Model reduction for nonlinear systems and nonlinear time-delay systems from input/output data

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Abstract—An algorithm for the estimation of the moments of nonlinear systems and nonlinear time-delay systems from input/output data is proposed. The estimate is exploited to construct a family of reduced order models. Conditions to enforce additional properties, *e.g.* matching with asymptotic stability, matching with prescribed relative degree, matching with prescribed zero dynamics, upon the reduced order model are provided. The use of the technique is illustrated by a few examples based on the averaged model of the DC-to-DC Ćuk converter.

I. INTRODUCTION

The increasing complexity of modern mathematical models has maintained high the demand for additional computational power [1]. A possible solution to this problem is represented by model reduction techniques which consist in determining *less complex* yet *meaningful* description (in a sense to be specified) of complex mathematical models. In the linear framework, in which the idea of complexity can be interpreted as the dimension of the state of the system, several techniques have been developed by the systems and control community. Some of these techniques are based on the singular value decomposition, see *e.g.* [2], [3], [4] which make use of Hankel operators and [5], [6], [7], [8] which make use of balanced realizations. Another family of techniques belongs to the Krylov projection theory, see *e.g.* [9], [10], [11], [12], [13], [14], [15], also called moment matching methods. For additional detail on linear model reduction techniques see [16] and references therein. One of the main drawbacks of the methods based on moment matching is the difficulty in enforcing or preserving important properties of the system to be reduced. This has been alleviated with the new steady-state description of moment given in [17] and [18]. Moreover, the new interpretation given in those papers led to further developments in the model reduction field, such as the extension of the model reduction theory to linear and nonlinear time-delay systems [19], [20], [21] see also [22], [23], [24], [25], [26].

All these methods assume the knowledge of the exact dynamics of the system to be reduced. However, in practice a model of the system to be reduced is not always available. The present paper extends the results given in [27] to the

nonlinear framework. Inspired by the learning algorithm given in [28], [29], [30] to solve a model-free adaptive dynamic programming problem (see also the references therein, *e.g.* [31], [32]), we propose an on-line algorithm for the model reduction of nonlinear systems and nonlinear time-delay systems from data. The obtained reduced order models asymptotically match the moment of the unknown system to be reduced. The main advantage of the proposed technique consists in the low computational power required for the process. In fact, instead of identifying a complex nonlinear system and applying the classical model reduction method which requires the computation of the solution of a partial differential equation, the proposed algorithm computes directly the moment of the system from the output and input data.

The rest of the paper is organized as follows. In Section II we recall the definition of moment for nonlinear systems and the model reduction techniques given in [18]. In Section III we develop the theoretical analysis to compute an approximation of the moment for nonlinear systems. This approximation is based on the estimation of the weight parameters of a family of basis functions. Then a moment estimation algorithm is provided and the convergence properties of this algorithm are formally proved. In Section IV we give a family of reduced order models for nonlinear systems, whereas in Section V we discuss how several properties, such as matching with asymptotic stability, matching with a passivity constraint, matching with L_2 -gain, can be enforced in the present scenario. In Section VI the technique is extended to nonlinear time-delay systems and a family of reduced order models for this class of systems is given. In Section VII we illustrate the results by means of two examples based on the model of the DC-to-DC Ćuk converter. Finally Section VIII contains some concluding remarks and future directions of investigation.

Notation. We use standard notation. $\mathbb{R}_{\geq 0}$ ($\mathbb{R}_{>0}$) denotes the set of non-negative (positive) real numbers; $\mathbb{R}_{\leq 0}$ denotes the set of non-positive real numbers. Given a set of delays $\{\tau_j\}$, the symbol $\mathfrak{R}_T^n = \mathfrak{R}_T^n([-T, 0], \mathbb{R}^n)$, with $T = \max_j \{\tau_j\}$, indicates the set of continuous functions mapping the interval $[-T, 0]$ into $\mathbb{R}^n$ with the topology of uniform convergence [33]. The symbol I denotes the identity matrix and $\sigma(A)$ denotes the spectrum of the matrix $A \in \mathbb{R}^{n \times n}$. The symbol $\otimes$ indicates the Kronecker product and $\|A\|$ indicates the induced Euclidean matrix norm. The vectorization of a matrix $A \in \mathbb{R}^{n \times m}$, denoted by $\text{vec}(A)$, is the $nm \times 1$ vector obtained by stacking the columns of the matrix A one on top of the other, namely $\text{vec}(A) = [a_1^\top, a_2^\top, \dots, a_m^\top]^\top$, where

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$a_i \in \mathbb{R}^n$ are the column of A and the superscript $\top$ denotes the transpose.

II. MODEL REDUCTION BY MOMENT MATCHING - RECALLED [18]

To render the paper self-contained in this section we recall the notion of moment for nonlinear systems as presented in [18]. Consider a nonlinear, single-input, single-output, continuous-time, system described by the equations

$$\dot{x} = f(x, u), \quad y = h(x), \quad (1)$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$, f and h smooth mappings, a signal generator described by the equations

$$\dot{\omega} = s(\omega), \quad u = l(\omega), \quad (2)$$

with $\omega(t) \in \mathbb{R}^v$, $\theta(t) \in \mathbb{R}$, s and l smooth mappings, and the interconnected system

$$\dot{\omega} = s(\omega), \quad \dot{x} = f(x, l(\omega)), \quad y = h(x). \quad (3)$$

In addition, suppose that $f(0, 0) = 0$, $s(0) = 0$, $l(0) = 0$ and $h(0) = 0$.

Assumption 1: There is unique mapping $\pi(\omega)$, locally defined in a neighborhood of $\omega = 0$, which solves the partial differential equation

$$\frac{\partial \pi}{\partial \omega} s(\omega) = f(\pi(\omega), l(\omega)). \quad (4)$$

Assumption 1 implies that the interconnected system (3) posses an invariant manifold described by the equation $x = \pi(\omega)$.

Assumption 2: The signal generator (2) is observable, i.e., for any pair of initial conditions $\omega_a(0)$ and $\omega_b(0)$, such that $\omega_a(0) \neq \omega_b(0)$, the corresponding output trajectories $l(\omega_a(t))$ and $l(\omega_b(t))$ are such that $l(\omega_a(t)) - l(\omega_b(t)) \neq 0$.

Definition 1: Consider system (1) and the signal generator (2). Suppose Assumptions 1 and 2 hold. The function $h(\pi)$, with π solution of equation (4), is the *moment of system (1) at (s, l)* .

Assumption 3: The zero equilibrium of the system $\dot{x} = f(x, 0)$ is locally exponentially stable and system (2) is Poisson stable¹ with $\omega(0) \neq 0$.

Theorem 1: [18] Consider system (1) and the signal generator (2). Suppose Assumptions 2 and 3 hold. Then Assumption 1 holds and the moment of system (1) at (s, l) coincides with the steady-state response of the output of the interconnected system (3).

Finally, as shown in [18], the system described by the equations

$$\dot{\xi} = s(\xi) - \delta(\xi)l(\xi) + \delta(\xi)u, \quad \psi = h(\pi(\xi)), \quad (5)$$

where δ is any mapping such that the equation

$$\frac{\partial p}{\partial \omega} s(\omega) = s(p(\omega)) - \delta(p(\omega))l(p(\omega)) + \delta(p(\omega))l(\omega),$$

has the unique solution $p(\omega) = \omega$, is a family of *reduced order models of (1) at (s, l)* .

III. ON-LINE MOMENT ESTIMATION FROM DATA

Solving equation (4) with respect to the mapping π is a difficult task even when there is perfect knowledge of the dynamics of the system, i.e. the mapping f . Moreover, solving numerically equation (4) has the additional drawback of requiring information on the state of the system. In practice, this is usually not the case and only the output y may be available, with the consequence that also the mapping h has to be known. However, note that given the exponential stability hypothesis on the system and Theorem 1, the equation

$$y(t) = h(\pi(\omega(t))) + \varepsilon(t), \quad (6)$$

where $\varepsilon(t)$ is an exponentially decaying signal, holds. We introduce the following assumption.

Assumption 4: The mapping $h \circ \pi$ belongs to the function space identified by the family of continuous basis functions $\varphi_j : \mathbb{R}^v \rightarrow \mathbb{R}$, with $j = 1, \dots, k$, i.e. there exist $\pi_j \in \mathbb{R}$, with $j = 1, \dots, k$, such that

$$h(\pi(\omega)) = \sum_{j=1}^k \pi_j \varphi_j(\omega),$$

for any ω .

Using a weighted sum of basis functions, equation (6) can be written as

$$y(t) = \sum_{j=1}^N \pi_j \varphi_j(\omega(t)) + e(t) + \varepsilon(t), \quad (7)$$

where $e(t)$ is the approximation error. Consider now the approximation

$$y(t) = \sum_{j=1}^N \tilde{\pi}_j \varphi_j(\omega(t)) = \widetilde{h \circ \pi}(\omega(t)), \quad (8)$$

which neglects the approximation error $e(t)$ and the transient error $\varepsilon(t)$, and that can be written as

$$y(t) = \widetilde{\Pi} \widetilde{\Phi}(\omega(t)), \quad (9)$$

with

$$\widetilde{\Pi} = [\tilde{\pi}_1 \quad \tilde{\pi}_2 \quad \dots \quad \tilde{\pi}_N],$$

$$\widetilde{\Phi}(\omega(t)) = [\varphi_1(\omega(t)) \quad \varphi_2(\omega(t)) \quad \dots \quad \varphi_N(\omega(t))]^\top.$$

Let $T_k^p = \{t_{k-p+1}, \dots, t_{k-1}, t_k\}$, with $0 \leq t_0 < t_1 < \dots < t_{k-p} < \dots < t_k < \dots < t_q$, with $p > 0$ and $q \geq p$, and define the time-snapshots $\tilde{R}_k \in \mathbb{R}^{w \times N}$ and $\tilde{Y}_k \in \mathbb{R}^w$ as

$$\tilde{R}_k = [\widetilde{\Phi}(\omega(t_{k-w+1})) \quad \dots \quad \widetilde{\Phi}(\omega(t_{k-1})) \quad \widetilde{\Phi}(\omega(t_k))]^\top$$

¹See [34, Chapter 8] for the definition of Poisson stability.

and

$$\tilde{\Upsilon}_k = \begin{bmatrix} y(t_{k-w+1}) & \dots & y(t_{k-1}) & y(t_k) \end{bmatrix}^\top.$$

Consider equation (7) and let Π_k be an on-line estimate of the matrix $\Pi = \begin{bmatrix} \pi_1 & \pi_2 & \dots & \pi_N \end{bmatrix}$ computed at T_k^p , namely computed at the time t_k using the last p instants of time t_i . Then

$$\text{vec}(\tilde{\Pi}_k) = (\tilde{R}_k^\top \tilde{R}_k)^{-1} \tilde{R}_k^\top \tilde{\Upsilon}_k, \quad (10)$$

is an approximation of the on-line estimate Π_k , if $\tilde{R}_k^\top \tilde{R}_k$ is full rank. To ensure that the approximation is well-defined for all k , we give an assumption in the spirit of persistence of excitation.

Assumption 5: For any $k \geq 0$, there exist $\bar{M} > 0$ and $\alpha > 0$ such that the elements of T_k^M , with $M > \bar{M}$, are such that

$$\frac{1}{M} \tilde{R}_k^\top \tilde{R}_k \geq \alpha I.$$

Note that if Assumption 5 holds (see [29] for a similar argument), $\tilde{R}_k^\top \tilde{R}_k$ is full rank. Equation (10) is a classic least-squares estimation formula. As already discussed in [27], adapting the results in [35], it is easy to derive a recursive least-squares approximation of Π_k . To this end, let

$$\begin{aligned} \chi_k &= (\tilde{R}_k^\top \tilde{R}_k)^{-1}, \\ \Psi_k &= (\tilde{R}_{k-1}^\top \tilde{R}_{k-1} + \tilde{\Phi}(\omega(t_k))\tilde{\Phi}(\omega(t_k))^\top)^{-1}. \end{aligned}$$

Then

$$\begin{aligned} \tilde{\Pi}_k &= \tilde{\Pi}_{k-1} + \chi_k \tilde{\Phi}(\omega(t_k))(y(t_k) - \tilde{\Phi}(\omega(t_k))^\top \tilde{\Pi}_{k-1}) \\ &\quad - \chi_k \tilde{\Phi}(\omega(t_{k-w}))(y(t_{k-w}) - \tilde{\Phi}(\omega(t_{k-w}))^\top \tilde{\Pi}_{k-1}), \end{aligned} \quad (11)$$

with

$$\begin{aligned} \chi_k &= \Psi_k - \Psi_k \tilde{\Phi}(\omega(t_{k-w})) \times \\ &\quad \times (I + \tilde{\Phi}(\omega(t_{k-w}))^\top \Psi_k \tilde{\Phi}(\omega(t_{k-w})))^{-1} \tilde{\Phi}(\omega(t_{k-w}))^\top \Psi_k \end{aligned} \quad (12)$$

and

$$\begin{aligned} \Psi_k &= \chi_{k-1} - \chi_{k-1} \tilde{\Phi}(\omega(t_k)) \times \\ &\quad \times (I + \tilde{\Phi}(\omega(t_k))^\top \chi_{k-1} \tilde{\Phi}(\omega(t_k)))^{-1} \tilde{\Phi}(\omega(t_k))^\top \chi_{k-1}. \end{aligned} \quad (13)$$

Note that for single-input, single-output systems the two matrix inversions in the definition of χ_k and Ψ_k are two divisions. Equations (11)-(12)-(13) can be used to compute a fast, on-line, estimate of Π_k , since the computational complexity of updating (11) is $\mathcal{O}(1)$. Thus, the implementation of equations (11)-(12)-(13) is preferred to equation (10).

The approximation $\tilde{\Pi}_k$ can be determined with the following algorithm.

Algorithm 1:

- 1) Set $k = 0$.
- 2) Let $k = k + 1$. Select² $w > 0$, construct the matrices $\tilde{R}_k$ and $\tilde{\Upsilon}_k$, and repeat Step 2, increasing w , until the matrix $\tilde{R}_k$ is full rank.

²Recall that w is the dimension of $\tilde{\Upsilon}_k$.

3) Compute $\tilde{\Pi}_k$ from equation (10) or (11).

4) Repeat Step 2 until

$$\|\tilde{\Pi}_k - \tilde{\Pi}_{k-1}\| \leq \frac{\eta}{t_k - t_{k-1}},$$

with η a small strictly positive number which has the role of threshold.

Theorem 2: Suppose Assumptions 2, 3, 4 and 5 hold. Then

$$\lim_{t \rightarrow \infty} (h(\pi(\omega(t))) - \lim_{N \rightarrow \infty} \tilde{\Pi}_k \tilde{\Phi}(\omega(t))) = 0.$$

Proof: Assumption 5 guarantees that the approximation $\tilde{\Pi}_k$ is well-defined for all k , whereas Assumptions 2 and 3 guarantee that Theorem 1 holds and then that $h \circ \pi$ is well-defined. The quantity $\|\varepsilon(t_k)\|$ vanishes exponentially to zero by Assumption 3. Hence, by Assumption 4, $\lim_{N \rightarrow \infty} \tilde{\Pi}_k \tilde{\Phi}(\omega(t))$ converges to $h(\pi(\omega(t)))$. ■

IV. FAMILIES OF REDUCED ORDER MODELS

Using the approximation given by Algorithm 1 a reduced order model of system (1) can be defined at each instant of time t_k .

Definition 2: Consider system (1) and the signal generator (2). Suppose Assumptions 2, 3, 4 and 5 hold. Then the system

$$\dot{\xi} = \phi_k(\xi, u), \quad \psi = \kappa_k(\xi), \quad (14)$$

with $\xi(t) \in \mathbb{R}^\nu$, $u(t) \in \mathbb{R}$, $\psi(t) \in \mathbb{R}$ and ϕ_k and κ_k smooth mappings, is a *model of system (1) at (s, l) at time t_k* , i.e. system (14) has the same moment of system (1) at (s, l) , if the equation

$$\frac{\partial p_k}{\partial \omega} s(\omega) = \phi_k(p_k(\omega), l(\omega)), \quad (15)$$

has a unique solution p_k such that

$$\tilde{\Pi}_k \tilde{\Phi}(\omega) = \kappa_k(p_k(\omega)), \quad (16)$$

where $\tilde{\Pi}_k$ is the unique solution of (10).

Remark 1: Let δ_k be such that, for all $k \geq 0$,

$$\frac{\partial p_k}{\partial \omega} s(\omega) = s(p_k(\omega)) - \delta_k(p_k(\omega))l(p_k(\omega)) + \delta_k(p_k(\omega))l(\omega),$$

has the unique solution $p_k(\omega) = \omega$ and select $\kappa_k(\omega) = \tilde{\Pi}_k \tilde{\Phi}(\omega)$. Then the system described by the equations

$$\begin{aligned} \dot{\xi} &= s(\xi) - \delta_k(\xi)l(\xi) + \delta_k(\xi)u, \\ \psi &= \tilde{\Pi}_k \tilde{\Phi}(\xi), \end{aligned} \quad (17)$$

is a model of system (1) at (s, l) for all t_k

V. PROPERTIES OF THE EXPONENTIALLY CONVERGING MODELS

In [18] the problem of enforcing additional properties and constraints on the reduced order model has been studied. In this section we briefly go through these properties to determine if, and under which conditions, they hold for the models (17).

A. Matching with Asymptotic Stability

Consider system (17) and the problem of determining at every k the mapping δ_k such that the model has an asymptotically stable equilibrium point. This is achievable if δ_k can be selected such that the zero equilibrium point of $\dot{\xi} = s(\xi) - \delta_k(\xi)l(\xi)$ is locally asymptotically stable. This is possible at every k if, for instance [18],

$$\left(\frac{\partial l(\xi)}{\partial \xi}, \frac{\partial s(\xi)}{\partial \xi} \right) \Big|_{\xi=0},$$

is observable (or detectable). Note that δ_k is independent from the estimate $\widetilde{h \circ \pi_k}$.

B. Matching with Prescribed Relative Degree

Consider system (17) and the problem of determining at every k the mapping δ_k such that the model has a given relative degree $r \in [1, \nu]$ at some point $\xi(0)$. The problem is solved selecting, at each k , δ_k such that the codistribution [18]

$$d\mathcal{O}_\nu = \text{span} \left\{ d\widetilde{h \circ \pi_k}(\xi), \dots, dL_s^{\nu-1} \widetilde{h \circ \pi_k}(\xi) \right\}, \quad (18)$$

has dimension ν at $\xi(0)$. Note that, since the partial derivatives are computed on the basis functions, there is no degradation on the computation of higher order derivatives caused by the numerical estimation of the coefficients. Even though the asymptotic value of $\widetilde{h \circ \pi_k}$ satisfies condition (18) there is no guarantee that the condition holds for all k and N . However, if the condition holds for the asymptotic value, there exist $\bar{k} \gg 0$ and $\bar{N} \gg 0$ such that for all $k \geq \bar{k}$ and $N \geq \bar{N}$ the model has relative degree r .

C. Matching with Prescribed Zero Dynamics

Consider system (17) and the problem of determining at every k the mapping δ_k such that the model has zero dynamics with specific properties. If $\xi(0)$ is an equilibrium of system (17), the problem is solved selecting, at each k , δ_k such that the codistribution (18) has dimension ν at $\xi(0)$ [18]. Then there is a δ_k such that the zero dynamics of system (17) have a locally exponentially stable equilibrium and there is a coordinate transformation such that the zero dynamics is described by the equations

$$\begin{aligned} \dot{z}_1 &= z_2 + \hat{\delta}_{1,k}(z)z_1, \\ \dot{z}_2 &= z_3 + \hat{\delta}_{2,k}(z)z_1, \\ &\vdots \\ \dot{z}_{\nu-r} &= \hat{f}(z) + \hat{\delta}_{\nu-r,k}(z)z_1, \end{aligned} \quad (19)$$

where the $\hat{\delta}_{i,k}$ are free functions and

$$\hat{f}(z) = \hat{f}(\mathcal{X})|_{\mathcal{X}=[0,\dots,0,z_1,\dots,z_{\nu-r}]^\top}.$$

If the condition holds for the asymptotic value, there exist $\bar{k} \gg 0$ and $\bar{N} \gg 0$ such that for all $k \geq \bar{k}$ and $N \geq \bar{N}$ the zero dynamics of the model can be described by (19).

D. Matching with a Passivity Constraint

Consider system (17) and the problem of determining at every k the mapping δ_k such that the model is passive. The problem is solved selecting, at each k , δ_k such that there exists a differentiable function V , locally positive definite around $\xi(0)$, such that [18]

$$dV(\xi)s(\xi) \leq \widetilde{h \circ \pi_k}(\xi)l(\xi), \quad (20)$$

$$d(dV(\xi))|_{\xi=\xi(0)} > 0. \quad (21)$$

If, in addition, there exists a positive definite function θ such that

$$dV(\xi)s(\xi) + \theta(\xi) \leq \widetilde{h \circ \pi_k}(\xi)l(\xi), \quad (22)$$

then the model is strictly passive. If the conditions (21)-(22) hold for the asymptotic value $\lim_{k \rightarrow \infty} \widetilde{h \circ \pi_k}$, there exist $\bar{k} \gg 0$ and $\bar{N} \gg 0$ such that for all $k \geq \bar{k}$ and $N \geq \bar{N}$ the family of models (17) is passive.

E. Matching with L_2 -Gain

Consider the problem of determining at every k the mapping δ_k such that the family of models (17) contains, locally around $\xi(0)$, a system with L_2 -gain not larger than $\gamma > 0$. The problem is solved selecting, at each k , δ_k such that there exists a differentiable function V , locally positive definite around $\xi(0)$, such that [18]

$$dV(\xi)s(\xi) + (\widetilde{h \circ \pi_k}(\xi))^2 \leq \gamma^2 l^2(\xi),$$

$$d(dV(\xi))|_{\xi=\xi(0)} > 0,$$

locally around $\xi(0)$. If for the asymptotic value $\lim_{k \rightarrow \infty} \widetilde{h \circ \pi_k}$ there exists a positive definite function θ such that

$$dV(\xi)s(\xi) + (\widetilde{h \circ \pi_k}(\xi))^2 + \theta(\xi) \leq \gamma^2 l^2(\xi),$$

then there exist $\bar{k} \gg 0$ and $\bar{N} \gg 0$ such that for all $k \geq \bar{k}$ and $N \geq \bar{N}$ there is a system in the family of models (17) with L_2 -gain not larger than $\gamma > 0$.

VI. NONLINEAR TIME-DELAY SYSTEMS

The results developed so far can be easily extended to nonlinear time-delay systems. In fact, consider a nonlinear, single-input, single-output, continuous-time, time-delay system described by the equations³

$$\begin{aligned} \dot{x} &= f(x, x_{\tau_1}, u, u_{\tau_2}), & y &= h(x), \\ x(\theta) &= \phi(\theta), & -T &\leq \theta \leq 0, \end{aligned} \quad (23)$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$, $\phi \in \mathfrak{R}_T^n$, $\tau_j \in \mathbb{R}_{>0}$ with $j = 1, 2$ and f and h smooth mappings. Consider the signal generator (2) and the interconnected system

$$\dot{\omega} = s(\omega), \quad \dot{x} = f(x, x_{\tau_1}, l(\omega), l(\omega_{\tau_2})), \quad y = h(x). \quad (24)$$

Suppose that $f(0, 0, 0, 0) = 0$, $s(0) = 0$, $l(0) = 0$ and $h(0) = 0$.

³We consider only one delay on the input and one on the state to ease the notation. The general case can be derived with a similar discussion, see [21].

Assumption 6: There exists a unique mapping $\pi(\omega)$, locally defined in a neighborhood of $\omega = 0$, which solves the partial differential equation

$$\frac{\partial \pi}{\partial \omega} s(\omega) = f(\pi(\omega), \pi(\bar{\omega}_{\tau_1}), l(\omega), l(\bar{\omega}_{\tau_2})), \quad (25)$$

where $\bar{\omega}_{\tau_i} = \Phi_{\tau_i}^s(\omega)$, with $i = 1, 2$, is the flow of the vector field s at τ_i , see [34].

Definition 3: [21] Consider system (23) and the signal generator (2). Suppose Assumptions 2 and 6 hold. The function $h(\pi)$, with π solution of equation (25), is the *moment of system (23) at (s, l)* .

Assumption 7: The zero equilibrium of the system $\dot{x} = f(x, x_{\tau_1}, 0, 0)$ is locally exponentially stable and system (2) is Poisson stable with $\omega(0) \neq 0$.

In [21] it has been shown that if Assumptions 2 and 7 hold, then Assumption 6 holds and the moment of system (23) at (s, l) coincides with the steady-state response of the output of the interconnected system (24). Thus the output $y(t)$ can be described by the equation (7) and Theorem 2 holds for the nonlinear time-delay system (23).

Definition 4: Consider system (23) and the signal generator (2). Suppose Assumptions 2, 4, 5 and 7 hold. Then the system

$$\dot{\xi} = \phi_k(\xi, \xi_{\chi_1}, u, u_{\chi_2}), \quad \psi = \kappa_k(\xi), \quad (26)$$

with $\xi(t) \in \mathbb{R}^\nu$, $u(t) \in \mathbb{R}$, $\psi(t) \in \mathbb{R}$, $\chi_j \in \mathbb{R}_{>0}$ with $j = 1, 2$ and ϕ_k and κ_k smooth mappings, is a *model of system (23) at (s, l) at time t_k* , i.e. system (26) has the same moment of system (23) at (s, l) , if the equation

$$\frac{\partial p_k}{\partial \omega} s(\omega) = \phi_k(p_k(\omega), p_k(\bar{\omega}_{\chi_1}), l(\omega), l(\bar{\omega}_{\chi_2})), \quad (27)$$

has a unique solution p_k such that

$$\tilde{\Pi}_k \tilde{\Phi}(\omega) = \kappa_k(p_k(\omega)), \quad (28)$$

where $\tilde{\Pi}_k$ is the unique solution of (10).

Remark 2: Let δ_k be such that, for all $k \geq 0$,

$$\begin{aligned} \frac{\partial p_k}{\partial \omega} s(\omega) &= s(p_k(\omega)) - \delta_{0,k}(p_k(\omega))l(p_k(\omega)) - \gamma_k(p_k(\bar{\omega}_{\chi_1})) \\ &\quad - \delta_{1,k}(p_k(\omega))l(p_k(\bar{\omega}_{\chi_2})) + \gamma_k(p_k(\omega_{\chi_1})) \\ &\quad + \delta_{0,k}(p_k(\omega))l(\omega) + \delta_{1,k}(p_k(\omega))l(\omega_{\chi_2}), \end{aligned}$$

has the unique solution $p_k(\omega) = \omega$ and select $\kappa_k(\omega) = \tilde{\Pi}_k \tilde{\Phi}(\omega)$. Then the system described by the equations

$$\begin{aligned} \dot{\xi} &= s(\xi) - \delta_{0,k}(\xi)l(\xi) - \delta_{1,k}(\xi)l(\bar{\xi}_{\chi_2}) \\ &\quad - \gamma_k(\bar{\xi}_{\chi_1}) + \gamma_k(\xi_{\chi_1}) + \delta_{0,k}(\xi)u + \delta_{1,k}(\xi)u_{\chi_2}, \quad (29) \\ \psi &= \tilde{\Pi}_k \tilde{\Phi}(\xi), \end{aligned}$$

where $\bar{\xi}_{\chi_j} = [\bar{\omega}_{\chi_j}]_{\omega=p_k^{-1}(\xi)}$, with $j = 1, 2$, is a model of system (23) at (s, l) for all t_k .

Fig. 1. The functions $h(\pi(\omega))$ (solid line) and $\widetilde{h \circ \pi}(\omega)$ (dotted line) for $\omega \in (0, 0.9]$. The seven data points are represented by squares.

Fig. 2. Top: time histories of the output of system (30) (solid line) and of the reduced order model (dotted line) for the set of input given by ω equal to 0.35, 0.85, 0.75, 0.55, 0.33, 0.15 and 0.45. Bottom: absolute error between the two outputs.

VII. SIMULATIONS

In this section we illustrate the results of the paper by means of two examples. Both examples are based on the averaged model of the DC-to-DC Ćuk converter [36]. We begin obtaining a scalar reduced order model which achieves moment matching at zero. In this case the exact expression of the mapping $h \circ \pi$ is known and we can compare it directly with the approximation obtained from a polynomial expansion. In the second part of the section we obtain a planar reduced order model for pseudo-sinusoidal inputs. In this case the exact expression of the mapping $h \circ \pi$ is not known and the results of the paper are used to obtain an approximation of the mapping $h \circ \pi$.

Fig. 3. Approximation $\widetilde{h \circ \pi}(\omega)$ (multicolored mesh) and data points used in the fitting (dotted lines).

Fig. 4. Top: time histories of the output of system (30) (solid line) and of the reduced order model (dotted line) for the input sequence represented in the bottom graph. The switching times are indicated by dotted vertical lines. Middle: absolute error between the two outputs.

A. Model of the DC-to-DC Ćuk converter

The averaged model of the DC-to-DC Ćuk converter is given by the equations [36]

$$\begin{aligned} L_1 \frac{d}{dt} i_1 &= -(1-u)v_2 + E, \\ C_2 \frac{d}{dt} v_2 &= (1-u)i_1 + ui_3, \\ L_3 \frac{d}{dt} i_3 &= -uv_2 - v_4, \\ C_4 \frac{d}{dt} v_4 &= i_3 - Gv_4, \\ y &= v_4, \end{aligned} \quad (30)$$

where $i_1(t) \in \mathbb{R}_{\geq 0}$ and $i_3(t) \in \mathbb{R}_{\leq 0}$ describe the currents, $v_2(t) \in \mathbb{R}_{\geq 0}$ and $v_4(t) \in \mathbb{R}_{\leq 0}$ the voltages, L_1, C_2, L_3, E and G positive parameters and $u(t) \in (0, 1)$ a continuous control signal which represents the slew rate of a pulse width modulation circuit used to control the switch position in the converter. In the remaining of the paper we used the numerical values given in [36] to simulate system (30).

B. Moment at zero

In [18], a scalar reduced order model of system (30) achieving moment matching at zero has been given. It has been shown that the moment of the system is described by

$$h(\pi(\omega)) = E \frac{\omega}{\omega - 1}.$$

We simulated system (30) with $u = \omega(0)$, where $\omega(0)$ switched every 0.05s between the values of 0.2, 0.3, 0.4, 0.5, 0.6, 0.7 and 0.8. We have then applied Algorithm 1 selecting the horizon length w equal to 1 and fixing the value of $\widehat{h \circ \pi}$ to the one just before the switching time. Then a cubic interpolation has been used to fit the seven points obtained from the simulations. Fig. 1 shows the function $h(\pi(\omega))$ (solid line) and the approximation $\widehat{h \circ \pi}(\omega)$ (dotted line) for values of $\omega \in (0, 0.9]$. The seven data points are represented by squares. Note that the approximation is close to the actual moment of the system for $\omega \in (0, 0.85]$.

The reduced order model is chosen as a linear system of order $\nu = 1$ with eigenvalue equal to -230 (to yield a reduced order model with “response time” comparable to the one of the system). The top plot in Fig. 2 shows the time histories of the output of system (30) (solid line) and of the reduced order model (dotted line) for the set of inputs given by ω equal to 0.35, 0.85, 0.75, 0.55, 0.33, 0.15 and 0.45. Note that no one of these values is one of the seven data points. The absolute error between the two outputs is shown in the bottom graph of Fig. 2. The figure shows an overall good approximation of system (30). The absolute error is comparable with the one shown in the simulation in [18] and it is due to the transient error $\varepsilon(t)$ caused by approximating a four dimensional nonlinear system with a scalar model.

C. Planar reduced order model

In the previous example the approximated moment of the system has been compared with the exact mathematical expression. However, the example is very simple since the

generator is scalar and the input is constant. In this section we present the case in which the signal generator is described by the equations

$$\dot{\omega} = \begin{bmatrix} 0 & -75 \\ 75 & 0 \end{bmatrix} \omega, \quad u = \max(0.15, [1 \ 0] \omega),$$

where the output mapping is selected to satisfy a positivity condition.

Simulations have been carried out on a few values of $\omega(0)$ selected in $[-0.6, 0]^2$. For each simulation, once reached the steady-state, the time-history of $y(t)$ has been extracted over one period and a map between $y(t)$ and each pair $(\omega_1(t), \omega_2(t))$ has been extrapolated. Then, repeating the procedure for different $\omega(0)$ and filling the gaps with a linear regression, an approximation of the moment of the system has been computed for $\omega \in [-0.8, 0.8]^2$. Fig. 3 shows the approximation $\widehat{h \circ \pi}(\omega)$ (multicolored mesh) and the data points used in the fitting (dotted lines). Each closed line shown in the figure is $\widehat{h \circ \pi}_k(\omega)$, for a selected $\omega(0)$, over one period. Note that the approximation fits well the data points for $\omega \in [-0.5, 0.5]^2$, whereas for the most external values of ω the approximation is far from the data points. This is caused by the low complexity of the fitting method (a linear regression).

The reduced order model is chosen as a system of dimension $\nu = 2$ with linear dynamics, with eigenvalues equal to -830 and -850 (to yield a reduced order model with “response time” comparable to the one of the system), and a nonlinear output map. The top plot in Fig. 4 shows the time histories of system (30) (solid line) and of the reduced order model (dotted line) for the input sequence represented in the bottom graph. The input is obtained switching $\omega(0)$ every 0.22s (the switching times are indicated by dotted vertical lines). $\omega(0)$ takes, in order, the values of $(-0.45, -0.45)$, $(-0.25, -0.45)$, $(0.15, 0.05)$, $(0.05, 0.35)$, $(0.55, 0.55)$, $(0.25, -0.3)$ and $(-0.1, 0.3)$. Fig. 4 (middle) shows the absolute error between the two outputs. Note that, excluding the fifth input signal, there is an overall good approximation. The poor approximation for the fifth input value is caused by the fact that the input signal lives in the area shown in Fig. 3 where $\widehat{h \circ \pi}(\omega)$ is not well-fitted.

Remark 3: In both examples the final value of $\widehat{h \circ \pi}_k$ is used to define the reduced order model instead of updating the model at every k . This is more illustrative for the type of time domain simulations performed. Note that there is no loss of generality with respect to the theoretical results of the paper. The reader is referred to [27] for an example where the reduced order model is updated at every iteration k .

VIII. CONCLUSION

An algorithm for the estimation of the moments of nonlinear systems and nonlinear time-delay systems from input/output data has been given. The obtained approximation has been used to construct reduced order models which asymptotically match the moment of the system to be reduced. Conditions to enforce additional properties, e.g.

matching with asymptotic stability, matching with prescribed relative degree, upon the reduced order model have been provided. A few examples based on the averaged model of the DC-to-DC Ćuk converter are used to illustrate the technique. Future research will consist in extending the theory to the case in which $u \neq l(\omega)$ and in testing the proposed algorithm on specific applications to validate the results of the paper.

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