Long-time Average Cost Control of Polynomial Systems: A Sum of Squares Approach*

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Abstract—This paper provides a numerically tractable approach for long-time average cost control of nonlinear dynamical systems with polynomials of system state on the right-hand side. First, a recently-proposed method of obtaining rigorous bounds of long-time average cost is outlined for the uncontrolled system, where the polynomial constraints are relaxed to be sum-of-squares and formulated as semi-definite programs. As such, it allows to use any general (polynomial) functions to optimize the bound. Then, a polynomial type state feedback controller design scheme is presented to further suppress the long-time average cost. The derivation of state feedback controller is given in terms of the solvability conditions of state-dependent bilinear matrix inequalities. Finally, the mitigation of oscillatory vortex shedding behind a cylinder is addressed to illustrate the validity of the proposed approach.

I. INTRODUCTION

Global stability theory plays a central role in systems theory and engineering [1], [2]. However, it is sometimes difficult or impossible to synthesize a global stabilizing controller due to controllability issues, input/output constraints, and/or the presence of external disturbances [3]. In many applications the full stabilization, while possible, carries high penalty due to the cost of the control. As a result, sometimes minimizing a long-time average cost functional might be a better alternative to full stabilization.

Analysis and optimal control of long-time averages of various characteristics of dynamical systems have been studied before. Systems that include some stochastic factors are often controlled in the sense of long-time average. In [4], a summary of long-time average cost problems for continuous-time Markov processes is given. When the control acts permanently in the system, the main idea is to study the induced Hamilton-Jacobi-Bellman equation directly or to study the asymptotic behavior of the discounted-cost problem indirectly. In [5], the long-time average control of a class of problems that arise in the modelling of semiactive suspension systems was considered, where the cost includes a term based on the local time process of a diffusion. The control problem is reformulated as a linear program over the set of invariant

distributions for the process, and Markov chain approximations are then used to reduce the infinite-dimensional linear programs to finite-dimensional linear programs. Note that the controller design methods proposed in [4], [5] are highly dependent on the stochastic property of dynamical systems.

In certain cases, as, for example, turbulent flows of fluid, calculating the time-averages is a big challenge even in the uncontrolled case, and developing the control, for example by using the receding horizon technique, leads to controllers too complicated for practical implementation [6]. To overcome this complexity, it was proposed [7] to use an upper bound for the long-time average cost instead of the long-time average cost itself in cases when such an upper bound is easier to calculate. The idea is based on the hope that the control reducing an upper bound for a quantity will also reduce the quantity itself.

For turbulent flows, methods of finding bounds on time-averaged momentum transport or, equivalently, the time-averaged energy dissipation rate, have long been known [8], [9], with the background-flow method being a more recent version of the approach [10]. Without going into technical details, it should be noted that these methods are based on optimizing quadratic functionals, which is a significant constraint resulting in the bounds often being far from tight. In [7] a new method of finding bounds for long-time averages was proposed, which allows a trade-off between the quality of bound and the complexity of its calculation. The method utilizes the use of the sum of squares (SOS) decomposition of polynomials and semidefinite programming (SDP).

The SOS methods apply to systems defined by a polynomial vector field. Such systems may either describe a wide variety of dynamics [11] or approximate a system defined by an analytical vector field [3]. A polynomial system can therefore yield a reliable model of a dynamical system globally or in larger regions than the linear approximation in the state-space [12]. Recent results on SOS decomposition have transformed the verification of non-negativity of polynomials into SDP, hence providing promising algorithmic procedures for stability analysis of polynomial systems. However, using SOS techniques for optimal control, as for example in [13], [14], [15], is subject to a generic difficulty: while the problem of optimizing the candidate Lyapunov function certifying the stability for a closed-loop system for a given controller and the problem of optimizing the controller for a given candidate Lyapunov function are reducible to an SDP and thus, are tractable, the problem of optimizing both the control and the Lyapunov function is non-convex. Naturally, iterative procedures are attempted for overcoming this difficulty.

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While optimization of an upper bound with control [7] does not involve a Lyapunov function, it does involve a similar function, and it shares the same difficulty of non-convexity. In the present work we present an iterative polynomial type state feedback controller design scheme for the long-time average upper-bound control. We apply it to an illustrative example and demonstrate that it does allow to reduce the long-time average cost even without fully stabilizing the system.

II. BACKGROUND

A. Sum of squares of polynomials

SOS techniques have been frequently used in the stability analysis and controller design for all kinds of systems, e.g., constrained ordinary differential equation systems [2], hybrid systems [17], time-delay systems [18], and partial differential equation systems [19], [20]. These techniques help to overcome the common drawback of approaches based on Lyapunov functions: before [13], there were no coherent and tractable computational methods for constructing Lyapunov functions.

A multivariate polynomial $f(\mathbf{x})$ is a SOS, if there exist polynomials $f_1(\mathbf{x}), \dots, f_m(\mathbf{x})$ such that

$$f(\mathbf{x}) = \sum_{i=1}^m f_i^2(\mathbf{x}). \quad (1)$$

If $f(\mathbf{x})$ is a SOS then $f(\mathbf{x}) \geq 0 \forall \mathbf{x}$. In the general multivariate case, however, $f(\mathbf{x}) \geq 0 \forall \mathbf{x}$ does not necessarily imply that $f(\mathbf{x})$ is SOS. While being stricter, the condition that $f(\mathbf{x})$ is SOS is much more computationally tractable than non-negativity [22]. At the same time, practical experience indicates that in many cases replacing non-negativity with the SOS property leads to satisfactory results.

In the present paper we will utilize the existence of efficient numerical methods and software [21], [24] for solving the optimization problems of the following type (although our derivation will result in more complicated SOS constraints that are BMIs): minimize the linear objective function

$$\mathbf{w}^T \mathbf{c} \quad (2)$$

where $\mathbf{c}$ is a vector formed from the (unknown) coefficients of the polynomials $p_i(\mathbf{x})$, for $i = 1, 2, \dots, \hat{N}$ and SOS $p_i(\mathbf{x})$, for $i = (\hat{N} + 1), \dots, N$ such that

$$a_{0,j}(\mathbf{x}) + \sum_{i=1}^N p_i(\mathbf{x}) a_{i,j}(\mathbf{x}) = 0, \quad j = 1, 2, \dots, \hat{J},$$

$$a_{0,j}(\mathbf{x}) + \sum_{i=1}^N p_i(\mathbf{x}) a_{i,j}(\mathbf{x}) \text{ are SOS, } j = (\hat{J} + 1), \dots, J.$$

B. Bound estimation of long-time average cost for uncontrolled systems

For the convenience of the reader we outline here the method of obtaining bounds for long-time averages proposed in [7] and make some remarks on it. Consider a system

$$\dot{\mathbf{x}} = f(\mathbf{x}), \quad (3)$$

where $f(\mathbf{x})$ is a vector of multivariate polynomials of the components of the state vector $\mathbf{x}$. The long-time average of a function of the state $\Phi(\mathbf{x})$ is defined as

$$\bar{\Phi} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \Phi(\mathbf{x}(t)) dt,$$

where $\mathbf{x}(t)$ is the solution of (3).

Define a polynomial function of the system state, $V(\mathbf{x})$, of order d_V , and containing unknown decision variables as its coefficients. The time derivative of V along the trajectories of system (3) is

$$\dot{V}(\mathbf{x}) = \nabla_{\mathbf{x}} V(\mathbf{x}) \cdot \dot{\mathbf{x}} = \nabla_{\mathbf{x}} V(\mathbf{x}) \cdot f(\mathbf{x}). \quad (4)$$

Consider the following quantity:

$$H(\mathbf{x}) \triangleq \dot{V}(\mathbf{x}) + \Phi(\mathbf{x}) = \nabla_{\mathbf{x}} V(\mathbf{x}) \cdot f(\mathbf{x}) + \Phi(\mathbf{x}). \quad (5)$$

The following result is from our prior work [7].

Lemma 1: For the system (3), assume that the state $\mathbf{x}$ is bounded in $\mathcal{D} \subseteq \mathbb{R}^n$. Then, $H(\mathbf{x}) \leq C$ implies $\bar{\Phi}(\mathbf{x}) \leq C, \forall \mathbf{x} \in \mathcal{D}$.

Hence, an upper bound of $\bar{\Phi}$ can be obtained by minimizing C over V under the constraint $H(\mathbf{x}) \leq C$, which can be formulated as a SOS optimization problem in the form:

$$\min_V C \quad (6)$$

$$\text{s.t. } -(\nabla_{\mathbf{x}} V(\mathbf{x}) \cdot f(\mathbf{x}) + \Phi(\mathbf{x}) - C) \text{ is SOS,} \quad (7)$$

which is a special case of (2). A better bound might be obtained by removing the requirement for $V(\mathbf{x})$ to be polynomial and replacing (7) with the requirement of non-negativeness, but solving the resulting problem would be too difficult.

Note that while V is similar to a Lyapunov function in a stability analysis, it is not required to be positive-definite. Note also that a lower bound of any long-time average cost of the system (3) can be analyzed in a similar way.

Remark 1: For many systems the boundedness of system state immediately follows from energy consideration. In general, if the system state is bounded, this can often be proven using the SOS approach. It suffices to check whether there exists a large but bounded global attractor, denoted by $\mathcal{D}_1$, which includes $\mathcal{D}$ as a subset. As an example, let $\mathcal{D}_1 = \{\mathbf{x} \mid 0.5\mathbf{x}^T \mathbf{x} \leq \beta\}$, where the constant β is sufficiently large. Then, the global attraction property of system in $\mathcal{D}_1$ may be expressed as

$$\mathbf{x}^T \dot{\mathbf{x}} = \mathbf{x}^T f(\mathbf{x}) \leq -(0.5\mathbf{x}^T \mathbf{x} - \beta). \quad (8)$$

Introducing a tunable polynomial $S(\mathbf{x})$ satisfying $S(\mathbf{x}) \geq 0 \forall \mathbf{x} \in \mathbb{R}^n$, (8) can be relaxed to

$$\begin{cases} -(\mathbf{x}^T f(\mathbf{x}) - S(\mathbf{x})(0.5\mathbf{x}^T \mathbf{x} - \beta)) \text{ is SOS,} \\ S(\mathbf{x}) \text{ is SOS.} \end{cases} \quad (9)$$

The generalized $\mathcal{S}$ -procedure [25], which provides a sufficient condition to test inclusions of sets defined by polynomials, can then be applied to obtain the checkable SOS conditions (9).

Remark 2: For a uniformly continuous cost function $\Phi(\mathbf{x})$, satisfying $\Phi(\mathbf{x}) \geq 0$, and $\Phi = 0$ if and only if $\mathbf{x} = \mathbf{0}$, a zero-bound of $\bar{\Phi}$ implies the asymptotic stability of system (3). A (trivial) proof is omitted for brevity.

III. PROBLEM FORMULATION

Consider a polynomial system with single input

$$\dot{\mathbf{x}} = f(\mathbf{x}) + g(\mathbf{x})u \quad (10)$$

where $f(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $g(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are polynomial functions of system state $\mathbf{x}$. The main results of this paper can easily be extended to multiple input systems. The control u is designed to minimize the upper bound of an average cost of the form:

$$\bar{\Phi} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \Phi(\mathbf{x}(t), u(t)) dt, \quad (11)$$

where $\mathbf{x}(t)$ is the closed-loop solution of the system (10) with the control $u(t)$. The continuous function Φ is a given non-negative polynomial cost in $\mathbf{x}$ and u . Choices of Φ depend on the designer's interest to the controlled system. For instance, if Φ is specified to be $\Phi_0(\mathbf{x}) + Ru^2$ with constant weight parameter R , then $\Phi_0(\mathbf{x})$ can be regarded as the system parameter we want to minimize and Ru^2 is the quadratic penalty on the control. If $\bar{\Phi} = \Phi_0(\mathbf{x}) + Rk(\mathbf{x})^T k(\mathbf{x})$ is associated with the form of control $u = k(\mathbf{x})^T \mathbf{x}$, then the penalty will be imposed on the nonlinear state-dependent feedback gain. Based on SOS decomposition and SDP, numerically tractable approaches will be presented to address the above control tasks.

IV. BOUND OPTIMIZATION OF LONG-TIME AVERAGE COST FOR CONTROLLED POLYNOMIAL SYSTEMS

In this section a polynomial type state feedback controller is designed to reduce the bound for the controlled polynomial system (10). While it is not guaranteed, it is reasonable to hope that a controller reducing the upper bound for the time-averaged cost will also reduce the time-averaged cost itself [7].

A. SOS based controller design

The controller takes the following form,

$$u = K(\mathbf{x}), \quad (12)$$

where $K(\mathbf{x})$ is a polynomial of order d_u in $\mathbf{x}$, and the associated cost function is

$$\Phi(\mathbf{x}, u) = \Phi_0(\mathbf{x}) + Ru^2, \quad (13)$$

where Φ_0 is the continuous non-negative part in system state, and vanishes if and only if $\mathbf{x} = \mathbf{0}$. The controlled closed-loop system is obtained by substituting (12) into (10),

$$\dot{\mathbf{x}} = f(\mathbf{x}) + g(\mathbf{x})K(\mathbf{x}). \quad (14)$$

Similar to the uncontrolled scenario, define a polynomial function of the system state $V(\mathbf{x})$, of order d_V , and containing unknown decision variables as its coefficients. In consequence, for any C , we have

$$\begin{aligned} H(\mathbf{x}) &\triangleq \dot{V}(\mathbf{x}) + \Phi(\mathbf{x}, K(\mathbf{x})) - C \\ &= \nabla_{\mathbf{x}} V \cdot (f + gK) + \Phi_0 + RK^2 - C. \end{aligned}$$

If the boundedness of the closed-loop state $\mathbf{x}$ is always guaranteed, then $H(\mathbf{x}) \leq 0$ implies that C is an upper bound of $\bar{\Phi}$. However, due to the existence of the quadratic term RK^2 , the induced inequality is not linear in K . Moreover, owing to the product of $\nabla_{\mathbf{x}} V$ and K , the corresponding optimization problem is not convex anymore. As a result, the convex-optimization-based SOS techniques are not applicable here. The difficulty may be overcome by considering more conservative inequality constraint as given below.

Theorem 1: $H(\mathbf{x}) \leq 0$ holds if

$$\Sigma_0(V, K, C) \triangleq \begin{bmatrix} E(V, C) & * \\ \frac{1}{2\sqrt{R}} \nabla_{\mathbf{x}} V \cdot g + \sqrt{R}K & -1 \end{bmatrix} \preceq 0, \quad (15)$$

where

$$E(V(\mathbf{x}), C) \triangleq \nabla_{\mathbf{x}} V \cdot f + \Phi_0 - C, \quad (16)$$

* denotes the symmetric entry of the symmetric matrix Σ_0 , and the symbol $\preceq$ indicates that Σ_0 is negative-semidefinite.

Proof. Rewrite the function $H(\mathbf{x})$ as follows,

$$\begin{aligned} H(\mathbf{x}) &= \nabla_{\mathbf{x}} V \cdot f + \Phi_0 - C + (\nabla_{\mathbf{x}} V \cdot g)K + RK^2 \\ &= E - \frac{1}{4R}(\nabla_{\mathbf{x}} V \cdot g)^2 \\ &\quad + \frac{1}{4R}(\nabla_{\mathbf{x}} V \cdot g)^2 + (\nabla_{\mathbf{x}} V \cdot g)K + RK^2. \end{aligned} \quad (17)$$

In (17), the argument $\mathbf{x}$ is omitted for brevity. Noticing

$$\begin{aligned} &\frac{1}{4R}(\nabla_{\mathbf{x}} V \cdot g)^2 + (\nabla_{\mathbf{x}} V \cdot g)K + RK^2 \\ &= \left(\frac{1}{2\sqrt{R}} \nabla_{\mathbf{x}} V \cdot g + \sqrt{R}K \right)^2, \end{aligned}$$

it follows that

$$H = E - \frac{1}{4R}(\nabla_{\mathbf{x}} V \cdot g)^2 + \left(\frac{1}{2\sqrt{R}} \nabla_{\mathbf{x}} V \cdot g + \sqrt{R}K \right)^2.$$

By the Schur complement formula, the inequality condition $H \leq 0$ is equivalent to the following matrix inequality

$$\begin{bmatrix} E - \frac{1}{4R}(\nabla_{\mathbf{x}} V \cdot g)^2 & * \\ \frac{1}{2\sqrt{R}} \nabla_{\mathbf{x}} V \cdot g + \sqrt{R}K & -1 \end{bmatrix} \preceq 0. \quad (18)$$

Due to the fact that $-\frac{1}{4R}(\nabla_{\mathbf{x}} V \cdot g)^2 \leq 0$, $H(\mathbf{x}) \leq 0$ if (15) holds. ■

Note that (15) is a linear matrix inequality (LMI) in V and K . It suffices to minimize C under the constraint $-\mathbf{v}^T \Sigma_0(V, K, C) \mathbf{v} \geq 0$ with $\mathbf{v} \in \mathbb{R}^2$ being a vector variable that is independent of $\mathbf{x}$. However, since the non-positive term $-\frac{1}{4R}(\nabla_{\mathbf{x}} V \cdot g)^2$ was discarded, the resulting bound might be not tight. Alternatively, an iterative algorithm is proposed in the following to check the feasibility of (18).

B. An iterative algorithm

Introduce a tunable vector variable $P(\mathbf{x}) \in \mathbb{R}^n$. Clearly,

$$((P - \nabla_{\mathbf{x}}V) \cdot g)^2 \geq 0, \quad (19)$$

or equivalently,

$$-(P \cdot g)^2 + 2(P \cdot g)(\nabla_{\mathbf{x}}V \cdot g) \leq (\nabla_{\mathbf{x}}V \cdot g)^2. \quad (20)$$

Hence, multiplying by $-1/4R$ on both sides of (20),

$$-\frac{1}{4R}(\nabla_{\mathbf{x}}V \cdot g)^2 \leq \frac{1}{4R}(P \cdot g)^2 - \frac{1}{2R}(P \cdot g)(\nabla_{\mathbf{x}}V \cdot g). \quad (21)$$

Combining (18) and (21) gives

$$\begin{aligned} H(\mathbf{x}) \leq & E + \frac{1}{4R}(P \cdot g)^2 - \frac{1}{2R}(P \cdot g)(\nabla_{\mathbf{x}}V \cdot g) \\ & + \left(\frac{1}{2\sqrt{R}}\nabla_{\mathbf{x}}V \cdot g + \sqrt{RK} \right)^2. \end{aligned} \quad (22)$$

Denote $E_1(V(\mathbf{x}), P(\mathbf{x}), C) = E + \frac{1}{4R}(P \cdot g)^2 - \frac{1}{2R}(P \cdot g)(\nabla_{\mathbf{x}}V \cdot g)$. Then, using the Schur complement formula, $H(\mathbf{x}) \leq 0$ if

$$\Sigma(V, K, P, C) \triangleq \begin{bmatrix} E_1(V, P, C) & * \\ \frac{1}{2\sqrt{R}}\nabla_{\mathbf{x}}V \cdot g + \sqrt{RK} & -1 \end{bmatrix} \preceq 0. \quad (23)$$

Note that control can make the system unbounded. Therefore, additional SOS constraints must be built in to make sure that the closed-loop system state is bounded. This can be done in the way similar to Remark 1. Since the control input is of polynomial type, the boundedness of system state implies the boundedness of input profile in the whole control process. Overall, the central task now is to solve the SDP:

$$\min_{V, K, S, P, C} \quad \text{s.t.} \quad (24)$$

$$\begin{cases} -\mathbf{v}^T \Sigma(V, K, P, C) \mathbf{v} \text{ is SOS,} \\ -(\mathbf{x}^T(f + gK) - S(\mathbf{x})(0.5\mathbf{x}^T \mathbf{x} - \beta)) \text{ is SOS,} \\ S(\mathbf{x}) \text{ is SOS.} \end{cases} \quad (25)$$

The second line in (25) ensures that the ball of the radius $\sqrt{2\beta}$ is an attractor. Note that (25) is linear in V, K, S and C , but quadratic in P . It also contains a product of terms containing P and V . It can be solved in the following way.

Given the vectors of monomials in $\mathbf{x}$ without repeated elements [26], $Z_i, i = 1, \dots, 3$, assign V, K, S to be $Q_1^T Z_1$, $Q_2^T Z_2$, and $Q_3^T Z_3$, respectively, where the parametric vectors $Q_i, i = 1, \dots, 3$ consist of tuning vector variables. The degrees of V, K and S are specified by the maximum degrees of monomials in $Z_i, i = 1, \dots, 3$, respectively.

The Iterative Algorithm to solve (24-25): **IA**

- (s0) Let $j = 0$ and set a threshold $\delta > 0$. Minimize C over V subject to the constraint $-E(V(\mathbf{x}), C)$ is SOS. Define $P_0 = \nabla_{\mathbf{x}}V$, and denote by $C_{0,opt}$ the optimized C .
- (s1) At the j -th iteration, $j = 1, 2, \dots$, fix $P = P_{j-1}$ and solve (24-25). Denote the minimized C by $C_{j,opt}$, and the corresponding V, K, S by $V_{j,opt}, K_{j,opt}$, and $S_{j,opt}$, respectively.

- (s2) If $|C_{j,opt} - C_{j-1,opt}| < \delta$, we obtain the optimal state feedback controller $u_{SOS} = K_{j,opt}$, the minimal upper bound $C_{SOS} = C_{j,opt}$, and the corresponding decision function $V_{SOS} = V_{j,opt}$. Otherwise, let $P_j = \nabla_{\mathbf{x}}V_{j,opt}$ and $j = j + 1$, then go back to step (s1).

Remark 3: It is worthy of noticing that, compared with some other iterative methods that deal with non-convexity of SOS decomposition, e.g., the augmented approach [14], [27], the computational cost is relatively low in **IA**, thus expediting the convergence speed of iterative search [28].

Remark 4: The strict convergence analysis of **IA** is of importance but beyond the scope of the present paper. The global extremum seeking algorithms for general BMI optimization are described in [29], [30] and references therein.

Remark 5: Currently, the only available direct solver for bilinear problems is PENBMI [31], a commercial solver that uses a local penalty method. However, as mentioned in [32], this software is still in an early stage and lacks both performance and reliability. If the solver PENBMI is unsuccessful in implementation, no conclusions about possible intractability of the problem can be drawn. **IA** is a further development in such case, especially for our particular problem formulation.

It would be beneficial to compare the following four upper bounds: 1) $C_{0,SOS}$, the guaranteed upper bound of the long-time average cost obtained by solving the SOS problem (6)-(7) for the uncontrolled case, 2) C_{SOS} , the guaranteed upper bound obtained by **IA** directly in the long-time average cost control, 3) $C_{u_{SOS}}$, the guaranteed upper bound obtained by substituting the controller u_{SOS} and solving the SOS optimization problem,

$$\min_V C, \quad \text{s.t.}$$

$$-(\nabla_{\mathbf{x}}V \cdot (f + gu_{SOS}) + \Phi_0 + Ru_{SOS}^2 - C) \text{ is SOS,}$$

and 4) C_{opt} , the globally minimal but unknown bound when restricting the controller to the set of all the polynomial functions of system state. It is understandable that C_{SOS} could be greater than C_{opt} , thus being a conservative upper bound of the long-time average cost. This is because only finite-order V and K are considered for (24)-(25), and because the positive-semidefinite conditions are replaced by more conservative SOS conditions. Moreover, it is also possible that $C_{u_{SOS}} < C_{SOS}$ since we solve (24)-(25) iteratively with a non-zero stop criterion δ . Although applying the algorithm **IA** does not necessarily ensure that a global extremum for the long-time average cost control can be found, **IA** is effective in the sense that $C_{SOS} < C_{0,SOS}$ usually. That is, compared with the uncontrolled case, a better bound of long-time average cost can be obtained via the designed controller.

C. SOS with minimal feedback gain

As an alternative to limiting the control input directly in the cost function (13), many control systems care about the input gain cost, namely, by the requirement that the feedback gain should be chosen as small as possible. This is because high gain feedback also amplifies the unmodelled part of

state output (e.g., measurement noise), thus degrading the control performance [33], and increasing the possibility of closed-loop instabilities when the approximated model is not accurate enough. Low-gain control results in a slow transient response, but it does not matter in optimizing the long-time average of cost.

Consider the SOS based controller design of the following form,

$$u = k(\mathbf{x})^T \mathbf{x}, \quad (26)$$

where $k(\mathbf{x})$ is a polynomial vector in $\mathbf{x}$. Now the objective is to minimize the long-time average of the cost function

$$\Phi(\mathbf{x}, k(\mathbf{x})) = \Phi_0(\mathbf{x}) + Rk(\mathbf{x})^T k(\mathbf{x}), \quad R > 0. \quad (27)$$

Parallel to the aforementioned scenario, it follows that

$$\begin{aligned} \dot{V} + \Phi - C &= E(V(\mathbf{x}), C) - \frac{1}{4R}(\nabla_{\mathbf{x}} V \cdot g)^2 \mathbf{x}^T \mathbf{x} \quad (28) \\ &+ \left(\frac{1}{2\sqrt{R}}(\nabla_{\mathbf{x}} V \cdot g)\mathbf{x} + \sqrt{R}k(\mathbf{x}) \right)^T \\ &\quad \left(\frac{1}{2\sqrt{R}}(\nabla_{\mathbf{x}} V \cdot g)\mathbf{x} + \sqrt{R}k(\mathbf{x}) \right), \end{aligned}$$

where E is already defined by (16). As such the inequality condition $\dot{V} + \Phi - C \leq 0$ is equivalent to

$$\begin{bmatrix} E(V(\mathbf{x}), C) - \frac{1}{4R}(\nabla_{\mathbf{x}} V \cdot g)^2 \mathbf{x}^T \mathbf{x} & * \\ \frac{1}{2\sqrt{R}}(\nabla_{\mathbf{x}} V \cdot g)\mathbf{x} + \sqrt{R}k(\mathbf{x}) & -I \end{bmatrix} \preceq 0,$$

where I is an identity matrix with appropriate dimension.

Further introduce another tunable vector variable $P(\mathbf{x})$. Denote $E_1(V, P, C) = E(V, C) + \frac{1}{4R}(P \cdot g)^2 \mathbf{x}^T \mathbf{x} - \frac{1}{2R}(P \cdot g)(\nabla_{\mathbf{x}} V \cdot g)\mathbf{x}^T \mathbf{x}$. Then, $\dot{V} + \Phi - C \leq 0$ if

$$\Sigma(V, P, k, C) \triangleq \begin{bmatrix} E_1(V, P, C) & * \\ \frac{1}{2\sqrt{R}}(\nabla_{\mathbf{x}} V \cdot g)\mathbf{x} + \sqrt{R}k(\mathbf{x}) & -I \end{bmatrix} \preceq 0.$$

The iterative algorithm **IA** can be used again to minimize the upper bound C over the functions V, k , and over the polynomial S , entering the necessary additional constraint (9) ensuring the boundedness of the system state.

V. ILLUSTRATIVE EXAMPLE: CONTROL OF CYLINDER WAKE

The ability to actively or passively manipulate a flow field to bring about a desired change is of immense technological importance. The potential benefits of improving flow control systems range from saving billions of dollars in fuel costs for land, air, and sea vehicles to achieving more economically competitive and environmentally sound industrial processes involving fluid flows [38]. Among the well-established control strategies, nonlinear control design is shown to be a critical enabler for robust model-based suppression of a flow instability.

As an illustrative example, SOS-based state feedback control is considered for a benchmark problem of fluid dynamics, namely, the mitigation of oscillatory vortex shedding behind a cylinder. Assuming a volume force in a control

volume downstream of the cylinder, the momentum equation including such volume force is

$$\frac{\partial \mathbf{w}}{\partial t} + (\mathbf{w} \cdot \nabla) \mathbf{w} = -\nabla p + \frac{1}{Re} \Delta \mathbf{w} + \mathbf{b}u, \quad (29)$$

where $\mathbf{w}$ is the vector of flow velocity, p is the pressure term, the Reynolds number Re represents the non-dimensionalized characteristic velocity, and $u \in \mathbb{R}$ describes the amplitude of the forcing on a compact support given by $\mathbf{b}$. Fix the Reynolds number at $Re = 100$. Using a Karhunen-Loève (KL) decomposition [39], where the first two KL modes and an additional shift mode are considered, a low-order Galerkin model of the cylinder flow is given as follows [40],

$$\begin{bmatrix} \dot{a}_1 \\ \dot{a}_2 \\ \dot{a}_3 \end{bmatrix} = \begin{bmatrix} \sigma_r & -\omega - \gamma a_3 & -\beta a_1 \\ \omega + \gamma a_3 & \sigma_r & -\beta a_2 \\ \alpha a_1 & \alpha a_2 & -\sigma_3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \\ 0 \end{bmatrix} u, \quad (30)$$

where $\sigma_r = 0.05439, \sigma_3 = 0.05347, \alpha = 0.02095, \beta = 0.02116, \gamma = -0.03504, \omega = 0.9232, g_1 = -0.15402$, and $g_2 = 0.046387$. More details on deriving the reduced-order model (30) are given in [42]. The system (30) possesses a unique equilibrium at the origin. Before conducting any controller design, the optimization problem (6)-(7) is solved to address the uncontrolled scenario. The minimal upper bound we can achieve is $C_{0,SOS} = 6.584$, which is verified by choosing the function

$$V = -96.63a_3 + 14.01a_1^2 + 14.01a_2^2 + 14.15a_3^2.$$

Due to the linear term of V , it is not a Lyapunov function. In this case increasing the degree of V cannot give a better bound. There exists a stable limit cycle in the phase space of (30), which satisfies $a_1^2 + a_2^2 = 6.5604, a_3 = 2.5704$, as shown in Fig. 1. Since $\Phi = 1/2 \mathbf{a}^T \mathbf{a} = 6.584$ on this limit cycle, the upper bound achieved by SOS optimization is tight.

In the following, SOS-based state feedback controllers are designed to suppress the long-time average cost of cylinder wake flow. The system state is always assumed to be available. In experiment, it can be estimated by designing state observer with some sensed output measurement at a typical position [42].

Consider state feedback control in the following form,

$$u = k(\mathbf{a})^T \mathbf{a}, \quad (31)$$

with the associated cost function

$$\Phi(\mathbf{a}, k(\mathbf{a})) = 1/2 \mathbf{a}^T \mathbf{a} + Rk(\mathbf{a})^T k(\mathbf{a}), \quad (32)$$

where $\mathbf{a} = [a_1 \ a_2 \ a_3]^T$ is the system state. Applying the proposed SOS-based optimal controller design approach with $\delta = 0.001$ in **IA**, the upper bounds of the long-time average cost Φ are evaluated with different degrees of V and u in $\mathbf{a}$, denoted by d_V and d_u , respectively.

Case 1: $d_V = 4, d_u = 1, R = 1/5$. The best bound achieved by SOS is $C_{1,SOS} = 0.113$, via the controller

Fig. 1. Solution of (30) with $u = 0$ and initial state $[0.1 \ 0.1 \ 2]^T$.

Fig. 2. Time average of $1/2\mathbf{a}^T\mathbf{a} + 1/5\mathbf{k}(\mathbf{a})^T\mathbf{k}(\mathbf{a})$ when the controller $u_{1,SOS}$ is applied. The long-time average is $C_{u_{1,SOS}} = 0.112$. Although the closed-loop system state converges to zero asymptotically, the non-zero gain $\mathbf{k}(\mathbf{a})$ at $\mathbf{a} = 0$ yields a nonzero bound of $\bar{\Phi}$.

$u_{1,SOS} = 0.7188a_1 - 0.2113a_2$. Using the obtained controller, the simulation results are given in Figs. 2-3, from which we can see that the cylinder wake is fully stabilized. Indeed, $C_{u_{1,SOS}} = R(0.7188^2 + 0.2113^2) = 0.112 \approx C_{1,SOS}$.

Case 2: $d_V = 6, d_u = 1, R = 1/5$. The minimal bound we can achieve is $C_{2,SOS} = 0.112$, almost same as in Case 1. The optimal bound might be further reduced by changing the stop threshold δ in **IA** from 0.001 to a smaller value. For instance, reset $\delta = 1 \times 10^{-4}$, and then the best bound achieved by SOS is 0.1, where the controller is $u_{2,SOS} = 0.6763a_1 - 0.2017a_2$.

In order to check that the minimal bound achieved by SOS almost coincides with the global optimal bound under linear control, we simulate all the controllers with the same structure, namely,

$$u = k_1a_1 + k_2a_2, \quad R(k_1^2 + k_2^2) \leq 0.1. \quad (33)$$

It is observed that the optimal controlled upper bound is $C_{opt} = 0.092$, achieved by the controller $u = 0.6314a_1 - 0.2284a_2$. Clearly, SOS based long-time average control almost achieves the global optimal bound.

To see the effect of the weight parameter R on result of SOS optimization, we consider a few more cases.

Fig. 3. Closed-loop system state profile with initial state $[0.1 \ 0.1 \ 2]^T$ when the controller $u_{1,SOS}$ is applied.

Fig. 4. Time average of $1/2\mathbf{a}^T\mathbf{a} + \mathbf{k}(\mathbf{a})^T\mathbf{k}(\mathbf{a})$ when the controller $u_{5,SOS}$ is applied. The long-time average is $C_{u_{5,SOS}} = 0.2585$.

Case 3: $d_V = 4, d_u = 1, R = 1$. The best bound is $C_{3,SOS} = 0.4688$, induced by the controller $u_{3,SOS} = 0.6528a_1 - 0.1960a_2$. Case 4: $d_V = 4, d_u = 2, R = 1$. The best bound is $C_{4,SOS} = 0.4656$, induced by the controller $u_{4,SOS} = 0.6499a_1 - 0.1946a_2 - 0.0415a_2a_3 + 0.0161a_1a_3$. Case 5: $d_V = 6, d_u = 3, R = 1$. The best result is $C_{5,SOS} = 0.2585$, induced by the controller

$$\begin{aligned} u_{5,SOS} &= 0.2785a_1 - 0.4165a_2 + 0.2627a_2a_3 - 0.3691a_1a_3 \\ &\quad + 5.427a_1^3 + 4.530a_1^2a_2 + 6.161a_1a_2^2 + 2.493a_2^3 \\ &\quad + 17.86a_1a_3^2 - 2.721a_2a_3^2. \end{aligned} \quad (34)$$

When the controller $u_{5,SOS}$ is applied, the time average of $\bar{\Phi}$ is shown in Fig. 4. Since $C_{u_{5,SOS}} = C_{5,SOS}$ in this case, the guaranteed upper bound is tight.

The results in Cases 3-5 reveal that increasing the degree of V and/or u may yield a lower upper bound of $\bar{\Phi}$. Moreover, notice that the cylinder flow is globally stabilized by the controllers $u_{3,SOS}$ and $u_{4,SOS}$, while it is not true for $u_{5,SOS}$. As can be seen from Fig. 5, the upper bound of $\bar{\Phi}$ obtained by $u_{5,SOS}$ is determined by the size of the invariant set on plane $a_3 = 0.02534$. Hence, when more penalty is put towards the input gain profile, the best controller that minimizes the long-time average cost may not be the one that fully stabilizes the controlled system.

Fig. 5. Solution of (30) with $u = u_{5,SOS}$ and initial state $[0.1 \ 0.1 \ 2]^T$. The upper bound of $\bar{\Phi}$ is determined by the size of the invariant set on plane $a_3 = 0.02534$.

VI. CONCLUSION

A numerically tractable approach is proposed to minimize the long-time average cost of polynomial dynamical systems via state feedback control. The controller design scheme is based on sum of squares decomposition of polynomials and semidefinite programming. An iterative algorithm has been presented to resolve the induced state-dependent non-convex matrix inequalities. The efficiency of the proposed control scheme has been verified in a class of cylinder wake flow stabilization problem. In the next research phase, we will consider long-time average control under modelling uncertainties as well as direct numerical simulations of the controlled fluid flows.

REFERENCES

- [1] H. K. Khalil, *Nonlinear Systems*. Prentice Hall, London, 2002.
- [2] A. Papachristodoulou and S. Prajna, On the construction of Lyapunov functions using the sum of squares decomposition. *Proc. the IEEE Conference on Decision and Control*, pp. 3482-3487, December 10-13, Las Vegas, NV, USA, 2002.
- [3] Q. Zheng and F. Wu, Regional stabilization of polynomial non-linear systems using rational Lyapunov functions. *Int. J. Control*, Vol. 82, No. 9, pp. 1605-1615, 2009.
- [4] M. Robin, Long-term average cost control problems for continuous time Markov processes: a survey. *Acta Applicandae Mathematicae*, Vol. 1, pp. 281-299, 1983.
- [5] M. S. Mendiondo and R. H. Stockbridge, Long term average control of a local time process. *Markov Processes and Controlled Markov Chains*, pp. 425-441, 2002.
- [6] T.R. Bewley, P. Moin, and R. Temam, DNS-based predictive control of turbulence: an optimal benchmark for feedback algorithms. *J. Fluid Mech.*, Vol. 447, pp. 179-225, 2001.
- [7] S. I. Chernyshenko, G. Paul, D. Huang, and A. Papachristodoulou, Polynomial sum of squares in fluid dynamics: a review with look. *Phil. Trans. R. Soc. A*, Vol. 372, 20130350, 2014.
- [8] L. N. Howard, Bounds on flow quantities. *Annual Review of Fluid Mechanics*, Vol. 4, pp. 473-494, 1972.
- [9] F. H. Busse, The optimum theory of turbulence. *Archive of Applied Mechanics*, Vol. 18, pp. 77-121, 1978.
- [10] C. R. Doering and P. Constantin, Variational bounds on energy dissipation in incompressible flows: shear flow. *Phys. Rev. E*, Vol. 49, pp. 4087-4099, 1994.
- [11] C. Ebenbauer and F. Allgöwer, Analysis and design of polynomial control systems using dissipation inequalities and sum of squares. *Comput. Chem. Eng.*, Vol. 30, pp. 1590-1602, 2006.
- [12] G. Valmorbida, S. Tarbouriech, and G. Garcia, Design of polynomial control laws for polynomial systems subject to actuator saturation. *IEEE Trans. Automat. Control*, Vol. 58, No. 7, pp. 1758-1770, 2013.
- [13] S. Prajna, A. Papachristodoulou, and F. Wu, Nonlinear control synthesis by sum of squares optimization: a Lyapunov-based approach. *Proc. of the 5th Asian Control Conference*, Vol. 1, pp. 157-165, 20-23 July, Melbourne, Victoria, Australia, 2002.
- [14] D. Zhao, J.-L. Wang, F. Liao, and E. K. Poh., Nonlinear optimal control for parameter-dependent polynomial nonlinear systems. *Proc. of the IEEE International Conference on Control and Automation*, Guangzhou, China, May 30- Jun 1, pp. 1710-1714, 2007.
- [15] C. Maier, C. Böhm, F. Deroo, and F. Allgöwer, Predictive control for polynomial systems subject to constraints using sum of squares. *Proc. of the 49th IEEE Conf. on Decision and Control*, Atlanta, GA, USA, Dec. 15-17, 2010.
- [16] M. Korda, D. Henrion, and C. N. Jones, Convex computation of the maximum controlled invariant set for polynomial control systems. *arXiv: 1303.6469v1*, 2013.
- [17] A. Papachristodoulou and S. Prajna, A tutorial on sum of squares techniques for system analysis. *Proc. of the 2005 American Control Conference*, pp. 2686-2700, June 8-10, Portland, OR, USA, 2005.
- [18] A. Papachristodoulou, Analysis of nonlinear time delay systems using the sum of squares decomposition. *Proc. of the American Control Conference*, pp. 4153-4158, June 30- July 2, Boston, MA, USA, 2004.
- [19] A. Papachristodoulou and M. Peet, On the analysis of systems described by classes of partial differential equations. *Proc. of the 45th IEEE Conference on Decision and Control*, pp. 747-752, December 13-15, San Diego, CA, USA, 2006.
- [20] H. Yu, K. Kashima, and J. Imura, Stability analysis of 2-dimensional fluid flow based on sum-of-squares relaxation. *Proc. of the SICE Annual Conference*, pp. 3321-3326, August 20-22, Chofu City, Tokyo, Japan, 2008.
- [21] S. Prajna, A. Papachristodoulou, P. Seiler, and P. A. Parrilo, *SOS-TOOLS: Sum of squares optimization toolbox for MATLAB*. Available from <http://www.cds.caltech.edu/sostools>, 2004.
- [22] P. A. Parrilo, *Structured Semidefinite Programs and Semialgebraic Geometry Methods in Robustness and Optimization*. PhD dissertation, California Institute of Technology, Pasadena, CA, USA, 2000.
- [23] J. Bochnak, M. Coste, and M.-F. Roy, *Real algebraic geometry*. Ergebnisse der Mathematik und ihrer Grenzgebiete 3, Folge 36, Springer-Verlag, New York, 1999.
- [24] J. Löfberg, Pre- and post-processing sum-of-squares programs in practise. *IEEE Trans. Automat. Control*, Vol. 54, No. 5, pp. 1007-1011, 2009.
- [25] W. Tan, Nonlinear control analysis and synthesis using sum-of-squares programming. Ph.D. dissertation, Univ. of California, Berkeley, CA, USA, 2006.
- [26] G. Chesi, A. Garulli, A. Tesi, and A. Vicino, *Homogeneous Polynomial Forms for Robustness Analysis of Uncertain Systems*. New York, NY, USA: Springer, 2009.
- [27] D. Zhao and J.-L. Wang, Robust static output feedback design for polynomial nonlinear systems. *Int. J. of Robust and Nonlinear Control*, 2009.
- [28] S. K. Nguang, M. Krug, and S. Saat, Nonlinear static output feedback controller design for uncertain polynomial systems: an iterative sums of squares approach. *Proc. of the IEEE Conference on Industrial Electronics and Applications*, pp. 979-984, 2011.
- [29] E. Beran, L. Vandenberghe, and S. Boyd, A global BMI algorithm based on the generalized benders decomposition. *Proc. of the European Control Conference*, Brussels, Belgium, 1997.
- [30] M. Fukuda and M. Kojima, Branch-and-cut algorithms for the bilinear matrix inequality eigenvalue problem. *Computational optimization and Applications*, Vol. 19, pp. 79-105, 2001.
- [31] M. Kócvára and M. Stingl, PENBMI Users Guide (Version 2.1). "www.penopt.com", Mar 05, 2006.
- [32] J. Theis, *Sum-of-Squares Applications in Nonlinear Controller Synthesis*. Research report, Technical University of Hamburg, 2012.
- [33] Y. Al-Sweiti and D. Söffker, Modeling and control of an elastic ship-mounted crane using variable gain model-based controller. *J. of Vibration and Control*, Vol. 13, pp. 657-685, 2007.
- [34] J.-X. Xu, Y. Tan, and T.-H. Lee, Iterative learning control design based on composite energy function with input saturation. *Automatica*, Vol. 40, 1371-1377, 2004.
- [35] A. Saberi, A. A. Stoorvogel, and P. Sannuti, *Internal and External Stabilization of Linear Systems with Constraints*. Springer, New York, 2012.
- [36] D. Huang and J.-X. Xu, Steady-state iterative learning control for a

- class of nonlinear PDE processes. *J. Process Control*, Vol. 21, pp. 1155-1163, 2011.
- [37] T. Gußner and J. Adamy, Controller design for polynomial systems with input constraints. *Proc. of the Joint 48th IEEE Conf. on Decision and Control and 28th Chinese Control Conference*, Shanghai, China, Dec. 16-18, 2009.
 - [38] M. Gad-el-Hak, *Flow control Passive, Active, and Reactive Flow Management*. Cambridge University Press, Cambridge, UK, 2000.
 - [39] B. R. Noack, K. Afanasiev, M. Morzyński, G. Tadmor, and F. Thiele, A hierarchy of low-dimensional models for the transient and post-transient cylinder wake. *J. Fluid Mech.*, Vol. 497, pp. 335-363, 2003.
 - [40] R. King, M. Seibold, O. Lehmann, B. R. Noack, M. Morzyński, and G. Tadmor, Nonlinear flow control based on a low dimensional model of fluid flow. *T. Meurer et al. (Eds): Control and Observer Design, LNCIS 322*, PP.369-386, 2005.
 - [41] P. J. Goulart and S. Chernyshenko. Global stability analysis of fluid flows using sum-of-squares. *Physica D*, Vol. 241, pp. 692-704, 2012.
 - [42] K. Aleksić-Roeßner, R. King, O. Lehmann, G. Tadmor, and M. Morzyński, On the need of nonlinear control for efficient model-based wake stabilization, *Theor. Comput. Fluid Dyn.*, Vol. 28, pp. 23-49, 2014.