On exponential consensus for time-varying non-cooperative nonlinear networks

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Abstract—In this paper we present new results on asymptotic consensus for continuous-time nonlinear time varying networks. A key feature in the following is that the monotonicity property (i.e. cooperativity) is not required, unlike most of existing literature on the subject. Moreover, we give an estimate of the exponential rate of convergence towards the agreement manifold.

keyword: nonlinear networks, Consensus, Multi agent systems.

I. INTRODUCTION

In recent years the scientific community has devoted considerable attention to the consensus problem (see [2], [1], [10], [3] [4] and references therein). The main feature of the above solutions is to allow every agent to autonomously converge to a common *agreement* value or trajectory about some variables of interest using only local information at the node and/or the information received from neighboring agents. In the literature different conditions have been proposed to assess consensus in discrete and continuous time ([6], [12], [13], [14], [5], [9], just to cite a few) and under different class of both nonlinear time invariant and switching/time varying networks as outlined in [18], [11], [21], [22], [19], [20]. Most of the above frameworks assumed (in implicit or explicit way) the property of cooperativity (that in turn implies monotonicity). Given a function $f(t, x) : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, piecewise continuous in t and locally Lipschitz continuous with respect to x , the associated system of differential equations $\dot{x}(t) = f(t, x(t))$, is called cooperative if for any $i \in \{1, 2, \dots, n\}$, $f_i(t, x)$ is non-decreasing with respect to x_j for all $j \neq i$. Notice that this condition implies monotonicity of the flow $\phi(t; t_0, x_0)$ with respect to initial conditions, namely, for all t_0 and all $t \geq t_0$, it holds $\phi(t; t_0, x_1) \geq \phi(t; t_0, x_2)$ if $x_1 \geq x_2$ (where “ $\geq$ ” is meant componentwise), [24]. This property is usually guaranteed by requiring the sign definiteness of off diagonal entries of the Jacobian matrix

$F(x)$ (i.e. [22]). An extension to the case of signed graph with both positive and negative entries of the Jacobian matrix is reported in [23], although the Jacobian has to fulfil the sign constant (i.e. $F_{ij}(x_1)F_{ij}(x_2) \geq 0$, $\forall x_1, x_2 \in \mathbb{R}^n$ and $i, j \in N$) and sign-symmetric (i.e. $F_{ij}(x)F_{ji}(x) \geq 0 \forall x \in \mathbb{R}^n$) assumptions. Additionally it is still required the system to be monotone ([24]) with respect to the partial order induced by some arbitrary orthant.

A. Paper Contribution

In a recent paper [16], we introduced a condition for asymptotic agreement (*state frozen integral connectivity*), suitable for nonlinear time varying monotone networks that extended to this scenario the notion of integral connectivity introduced by Moreau for linear networks [15] with the additional merit to be frozen in state variables and therefore of simpler verification. Herein we undertake a non trivial further step that greatly extends the class of considered systems by removing the cooperativity assumption (and therefore the system flow monotonicity). Specifically the paper contributions are: i) guaranteed exponential consensus under weak connectivity properties (just existence of a spanning tree for a suitable averaged graph is required) for a large class of nonlinear time-varying non cooperative networks. This encompasses most of the agents models normally adopted in the literature in the linear and non linear time varying setting. Moreover, both the dynamic at the node (self-feedback) and the coupling can be time varying and state dependent with the notable feature that the strength of attraction between two agents may vanish as so as the distance between the state values becomes larger. This is representative of many interactions occurring in several network scenarios: for instance in social networks, agents with different opinions have a low mutual influence when their distance increases; in sensor networks the data coming from nodes at higher distance may be less weighted in data fusion operations at the node; ii) we extend the use of a “State Frozen” concept [16] and integral connectivity to this non-trivial scenario of non cooperative networks by introducing a suitable agents connectivity property (later called “Frozen integral connectivity”). This has the merit to avoid the circular argument by which solutions depend

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on the connectivity and the latter is in turn influenced by state evolutions. This type of circular argument normally makes up for conditions that can hardly be tested in the case of time-varying nonlinear agent dynamics and coupling without explicit apriori knowledge of solutions; iii) for the described class of systems, we provide an estimate of the exponential rate of convergence towards the agreement manifold.

II. NOTATION AND PROBLEM STATEMENT

Throughout the paper all vectors are assumed to be column vectors. To denote vectors we write $x = [x_1, \dots, x_n]$ for the column vector $x \in \mathbb{R}^n$. $|x|$ denotes the Euclidean norm of x . $\mathbf{1}$ is the vector of all ones and e_j is the j -th element of the canonical basis of $\mathbb{R}^n$, where n should normally be clear from the context. The integer interval $N = \{1, 2, \dots, n\}$ will be identified with the set of interacting agents.

Consider a network of agents as described by the following system of nonlinear differential equations:

$$\dot{x}(t) = f(t, x(t)) \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $t \in \mathbb{R}_+$ denotes time and f is a vector field $f : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ describing the dynamics of the interaction between agents.

We assume: i) f is locally Lipschitz continuous with respect to x uniformly in time, viz. for all compacts $K \in \mathbb{R}^n$ there exists $L_K > 0$, such that, for all $x_a, x_b \in K$ and all $t \geq 0$ it holds $|f(t, x_a) - f(t, x_b)| \leq L_K |x_a - x_b|$;¹ ii) that f admits an agreement equilibrium set, that is:

$$\mathcal{E} = \text{span}\{\mathbf{1}\} \subseteq \{x \in \mathbb{R}^n : f(t, x) = 0 \forall t \in \mathbb{R}_+\}. \quad (2)$$

The assumptions on f , imply the local existence and the unicity of the system's solution on some maximally extended open interval of definition.

Let $x(t)$ denote a solution of (1). At any time instant t the following quantities are of interest:

$$x_{\max}(t) = \max_{k \in N} \{x_k(t)\}, \quad x_{\min}(t) = \min_{k \in N} \{x_k(t)\}$$

and $\delta_k(t) = |x_{\max}(t) - x_k(t)|$ for all $k \in N$ (or symmetrically $\bar{\delta}_k(t) = |x_{\min}(t) - x_k(t)|$).

Fixed an arbitrary solution $x(\cdot)$ and an arbitrary time t we define a time-dependent permutation $p_j(t)$ of indices $j \in N$ such that it fulfills

$$x_{p_1(t)}(t) \leq x_{p_2(t)}(t) \leq x_{p_3(t)}(t) \leq \dots \leq x_{p_n(t)}(t).$$

Notice that, if two or more entries of x take some given value, then the permutation is not uniquely defined. Nevertheless the permutation always exists and the value $x_{p_i(t)}$ is independent of how it is selected. Therefore, for

¹This holds, for instance, when the Jacobian is uniformly bounded as a function of time.

any solution $x(t)$ of (1) we can define the corresponding re-ordered solution as $x_{p_i(t)}$. At any time instant t the following quantities are of interest:

$$x_{\max}(t) = \max_{k \in N} \{x_k(t)\}; \quad x_{\min}(t) = \min_{k \in N} \{x_k(t)\}.$$

III. MAIN RESULTS

Next we state our connectivity assumption which is crucial to attain asymptotic consensus.

Assumption 1 (Frozen Integral Connectivity) *Given network (1), we say that Integral frozen connectivity holds, provided for all compact intervals $K \subset \mathbb{R}$ there exist some sufficiently large $T > 0$, positive ε_T , ε_K , a root node $r \in N$, a rooted spanning tree $\mathcal{T}_r \subset N \times N$ and associated connectivity sets $\psi_{(i,j)} \subset \mathbb{R}$ (defined for $(i, j) \in \mathcal{T}_r$) such that for all $x \in K^n$ and any $t \geq 0$*

$$\begin{aligned} \text{sign}(x_j - x_i)[f_i(t, x) - f_i(t, x + (x_i - x_j)e_j)] \\ \geq \Psi_{ij}(t)\varepsilon_K|x_j - x_i|, \end{aligned} \quad (3)$$

with

$$\Psi_{ij}(t) = \begin{cases} 1 & \text{if } t \in \psi_{(i,j)}, (i, j) \in \mathcal{T}_r; \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\int_{\psi_{(i,j)} \cap [t, t+T]} d\tau \geq \varepsilon_T, \quad (4)$$

Remark 1 *Notice that if equation (4) holds for some T , it holds a fortiori for all $\bar{T} > T$.*

Notice that, in the light of equation (4) this is an assumption of averaged weak connectedness across uniform time intervals, while by condition (3), the node interaction property is defined on frozen state variables across the same interval, making its verification straightforward. The following fact is well-known for the existing consensus conditions and it holds under the proposed assumption. The prove is omitted for sake of brevity and may be find elsewhere ([17]).

Lemma 1 *The functions $x_{\max}(t)$ and $x_{\min}(t)$ are (respectively) monotonically non-increasing and non-decreasing.*

In what follows we will present a key lemma which will allow us to later prove exponential asymptotic consensus.

Lemma 2 *Let $r \in N$ be the root of the spanning tree as from Assumption 1. For all initial conditions $x(0) \in \mathbb{R}^n$, there exists a finite positive integer $\bar{k}$ and $\mu > 0$ (uniform*

in time) such that, for all $t \geq 0$, the following holds along the solutions of (1):

$$x_{\max}(t + \bar{k}T) \leq x_{\max}(t) - \mu|x_{\max}(t) - x_r(t)| \quad (5)$$

and:

$$x_{\min}(t + \bar{k}T) \geq x_{\min}(t) + \mu|x_{\min}(t) - x_r(t)|. \quad (6)$$

Proof: We prove the Lemma for $x_{\max}(t)$, a similar argument holds for $x_{\min}(t)$. Let $\varepsilon = \varepsilon_T \varepsilon_x$, and $d(q) : N \rightarrow \mathbb{N}$ denote the distance in the spanning tree of node q from the root r of the tree $\mathcal{T}_r$ as in Assumption 1. Let us deal first with nodes q at distance $d(q) = 1$. We consider different cases.

Case $a)$: $x_r(\tau) \leq x_q(\tau)$, for all $\tau \in [t, t + 2T]$.

To come up with a suitable estimate we further need to consider the following subcases.

Subcase $a_1)$ $\delta_q(t) \leq \frac{1}{2}\delta_r(t)$. Define $\bar{q}(\tau) \in \{1, \dots, n\}$ so as to fulfill $p_{\bar{q}(\tau)}(\tau) = q$. In the following expressions, the time dependence of $\bar{q}$ will be omitted for the sake of simplicity of notation. Then, for any node q at distance 1 from the root it holds for all $\tau \in [t, t + 2T]$:

$$\begin{aligned} x_q(\tau) - x_q(t) &= \int_t^\tau f_q(\theta, x(\theta)) d\theta = \\ &\int_t^\tau \left([f_q(\theta, x(\theta)) - f_q(\theta, x(\theta) + (x_q(\theta) - x_{p_{\bar{q}-1}(\theta)}(\theta))e_{p_{\bar{q}-1}(\theta)})] + [f_q(\theta, x(\theta) + \right. \\ &(x_q(\theta) - x_{p_{\bar{q}-1}(\theta)}(\theta))e_{p_{\bar{q}-1}(\theta)}) - f_q(\theta, x(\theta) + \\ &(x_q(\theta) - x_{p_{\bar{q}-1}(\theta)}(\theta))e_{p_{\bar{q}-1}(\theta)}) + (x_q(\theta) - x_{p_{\bar{q}-2}(\theta)}(\theta))e_{p_{\bar{q}-2}(\theta)})] + [f_q(\theta, x(\theta) + (x_q(\theta) - \\ &x_{p_{\bar{q}-1}(\theta)}(\theta))e_{p_{\bar{q}-1}(\theta)}) \\ &+ (x_q(\theta) - x_{p_{\bar{q}-2}(\theta)}(\theta))e_{p_{\bar{q}-2}(\theta)}) - \\ &f_q(\theta, x(\theta) + (x_q(\theta) - x_{p_{\bar{q}-1}(\theta)}(\theta))e_{p_{\bar{q}-1}(\theta)}) \\ &+ (x_q(\theta) - x_{p_{\bar{q}-2}(\theta)}(\theta))e_{p_{\bar{q}-2}(\theta)}) \\ &+ (x_q(\theta) - x_{p_{\bar{q}-3}(\theta)}(\theta))e_{p_{\bar{q}-3}(\theta)})] + \\ &\dots + [f_q(\theta, x(\theta) + (x_q(\theta) - x_{p_{\bar{q}-1}(\theta)}(\theta))e_{p_{\bar{q}-1}(\theta)}) + \dots \\ &+ (x_q(\theta) - x_{p_2(\theta)}(\theta))e_{p_2(\theta)}) \\ &- f_q(\theta, x(\theta) + (x_q(\theta) - x_{p_{\bar{q}-1}(\theta)}(\theta))e_{p_{\bar{q}-1}(\theta)}) \\ &+ \dots + (x_q(\theta) - x_{p_1(\theta)}(\theta))e_{p_1(\theta)})] \\ &+ f_q(\theta, x(\theta) + (x_q(\theta) - x_{p_{\bar{q}-1}(\theta)}(\theta))e_{p_{\bar{q}-1}(\theta)}) \\ &+ \dots + (x_q(\theta) - x_{p_1(\theta)}(\theta))e_{p_1(\theta)}) \Big) d\theta \end{aligned}$$

The application of Assumption 1 to each of the terms in the integrand of the previous expression (except for the last one) leads to:

$$x_q(\tau) - x_q(t) \leq$$

$$\begin{aligned} &- \int_t^\tau \sum_{j: x_j(\theta) < x_q(\theta)} \Psi_{qj}(\theta) \varepsilon_{\mathcal{K}} |x_q(\theta) - x_j(\theta)| d\theta \\ &+ \int_t^\tau f_q(\theta, x(\theta) + (x_q(\theta) - x_{p_{\bar{q}-1}(\theta)}(\theta))e_{p_{\bar{q}-1}(\theta)}) \\ &+ \dots + (x_q(\theta) - x_{p_1(\theta)}(\theta))e_{p_1(\theta)}) d\theta \end{aligned}$$

The former calculations are instrumental for the subsequent exploitation of uniform Lipschitz continuity of f as detailed below:

$$\begin{aligned} x_q(\tau) - x_q(t) &\leq \\ &- \int_t^\tau \sum_{j: x_j(\theta) < x_q(\theta)} \Psi_{qj}(\theta) \varepsilon_{\mathcal{K}} |x_q(\theta) - x_j(\theta)| d\theta \\ &+ \int_t^\tau \left([f_q(\theta, x(\theta) + (x_q(\theta) - x_{p_{\bar{q}-1}(\theta)}(\theta))e_{p_{\bar{q}-1}(\theta)} + \right. \\ &\dots + (x_q(\theta) - x_{p_1(\theta)}(\theta))e_{p_1(\theta)}) - f_q(\theta, x_q(\theta)\mathbf{1})] \\ &\left. + f_q(\theta, x_q(\theta)\mathbf{1}) \right) d\theta. \end{aligned}$$

Being $f_q(\theta, x_q(\theta)\mathbf{1}) = 0$, it results:

$$\begin{aligned} x_q(\tau) - x_q(t) &\leq \\ &- \int_t^\tau \sum_{j: x_j(\theta) < x_q(\theta)} \Psi_{qj}(\theta) \varepsilon_{\mathcal{K}} |x_q(\theta) - x_j(\theta)| d\theta \\ &- L \int_t^\tau \sum_{j: x_j(\theta) \geq x_q(\theta)} [x_q(\theta) - x_j(\theta)] d\theta \\ &\leq - \int_t^\tau \sum_{j: x_j(\theta) < x_q(\theta)} \Psi_{qj}(\theta) \varepsilon_{\mathcal{K}} |x_q(\theta) - x_j(\theta)| d\theta \\ &- L \int_t^\tau \sum_{j: x_j(\theta) \geq x_q(\theta)} [x_q(\theta) - x_{\max}(t)] d\theta \\ &\leq - \int_t^\tau \sum_{j: x_j(\theta) < x_q(\theta)} \Psi_{qj}(\theta) \varepsilon_{\mathcal{K}} |x_q(\theta) - x_j(\theta)| d\theta \\ &- (n-1)L \int_t^\tau [x_q(\theta) - x_{\max}(t)] d\theta, \end{aligned}$$

with L denotes the (time-independent) Lipschitz constant of f_q . In particular for all $\tau \in [t + T, t + 2T]$ we see that:

$$\begin{aligned} x_q(\tau) - x_{\max}(t) &\leq x_q(\tau) - x_q(t) \quad (7) \\ &\leq - \int_t^\tau \sum_{j: x_j(\theta) < x_q(\theta)} \Psi_{qj}(\theta) \varepsilon_{\mathcal{K}} |x_q(\theta) - x_j(\theta)| d\theta \\ &- (n-1)L \int_t^\tau [x_q(\theta) - x_{\max}(t)] d\theta \\ &\leq - \int_t^\tau \Psi_{qr}(\theta) \varepsilon_{\mathcal{K}} |x_q(\theta) - x_r(\theta)| d\theta \\ &- (n-1)L \int_t^\tau [x_q(\theta) - x_{\max}(t)] d\theta. \end{aligned}$$

By the triangular inequality it holds:

$$-|x_q(\theta) - x_r(\theta)| \leq -|x_{\max}(t) - x_r(\theta)| + |x_{\max}(t) - x_q(\theta)|,$$

moreover, by Lipschitz continuity of f_r , we may infer:

$$|x_{\max}(t) - x_r(\theta)| \geq e^{-L(\theta-t)} |x_{\max}(t) - x_r(t)|.$$

Combining the above inequalities, we may restate the bound for $x_q(\tau) - x_{\max}(t)$ expressed in (7) as detailed below:

$$\begin{aligned} x_q(\tau) - x_{\max}(t) &\leq \\ &- \int_t^\tau \Psi_{qr}(\theta) \varepsilon_K e^{-L(\theta-t)} |x_{\max}(t) - x_r(t)| d\theta \\ &- \int_t^\tau \Psi_{qr}(\theta) \varepsilon_K [x_q(\theta) - x_{\max}(t)] d\theta \\ &- (n-1)L \int_t^\tau [x_q(\theta) - x_{\max}(\theta)] d\theta \\ &\leq -\varepsilon_K e^{-2LT} |x_{\max}(t) - x_r(t)| \int_t^\tau \Psi_{qr}(\theta) d\theta \\ &- \varepsilon_K \int_t^\tau [x_q(\theta) - x_{\max}(t)] d\theta \\ &- (n-1)L \int_t^\tau [x_q(\theta) - x_{\max}(t)] d\theta \\ &\leq -\varepsilon e^{-2LT} |x_{\max}(t) - x_r(t)| \\ &- \varepsilon_K \int_t^\tau [x_q(\theta) - x_{\max}(t)] d\theta \\ &- (n-1)L \int_t^\tau [x_q(\theta) - x_{\max}(t)] d\theta \end{aligned} \quad (8)$$

with $\varepsilon = \varepsilon_K \varepsilon_T$. By defining $\Delta(\tau) = \int_t^\tau [x_q(\theta) - x_{\max}(t)] d\theta$ we can recast equation (8) as:

$$\begin{aligned} \frac{d}{d\tau} \Delta(\tau) &\leq -\varepsilon e^{-2LT} |x_{\max}(t) - x_r(t)| \\ &- ((n-1)L + \varepsilon_K) \Delta(\tau), \end{aligned}$$

which holds for all $\tau \in [t+T, t+2T]$. Since $\Delta(t+T) \leq 0$, by a standard comparison principle we see that:

$$\Delta(\tau) \leq -\mu_\Delta(\tau) |x_{\max}(t) - x_r(t)|, \quad (9)$$

with

$$\mu_\Delta(\tau) = e^{-2LT} \frac{\varepsilon [1 - e^{-((n-1)L + \varepsilon_K)(\tau-T-t)}]}{((n-1)L + \varepsilon_K)},$$

which holds for all $\tau \in [t+T, t+2T]$. In particular, for $\tau = t+2T$ equation (9) yields:

$$\Delta(t+2T) \leq -\mu_\Delta |x_{\max}(t) - x_r(t)|, \quad (10)$$

with

$$\mu_\Delta = \frac{\varepsilon}{((n-1)L + \varepsilon_K)} e^{-2LT} [1 - e^{-((n-1)L + \varepsilon_K)T}].$$

From the mean value theorem it results:

$$\exists t^* \in [t, t+2T] : x_q(t^*) - x_{\max}(t) = \frac{\Delta(t+2T)}{2T}. \quad (11)$$

By Lipschitz continuity of f , convergence of $x_q(t)$ towards $x_{\max}(t)$ is at most exponential in time and therefore we may infer:

$$x_q(t+2T) - x_{\max}(t) \leq (x_q(t^*) - x_{\max}(t)) e^{-2LT}. \quad (12)$$

From (11) and (12) it results:

$$x_q(t+2T) - x_{\max}(t) \leq \frac{\Delta(t+2T)}{2T} e^{-2LT} \quad (13)$$

Finally, in order to derive an estimate of how decreasing is $x_q(t)$ which is uniform in time we combine (13) and (10) and obtain:

$$x_q(t+2T) - x_{\max}(t) \leq -\mu_{a_1} \delta_r(t),$$

with $\mu_{a_1} = e^{-4LT} \frac{\varepsilon [1 - e^{-((n-1)L + \varepsilon_K)T}]}{2((n-1)L + \varepsilon_K)T}$ and $\delta_r(t) = |x_{\max}(t) - x_r(t)|$.

Subcase a_2) $\delta_q(t) > \frac{1}{2} \delta_r(t)$

In this scenario, by Lipschitz continuity of f , convergence of $x_q(\tau)$ towards the value $x_{\max}(t)$ is at most exponential, and therefore we may infer:

$$|x_{\max}(t) - x_q(t+2T)| \geq e^{-2LT} |x_{\max}(t) - x_q(t)|.$$

that yields:

$$\begin{aligned} x_q(t+2T) &\leq x_{\max}(t) - e^{-2LT} |x_{\max}(t) - x_q(t)| \\ &= x_{\max}(t) - e^{-2LT} \delta_q(t) \\ &\leq x_{\max}(t) - \mu_{a_2} \delta_r(t). \end{aligned}$$

with $\mu_{a_2} = \frac{1}{2} e^{-2LT}$ and $\delta_r(t) = |x_{\max}(t) - x_r(t)|$.

Case b): $x_r(\tau) \geq x_q(\tau)$ for all $\tau \in [t, t+2T]$

In this case considering that $\delta_q(t) \geq \delta_r(t)$ and exploiting Lipschitz continuity of f , we may infer:

$$\begin{aligned} x_q(t+2T) &\leq x_{\max}(t) - e^{-2LT} |x_{\max}(t) - x_q(t)| \\ &= x_{\max}(t) - e^{-2LT} \delta_q(t) \leq x_{\max}(t) - \mu_b \delta_r(t), \end{aligned}$$

with $\mu_b = e^{-2LT}$.

Case c): $\exists \bar{\tau} \in (0, 2T]$ such that $x_q(t+\bar{\tau}) = x_r(t+\bar{\tau})$.

By Lipschitz continuity of f , convergence of x_r and x_q towards the value $x_{\max}(t)$ is at most exponential. This, along with assumption $x_q(t+\bar{\tau}) = x_r(t+\bar{\tau})$, yields:

$$\begin{aligned} |x_{\max}(t) - x_q(t+2T)| &\geq \\ &e^{-L(2T-\bar{\tau})} |x_{\max}(t) - x_q(t+\bar{\tau})| \\ &= e^{-L(2T-\bar{\tau})} |x_{\max}(t) - x_r(t+\bar{\tau})| \\ &\geq e^{-2LT} |x_{\max}(t) - x_r(t)|, \end{aligned}$$

and therefore $x_q(t+2T) \leq x_{\max}(t) - \mu_c \delta_r(t)$ with $\mu_c = e^{-2LT}$.

Therefore, in any of cases a , b and c it results:

$$x_q(t+2T) \leq x_{\max}(t) - \mu_1 \delta_r(t) \quad (14)$$

or in other terms:

$$|x_{\max}(t) - x_q(t + 2T)| \geq \mu_1 \delta_r(t) \quad (15)$$

with $\mu_1 = \min\{\mu_{a_1}, \mu_{a_2}, \mu_b, \mu_c\}$ and $\delta_r(t) = |x_{\max}(t) - x_r(t)|$. Next we deal with nodes $k \in N$ with $d(k) = 2$. Let q be such that $d(q) = 1$ and $(q, k) \in \mathcal{T}_r$. We consider different cases.

Case a): $x_k(t + \tau) \geq x_q(t + \tau)$, for all $\tau \in [2T, 4T]$

Subcase a_1): $\delta_k(t + 2T) \leq \frac{1}{2}\delta_q(t + 2T)$. For node k , let $\bar{k}(\tau)$ be such that $p_{\bar{k}(\tau)}(\tau) = k$. In the following we omit the time dependence of $\bar{k}$ for the sake of simplicity of notation. Then, the following holds for all $\tau \in [t + 2T, t + 4T]$:

$$\begin{aligned} x_k(\tau) - x_k(t + 2T) &= \int_{t+2T}^{\tau} f_k(\theta, x(\theta)) d\theta \\ &= \int_{t+2T}^{\tau} \left([f_k(\theta, x(\theta)) - f_k(\theta, x(\theta) \right. \\ &\quad + (x_k(\theta) - x_{p_{\bar{k}-1}(\theta)}(\theta))e_{p_{\bar{k}-1}(\theta)}] \\ &\quad + [f_k(\theta, x(\theta) + (x_k(\theta) - x_{p_{\bar{k}-1}(\theta)}(\theta))e_{p_{\bar{k}-1}(\theta)}) \\ &\quad - f_k(\theta, x(\theta) + (x_k(\theta) - x_{p_{\bar{k}-1}(\theta)}(\theta))e_{p_{\bar{k}-1}(\theta)} \\ &\quad + (x_k(\theta) - x_{p_{\bar{k}-2}(\theta)}(\theta))e_{p_{\bar{k}-2}(\theta)}] \\ &\quad + [f_k(\theta, x(\theta) + (x_k(\theta) - x_{p_{\bar{k}-1}(\theta)}(\theta))e_{p_{\bar{k}-1}(\theta)} \\ &\quad + (x_k(\theta) - x_{p_{\bar{k}-2}(\theta)}(\theta))e_{p_{\bar{k}-2}(\theta)}) \\ &\quad - f_k(\theta, x(\theta) + (x_k(\theta) - x_{p_{\bar{k}-1}(\theta)}(\theta))e_{p_{\bar{k}-1}(\theta)} + \\ &\quad (x_k(\theta) - x_{p_{\bar{k}-2}(\theta)}(\theta))e_{p_{\bar{k}-2}(\theta)} \\ &\quad + (x_k(\theta) - x_{p_{\bar{k}-3}(\theta)}(\theta))e_{p_{\bar{k}-3}(\theta)}] \\ &\quad + \dots + [f_k(\theta, x(\theta) + (x_k(\theta) - x_{p_{\bar{k}-1}(\theta)}(\theta))e_{p_{\bar{k}-1}(\theta)} + \\ &\quad \dots + (x_k(\theta) - x_{p_2(\theta)}(\theta))e_{p_2(\theta)}) \\ &\quad - f_k(\theta, x(\theta) + (x_k(\theta) - x_{p_{\bar{k}-1}(\theta)}(\theta))e_{p_{\bar{k}-1}(\theta)} + \\ &\quad \dots + (x_k(\theta) - x_{p_1(\theta)}(\theta))e_{p_1(\theta)}] \\ &\quad + f_k(\theta, x(\theta) + (x_k(\theta) - x_{p_{\bar{k}-1}(\theta)}(\theta))e_{p_{\bar{k}-1}(\theta)} + \\ &\quad \dots + (x_k(\theta) - x_{p_1(\theta)}(\theta))e_{p_1(\theta)}) \Big) d\theta \\ &\leq - \int_{t+2T}^{\tau} \sum_{j: x_j(\theta) < x_k(\theta)} \Psi_{kj}(\theta) \varepsilon_{\mathcal{K}} |x_k(\theta) - x_j(\theta)| d\theta \\ &\quad + \int_{t+2T}^{\tau} f_k(\theta, x(\theta) + (x_k(\theta) - x_{p_{\bar{k}-1}(\theta)}(\theta))e_{p_{\bar{k}-1}(\theta)} \\ &\quad + \dots + (x_k(\theta) - x_{p_1(\theta)}(\theta))e_{p_1(\theta)}) d\theta \\ &\leq - \int_{t+2T}^{\tau} \sum_{j: x_j(\theta) < x_k(\theta)} \Psi_{kj}(\theta) \varepsilon_{\mathcal{K}} |x_k(\theta) - x_j(\theta)| d\theta + \\ &\quad \int_{t+2T}^{\tau} \left([f_k(\theta, x(\theta) + (x_k(\theta) - x_{p_{\bar{k}-1}(\theta)}(\theta))e_{p_{\bar{k}-1}(\theta)} + \right. \\ &\quad \dots + (x_k(\theta) - x_{p_1(\theta)}(\theta))e_{p_1(\theta)}) - f_k(\theta, x_k(\theta)\mathbf{1}) \\ &\quad \left. + f_k(\theta, x_k(\theta)\mathbf{1}) \right) d\theta \end{aligned}$$

Being $f_k(\theta, x_k(\theta)\mathbf{1}) = 0$, it results:

$$\begin{aligned} x_k(\tau) - x_k(t) & \quad (16) \\ &\leq - \int_{t+2T}^{\tau} \sum_{j: x_j(\theta) < x_k(\theta)} \Psi_{kj}(\theta) \varepsilon_{\mathcal{K}} |x_k(\theta) - x_j(\theta)| d\theta \\ &\quad - L \int_{t+2T}^{\tau} \sum_{j: x_j(\theta) \geq x_k(\theta)} [x_k(\theta) - x_j(\theta)] d\theta \\ &\leq - \int_{t+2T}^{\tau} \sum_{j: x_j(\theta) < x_k(\theta)} \Psi_{kj}(\theta) \varepsilon_{\mathcal{K}} |x_k(\theta) - x_j(\theta)| d\theta \\ &\quad - L \int_{t+2T}^{\tau} \sum_{j: x_j(\theta) \geq x_k(\theta)} [x_k(\theta) - x_{\max}(t)] d\theta \\ &\leq - \int_{t+2T}^{\tau} \sum_{j: x_j(\theta) < x_k(\theta)} \Psi_{kj}(\theta) \varepsilon_{\mathcal{K}} |x_k(\theta) - x_j(\theta)| d\theta \\ &\quad - (n-1)L \int_{t+2T}^{\tau} [x_k(\theta) - x_{\max}(t)] d\theta \end{aligned}$$

where again L denotes the (time-independent) Lipschitz constant of f_k .

In particular, exploiting:

$$x_k(t + 2T) \leq x_{\max}(t + 2T) \leq x_{\max}(t),$$

and resuming the series of inequalities in (16), we see that for all $\tau \in [t + 3T, t + 4T]$ it holds:

$$\begin{aligned} x_k(\tau) - x_{\max}(t) &\leq x_k(\tau) - x_k(t + 2T) \quad (17) \\ &\leq - \int_{t+2T}^{\tau} \sum_{j: x_j(\theta) < x_k(\theta)} \Psi_{kj}(\theta) \varepsilon_{\mathcal{K}} |x_k(\theta) - x_j(\theta)| d\theta \\ &\quad - (n-1)L \int_{t+2T}^{\tau} [x_k(\theta) - x_{\max}(t)] d\theta \\ &\leq - \int_{t+2T}^{\tau} \Psi_{kq}(\theta) \varepsilon_{\mathcal{K}} |x_k(\theta) - x_q(\theta)| d\theta \quad (18) \\ &\quad - (n-1)L \int_{t+2T}^{\tau} [x_k(\theta) - x_{\max}(t)] d\theta. \end{aligned}$$

By the triangular inequality:

$$-|x_k(\theta) - x_q(\theta)| \leq -|x_{\max}(t) - x_q(\theta)| + |x_{\max}(t) - x_k(\theta)|,$$

moreover, by Lipschitz continuity of f , we may infer: $|x_{\max}(t) - x_q(\theta)| \geq e^{-L(\theta - (t+2T))} |x_{\max}(t) - x_q(t + 2T)|$. Therefore, substituting the previous inequalities in (17), we have:

$$\begin{aligned} x_k(\tau) - x_{\max}(t) &\leq \\ &- \int_{t+2T}^{\tau} \Psi_{kq}(\theta) \varepsilon_{\mathcal{K}} e^{-L(\theta - t)} |x_{\max}(t) - x_q(t + 2T)| d\theta \\ &- \int_{t+2T}^{\tau} \Psi_{kq}(\theta) \varepsilon_{\mathcal{K}} [x_k(\theta) - x_{\max}(t)] d\theta \\ &- (n-1)L \int_{t+2T}^{\tau} [x_k(\theta) - x_{\max}(t)] d\theta \end{aligned}$$

$$\begin{aligned}
&\leq -\varepsilon_{\mathcal{K}} e^{-2LT} |x_{\max}(t) - x_q(t+2T)| \int_{t+2T}^{\tau} \Psi_{kq}(\theta) d\theta \\
&- \varepsilon_{\mathcal{K}} \int_{t+2T}^{\tau} [x_k(\theta) - x_{\max}(t)] d\theta \\
&- (n-1)L \int_{t+2T}^{\tau} [x_k(\theta) - x_{\max}(t)] d\theta \\
&\leq -\varepsilon e^{-2LT} |x_{\max}(t) - x_q(t+2T)| \\
&- \varepsilon_{\mathcal{K}} \int_{t+2T}^{\tau} [x_k(\theta) - x_{\max}(t)] d\theta \\
&- (n-1)L \int_{t+2T}^{\tau} [x_k(\theta) - x_{\max}(t)] d\theta
\end{aligned}$$

By defining $\Delta(\tau) = \int_{t+2T}^{\tau} [x_k(\theta) - x_{\max}(t)] d\theta$ we can recast the above equation as:

$$\begin{aligned}
\frac{d}{d\tau} \Delta(\tau) &\leq -((n-1)L + \varepsilon_x) \Delta(\tau) \\
&- \varepsilon e^{-2LT} |x_{\max}(t) - x_q(t+2T)|,
\end{aligned}$$

which holds for all $\tau \in [t+3T, t+4T]$. Since $\Delta(t+3T) \leq 0$, by a standard comparison principle we see that:

$$\Delta(\tau) \leq -\mu_{\Delta}(\tau) |x_{\max}(t) - x_q(t+2T)|, \quad (19)$$

with

$$\mu_{\Delta}(\tau) = e^{-2LT} \frac{\varepsilon [1 - e^{-((n-1)L + \varepsilon_{\mathcal{K}})(\tau - T - t)}]}{((n-1)L + \varepsilon_{\mathcal{K}})},$$

which holds for all $\tau \in [t+3T, t+4T]$. In particular, for $\tau = t+4T$ equation (19) yields:

$$\Delta(t+4T) \leq -\mu_{\Delta} |x_{\max}(t) - x_q(t+2T)|, \quad (20)$$

with

$$\mu_{\Delta} = \frac{\varepsilon}{((n-1)L + \varepsilon_{\mathcal{K}})} e^{-2LT} [1 - e^{-((n-1)L + \varepsilon_{\mathcal{K}})T}].$$

By applying the mean value theorem and exploiting Lipschitz continuity, it results:

$$\begin{aligned}
x_k(t+4T) - x_{\max}(t) &\leq \frac{\Delta(t+4T)}{2T} e^{-2LT} \quad (21) \\
&\leq -\mu_{a_1} |x_{\max}(t) - x_q(t+2T)|
\end{aligned}$$

$$\text{with } \mu_{a_1} = e^{-4LT} \frac{\varepsilon [1 - e^{-((n-1)L + \varepsilon_{\mathcal{K}})T}]}{2((n-1)L + \varepsilon_{\mathcal{K}})T}.$$

Subcase a_2): $\delta_k(t+2T) \geq \frac{1}{2}\delta_q(t+2T)$. In this scenario taking into account that by Lipschitz continuity of f it results $|x_{\max}(t+4T) - x_k(t+4T)| \geq e^{-2LT} |x_{\max}(t+2T) - x_k(t+2T)|$, we may infer:

$$\begin{aligned}
x_k(t+4T) - x_{\max}(t) &\quad (22) \\
&= x_k(t+4T) - x_{\max}(t+2T) \\
&- [x_{\max}(t) - x_{\max}(t+2T)] \\
&\leq x_k(t+4T) - x_{\max}(t+4T) \\
&- [x_{\max}(t) - x_{\max}(t+2T)]
\end{aligned}$$

$$\begin{aligned}
&\leq -e^{-2LT} |x_{\max}(t+2T) - x_k(t+2T)| \quad (23) \\
&- [x_{\max}(t) - x_{\max}(t+2T)] \\
&= -e^{-2LT} \delta_k(t+2T) - [x_{\max}(t) - x_{\max}(t+2T)] \\
&\leq -e^{-2LT} \frac{1}{2} \delta_q(t+2T) \\
&- [x_{\max}(t) - x_{\max}(t+2T)] \\
&= -e^{-2LT} \frac{1}{2} [x_{\max}(t+2T) - x_q(t+2T)] \\
&- [x_{\max}(t) - x_{\max}(t+2T)] \\
&\leq -e^{-2LT} \frac{1}{2} [x_{\max}(t) - x_q(t+2T)] \\
&+ e^{-2LT} \frac{1}{2} [x_{\max}(t) - x_{\max}(t+2T)] \\
&- [x_{\max}(t) - x_{\max}(t+2T)] \\
&= -\mu_{a_2} [x_{\max}(t) - x_q(t+2T)] \\
&- (1 - \mu_{a_2}) [x_{\max}(t) - x_{\max}(t+2T)] \\
&\leq -\mu_{a_2} [x_{\max}(t) - x_q(t+2T)]
\end{aligned}$$

being $\mu_{a_2} = e^{-2LT} \frac{1}{2} < 1$.

Case b): $x_q(t+\tau) \geq x_k(t+\tau) \quad \tau \in [2T, 4T]$

In this case, we may infer:

$$\begin{aligned}
x_k(t+4T) &\leq x_{\max}(t) - e^{-2LT} |x_{\max}(t) - x_k(t+2T)| \\
&\leq x_{\max}(t) - \mu_b |x_{\max}(t) - x_q(t+2T)|,
\end{aligned}$$

with $\mu_b = e^{-2LT}$.

Case c): $x_q(t+\bar{\tau}) = x_k(t+\bar{\tau})$ for some $\bar{\tau} \in (2T, 4T]$.

By Lipschitz continuity of f , it results:

$$\begin{aligned}
&|x_{\max}(t) - x_k(t+4T)| \geq \\
&e^{-L(4T-\bar{\tau})} |x_{\max}(t) - x_k(t+\bar{\tau})| \\
&e^{-L(4T-\bar{\tau})} |x_{\max}(t) - x_q(t+\bar{\tau})| \\
&\geq e^{-2LT} |x_{\max}(t) - x_q(t+2T)| \\
&= \mu_c |x_{\max}(t) - x_q(t+2T)|.
\end{aligned}$$

with $\mu_c = e^{-2LT}$.

Therefore, in any of cases a , b and c it results:

$$x_k(t+4T) \leq x_{\max}(t) - \mu_2 |x_{\max}(t) - x_q(t+2T)| \quad (24)$$

with $\mu_2 = \min\{\mu_{a_1}, \mu_{a_2}, \mu_b, \mu_c\}$.

Consequently, in order to derive an estimate of how decreasing is $x_k(t)$ which is uniform in time by combining (24) and (15) we obtain:

$$x_k(t+4T) - x_{\max}(t) \leq -\mu_1 \mu_2 |x_{\max}(t) - x_r(t)|. \quad (25)$$

A similar procedure can be used to construct an estimate of the convergence rate for an arbitrary node at distance $d(k)+1$ based on the estimate for nodes at distance $d(k)$. By induction, for any node k at distance $d(k)$ from the

root, the following inequality holds:

$$\begin{aligned} & x_k(t + 2d(k)T) - x_{\max}(t) \\ & \leq - \left(\prod_{i=1}^{d(k)} \mu_i \right) |x_{\max}(t) - x_r(t)| \\ & = -\mu(d(k)) |x_{\max}(t) - x_r(t)|, \end{aligned} \quad (26)$$

with $\mu(d(k)) = \prod_{i=1}^{d(k)} \mu_i$ being a positive constant for any fixed $d(k)$.

Given the fact that only a finite number of agents are present and by Assumption 1 every agent k has a finite distance from the root, a uniform estimate of the convergence rate can be provided.

Estimate (26) is still not of the form needed to prove our claim as the estimated rate of contraction is $d(k)$ -dependent and the number of T intervals needed in order to guarantee such decrease in $x_{\max}$ is proportional to $d(k)$.

Nevertheless, $\mu(d(k)) \in (0, 1)$ for any $d(k)$ and by monotonicity of $x_{\max}(t)$ (see Lemma 1) and finiteness of the number of agents, one can take without loss of generality $\bar{k} := 2(n-1)$ and $\mu = \mu(\bar{k}/2) = \mu(n-1)$. This concludes the proof of the Lemma. ■

Theorem 1 *Consider the network modeled by equations (1), if Assumptions 1 hold, then the equilibrium set is uniformly exponentially stable and, for any initial condition $x(0)$, $x(t)$ converges to an agreement equilibrium state with the following contraction rate:*

$$V(x(t + s\bar{k}T)) \leq (1 - \hat{\mu})^s V(x(t)). \quad (27)$$

with $\hat{\mu} = \mu(n-1)$.

The result is proven by using Lemma 1, Lemma 2 and Lyapunov argument. The proof is omitted for sake of brevity and may be find elsewhere ([17]). An extended version of the paper including the omitted proofs and numerical examples and discussions has been submitted ([17]).

IV. CONCLUSIONS

In this paper, we introduced a criterion for asymptotic exponential consensus avoiding the assumption of cooperativity. This is widely adopted in the literature, either explicitly or implicitly, and implies monotonicity of the system's flow with respect to initial conditions. The criterion extends to this class of nonlinear scenario the possibility of guaranteeing consensus through an averaged notion of connectivity.

REFERENCES

- [1] R.M. Murray, *Recent Research in Cooperative Control of Multi-vehicle Systems*, Journal of Dynamic Systems, Measurement, and Control, 129, 5, 2007
- [2] W. Ren, R.W. Beard, and E. M. Atkins, *Information consensus in multivehicle control*, IEEE Control Syst. Mag., 27, 2, 2007.
- [3] Sarlette, A.; Bonnabel, S.; Sepulchre, R. *Coordinated Motion Design on Lie Groups*, IEEE Transactions on Automatic Control, 55, 5, 2010
- [4] Sarlette, A.; Bonnabel, S.; Sepulchre, R. *Degree Fluctuations and the Convergence Time of Consensus Algorithms*, IEEE Transactions on Automatic Control, 58, 10, 2013
- [5] Saber R.O., Murray R.M. *Consensus Problems in Networks of Agents with Switching Topology and Time-Delays*, IEEE Transactions on Automatic Control, 49, 9 2004.
- [6] Jadbabaie, A., Lin, J. and Morse A.S. *Coordination of groups of mobile agents using nearest neighbor rules*. IEEE Transactions on Automatic Control, 48, 6, 2003
- [7] F.H. Clarke, Y.S. Ledyaev, R.J. Stern and P.R. Wolenski, *Nonsmooth Analysis and Control Theory*, Springer, 1998.
- [8] F. Blanchini and S. Miani, *Set Theoretic Methods in Control*, Systems and Control Foundations and Applications, Birkhauser, 2008.
- [9] Scardovi, L., Sepulchre, R. *Synchronization in networks of identical linear systems* Automatica, 45, 11, 2009
- [10] Li, Z., and Chen, G. *Global synchronization and asymptotic stability of complex dynamical networks*, IEEE Transactions on Circuits and Systems II, 53, 1, 2006
- [11] Chopra, N., Spong, M. *Passivity Based Control of Multi-Agent Systems*. Advances in Robot Control From Everyday Physics in Human-Like Movements, 107-134, 2007.
- [12] L. Moreau. *Stability of multiagent systems with time-dependent communication links*. IEEE Transactions on Automatic Control, 50, 2, 2005.
- [13] V. D. Blondel, J. M. Hendrickx, A. Olshevsky, and J. N. Tsitsiklis. *Convergence in multiagent coordination, consensus, and flocking*. In IEEE Conference Decision and Control and European Control Conference, 2005.
- [14] D. Angeli, P.A. Bliman, *Tight estimates for convergence of some non-stationary consensus algorithms*, Systems & Control Letters, 57, 12, 2008
- [15] L. Moreau. *Stability of continuous-time distributed consensus algorithms*, Proc. 43rd IEEE Conf. Decision and Control, 2004.
- [16] S. Manfredi and D. Angeli. *Frozen State Conditions for Asymptotic Consensus of Time-Varying Cooperative Nonlinear Networks*. 52nd IEEE Conference on Decision and Control, pp. 1325 - 1330, ISSN:0743-1546, DOI:10.1109/CDC.2013.6760066, December 10-13, 2013, Firenze, Italy
- [17] S. Manfredi and D. Angeli. *A Criterion for exponential consensus of time-varying non-cooperative nonlinear networks*. submitted to Journal, 2015
- [18] J. M. Hendrickx and J. N. Tsitsiklis. *Convergence of type-symmetric and cut-balanced consensus seeking systems*. IEEE Transactions on Automatic Control, 58, 1, 2013
- [19] Z. Qu, J. Chunyu, and J. Wang. *Nonlinear cooperative control for consensus of nonlinear and heterogeneous systems*. In Proceedings of the 46th IEEE Conference on Decision and Control, pages 2301-2308, New Orleans, USA, 2007.
- [20] U. Munz, A. Papachristodoulou, and F. Allgower, *Robust Consensus Controller Design for Nonlinear Relative Degree Two Multi-Agent Systems With Communication Constraints*, IEEE Transactions on Automatic Control, 56, 1, 2011.
- [21] Y. Cao and W. Ren, *Distributed Multi-Agent Coordination: A Comparison Lemma Based Approach*, in Proceedings of the IEEE American Control Conference, San Francisco, CA, June 2011.
- [22] J.-J. E. Slotine and W. Wang, *A study of synchronization and group cooperation using partial contraction theory*, in Cooperative Control (V. Kumar, N. E. Leonard, and A. S. Morse, eds.), vol. 309, Springer-Verlag Series: Lecture Notes in Control and Information Sciences, 2004.
- [23] C. Altafini, *Consensus problems on networks with antagonistic interactions*, IEEE Transactions on Automatic Control, Vol. 58, No. 4, 2013.
- [24] H. L. Smith. *Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems*, Volume 41 of Mathematical Surveys and Monographs. AMS, Providence, RI, 1995.