T = N_2$ which gives $q = M_2$. The first condition is straightforward. If $qN_1 \leq (T - q)N_2$ the second condition is satisfied similar to the previous case. If $qN_1 > (T - q)N_2$, substituting for T, d_1, d_2, q , this condition simplifies as:

$$M_1(N_2 - M_2) + M_2N_2 \leq (N_2 - M_2)N_1 + M_2^2 + \min(M_1M_2, N_2(N_2 - M_2)).$$

Next we show that this condition holds: It is easy to see that $(M_2 + M_1 - N_1) \leq M_2 < N_2$, hence

$$(M_2 + M_1 - N_1)(N_2 - M_2) < \min(M_1M_2, N_2(N_2 - M_2)).$$

This simplifies to

$$M_1(N_2 - M_2) + M_2N_2 \leq (N_2 - M_2)N_1 + M_2^2 + \min(M_1M_2, N_2(N_2 - M_2)).$$

Clearly the third condition is satisfied with equality.

APPENDIX C

PROOF OF LEMMA 4

In order to prove the theorem we need to verify the conditions $\bar{C}.I$ and $\bar{C}.II$ for group G_2 , which only contains the case A.I.3. We have $M_1 > N_1 \geq N_2$ and $M_2 > N_2$ and $M_2 > N_1$. The corner point of the region is $d_1 = \frac{N_1M_1''(M_2'' - N_2)}{N_1(M_2'' - N_2) + M_2''(M_1'' - N_1)}$ and $d_2 = \frac{N_2M_2''(M_1'' - N_1)}{N_1(M_2'' - N_2) + M_2''(M_1'' - N_1)}$, where $M_i'' =$

$\min(M_i, N_1 + N_2)$. If we choose the number of time slots equal to the denominator we have $T = N_1(M_2'' - N_2) + M_2''(M_1'' - N_1)$ which gives $d_1^* = N_1 M_1''(M_2'' - N_2)$ and $d_2^* = N_2 M_2''(M_1'' - N_1)$.

Therefore we have $q_1 = \frac{d_1^*}{N_1} = M_1''(M_2'' - N_2)$ and $q_2 = \frac{d_2^*}{N_2} = M_2''(M_1'' - N_1)$.

We first prove $\bar{C}.I$: Since we have $M_2 > N_1$ this gives

$$\begin{aligned} N_2(M_1'' - N_1)(M_2'' - N_1) &> 0 \Rightarrow M_1''N_2M_2'' - M_1''N_1N_2 - M_2''N_1N_2 > -N_1^2N_2 \\ &\Rightarrow M_1''N_2M_2'' - M_1''N_1N_2 - M_2''N_1N_2 + M_1''N_1M_2'' > M_1''N_1M_2'' - N_1^2N_2 \\ &\Rightarrow d_1^* + d_2^* > TN_1 \Rightarrow d_1 + d_2 > N_1. \end{aligned}$$

which also implies $d_1 + d_2 > N_2$ since $N_1 \geq N_2$. The proof of $\bar{C}.II$ is straightforward: $d_j^* = N_j M_j''(M_i'' - N_i) = (T - q_i)M_j'' \leq (T - q_i)M_j$.

APPENDIX D

PROOF OF THEOREM 3

This proof uses similar techniques and ideas as the proofs of Theorems 1 and 2. According to the construction of $\mathbf{V}_j^{\text{ph2}}$ we have $\mathbf{H}_{ij}^{\text{ph2}} \mathbf{V}_j^{\text{ph2}} = \mathbf{H}_{ij}^{\text{ph2}} \mathbf{V}_{j,1}^{\text{ph2}} \mathbf{H}_{ij}^{\text{ph1}} \mathbf{V}_j^{\text{ph1}}$. Therefore $\mathbf{R}_i$ can be written as $\mathbf{R}_i = \begin{bmatrix} \mathbf{R}_i^S & \mathbf{R}_i^I \end{bmatrix}$ with $\mathbf{R}_i^S = \begin{bmatrix} \mathbf{H}_{ii}^{\text{ph1}} \mathbf{V}_i^{\text{ph1}} \\ \mathbf{H}_{ii}^{\text{ph2}} \mathbf{V}_{i,1}^{\text{ph2}} \mathbf{H}_{ji}^{\text{ph1}} \mathbf{V}_i^{\text{ph1}} \end{bmatrix}$ and $\mathbf{R}_i^I = \begin{bmatrix} \mathbf{H}_{ij}^{\text{ph1}} \mathbf{V}_j^{\text{ph1}} \\ \mathbf{H}_{ij}^{\text{ph2}} \mathbf{V}_{j,1}^{\text{ph2}} \mathbf{H}_{ij}^{\text{ph1}} \mathbf{V}_j^{\text{ph1}} \end{bmatrix}$.

There exist two possible cases: 1) $q_i \leq q_j$ and 2) $q_i > q_j$. Since the proof of both cases is based on similar ideas, due to lack of space, we only present the proof for one scenario. We consider $q_i \leq q_j$. In this case the top block in $\mathbf{R}_i^I$ is taller than the top block in $\mathbf{R}_i^S$. We replace the first $(q_j - q_i)N_i$ rows of the bottom block in $\mathbf{R}_i^S$ with a zero matrix. This is done by removing the first $q_j - q_i$ blocks of $\mathbf{H}_{ii}^{\text{ph2}}$, which yields the reduced matrix $\bar{\mathbf{H}}_{ii}^{\text{ph2}} \in \mathbb{C}^{q_i N_i \times q_j M_i}$. Therefore we have

$$\text{rank}(\mathbf{R}_i) \geq \text{rank}(\mathbf{R}_i')$$

with

$$\mathbf{R}'_i = \left[\begin{array}{c|c} \mathbf{H}_{ii}^{\text{ph1}} \mathbf{V}_i^{\text{ph1}} & \mathbf{H}_{ij}^{\text{ph1}} \mathbf{V}_j^{\text{ph1}} \\ \hline \mathbf{0} & \\ \hline \bar{\mathbf{H}}_{ii}^{\text{ph2}} \mathbf{V}_{i,1}^{\text{ph2}} \mathbf{H}_{ji}^{\text{ph1}} \mathbf{V}_i^{\text{ph1}} & \mathbf{H}_{ij}^{\text{ph2}} \mathbf{V}_{j,1}^{\text{ph2}} \mathbf{H}_{ij}^{\text{ph1}} \mathbf{V}_j^{\text{ph1}} \end{array} \right]$$

where we have used the familiar argument that adding an independent matrix with zero or random elements does not reduce the rank.

$\mathbf{R}'_i$ can be written as

$$\mathbf{R}'_i = \begin{bmatrix} \mathbf{I}_{(T-q_i)N_i} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{H}}_{ii}^{\text{ph2}} & \bar{\mathbf{H}}_{ij}^{\text{ph2}} \end{bmatrix} \left[\begin{array}{c|c} \mathbf{H}_{ii}^{\text{ph1}} \mathbf{V}_i^{\text{ph1}} & \mathbf{H}_{ij}^{\text{ph1}} \mathbf{V}_j^{\text{ph1}} \\ \hline \mathbf{0} & \\ \hline \mathbf{V}_{i,1}^{\text{ph2}} \mathbf{H}_{ji}^{\text{ph1}} \mathbf{V}_i^{\text{ph1}} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{V}_{j,1}^{\text{ph2}} \mathbf{H}_{ij}^{\text{ph1}} \mathbf{V}_j^{\text{ph1}} \end{array} \right]$$

Let $\alpha_i = (q_i + q_j - T)N_i$. Using the same method as in the proofs of Theorems 1 and 2, we have

$$\text{rank}(\mathbf{R}'_i) \geq \text{rank}(\mathbf{R}_{i,1} \mathbf{R}_{i,2})$$

where

$$\mathbf{R}_{i,1} = \begin{bmatrix} \mathbf{I}_{(T-q_i)N_i} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \bar{\bar{\mathbf{H}}}_{ii}^{\text{ph2}} & \bar{\bar{\mathbf{H}}}_{ij}^{\text{ph2}} \end{bmatrix}, \quad \mathbf{R}_{i,2} = \left[\begin{array}{c|c} \mathbf{H}_{ii}^{\text{ph1}} \mathbf{V}_i^{\text{ph1}} & \mathbf{H}_{ij}^{\text{ph1}} \mathbf{V}_j^{\text{ph1}} \\ \hline \mathbf{0} & \\ \hline \bar{\mathbf{V}}_{i,1}^{\text{ph2}} \mathbf{H}_{ji}^{\text{ph1}} \mathbf{V}_i^{\text{ph1}} & \mathbf{0} \\ \hline \mathbf{0} & \bar{\mathbf{V}}_{j,1}^{\text{ph2}} \mathbf{H}_{ij}^{\text{ph1}} \mathbf{V}_j^{\text{ph1}} \end{array} \right]$$

and $\bar{\bar{\mathbf{H}}}_{ii}^{\text{ph2}} \in \mathbb{C}^{q_i N_i \times \alpha_i}$ and $\bar{\bar{\mathbf{H}}}_{ij}^{\text{ph2}} \in \mathbb{C}^{q_i N_i \times (q_i N_i - \alpha_i)}$ are constructed by removing columns from $\bar{\mathbf{H}}_{ii}^{\text{ph2}}$ and $\bar{\mathbf{H}}_{ij}^{\text{ph2}}$, respectively, and $\bar{\mathbf{V}}_{i,1}^{\text{ph2}} \in \mathbb{C}^{\alpha_i \times (T-q_j)N_j}$ is constructed by removing the corresponding rows from $\mathbf{V}_{i,1}^{\text{ph2}}$ and $\bar{\mathbf{V}}_{j,1}^{\text{ph2}} \in \mathbb{C}^{(q_i N_i - \alpha_i) \times (T-q_i)N_i}$ is constructed by removing the corresponding rows from $\mathbf{V}_{j,1}^{\text{ph2}}$.

Next we show that this removal is feasible:

$$\alpha_i = (q_i + q_j - T)N_i \leq q_i N_i \leq q_j N_i \leq q_j M_i$$

$$q_i N_i - \alpha_i \leq q_i N_i \leq q_i M_j$$

Note that the removal of columns from $\bar{\mathbf{H}}_{ii}^{\text{ph2}}$ and $\mathbf{H}_{ij}^{\text{ph2}}$ can be done in a way that the sum of the number of columns on the l th block of $\bar{\mathbf{H}}_{ii}^{\text{ph2}}$ and $\bar{\mathbf{H}}_{ij}^{\text{ph2}}$ is equal to N_1 , for $l = 1, \dots, q_i$. With this removal $\mathbf{R}_{i,1}$ will be square and full rank, which yields

$$\text{rank}(\mathbf{R}_{i,1}\mathbf{R}_{i,2}) = \text{rank}(\mathbf{R}_{i,2}).$$

$\mathbf{R}_{i,2}$ can be further written as

$$\mathbf{R}_{i,2} = \begin{bmatrix} \mathbf{I}_{(T-q_i)N_i} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{V}}_{i,1}^{\text{ph2}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{V}}_{j,1}^{\text{ph2}} \end{bmatrix} \begin{bmatrix} \mathbf{H}_{ii}^{\text{ph1}} \mathbf{V}_i^{\text{ph1}} & \mathbf{H}_{ij}^{\text{ph1}} \mathbf{V}_j^{\text{ph1}} \\ \mathbf{0} & \\ \mathbf{H}_{ji}^{\text{ph1}} \mathbf{V}_i^{\text{ph1}} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{ij}^{\text{ph1}} \mathbf{V}_j^{\text{ph1}} \end{bmatrix}.$$

Therefore we have

$$\text{rank}(\mathbf{R}_{i,2}) \geq \text{rank}(\mathbf{R}_{i,2,1}\mathbf{R}_{i,2,2})$$

where

$$\mathbf{R}_{i,2,1} = \begin{bmatrix} \mathbf{I}_{(T-q_i)N_i} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{V}}_{i,1}^{\text{ph2}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{V}}_{j,1}^{\text{ph2}} \end{bmatrix}, \quad \mathbf{R}_{i,2,2} = \begin{bmatrix} \mathbf{H}_{ii}^{\text{ph1}} \mathbf{V}_i^{\text{ph1}} & \mathbf{H}_{ij}^{\text{ph1}} \mathbf{V}_j^{\text{ph1}} \\ \mathbf{0} & \\ \bar{\mathbf{H}}_{ji}^{\text{ph1}} \mathbf{V}_i^{\text{ph1}} & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{H}}_{ij}^{\text{ph1}} \mathbf{V}_j^{\text{ph1}} \end{bmatrix}$$

and $\bar{\mathbf{H}}_{ji}^{\text{ph1}} \in \mathbb{C}^{\alpha_i \times (T-q_j)M_i}$ and $\bar{\mathbf{H}}_{ij}^{\text{ph1}} \in \mathbb{C}^{(q_i N_i - \alpha_i) \times (T-q_i)M_j}$ are constructed by removing rows from $\mathbf{H}_{ji}^{\text{ph1}}$ and $\mathbf{H}_{ij}^{\text{ph1}}$ respectively and $\bar{\mathbf{V}}_{i,1}^{\text{ph2}} \in \mathbb{C}^{\alpha_i \times \alpha_i}$ is constructed by removing the corresponding columns from $\bar{\mathbf{V}}_{i,1}^{\text{ph2}}$ and $\bar{\mathbf{V}}_{j,1}^{\text{ph2}} \in \mathbb{C}^{(q_i N_i - \alpha_i) \times (q_i N_i - \alpha_i)}$ is constructed by removing the corresponding columns from $\bar{\mathbf{V}}_{j,1}^{\text{ph2}}$. This removal is feasible since

$$q_i N_i - \alpha_i = (T - q_j)N_i \leq (T - q_i)N_i$$

$$\alpha_i = q_i N_i - (T - q_j)N_i = (T - q_j)M_i'' - (T - q_j)N_i = (T - q_j)(M_i'' - N_i) \leq (T - q_j)N_j$$

where we have used $\bar{\text{C.II}}$ and the inequality $M_i'' \leq N_i + N_j$. The process of removing the rows from $\mathbf{H}_{ji}^{\text{ph1}}$ is such that we start by removing one row from the first block and then the second block and so on (if necessary going back to the first block and removing further rows) until we reach the desired

number of rows. In this way it is easy to see that eventually the k th block of $\bar{\mathbf{H}}_{ji}^{\text{ph1}}$ will have $s_k = g_1$ or $s_k = g_1 - 1$ rows for some $g_1 \leq N_j$ and $\sum_{k=1}^{T-q_j} s_k = \alpha_i$.

We construct $\bar{\mathbf{H}}_{ij}^{\text{ph1}}$ from the first $q_i N_i - \alpha_i$ rows of $\mathbf{H}_{ij}^{\text{ph1}}$. Since $q_i N_i - \alpha_i = (T - q_j) N_i$, if we subtract the first $(T - q_j) N_i$ rows of $\mathbf{R}_{i,2,2}$ from the last $(T - q_j) N_i$ rows, then the resulting matrix $\mathbf{R}'_{i,2,2}$ will have the same rank as $\mathbf{R}_{i,2,2}$, i.e., $\text{rank}(\mathbf{R}'_{i,2,2}) = \text{rank}(\mathbf{R}_{i,2,2})$ with

$$\mathbf{R}'_{i,2,2} = \left[\begin{array}{c|c} \mathbf{H}_{ii}^{\text{ph1}} \mathbf{V}_i^{\text{ph1}} & \mathbf{H}_{ij}^{\text{ph1}} \mathbf{V}_j^{\text{ph1}} \\ \hline \mathbf{0} & \\ \hline \bar{\mathbf{H}}_{ji}^{\text{ph1}} \mathbf{V}_i^{\text{ph1}} & \mathbf{0} \\ \hline -\mathbf{H}_{ii}^{\text{ph1}} \mathbf{V}_i^{\text{ph1}} & \mathbf{0} \end{array} \right].$$

From the group condition, $d_i^* + d_j^* > T N_i$, we have $d_j^* > (T - q_i) N_i$. Therefore

$$\text{rank}(\mathbf{H}_{ij}^{\text{ph1}} \mathbf{V}_j^{\text{ph1}}) = \min(d_j^*, (T - q_i) N_i, (T - q_i) M_j) = (T - q_i) N_i$$

where we also used the fact that $M_j \geq N_i$ for this group. Hence

$$\begin{aligned} \text{rank}(\mathbf{R}'_{i,2,2}) &\geq \text{rank} \left(\begin{bmatrix} \bar{\mathbf{H}}_{ji}^{\text{ph1}} \mathbf{V}_i^{\text{ph1}} \\ -\mathbf{H}_{ii}^{\text{ph1}} \mathbf{V}_i^{\text{ph1}} \end{bmatrix} \right) + \text{rank}(\mathbf{H}_{ij}^{\text{ph1}} \mathbf{V}_j^{\text{ph1}}) \\ &= \text{rank} \left(\begin{bmatrix} \bar{\mathbf{H}}_{ji}^{\text{ph1}} \\ -\mathbf{H}_{ii}^{\text{ph1}} \end{bmatrix} \mathbf{V}_i^{\text{ph1}} \right) + (T - q_i) N_i \\ &= \text{rank} \left(\begin{bmatrix} \bar{\bar{\mathbf{H}}}_{ji}^{\text{ph1}} \\ -\bar{\mathbf{H}}_{ii}^{\text{ph1}} \end{bmatrix} \bar{\mathbf{V}}_i^{\text{ph1}} \right) + (T - q_i) N_i \\ &= \text{rank}(\bar{\mathbf{V}}_i^{\text{ph1}}) + (T - q_i) N_i \\ &= q_i N_i + (T - q_i) N_i \\ &= T N_i \end{aligned}$$

where we construct $\bar{\bar{\mathbf{H}}}_{ji}^{\text{ph1}} \in \mathbb{C}^{\alpha_i \times q_i N_i}$ and $\bar{\mathbf{H}}_{ii}^{\text{ph1}} \in \mathbb{C}^{(q_i N_i - \alpha_i) \times q_i N_i}$ by removing columns from $\bar{\mathbf{H}}_{ji}^{\text{ph1}}$ and $\mathbf{H}_{ii}^{\text{ph1}}$ respectively and $\bar{\mathbf{V}}_i^{\text{ph1}} \in \mathbb{C}^{q_i N_i \times q_i N_i}$ by removing the corresponding rows from $\mathbf{V}_i^{\text{ph1}}$. The columns are removed in a way that the square matrix $\begin{bmatrix} \bar{\bar{\mathbf{H}}}_{ji}^{\text{ph1}} \\ -\bar{\mathbf{H}}_{ii}^{\text{ph1}} \end{bmatrix}$ has full rank. This is possible if we leave $N_i + s_k$ columns in the k th blocks of $\bar{\mathbf{H}}_{ji}^{\text{ph1}}$ and $\mathbf{H}_{ii}^{\text{ph1}}$ and remove the remaining columns. Through

a simple row permutation, the matrix $\begin{bmatrix} \bar{\bar{\mathbf{H}}}_{ji}^{\text{ph1}} \\ -\bar{\mathbf{H}}_{ii}^{\text{ph1}} \end{bmatrix}$ can be turned into a block-diagonal matrix where the k th block of the resulting matrix is a square matrix of size $(N_i + s_k) \times (N_i + s_k)$. Next we verify that the removal is feasible, i.e., $N_i + s_k \leq M_i$. We have

$$\sum_{k=1}^{T-q_j} (N_i + s_k) = (T - q_j)N_i + \sum_{k=1}^{T-q_j} s_k = (T - q_j)N_i + \alpha_i = q_i N_i.$$

Since $q_i N_i \leq (T - q_j)M_i$,

$$\sum_{k=1}^{T-q_j} (N_i + s_k) \leq (T - q_j)M_i.$$

Since we know that $s_k = g_1$ or $s_k = g_1 - 1$, $N_i + s_k$ values cannot differ by more than 1 for different k . Thus

$$(N_i + s_k) \leq M_i, \quad \forall k.$$

This means $\begin{bmatrix} \bar{\bar{\mathbf{H}}}_{ji}^{\text{ph1}} \\ -\bar{\mathbf{H}}_{ii}^{\text{ph1}} \end{bmatrix}$ is a square and full rank matrix which yields

$$\text{rank} \left(\begin{bmatrix} \bar{\bar{\mathbf{H}}}_{ji}^{\text{ph1}} \\ -\bar{\mathbf{H}}_{ii}^{\text{ph1}} \end{bmatrix} \bar{\mathbf{V}}_i^{\text{ph1}} \right) = \text{rank}(\bar{\mathbf{V}}_i^{\text{ph1}}).$$

Finally we can trace back the inequalities to get

$$\text{rank}(\mathbf{R}_i) \geq TN_i$$

and since $\text{rank}(\mathbf{R}_i) \leq TN_i$, this yields our final result

$$\text{rank}(\mathbf{R}_i) = TN_i.$$

REFERENCES

- [1] M. Rezaee, P. Schreier, M. Guillaud, and B. Clerckx, "A simple DoF-achievable scheme for the gaussian interference channel with delayed CSIT," *to appear in IEEE Globecom 2015*.
- [2] M. A. Maddah-Ali and D. Tse, "Completely stale transmitter channel state information is still very useful," *IEEE Trans. Inf. Theory*, vol. 58, no. 7, pp. 4418–4431, Jul. 2012.
- [3] C. S. Vaze and M. K. Varanasi, "Degrees of freedom region for the two-user MIMO broadcast channel with delayed CSIT," in *Proc. IEEE Int. Symp. Information Theory (ISIT)*, Russia, Aug. 2011.

- [4] H. Maleki, S. A. Jafar, and S. Shamai, "Retrospective interference alignment over interference networks," *IEEE Journal of Selected Topics in Signal Processing*, vol. 6, no. 3, pp. 228–240, Jun. 2012.
- [5] A. Ghasemi, A. S. Motahari, and A. K. Khandani, "On the degrees of freedom of X channel with delayed CSIT," in *Proc. IEEE Int. Symp. Information Theory (ISIT)*, Russia, Aug. 2011.
- [6] M. J. Abdoli, A. Ghasemi, and A. K. Khandani, "On the degrees of freedom of the K-user SISO interference and X channels with delayed CSIT," *IEEE Trans. Inf. Theory*, vol. 59, no. 10, pp. 6542–6561, Jun. 2013.
- [7] A. Ghasemi, A. S. Motahari, and A. K. Khandani, "Interference alignment for the MIMO interference channel with delayed local CSIT," *submitted to IEEE Transactions on Information Theory*, [Online]:[arXiv:1102.5673](http://arxiv.org/abs/1102.5673), Feb. 2011.
- [8] S. Lashgari, A. S. Avestimehr, and C. Suh, "Linear degrees of freedom of the X-channel with delayed CSIT," *IEEE Trans. Inf. Theory*, vol. 60, no. 4, pp. 2180 – 2189, Apr. 2014.
- [9] M. G. Kang and W. Choi, "Ergodic interference alignment with delayed feedback," *IEEE Signal Processing Letters*, vol. 20, no. 5, pp. 512 – 514, May. 2013.
- [10] C. S. Vaze and M. K. Varanasi, "The degrees of freedom region and interference alignment for the MIMO interference channel with delayed CSIT," *IEEE Trans. Inf. Theory*, vol. 58, no. 7, pp. 4396–4417, Jul. 2012.
- [11] S. Yang, M. Kobayashi, D. Gesbert, and X. Yi, "On the degrees of freedom of time correlated MISO broadcast channel with delayed CSIT," *IEEE Trans. Inf. Theory*, vol. 59, no. 1, pp. 315 – 328, Jan. 2013.
- [12] X. Yi, S. Yang, D. Gesbert, and M. Kobayashi, "The degrees of freedom region of temporally correlated MIMO networks with delayed CSIT," *IEEE Trans. Inf. Theory*, vol. 60, no. 1, pp. 494 – 514, Jan. 2014.
- [13] N. Lee and R. W. Heath, Jr, "Space-time interference alignment and degrees of freedom regions for the MISO broadcast channel with periodic CSI feedback," *IEEE Trans. Inf. Theory*, vol. 60, no. 1, pp. 515 – 528, Jan. 2014.
- [14] R. Tandon, S. A. Jafar, S. Shamai, and H. V. Poor, "On the synergistic benefits of alternating CSIT for the MISO-BC," *IEEE Trans. Inf. Theory*, vol. 59, no. 7, pp. 4106 – 4128, Jul. 2013.
- [15] T. Gou and S. A. Jafar, "Optimal use of current and outdated channel state information - degrees of freedom of the MISO BC with mixed CSIT," *IEEE Communications Letters*, vol. 16, no. 7, pp. 1084 – 1087, Jul. 2012.
- [16] B. Yang, D. Wang, J. Sun, and Y. Liu, "On the degrees of freedom region of general MIMO broadcast channel with mixed CSIT," in *Proc. IEEE Int. Symp. Information Theory (ISIT)*, Jul. 2013.
- [17] B. Rassouli, C. Hao, and B. Clerckx, "DoF analysis of the K-user MISO broadcast channel with hybrid CSIT," *accepted to IEEE ICC 2015*, <http://arxiv.org/abs/1503.00916>.
- [18] C. Hao and B. Clerckx, "Degrees-of-freedom of K-user MISO interference channel with delayed local CSIT," *accepted to IEEE ICC 2015*, <http://arxiv.org/abs/1502.07123>.
- [19] J. H. Conway, R. H. Hardin, and N. J. Sloane, "Packing lines, planes, etc.: Packings in grassmannian spaces." *Experimental Mathematics*, vol. 5, no. 2, pp. 139–159, 1996.