Nonlinear Analysis of Laminated Shells with Alternating Stiff/Soft Lay-up

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Abstract

This paper proposes a multi-layer formulation for the nonlinear analysis of laminated shells with an alternating stiff/soft lay-up. The zigzag variation of planar displacements is taken into account by adding to the Reissner-Mindlin formulation a specific set of zigzag function which is effective for the considered lay-up. Furthermore, a piecewise linear through-thickness distribution of the material transverse shear strain is assumed, which agrees well with the real distribution. The proposed lamination model with a low-order nonlinear strain-displacement relationship is incorporated within a co-rotational framework for geometric nonlinear analysis, thus upgrading the low-order local element formulation to large displacement analysis with relative ease. In addition, a local shell system is employed for direct definition of the additional zigzag displacement fields and associated parameters, which are thus excluded from the large displacement co-rotational transformations. The application of the proposed laminated shell modelling approach is illustrated in this paper for a 9-noded co-rotational shell element, which utilises the Mixed Interpolation of Tensorial Components (MITC) method in the local system for overcoming locking effects. Several linear and nonlinear numerical examples of multi-layer shell structures with alternating stiff/soft lay-ups are used to illustrate the effectiveness and efficiency of the proposed modelling approach.

Keywords: multi-layer; alternating stiff/soft lay-up; zigzag effect; transverse shear strain; large displacement.

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1 INTRODUCTION

Laminated structures, which are composed of different material layers or of the same material layers but with different fibre orientations, have been widely employed in many engineering applications. Due to the change in material properties through the thickness, such structures are associated with piecewise forms of displacement fields and transverse stresses in the thickness direction. 3D continuum models for such structures are associated with huge computational cost, although they can capture both the global and local response of a composite laminate. An enormous amount of research effort has therefore been made in the establishment of plate theory-based lamination models for a close estimation of the structural behaviour at reduced computational demands. Layer-wise models (LWMs) [1-4] and zigzag models (ZZMs) [5-9] achieve the reproduction of piecewise continuous displacement fields in the thickness direction by employing the first-order shear deformation theory (FSDT) or higher-order shear deformation theories (HSDTs) at each constitutive layer and imposing C^0 continuity constraints at laminar interfaces. The Equivalent Single Layer Models (ESLMs) [10], which follow the idea of Murakami [11] with further developments, obtain piecewise displacements by simply adding Murakami's zigzag function [11] to a classical Taylor type expansion defined along the whole plate thickness.

For thin-to-moderately thick laminations, the accuracy of a model depends significantly on whether the displacement variables are able to capture the zigzag effect, whereas its efficiency is a function of the number of independent displacement variables as well as the sparsity of the resulting stiffness matrix. The number of displacement variables for LWMs is dependent on the number of constitutive layers, which results in the size of the stiffness matrix increasing with layers. Further C^0 continuity constraints on transverse stresses distinguish the ZZMs from the LWMs, which leads to continuous through-thickness distribution of transverse stresses with a reduced number of independent displacement variables. The displacement-based formulations of the ESLMs, denoted as 'EDZ' models [10], have the number of displacement variables independent of the constitutive layers without the requirement of imposing transverse stress constraints. Nevertheless, usually high-order Taylor expansions may be required to enrich the through-thickness displacement

variation for a good approximation of the zigzag effect [10, 12-14]. There are also a few mixed formulations of the ESLMs denoted as ‘EMZC’ models [10, 15-16], which, on the basis of the EDZ models, assume a piecewise continuous distribution of transverse stresses and employ Reissner variational principle [17] for a better description of transverse stresses. ZZMs and EMZCs satisfy C^0 -continuity requirements for both displacements and transverse stresses, though this imposes additional computational demands which can become significant in the presence of material nonlinearity due to the need for iterations to satisfy shear stress continuity.

There is a special group of laminations which have an alternating stiff/soft stacking sequence, including for example engineering the laminated glass and polymer-metal composites. These laminated structures are characterised by a very large stiffness ratio between the stiff layers and the soft layers, which induces a specific zigzag pattern for planar displacements and a distinct distribution of transverse shear strains. Following the characteristics of the displacements and transverse shear strains, an efficient lamination model is proposed in this paper for the analysis of structures with this alternating stiff/soft lay-up. The proposed model sits somewhere between a LWM and an EDZ model. Similar to the EDZ model, a set of zigzag functions is defined over the full plate thickness, which enriches the classical Reissner-Mindlin formulation [18-19] to allow for cross-sectional warping and satisfies the continuity of displacements at laminar-interfaces a priori via the assumed zigzag modes. However, in the proposed approach, the number of additional displacement variables depends on the number of soft layers, which resembles LWMs but generates a smaller number of independent displacement variables. On the other hand, a piecewise linear transverse shear strain distribution is assumed for the lamination model, which imposes no constraints on transverse shear stresses but is shown to provide an accurate representation of the actual distribution without sacrificing computational efficiency.

In formulating large displacement finite elements for small strain problems, the relationship between the strain and displacement fields is highly nonlinear and complex if the displacement fields are referred to a fixed coordinate system, where the nonlinear strain terms arise mainly from the element rigid body rotations. The co-rotational approach, which

decomposes the element motion into rigid body and strain-inducing parts via the use of a local co-rotational system, allows the employment of low-order, even linear, relationships between the strain and local displacement fields, thus upgrading linear local element formulations of different sophistication to geometric nonlinear analysis with relative ease [20-21]. Furthermore, the additional zigzag displacement variables are associated with local cross-sectional warping only; hence, a 2D ‘shell’ coordinate system [22] is employed in this paper for direct definition of these additional variables, which effectively minimises the required co-rotational transformations and enhances computational efficiency.

Another issue related to low-order shell elements is the locking phenomenon, in which the element can exhibit an over-stiff response, resulting from its inability to correctly model lower-order modes. Numerous locking-elimination methods have been developed, among which are reduced/selective integration methods [23-24], enhanced strain methods [25], and strain mapping methods [26-28]. Most of these methods can be equally applied to generalised strains at the level of overall cross-section or individual layers, where the latter is particularly advantageous for the present work, since the overall laminated shell response is obtained from contributions of layers that are processed individually as monolithic shells.

Herein, the application of the proposed laminated shell model is illustrated for a 9-noded shell element [29-30], which employs a bisector co-rotational system [20] for modelling geometric nonlinearity. The basic local displacement variables consistent with the Reissner-Mindlin formulation are related to the global parameters according to discrete nonlinear co-rotational transformations. On the other hand, the additional zigzag displacement variables are defined in a 2D curvilinear shell system which follows the element local system according to a fixed relationship, thus avoiding the need for nonlinear co-rotational transformation of the additional zigzag variables. The Mixed Interpolation Tensorial Components (MITC) method [27, 31-32] is employed for each constitutive layer of the 9-noded shell element to overcome the locking phenomenon

The paper proceeds with highlighting the principal characteristics of laminations with an alternating stiff/soft lay-up, which is followed by presenting the proposed laminated shell model. The application of the proposed laminated shell model is subsequently illustrated for a

9-noded co-rotational element, and several linear and nonlinear numerical examples are finally presented to demonstrate the accuracy and efficiency of the proposed approach.

2 CHARACTERISTICS OF ALTERNATING STIFF/SOFT LAMINATIONS

Laminations with an alternating stiff/soft lay-up have two main characteristics which distinguish them from other laminations. Firstly, the large stiffness ratio (denoted as ‘SR’ hereafter) between the stiff layer and the soft layer plays an important role in the through-thickness distribution of the transverse shear stresses and strains. To illustrate this point, multi-layer beams with an alternating stiff/soft lay-up have been modelled under bending with 2D plane-stress analysis, where Figures 1-3 depict the schematic distributions of the transverse shear stress and strain with different number of constitutive layers and various SRs. Clearly, for each of the laminations considered, the distribution of the transverse shear stress changes significantly with different SR values. However, the transverse shear strain distribution for such kind of laminations shows that for the considered SR range the softer layers sustain much larger strains than the stiffer layers and exhibit a near constant distribution through the constitutive layer.

Secondly, a large SR induces significant zigzag effect in such laminations. The variation of planar displacements under bending is investigated by performing a 2D plane-stress analysis of a multi-layer beam with an alternating stiff/soft lay-up, which indicates that all stiff layers have almost identical rotations of their respective normals, whereas the soft layers can have different rotations.

3 KINEMATIC DESCRIPTION FOR LAMINATED SHELLS

Following on from the above noted characteristics, a laminated shell model with an alternating stiff/soft lay-up is proposed. Figure 4 depicts the lamination model for a plate along with the local coordinates, where the x- and y-axes are located at the middle surface while the z-axis is normal to the plate, and where each layer is identified by a unique index. It is important to note that while the kinematic descriptions is presented for a plate problem, it

is equally applicable to local formulations of shallow shells and, through incorporation within a co-rotational framework, to the nonlinear analysis of general curved shells, as presented in Section 4 and demonstrated by the numerical examples of Section 6.

3.1 Zigzag displacement fields

In this lamination model, a piecewise linear variation of planar displacements in the z direction is assumed and C^0 -continuity constraints are imposed at laminar interfaces. Based on the summarised pattern of the zigzag displacements, it is assumed that all stiff sheets have identical rotations of the normal, whereas the soft sheets allow different rotations. Accordingly, for a lamination consisting of (N_c+1) stiff layers bonded by N_c soft cores (the subscript 'c' denotes core), the through-thickness distribution of the planar displacements can be decomposed into a constant and a linear mode, in accordance with the Reissner-Mindlin kinematic hypothesis, as well as N_c zigzag modes, denoted by $\Lambda_j^o(z)$ ($j=1 \rightarrow N_c$), accounting for the zigzag effect. Each zigzag mode can be initially expressed as (Figure 5.b):

$$\Lambda_j^o(z) = \begin{cases} 0 & z \in [h_-, h_{2j-}] \\ \frac{1}{h_{2j}}(z - h_{2j-}) & z \in [h_{2j-}, h_{2j+}] \\ 1 & z \in [h_{2j+}, h_+] \end{cases} \quad (j=1 \rightarrow N_c) \quad (1)$$

in which h_- and h_+ denote the values of z at the bottom and top of the cross-section, respectively; h_{k-} and h_{k+} refer to the values of z at the bottom and top of layer (k), respectively; h_k is the thickness of layer (k).

By orthogonalising each zigzag mode $\Lambda_j^o(z)$ with respect to the constant and linear modes, with the addition of constant and linear terms, the zigzag mode becomes (Figure 5.c):

$$\Lambda_j(z) = \Lambda_j^o(z) + \alpha_{0,j} + \alpha_{1,j}z \quad (2)$$

$$\alpha_{0,j} = \frac{h_{2j-} + h_{2j+}}{2h} - \frac{1}{2}, \quad \alpha_{1,j} = \frac{4h_{2j-}^2 + 4h_{2j-}h_{2j+} + 4h_{2j+}^2 - 3h^2}{2h^3} \quad (3)$$

The resulting planar displacement fields are expressed as:

$$u_\alpha(x, y, z) = u_{\alpha 0}(x, y) + z\theta_\alpha(x, y) + \sum_{j=1}^{N_c} \Lambda_j(z) \vartheta_\alpha^j(x, y) \quad (\alpha = x, y) \quad (4)$$

where $u_{\alpha 0}$ are the planar displacement fields along the x- or y-axis evaluated on the middle surface; θ_{α} are the components of the normal vector along the x- or y-axis in the absence of zigzag displacements; $\mathfrak{g}_{\alpha}^j$ are the additional fields associated with the proposed zigzag functions along the x- or y-axis. The transverse displacement is assumed to be constant through the plate thickness, and is thus denoted by $u_{z0}(x, y)$.

3.2 Kinematics of constitutive layer

Each constitutive layer of the lamination model is regarded as a pseudo plate. At layer (k) ($k=1 \rightarrow N_l$, where N_l denotes the number of constitutive layers), the translational displacements on the layer mid-surface are obtained as:

$$u_{\alpha}^{(k)} = u_{\alpha 0} + z^{(k)}\theta_{\alpha} + \sum_{j=1}^{N_c} \Lambda_j^{(k)} \mathfrak{g}_{\alpha}^j \quad (\alpha = x, y) \quad (5a)$$

$$u_z^{(k)} = u_{z0} \quad (5b)$$

where $\Lambda_j^{(k)} \equiv \Lambda_j(z^{(k)})$ represents the extracted value of the zigzag function $\Lambda_j(z)$ on the middle surface of layer (k); $z^{(k)}$ is the extracted value of z on the middle surface of layer (k). The rotational displacements of layer (k) are derived by taking the first derivatives of the planar displacements with respect to z :

$$\theta_{\alpha}^{(k)} = \left. \frac{\partial u_{\alpha}^{(k)}}{\partial z} \right|_{z^{(k)}} = \theta_{\alpha} + \sum_{j=1}^{N_c} \lambda_j^{(k)} \mathfrak{g}_{\alpha}^j, \quad \lambda_j^{(k)} = \left. \frac{\partial \Lambda_j}{\partial z} \right|_{z^{(k)}} \quad (\alpha = x, y) \quad (6)$$

Accordingly, the following relationship holds at each layer:

$$\mathbf{u}^{(k)} = \mathbf{T}_c^{(k)} \mathbf{u}_c + \mathbf{T}_a^{(k)} \mathbf{u}_a \quad (7)$$

$$\mathbf{T}_c^{(k)} = \begin{bmatrix} 1 & 0 & 0 & z^{(k)} & 0 \\ 0 & 1 & 0 & 0 & z^{(k)} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (8a)$$

$$\mathbf{T}_a^{(k)} = \begin{bmatrix} \Lambda_1^{(k)} & 0 & \dots & \Lambda_j^{(k)} & 0 & \dots & \Lambda_{N_c}^{(k)} & 0 \\ 0 & \Lambda_1^{(k)} & \dots & 0 & \Lambda_j^{(k)} & \dots & 0 & \Lambda_{N_c}^{(k)} \\ \hline 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ \hline \lambda_1^{(k)} & 0 & \dots & \lambda_j^{(k)} & 0 & \dots & \lambda_{N_c}^{(k)} & 0 \\ 0 & \lambda_1^{(k)} & \dots & 0 & \lambda_j^{(k)} & \dots & 0 & \lambda_{N_c}^{(k)} \end{bmatrix}_{(5 \times 2N_c)} \quad (8b)$$

where $\mathbf{u}_c = \langle u_{x0}, u_{y0}, u_{z0}, \theta_x, \theta_y \rangle^T$ are the basic local displacement fields consistent with the Reissner-Mindlin formulation; $\mathbf{u}_a = \langle \mathfrak{g}_x^1, \mathfrak{g}_y^1 \dots \mathfrak{g}_x^{N_c}, \mathfrak{g}_y^{N_c} \rangle^T$ are the additional displacement fields associated with the zigzag functions $\Lambda_j^o(z)$ ($j = 1 \rightarrow N_c$) ; $\mathbf{u}^{(k)} = \langle u_x^{(k)}, u_y^{(k)}, u_z^{(k)}, \theta_x^{(k)}, \theta_y^{(k)} \rangle^T$ are the displacement fields at layer (k).

The strain state within each layer (k) is fully determined by the membrane strains $\boldsymbol{\varepsilon}_m^{(k)}$, bending generalised strains $\boldsymbol{\varepsilon}_b^{(k)}$, and transverse shear strains $\boldsymbol{\varepsilon}_s^{(k)}$. The various generalised strains are obtained as follows:

$$\boldsymbol{\varepsilon}_m^{(k)} = \begin{Bmatrix} \varepsilon_x^{(k)} \\ \varepsilon_y^{(k)} \\ \gamma_{xy}^{(k)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u_x^{(k)}}{\partial x} \\ \frac{\partial u_y^{(k)}}{\partial y} \\ \frac{\partial u_x^{(k)}}{\partial y} + \frac{\partial u_y^{(k)}}{\partial x} \end{Bmatrix} + \begin{Bmatrix} \frac{1}{2} \left(\frac{\partial z_0}{\partial x} + \frac{\partial u_z^{(k)}}{\partial x} \right)^2 - \frac{1}{2} \left(\frac{\partial z_0}{\partial x} \right)^2 \\ \frac{1}{2} \left(\frac{\partial z_0}{\partial y} + \frac{\partial u_z^{(k)}}{\partial y} \right)^2 - \frac{1}{2} \left(\frac{\partial z_0}{\partial y} \right)^2 \\ \left(\frac{\partial z_0}{\partial x} + \frac{\partial u_z^{(k)}}{\partial x} \right) \left(\frac{\partial z_0}{\partial y} + \frac{\partial u_z^{(k)}}{\partial y} \right) - \left(\frac{\partial z_0}{\partial x} \right) \left(\frac{\partial z_0}{\partial y} \right) \end{Bmatrix} \quad (9a)$$

$$\boldsymbol{\varepsilon}_b^{(k)} = \begin{Bmatrix} \kappa_x^{(k)} \\ \kappa_y^{(k)} \\ \kappa_{xy}^{(k)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \theta_x^{(k)}}{\partial x} \\ \frac{\partial \theta_y^{(k)}}{\partial y} \\ \frac{\partial \theta_x^{(k)}}{\partial y} + \frac{\partial \theta_y^{(k)}}{\partial x} \end{Bmatrix} \quad (9b)$$

$$\boldsymbol{\varepsilon}_s^{(k)} = \begin{Bmatrix} \gamma_{xz}^{(k)} \\ \gamma_{yz}^{(k)} \end{Bmatrix} = \begin{Bmatrix} \theta_x^{(k)} + \frac{\partial u_z^{(k)}}{\partial x} \\ \theta_y^{(k)} + \frac{\partial u_z^{(k)}}{\partial y} \end{Bmatrix} \quad (9c)$$

in which z_0 represents the offset of the shell mid-surface along the z -axis, thus generalising the kinematics of flat plates to shallow shells; in this respect, the kinematic expressions presented previously remain unaffected for a shallow shell with z taken as zero along the shell mid-surface.

It is worth noting that quadratic terms of the membrane strains in (9a) take into account the effect of shell curvature, which are not necessary within a co-rotational approach but enable better accuracy with coarser meshes.

3.3 Through-thickness distribution of transverse shear strains

The ZZTs and EMZC models, which impose inter-laminar constraints on both displacements and transverse stresses, fully fulfil the continuity requirement and allow a continuous variation of transverse stresses, compared with other theories. Nevertheless, when material nonlinearity is considered, the continuity requirement on transverse shear stresses necessitates an iterative process which impose additional computational demands. On the other hand, in the proposed approach, continuity constraints are not imposed on transverse shear stresses. Rather, based on the aforementioned pattern of transverse shear strains, as depicted in Figures 1-3, it is assumed that in the external stiff sheets the shear strain varies linearly from zero at the outer surface, whereas for each internal layer the shear strain remains constant. The through-thickness distribution of the assumed transverse shear strain can thus be expressed as follows:

$$\boldsymbol{\epsilon}_{s,AS}^{(k)} = \omega^{(k)} F_k(z) \boldsymbol{\epsilon}_s^{(k)} \quad z \in [h_{k-}, h_{k+}] \quad (10)$$

$$F_1(z) = 1 + \frac{2(z - z^{(1)})}{h_1}, \quad F_{N_1}(z) = 1 - \frac{2(z - z^{(N_1)})}{h_{N_1}}, \quad F_k(z) = 1 \quad (k = 2 \rightarrow N_1 - 1) \quad (11)$$

where $F_k(z)$ is the assumed distribution of transverse shear strains at layer (k) , and $\omega^{(k)}$ is the shear correction factor of layer (k) , which can be derived from energy equivalence at the generalised stress/strain and material stress/strain levels. Considering the equivalence of the generalised shear stresses and the resultant shear forces from equilibrium considerations, this

results in $\omega^{(1)} = \omega^{(N_1)} = \frac{3}{4}$ and $\omega^{(k)} = 1 \quad (k = 2 \rightarrow N_1 - 1)$.

In the range of large stiffness ratios (SR), this assumed distribution for the transverse shear strain offers a realistic representation of the exact solution, albeit with discontinuous transverse shear stresses. This is besides the fact that an assumed strain distribution is much more practical than an assumed stress distribution when considering material nonlinearity. The exclusion of stress coupling between layers leads to a lamination formulation that achieves good levels of accuracy with high computational efficiency, which becomes even more pronounced for laminated shell structures with more layers.

4 ENHANCEMENTS FOR LARGE DISPLACEMENT ANALYSIS

In this section, enhancements are presented for the laminated plate/shallow shell model, which upgrade local element formulations based on this model to consider geometric nonlinearity with large displacements using a co-rotational approach. The merits of the co-rotational approach in the context of laminated shell modelling are first discussed, which is followed by proposing a 2D curvilinear system, so-called ‘shell’ coordinate system, which enables the effective and efficient consideration of the additional zigzag displacement fields. Nonlinear transformations between the global coordinate system and the local co-rotational system, as well as the required linear transformations between the shell coordinate system and the local co-rotational system, are finally presented.

4.1 Co-rotational approach

In formulating large displacement finite elements for small-strain problems, the relationship between the strain and displacement fields is highly nonlinear and complex if the displacement fields are referred to a fixed coordinate system, where the nonlinear strain terms arise mainly from the element rigid body rotations. In this respect, the use of a co-rotational approach [20-21] overcomes this complexity, enabling the use of a low-order strain displacement relationship at the local level, as presented for layers of shallow shells in (9), and addressing geometric nonlinearity through transformations between the local and global systems that are applied at the level of discrete element parameters.

A co-rotational system follows the current element configuration throughout the analysis, as illustrated in Section 5 for a 9-noded element, and decomposes the element motion into

rigid body and strain-inducing parts, leading to an explicit relationship between the global and the local displacement parameters. The embedment of a monolithic Reissner-Mindlin formulation into the co-rotational framework is usually achieved by relating 5 local nodal displacement parameters, 3 translations and 2 rotations, to their counterparts in the global system; the exception would be where adjacent shell elements meet at an angle, in which case 3 rotational parameters would be used [30]. Since the co-rotational system follows the element configuration throughout the large displacement analysis, the transformations between the global and local element systems are nonlinear and vary from step to step.

4.2 Shell coordinate system

For a laminated shell element formulation, if the continuity of the zigzag displacement fields is enforced via additional parameters defined in the global coordinate system, similar to the basic nodal displacement parameters, then these would be subject to co-rotational transformations to the local system, thus imposing further computational demands that increase with the number of additional displacements and hence the number of layers. Noting that the zigzag fields describe the local effect of cross-sectional warping, it is proposed that they are defined in a 2D orthogonal curvilinear coordinate system over the shell structure, denoted as the ‘shell’ coordinate system, which thus follows the local co-rotational system at the element level. With the associated additional zigzag parameters defined in this shell coordinate system, continuity of the zigzag fields is ensured. Importantly, the element response associated with the zigzag parameters can thus be evaluated via a fixed linear kinematic transformation between the shell and local element systems, as elaborated in Section 4.3, rather than a varying nonlinear co-rotational transformation, which enhances the computational efficiency of the geometric nonlinear analysis of laminated shells. Another main benefit of using a shell coordinate system relates to defining fibre orientation for composite materials [22].

In order to ensure continuity of the zigzag fields, a key requirement is that the 2D shell coordinate system must be associated with a unique orientation of its orthogonal directional vectors at an arbitrary point on the shell mid-surface. Besides this fundamental requirement

of uniqueness, it is desirable for the 2D curvilinear shell system to be defined in a continuous manner, as illustrated by the dotted contour lines in Figure 6. For a smooth shell structure, a continuous definition of the shell system can be obtained in different ways, provided the shell surface is open. On the other hand, for a closed shell surface (e.g. a sphere), a discontinuous definition of the 2D shell coordinate system would be necessary, where the discontinuity may be localised to a single point or line. For shell structures with a folded edge, the shell system would not be uniquely defined along the fold line, though there is no requirement for continuity of the zigzag fields in such locations; a typical realistic treatment would be to restrain the additional zigzag parameters at fold lines, though a more relaxed treatment based on a free natural boundary condition for the associated zigzag forces can also be considered with the use of element-specific zigzag parameters along the folds.

With reference to the 2D curvilinear shell system (r,s) shown in Figure 6, the additional displacement zigzag parameters of an arbitrary element can be defined along the two curvilinear directions at the node level (refer to Element I). Although the relative orientation of the shell coordinate system and local element system can vary over one element, a constant relative orientation may also be considered at the element level (refer to Element II), where all additional zigzag parameters would be assumed to accord with the surface vectors at the element centre, provided the 2D shell system is continuous. While this assumption is associated with some inaccuracy, especially for a coarse mesh, it simplifies the determination of the additional displacement fields over the element, and importantly it retains the convergence property with mesh refinement. For small-strain problems, the relative orientation of the shell coordinate system and the element local system can be assumed to remain constant throughout the analysis; hence this orientation can be established at the start of nonlinear analysis in terms of a fixed angle β for each element denoting the rotation from $\mathbf{c}_r$ to $\mathbf{c}_x$ (Figure 7).

Different methods for defining a unique and continuous 2D curvilinear shell system over a smooth shell structure with a continuous surface can be found elsewhere [22]. One such definition, which is used for the examples presented in Section 6, considers $\mathbf{c}_s^0$ in the initial undeformed configuration to be a projection of a user-defined vector $\mathbf{n}$ on the shell surface,

as illustrated in Figure 8. This definition can be used to generate a continuous 2D shell system provided a vector $\mathbf{n}$ can be specified which is not orthogonal to the shell surface at any point. For some curved shells with open surfaces, such as a hemi-spherical shell, this definition cannot be applied, though alternative definitions of a continuous 2D shell system can be employed instead, as elaborated in [22].

4.3 Kinematic transformations between global, local and shell systems

As already noted, the present work proposes the use of a co-rotational framework for upgrading the low-order laminated plate/shallow shell model to geometrically nonlinear analysis, where the nonlinear kinematic transformations between the global and local element systems are conveniently restricted to the basic nodal displacement and rotational parameters. On the other hand, the additional zigzag displacement parameters, which describe the local cross-sectional warping behaviour only, are defined in a specific shell system which follow the local element system at a constant orientation, and are therefore excluded from the co-rotational transformations.

The kinematic relationship between local displacement variables and their global counterparts depends on the employed definition of the co-rotational approach and the sequence of nodal numbering. This is illustrated for a 9-noded shell element using the bisector co-rotational system definition [20,30] in Section 5.1. On the other hand, the relationship between the zigzag displacements defined in the shell and local systems is linear for small-strain problems, where the following is employed to transform the additional fields from the shell system to the local element system:

$$\begin{Bmatrix} g_x^j \\ g_y^j \end{Bmatrix} = \begin{bmatrix} \hat{c} & \hat{s} \\ -\hat{s} & \hat{c} \end{bmatrix} \begin{Bmatrix} g_r^j \\ g_s^j \end{Bmatrix}, \quad \hat{c} = \cos(\beta), \quad \hat{s} = \sin(\beta) \quad (j = 1 \rightarrow N_c) \quad (12)$$

where $\langle g_x^j, g_y^j \rangle$ ($j = 1 \rightarrow N_c$) are additional zigzag displacement fields in the element local system; $\langle g_r^j, g_s^j \rangle$ ($j = 1 \rightarrow N_c$) are the associated fields defined in the curvilinear shell system; and angle β is the relative orientation of the two systems obtained at the start of analysis.

Note that (12) is most effectively accounted for in the kinematic description of (7) and (8) by re-defining the additional zigzag fields $\mathbf{u}_a$ in the shell coordinate system, i.e. $\mathbf{u}_a = \langle \vartheta_r^1, \vartheta_s^1 \dots \vartheta_r^{N_c}, \vartheta_s^{N_c} \rangle^T$, and adjusting the transformation matrix $\mathbf{T}_a^{(k)}$ to:

$$\mathbf{T}_a^{(k)} = \begin{bmatrix} \hat{c} \Lambda_1^{(k)} & \hat{s} \Lambda_1^{(k)} & \dots & \hat{c} \Lambda_j^{(k)} & \hat{s} \Lambda_j^{(k)} & \dots & \hat{c} \Lambda_{N_c}^{(k)} & \hat{s} \Lambda_{N_c}^{(k)} \\ -\hat{s} \Lambda_1^{(k)} & \hat{c} \Lambda_1^{(k)} & \dots & -\hat{s} \Lambda_j^{(k)} & \hat{c} \Lambda_j^{(k)} & \dots & -\hat{s} \Lambda_{N_c}^{(k)} & \hat{c} \Lambda_{N_c}^{(k)} \\ \hline 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ \hline \hat{c} \lambda_1^{(k)} & \hat{s} \lambda_1^{(k)} & \dots & \hat{c} \lambda_j^{(k)} & \hat{s} \lambda_j^{(k)} & \dots & \hat{c} \lambda_{N_c}^{(k)} & \hat{s} \lambda_{N_c}^{(k)} \\ -\hat{s} \lambda_1^{(k)} & \hat{c} \lambda_1^{(k)} & \dots & -\hat{s} \lambda_j^{(k)} & \hat{c} \lambda_j^{(k)} & \dots & -\hat{s} \lambda_{N_c}^{(k)} & \hat{c} \lambda_{N_c}^{(k)} \end{bmatrix}_{(5 \times 2N_c)} \quad (13)$$

This works well provided the shell system is continuous over the element, in which case the response is convergent with mesh refinement even where any variation in the relative orientation of the shell and local element system is ignored for curved shells, with $(\hat{c}, \hat{s})$ assumed constant over the element. On the other hand, when the local shell system is discontinuous, as would be the case at specific locations for a closed shell structure, the most effective approach would be to transform the nodal zigzag displacement parameters from the node-specific shell system to the local element system, with the local parameters then used to define the local zigzag fields $\langle \vartheta_x^j, \vartheta_y^j \rangle$ directly. The latter approach is utilised for generality in the following application to a 9-noded laminated shell element.

5 APPLICATION TO 9-NODED SHELL ELEMENT

In this section, the application of the proposed lamination model is illustrated for a 9-noded co-rotational shell element [29-30], noting that it can also be similarly applied to other shell elements of different order and shape. A bisector co-rotational framework, shown in Figure 9, is employed for large displacement analysis, where basic local displacements $\mathbf{u}_c$ are related to the global parameters according to nonlinear kinematic transformations that exclude the influence of rigid body rotations at the local element level. Additional zigzag displacements $\mathbf{u}_a$ are defined in the 2D curvilinear shell system, and their associated resistance forces are readily assembled without the need for co-rotational transformation. At each layer (k) , the Mixed Interpolation of Tensorial Components (MITC) method [31-32] is employed for alleviating locking. The development of the 9-noded element within the

bisector co-rotational framework for application to laminated shells according to the proposed approach is elaborated in the following sub-sections.

5.1 Bisector co-rotational system

Figure 9 illustrates the co-rotational framework used for the 9-noded shell element, where (X, Y, Z) and (x, y, z) refer to the global and local co-rotational coordinate systems, respectively. The x - and y -axes of the local co-rotational system are defined such that they coincide with the bisectors of the diagonal vectors generated from the four corner nodes in the current unknown configuration [20]. Three translational and two rotational global parameters are defined in the global coordinate system, which are then transformed to five basic parameters $(u_{x0}, u_{y0}, u_{z0}, \theta_x, \theta_y)$ in the co-rotational system. The triad of this co-rotational system and the relevant global-local displacement transformations are provided in Appendix A. By filtering out the rigid body rotations, the co-rotational approach shifts a large-displacement/small-strain problem to a small-displacement/small-strain problem at the local element level, thus allowing the employment of low-order kinematics, as proposed in Section 3 for laminated shells.

5.2 Local element formulation

Local and additional parameters are respectively defined as $\mathbf{U}_C = \langle \mathbf{U}_{C1}^T, \dots, \mathbf{U}_{Ci}^T, \dots, \mathbf{U}_{C9}^T \rangle^T$ and $\mathbf{U}_A = \langle \mathbf{U}_{A1}^T, \dots, \mathbf{U}_{Ai}^T, \dots, \mathbf{U}_{A9}^T \rangle^T$, where $\mathbf{U}_{Ci}$ and $\mathbf{U}_{Ai}$ contain respectively five local co-rotational nodal parameters and $2N_c$ additional zigzag nodal parameters, which are expressed as $\mathbf{U}_{Ci} = \langle u_{x0,i}, u_{y0,i}, u_{z0,i}, \theta_{x,i}, \theta_{y,i} \rangle^T$ and $\mathbf{U}_{Ai} = \langle \vartheta_{r,i}^1, \vartheta_{s,i}^1, \dots, \vartheta_{r,i}^j, \vartheta_{s,i}^j, \dots, \vartheta_{r,i}^{N_c}, \vartheta_{s,i}^{N_c} \rangle^T$. The pseudo nodal parameters at layer (k) , which are defined as $\mathbf{U}^{(k)} = \langle \mathbf{U}_1^{(k)T}, \dots, \mathbf{U}_i^{(k)T}, \dots, \mathbf{U}_9^{(k)T} \rangle^T$ with $\mathbf{U}_i^{(k)} = \langle u_{x,i}^{(k)}, u_{y,i}^{(k)}, u_{z,i}^{(k)}, \theta_{x,i}^{(k)}, \theta_{y,i}^{(k)} \rangle^T$, can be obtained from the following relationship:

$$\mathbf{U}^{(k)} = \mathbf{T}_C^{(k)} \mathbf{U}_C + \mathbf{T}_A^{(k)} \mathbf{U}_A \quad (14)$$

$$\mathbf{T}_C^{(k)} = \begin{bmatrix} \mathbf{T}_c^{(k)} & \mathbf{0} \\ & \ddots \\ \mathbf{0} & \mathbf{T}_c^{(k)} \end{bmatrix}_{(45 \times 45)}, \quad \mathbf{T}_A^{(k)} = \begin{bmatrix} \mathbf{T}_a^{(k)} & \mathbf{0} \\ & \ddots \\ \mathbf{0} & \mathbf{T}_a^{(k)} \end{bmatrix}_{(45 \times 18N_c)} \quad (15)$$

where $\mathbf{T}_c^{(k)}$ and $\mathbf{T}_a^{(k)}$ are given in (8a) and (13), respectively. Note that $\mathbf{T}_A^{(k)}$ applies to a continuous shell system definition, ignoring the change of $(\hat{c}, \hat{s})$ over the element, but it can be easily modified to account for different shell orientation vectors at individual nodes by adjust the component diagonal $\mathbf{T}_a^{(k)}$ sub-matrices accordingly.

Isoparametric mapping of element geometry and pseudo displacement fields is performed with the use of quadratic Lagrangian shape functions in the natural coordinate system (ξ, η, ζ) :

$$\phi_i(\xi, \eta) = \frac{(\xi - \xi'_i)(\xi - \xi''_i)}{(\xi_i - \xi'_i)(\xi_i - \xi''_i)} \cdot \frac{(\eta - \eta'_i)(\eta - \eta''_i)}{(\eta_i - \eta'_i)(\eta_i - \eta''_i)} \quad (i = 1 \rightarrow 9) \quad (16)$$

in which $(\xi_i \neq \xi'_i \neq \xi''_i) = -1, 0, 1$; $(\eta_i \neq \eta'_i \neq \eta''_i) = -1, 0, 1$; (ξ_i, η_i) represent the natural coordinates of node i . With the mapped pseudo displacement fields, the generalised strains of each layer (k) are calculated via (9).

For isotropic materials, the material stresses are obtained from the following equations:

$$\boldsymbol{\sigma}_p^{(k)} = \mathbf{C}_p^{(k)} \boldsymbol{\epsilon}_p^{(k)}, \quad \boldsymbol{\sigma}_{s,AS}^{(k)} = \mathbf{C}_s^{(k)} \boldsymbol{\epsilon}_{s,AS}^{(k)} \quad (17)$$

where $\boldsymbol{\epsilon}_p^{(k)}$ are planar material strains of layer (k) , given as:

$$\boldsymbol{\epsilon}_p^{(k)} = \boldsymbol{\epsilon}_m^{(k)} + \frac{h_k}{2} \zeta^{(k)} \boldsymbol{\epsilon}_b^{(k)}, \quad \zeta^{(k)} = \frac{2(z - z^{(k)})}{h_k} \quad z \in [h_{k-}, h_{k+}] \quad (18)$$

$\boldsymbol{\epsilon}_{s,AS}^{(k)}$ represents the assumed transverse shear strains as presented in Section 3.3; $\mathbf{C}_p^{(k)}$ and $\mathbf{C}_s^{(k)}$ are material constitutive matrices for planar and transverse shear stresses/strains of layer (k) , which are given as:

$$\mathbf{C}_p^{(k)} = \frac{E^{(k)}}{1 - \nu^{(k)2}} \begin{bmatrix} 1 & \nu^{(k)} & 0 \\ \nu^{(k)} & 1 & 0 \\ 0 & 0 & \frac{1 - \nu^{(k)}}{2} \end{bmatrix}, \quad \mathbf{C}_s^{(k)} = \frac{E^{(k)}}{2(1 + \nu^{(k)})} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (19)$$

with $E^{(k)}$ and $\nu^{(k)}$ representing respectively the Young's modulus and Poisson's ratio of layer (k) . Note that although only isotropic materials are considered in this paper, other material models may also be used.

Local resistance forces of the laminated shell element are obtained from the internal virtual work of the element, which is expressed as:

$$\delta \mathbf{U}_C^T \mathbf{f}_C + \delta \mathbf{U}_A^T \mathbf{f}_A = \sum_{k=1}^{N_l} \int_{\Omega^e} \left(\int_{h_{k-}}^{h_{k+}} \left(\delta \boldsymbol{\epsilon}_p^{(k)T} \boldsymbol{\sigma}_p^{(k)} + \delta \boldsymbol{\epsilon}_{s,AS}^{(k)T} \boldsymbol{\sigma}_{s,AS}^{(k)} \right) dz \right) d\Omega^e \quad \forall (\delta \mathbf{U}_C, \delta \mathbf{U}_A) \quad (20)$$

where integration is performed over the local element domain Ω^e ; $\mathbf{f}_C$ and $\mathbf{f}_A$ are resistance forces with respect to basic parameters $\mathbf{U}_C$ and additional parameters $\mathbf{U}_A$, respectively.

By defining the generalised membrane, bending, and transverse shear stresses as follows:

$$\mathbf{F}^{(k)} = \int_{h_{k-}}^{h_{k+}} \boldsymbol{\sigma}_p^{(k)} dz = h_k \mathbf{C}_p^{(k)} \boldsymbol{\epsilon}_m^{(k)} \quad (21)$$

$$\mathbf{M}^{(k)} = \int_{h_{k-}}^{h_{k+}} \boldsymbol{\sigma}_p^{(k)} (t - t^{(k)}) dz = \frac{1}{12} h_k^3 \mathbf{C}_p^{(k)} \boldsymbol{\epsilon}_b^{(k)} \quad (22)$$

$$\mathbf{Q}_{AS}^{(k)} = \int_{h_{k-}}^{h_{k+}} \boldsymbol{\sigma}_{s,AS}^{(k)} dz = \omega^{(k)} h_k \mathbf{C}_s^{(k)} \boldsymbol{\epsilon}_s^{(k)} \quad (23)$$

Equation (20) is expressed in the following form:

$$\delta \mathbf{U}_C^T \mathbf{f}_C + \delta \mathbf{U}_A^T \mathbf{f}_A = \sum_{k=1}^{N_l} \int_{\Omega^e} \left(\delta \boldsymbol{\epsilon}_m^{(k)T} \mathbf{F}^{(k)} + \delta \boldsymbol{\epsilon}_b^{(k)T} \mathbf{M}^{(k)} + \delta \boldsymbol{\epsilon}_s^{(k)T} \mathbf{Q}_{AS}^{(k)} \right) d\Omega^e \quad \forall (\delta \mathbf{U}_C, \delta \mathbf{U}_A) \quad (24)$$

Equation (24) can be further manipulated to:

$$\begin{aligned} \delta \mathbf{U}_C^T \mathbf{f}_C + \delta \mathbf{U}_A^T \mathbf{f}_A = \\ \sum_{k=1}^{N_l} \left(\delta \mathbf{U}_C^T \mathbf{T}_C^{(k)T} + \delta \mathbf{U}_A^T \mathbf{T}_A^{(k)T} \right) \int_{\Omega^e} \left(\mathbf{B}_m^{(k)T} \mathbf{D}_m^{(k)} \boldsymbol{\epsilon}_m^{(k)} + \mathbf{B}_b^{(k)T} \mathbf{D}_b^{(k)} \boldsymbol{\epsilon}_b^{(k)} + \mathbf{B}_s^{(k)T} \mathbf{D}_s^{(k)} \boldsymbol{\epsilon}_s^{(k)} \right) d\Omega^e \quad (25) \\ \forall (\delta \mathbf{U}_C, \delta \mathbf{U}_A) \end{aligned}$$

where $\mathbf{B}_m^{(k)}$, $\mathbf{B}_b^{(k)}$ and $\mathbf{B}_s^{(k)}$ are the first derivatives of the generalised strains at layer (k) with respect to pseudo parameters $\mathbf{U}^{(k)}$; $\mathbf{D}_m^{(k)}$, $\mathbf{D}_b^{(k)}$, and $\mathbf{D}_s^{(k)}$ are generalised constitutive matrices at layer (k), which are expressed as:

$$\mathbf{D}_m^{(k)} = h_k \mathbf{C}_p^{(k)}, \quad \mathbf{D}_b^{(k)} = \frac{1}{12} h_k^3 \mathbf{C}_p^{(k)}, \quad \mathbf{D}_s^{(k)} = \omega^{(k)} h_k \mathbf{C}_s^{(k)} \quad (26)$$

5.3 MITC9 locking-elimination technique

In order to address the locking phenomenon in the 9-noded shell element, the MITC9 [31-32] strain-mapping approach is employed in the local formulation of each constitutive layer. Figure 10 shows the positions of the tying points for different strain components, and

the associated interpolation functions for the covariant strains can be found elsewhere [31]. In order to allow the MITC9 element to pass the patch tests, a constant Jacobian matrix is required in the strain-mapping steps [32]. Accordingly, the adopted MITC9 strain-mapping procedure can be summarised as follows:

- (i) Evaluate Green strains directly from displacement fields at a set of prescribed tying points;
- (ii) Transform the extracted Green strains into corresponding covariant strain components:

$$\boldsymbol{\varepsilon}_2 = \mathbf{J}_C^T \mathbf{E}_2 \mathbf{J}_C \quad (27)$$

where $\mathbf{E}_2$ is the Green strain tensor; $\boldsymbol{\varepsilon}_2$ is the covariant strain tensor; $\mathbf{J}_C$ is the Jacobian matrix $\mathbf{J}$ evaluated at the element centre ($\xi = 0, \eta = 0$), with $\mathbf{J}$ expressed as:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \quad (28)$$

- (iii) Interpolate covariant strain fields with the use of the extracted covariant strains:

$$\boldsymbol{\varepsilon}_{ij}^{AS}(\xi, \eta, \zeta) = \mathbf{H}_{ij} \boldsymbol{\varepsilon}_{ij,T}^{DI} \quad (29)$$

where: $i, j = (\xi, \eta, \zeta)$; the superscript ‘AS’ and ‘DI’ refers respectively to the assumed strain distribution and the distribution obtained directly from displacement fields; $\mathbf{H}_{ij}$ is a row vector of interpolation functions associated with the tying points; $\boldsymbol{\varepsilon}_{ij,T}^{DI}$ consists of the covariant strain values extracted at the tying points.

- (iv) Transform the assumed covariant strain fields back to the corresponding Green strain fields in terms of real coordinates:

$$\mathbf{E}_2^{AS} = \mathbf{J}_C^{-T} \boldsymbol{\varepsilon}_2^{AS} \mathbf{J}_C^{-1} \quad (30)$$

- (v) Replace displacement-based strains with the assumed strain distributions obtained in the element formulation.

Before employing the tying schemes to each constitutive layer, further assumptions are made that the element is thin and shallow in the local system, so that the natural coordinate axis ζ is taken to have an identical orientation to the local z-axis, and the transverse normal strain is ignored. Accordingly, the Jacobian matrix for each layer (k) is simplified to:

$$\mathbf{J}^{(k)} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & 0 \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & 0 \\ 0 & 0 & \frac{h_k}{2} \end{bmatrix} \quad (31)$$

Equation (31) enables decoupled relationships between real strains and covariant strains, with each set of the generalised real strains related to their covariant counterparts only, which are expressed thus as:

$$\begin{bmatrix} \varepsilon_{\xi\xi}^{(k)} & \varepsilon_{\xi\eta}^{(k)} \\ \varepsilon_{\xi\eta}^{(k)} & \varepsilon_{\eta\eta}^{(k)} \end{bmatrix} = \mathbf{J}_p^T \begin{bmatrix} \varepsilon_{11}^{(k)} & \frac{1}{2}\gamma_{12}^{(k)} \\ \frac{1}{2}\gamma_{12}^{(k)} & \varepsilon_{22}^{(k)} \end{bmatrix} \mathbf{J}_p \quad (32a)$$

$$\begin{bmatrix} \kappa_{\xi\xi}^{(k)} & \kappa_{\xi\eta}^{(k)} \\ \kappa_{\xi\eta}^{(k)} & \kappa_{\eta\eta}^{(k)} \end{bmatrix} = \mathbf{J}_p^T \begin{bmatrix} \kappa_{11}^{(k)} & \frac{1}{2}\kappa_{12}^{(k)} \\ \frac{1}{2}\kappa_{12}^{(k)} & \kappa_{22}^{(k)} \end{bmatrix} \mathbf{J}_p \quad (32b)$$

$$\begin{Bmatrix} \varepsilon_{\xi\zeta}^{(k)} \\ \varepsilon_{\eta\zeta}^{(k)} \end{Bmatrix} = \frac{h_k}{2} \mathbf{J}_p^T \begin{Bmatrix} \frac{1}{2}\gamma_{13}^{(k)} \\ \frac{1}{2}\gamma_{23}^{(k)} \end{Bmatrix} \quad (32c)$$

where $\mathbf{J}_p$ is a sub-matrix of the Jacobian matrix:

$$\mathbf{J}_p = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix} \quad (33)$$

By using (32a-c) and evaluating $\mathbf{J}_p$ at the element centroid, the strains at each tying point are readily transformed to the covariant strains, which are then used in mapping the assumed covariant strain fields. Once the distribution of the covariant strain fields is obtained, these

are transformed back to real assumed strains in the local co-rotational system via the use of inverse of (32a-c). After the application of the tying scheme at each constitutive layer, the conforming strains $\boldsymbol{\epsilon}_m^{(k)}$, $\boldsymbol{\epsilon}_b^{(k)}$ and $\boldsymbol{\epsilon}_s^{(k)}$, and the matrices $\mathbf{B}_m^{(k)}$, $\mathbf{B}_b^{(k)}$ and $\mathbf{B}_s^{(k)}$ in (25) are replaced with $\tilde{\boldsymbol{\epsilon}}_m^{(k)}$, $\tilde{\boldsymbol{\epsilon}}_b^{(k)}$, $\tilde{\boldsymbol{\epsilon}}_s^{(k)}$, $\tilde{\mathbf{B}}_m^{(k)}$, $\tilde{\mathbf{B}}_b^{(k)}$ and $\tilde{\mathbf{B}}_s^{(k)}$, where ‘ $\tilde{\cdot}$ ’ denotes the application of the MITC9 strain-mapping procedure at each individual layer.

Considering (25), the total resistance forces of the shell element associated with the local nodal parameters $\mathbf{U}_C$ and the additional parameters $\mathbf{U}_A$ are thus obtained as:

$$\mathbf{f}_C = \sum_{k=1}^{N_l} \left(\mathbf{T}_C^{(k)T} \mathbf{f}^{(k)} \right), \quad \mathbf{f}_A = \sum_{k=1}^{N_l} \left(\mathbf{T}_A^{(k)T} \mathbf{f}^{(k)} \right) \quad (34)$$

where $\mathbf{f}^{(k)}$ is the vector of pseudo nodal forces at layer (k), expressed as:

$$\mathbf{f}^{(k)} = \int_{\Omega^e} \left(\tilde{\mathbf{B}}_m^{(k)T} \mathbf{D}_m^{(k)} \tilde{\boldsymbol{\epsilon}}_m^{(k)} + \tilde{\mathbf{B}}_b^{(k)T} \mathbf{D}_b^{(k)} \tilde{\boldsymbol{\epsilon}}_b^{(k)} + \tilde{\mathbf{B}}_s^{(k)T} \mathbf{D}_s^{(k)} \tilde{\boldsymbol{\epsilon}}_s^{(k)} \right) d\Omega^e \quad (35)$$

Furthermore, the local tangent stiffness matrices of the element are obtained as:

$$\mathbf{k}_C = \frac{\partial \mathbf{f}_C}{\partial \mathbf{U}_C^T} = \sum_{k=1}^{N_l} \left(\mathbf{T}_C^{(k)T} \mathbf{k}^{(k)} \mathbf{T}_C^{(k)} \right), \quad \mathbf{k}_A = \frac{\partial \mathbf{f}_A}{\partial \mathbf{U}_A^T} = \sum_{k=1}^{N_l} \left(\mathbf{T}_A^{(k)T} \mathbf{k}^{(k)} \mathbf{T}_A^{(k)} \right) \quad (36a)$$

$$\mathbf{k}_{CA} = \mathbf{k}_{AC}^T = \frac{\partial \mathbf{f}_C}{\partial \mathbf{U}_A^T} = \sum_{k=1}^{N_l} \left(\mathbf{T}_C^{(k)T} \mathbf{k}^{(k)} \mathbf{T}_A^{(k)} \right) \quad (36b)$$

where $\mathbf{k}^{(k)}$ is the pseudo local stiffness of layer (k), which is expressed as:

$$\mathbf{k}^{(k)} = \int_{\Omega^e} \left(\tilde{\mathbf{B}}_m^{(k)T} \mathbf{D}_m^{(k)} \tilde{\mathbf{B}}_m^{(k)} + \tilde{\mathbf{B}}_b^{(k)T} \mathbf{D}_b^{(k)} \tilde{\mathbf{B}}_b^{(k)} + \tilde{\mathbf{B}}_s^{(k)T} \mathbf{D}_s^{(k)} \tilde{\mathbf{B}}_s^{(k)} + \frac{\partial^2 \tilde{\boldsymbol{\epsilon}}_m^{(k)T}}{\partial \mathbf{U}^{(k)} \partial \mathbf{U}^{(k)T}} \mathbf{D}_m^{(k)} \tilde{\boldsymbol{\epsilon}}_m^{(k)} \right) d\Omega^e \quad (37)$$

5.4 Co-rotational transformation of resistance forces and stiffness

In accordance with the co-rotational approach, the local resistance forces and stiffness matrices of the laminated shell element are transformed to the corresponding global system entities before assembly at the overall structural level. It is important to note that the relationship between additional parameters defined in the shell system and their counterparts in the element local system is directly considered by incorporating $(\hat{c}, \hat{s})$ into $\mathbf{T}_a^{(k)}$, as given in (13). Furthermore, the resistant force vector $\mathbf{f}_A$ and the stiffness matrix $\mathbf{k}_A$ are excluded

from the co-rotational transformations, since the associated zigzag parameters are defined in the shell system at the overall structural level.

The transformation of the resistant forces and stiffness matrices to the global coordinate system are given as:

$$\mathbf{f}_G = \mathbf{T}^T \mathbf{f}_C \quad (38)$$

$$\mathbf{k}_G = \frac{\partial \mathbf{f}_G}{\partial \mathbf{U}_G^T} = \mathbf{T}^T \mathbf{k}_C \mathbf{T} + \frac{\partial^2 \mathbf{U}_C^T}{\partial \mathbf{U}_G \partial \mathbf{U}_G^T} \mathbf{f}_C \quad (39a)$$

$$\mathbf{k}_{GA} = \mathbf{k}_{AG}^T = \frac{\partial \mathbf{f}_G}{\partial \mathbf{U}_A^T} = \mathbf{T}^T \mathbf{k}_{CA} \quad (39b)$$

in which $\mathbf{T}$ is the nodal displacement transformation matrix from global parameters $\mathbf{U}_G$ to co-rotational parameters $\mathbf{U}_C$ [20, 30], defined as:

$$\mathbf{T} = \frac{\partial \mathbf{U}_C}{\partial \mathbf{U}_G^T} \quad (40)$$

6 NUMERICAL EXAMPLES

The proposed laminated shell modelling approach for an alternating stiff/soft lay-up, instantiated for the 9-noded element and referred to as ‘LN-MITC9’ (‘N’ stands for the number of constitutive layers), has been implemented in ADAPTIC [33] v2.14.2, which is used hereafter in several verification examples to demonstrate the accuracy of the newly developed approach.

6.1 Laminated plate under bidirectional sinusoidal loading

A laminated plate, which has a length-to-width ratio $b/a = 3$ and a plate thickness h , is simply supported on all four edges and transversely loaded with a bidirectional sinusoidal pressure $p = p_0 \sin(\pi x/a) \sin(\pi y/b)$ on its top surface, as shown in Figure 11. Four scenarios are considered in this linear elastic problem to assess the accuracy of the proposed multi-layer shell element:

Scenario 1: a 3-layer, asymmetrically laminated plate;

Scenario 2: a 5-layer, asymmetrically laminated plate;

Scenario 3: a 7-layer, symmetrically laminated plate with different thicknesses of stiff sheets;

Scenario 4: 3-, 5-, 7-, 9- and 11-layer, symmetrically laminated plate with the same thicknesses for all stiff sheets.

The elasticity solution for a general laminated plate loaded with a transverse bi-directional sinusoidal loading has been given by Demasi [34-35]. Brischetto et al. [14] have presented closed form solutions with EDZ models for Scenario 1, where the zigzag effect of displacements is considered by adding Murakami's zigzag function to Taylor expansions (Appendix B). The results of EDZ models are compared with the results by using the LN-MITC9 elements. On the other hand, 'EDZ*' formulations, which are based on EDZ models but with further simplifications, are also implemented with the 9-noded co-rotational element for comparison against the LN-MITC9 elements in Scenario 2 to 4. It is important to note that three assumptions have been made which distinguish the implemented EDZ* formulations from the original EDZ models [14]. Firstly the zigzag effect is considered in planar displacements only. Secondly, to facilitate the implementation of the EDZ*-MITC9 element, Taylor expansions are approximated with a piecewise linear curve based on values at the laminar interfaces, as illustrated in Figure 12. Thirdly, the proposed transverse shear strain distribution is used for EDZ*. Nevertheless, the aim of providing EDZ* results is to demonstrate the efficiency of the zigzag modes proposed in the present work for laminates with alternating stiff/soft lay-up. In this respect, the only difference between the EDZ*-MITC9 models and the LN-MITC9 models is the employed zigzag functions, which facilitates the comparison between both sets of additional displacement variables in modelling the considered laminations. Table 1 lists the number of displacement fields for the considered lamination models.

Due to symmetry, a quarter of the plate is modelled with an 8×8 mesh of the LN-MITC9 elements, which provides a convergent solution for all four scenarios. In this model, the shell system can be obtained according to the approach illustrated in Figure 8 with the reference vector $\mathbf{n}$ aligned with the global y-axis, in which case the curvilinear shell triad maintains the same (x,y) directions for all elements. The results of the LN-MITC9 model are compared

against those of the EDZ or EDZ* models as well as the elasticity solution [34] in terms of non-dimensional displacement, stress and strain values defined as follows:

$$\bar{u}_z = \frac{100u_z E_{(C)}}{p_0 h (a/h)^4}, \quad \bar{\sigma}_x = \frac{\sigma_x}{p_0 (a/h)^2}, \quad \bar{\sigma}_{xz} = \frac{\sigma_{xz}}{p_0 (a/h)}, \quad \bar{\gamma}_{xz} = \frac{E_{(C)}}{2(1+\nu_{(C)}) p_0 (a/h)} \gamma_{xz}$$

where $E_{(C)}$ and $\nu_{(C)}$ are the Young's modulus and Poisson's ratio of soft core layers.

Scenario 1: Three-layer, asymmetrically laminated plate

The thicknesses of three layers are given as: $h_1 = h/10$, $h_2 = 7h/10$, and $h_3 = 2h/10$. All layers are made of isotropic material. The stiffness ratios between the constitutive layers are given as $E^{(1)}/E^{(3)} = 5/4$, and $SR \equiv E^{(1)}/E^{(2)} = 10, 10^5$. Poisson's ratios for all layers are 0.34. Two length-to-thickness ratios have been considered by Brischetto et al. with EDZ models [14]: $a/h=4, 100$. Hereafter, the results of the L3-MITC9 models are compared with the FSDT and EDZ models.

Table 2 gives the predictions on central deflection $\bar{u}_z(a/2, b/2)$ at the bottom of the upper sheet with the considered models. Clearly, the L3-MITC9 model provides a much closer estimation of deflection than the EDZ1 model and even better results than the EDZ4 predictions, which utilise more displacement fields as indicated in Table 1, except where $a/h=4$ and $SR = 10^5$, in which case the transverse elastic deformation for such a thick plate with very soft core is too significant to be neglected. Since the proposed model is intended for analysis of thin-to-moderately thick plates and shells, the neglect of the through-thickness variation in the transverse displacement still yields good results within the scope of interest.

The through-thickness variations of the planar stress $\bar{\sigma}_x(a/2, b/2)$ for the cases where $a/h=100$ (thin plate) and $SR = 10, 10^5$ are depicted in Figure 13, which highlight the accuracy of the L3-MITC9 model for a wide range of SR values. The noticeable deviation of the EDZ1 curve in Figure 13.b implies the inaccuracy of Murakami's function in capturing the zigzag effect. This deviation is alleviated with the use of higher-order EDZ models.

Figure 14 shows the through-thickness variations of the transverse shear stress $\bar{\sigma}_{xz}(0, b/2)$ for the cases where $a/h=4$ (thick plate) and $SR = 10, 10^5$. Clearly, the continuous transverse shear stress predicted by the EDZ4 model posts a close approximation of the elasticity

solution. On the other hand, the L3-MITC9 model, which assumes a piecewise linear-constant-linear transverse shear strain pattern, provides an accurate prediction of transverse shear stresses in the core, though discrepancies arise in the face sheets. With further manipulation, the through-thickness variation of the transverse shear strain $\bar{\gamma}_{xz}(0, b/2)$ for the case $a/h=4$ and $SR=10$ can be obtained for each model, as depicted in Figure 15. Clearly, the transverse shear strains in the face sheets are much smaller than the strain in the soft layer, which indicates negligible influence of the stiff layers on the overall transverse shear strain energy.

Scenario 2: Five-layer, asymmetrically laminated plate

The layer thicknesses and materials of the five-layer laminated panel are given in Table 3, with $E_{(B)}/E_{(A)} = 4/5$, and $\nu_{(A)} = \nu_{(B)} = \nu_{(C)} = 0.34$.

Different span-to-thickness ratios ($a/h=10,100$) and stiffness ratios ($SR \equiv E_{(A)}/E_{(C)} = 10, 10^3, 10^5$) are considered to investigate the performance of the shell element. Estimations of the non-dimensional central deflection $\bar{u}_z(a/2, b/2)$ at the bottom of the top layer with the L5-MITC9 and EDZ2*-MITC9 models, which have the same number of additional displacement variables, are listed in Table 4, compared against the elasticity solution. A FSDT solution is also available by restraining all the additional DOFs of the L5-MITC9 model in the analysis, although the assumed distribution of transverse shear strains is employed. Clearly, both models provide comparable accuracy for a relatively small $SR=10$. However, the zigzag effect becomes significant as the SR increases, evident from the resulting large relative error of FSDT-MITC9 results. Although the EDZ2*-MITC9 model improves the FSDT-MITC9 results somewhat, still significant inaccuracy remains, hence requiring higher-order Taylor expansions for better accuracy. On the other hand, the L5-MITC9 model maintains high accuracy with a wide range of SR s owing to the efficiency of the selected zigzag displacement modes for the analysis of such laminations.

The through-thickness distribution of the non-dimensional planar stress $\bar{\sigma}_x$ at the plate centre ($a/2, b/2$) for the cases where $a/h=100$ (thin plate) and $SR=10, 10^3$ with the L5-MITC9 model is depicted in Figure 16, and the elasticity results using Demasi's solution [34]

is also depicted for comparison. The figure highlights the accuracy of the L5-MITC9 model for a wide range of the SR values. Figure 17 depicts the non-dimensional transverse shear strain $\bar{\gamma}_{xz}$ of the L5-MITC9 model at location $(0, b/2)$, compared against the elasticity results. The results also indicate the adequacy of the proposed transverse shear strain distribution for the problem considered.

Scenario 3: Seven-layer, symmetrically laminated plate with thicker external layers

In this scenario, the effect of different thicknesses on the accuracy of the zigzag functions is investigated. The layer thicknesses and materials of the seven-layer laminated plate are given in Table 5, with $SR \equiv E_{(F)}/E_{(C)}$, and $\nu_{(F)} = \nu_{(C)} = 0.34$. Different span-to-thickness ratios ($a/h=10, 100$) and stiffness ratios ($SR = 10, 10^3, 10^5$) are considered to investigate the performance of the shell element.

Estimations of the non-dimensional central deflection $\bar{u}_z(a/2, b/2)$ at the bottom of the top layer with various models are listed in Table 6, compared against the elasticity solution. Similar to Scenario 2, the L7-MITC9 and EDZ3*-MITC9 models provide comparable accuracy for a relatively small $SR=10$. As the stiffness mismatch becomes very significant, the L7-MITC9 model shows better accuracy than the EDZ3*-MITC9 model, which have the same number of zigzag displacement fields.

The through-thickness distribution of the non-dimensional $\bar{\sigma}_x(a/2, b/2)$ and $\bar{\gamma}_{xz}(0, b/2)$ for the L7-MITC9 model are depicted in Figures 18 and 19, respectively, compared against the elasticity results using Demasi's solution [34], where the comparison confirms the accuracy of the L7-MITC9 model for a wide range of SR.

Scenario 4: Symmetrically laminated plate with same face sheets and core sheets

In this scenario, the laminated plate is composed of the same face sheets and the same cores in an (F/C/F/.../F/C/F) lay-up. The material properties of the face sheet and the core are given as $E_{(F)}/E_{(C)} = 1000$, and $\nu_{(F)} = \nu_{(C)} = 0.34$, while the span-to-thickness ratio is ($a/h=10$). This problem is analysed for the cases of 5, 7, 9 and 11 layers, in all of which the plate thickness remains h . Two stiff-to-soft layer thickness ratios ($h_{(F)}/h_{(C)} = 1/5, 5$) are considered to investigate the performance of the shell models. The estimated non-

dimensional central deflection $\bar{u}_z(a/2, b/2)$ at the bottom of the top layer with LN-MITC9 and EDZ*-MITC9 models for different lay-ups are given in Table 7, compared with the elasticity solution. It is clear that as the number of layers increases, the LN-MITC9 model provides better accuracy than the EDZ*-MITC9 model, which verifies the efficiency of the proposed zigzag modes in the analysis of laminations with an alternating stiff/soft lay-up. Note that the L3-MITC9 and the EDZ1* models for a 3-layer case are identical, and therefore not presented, due to the fact that the proposed zigzag function for a 3-layer model becomes identical to Murakami's zigzag function when the two external layers are of identical thickness.

6.2 Circular laminated plate under uniform pressure

The geometrically nonlinear response of an 11-layer circular plate is considered here, where the plate is fully clamped along its edge and is subjected to a uniformly distributed transverse loading p , as shown in Figure 20. The geometric parameters are given as $R = 20$ and $h = 1.0$. The laminated plate is composed of the same stiff sheets, denoted by 'F', and the same soft sheets, denoted by 'C', with an alternating stiff/soft lay-up. Two stiff-to-soft layer thickness ratios ($h_{(F)}/h_{(C)} = 1/5, 5$), denoted as scheme (1) and (2) respectively, are considered to investigate the performance of the shell models. The material parameters are given as $E_{(F)} = 1.0 \times 10^7$, $E_{(C)} = 3750$, and $\nu_{(F)} = \nu_{(C)} = 0.25$. Due to symmetry, a quarter of the circular plate is modelled with a mesh of 9-noded laminated shell elements, which provides a convergent solution. The mesh is depicted in Figure 21, where the quarter model is divided into three sections, with each section discretised into 6×6 L11-MITC9 elements. In this model, the shell system is aligned with the (x, y) planar coordinate system. By restraining all the additional DOFs in the model, a FSDT solution is also available. The reference solution is obtained from a 3D continuum model using a fine mesh of standard 20-noded quadratic brick elements [36], denoted as 'BK20', where on the planar surface each of the three sections is meshed with 24×24 BK20 elements, and two elements are employed for each constitutive layer. The load-deflection curves at the plate centre, point O, obtained with various models are depicted in Figure 22. Clearly, the disparity of the FSDT-MITC9 results

from the others indicates the significance of the zigzag effect. The L11-MITC9 model matches well with the solid model, confirming its high accuracy for both lay-up schemes.

6.3 Multi-layer hemispherical shell with 18° cut-off

Another large displacement problem is considered here, where a five-layer hemispherical shell with a circular cut-off at its top is subjected to symmetric concentrated forces at its base, as shown in Figure 23. The hole aperture is 18°, the sphere radius is 10, and the shell thickness is $h=0.075$. Three alternative lay-ups are considered to investigate the performance of the multi-layer shell element in problems involving both symmetric and asymmetric cross-sections. The layer material type and thickness for each scenario are listed in Table 8, where layer (1) corresponds to the interior layer of the hemispherical shell. The material parameters for the stiff layers (F) and soft layers (C) are given as: $E_{(F)} = 1.0 \times 10^7$, $E_{(C)} = 5.0 \times 10^3$, and $\nu_{(F)} = \nu_{(C)} = 0.2$.

Note that in this model, the shell system is defined to follow the longitudinal and latitudinal lines of the sphere, which can be easily realised with the use of the approach illustrated in Figure 8 by aligning the reference vector $\mathbf{n}$ with the axis of symmetry. Due to the curved geometry, the curvilinear shell triads vary in orientation between adjacent elements, though any inaccuracy diminishes with mesh refinement, as confirmed in the presented results. Due to symmetry, only a quarter of the hemispherical shell is modelled with a 16×16 mesh of the L5-MITC9 and EDZ2*-MITC9 elements, both of which provide convergent solutions. By restraining all additional zigzag parameters, the corresponding FSDT-MITC9 results are also available. The results of a $64 \times 64 \times 10$ solid model using the BK20 element are utilised for a reference solution. Figures 24-26 depict the equilibrium paths of the radial displacements at points A and B for the respective lay-up schemes. The deviation of the FSDT results from the reference solution indicates the significance of the zigzag effect for this problem. It is interesting to note that the EDZ2*-MITC9 results agree with the reference solution for lay-up (1) but is as inaccurate as the FSDT-MITC9 solution for the other two lay-up scenarios. This is attributed to the inadequacy of Murakami's zigzag function in capturing the real zigzag mode for the last two lay-ups, hence requiring higher-

order Taylor expansions with more additional displacement variables for improved estimation. On the other hand, the results of the L5-MITC9 shell element present an excellent match against the results obtained from the 3D continuum solid model for all of the considered lay-up schemes.

7 CONCLUSIONS

This paper presents a multi-layer shell modelling approach for the nonlinear analysis of thin to moderately thick laminated plates and shells with an alternating stiff/soft lay-up, which is instantiated with a 9-noded co-rotational shell element for large displacement analysis of laminated shell structures.

The laminations of interest are widely employed in engineering, including for example laminated glass and polymer-metal composites, and are characterised by a significantly large stiffness mismatch through the laminate thickness. Under transverse loading, the stiff sheets have almost identical rotations of the normal. Furthermore, the transverse shear strain exhibits a near constant through-thickness distribution in each of the soft layers, while it is much smaller in the stiff layers. Based on these distinct characteristics, a simple yet effective lamination model is proposed, in which the zigzag effect of planar displacements is taken into account by adding to the Reissner-Mindlin formulation a set of zigzag fields, the number of which depends on the number of soft layers. Furthermore, a piecewise linear distribution of the transverse shear strain, approximating the observed pattern of strain distribution, is assumed in the thickness direction, which leads to an effective representation of the real distribution without imposing stress constraints at laminar interfaces.

Importantly, the proposed lamination model developed for shallow shell finite elements can be applied in large displacement analysis with the use of a co-rotational approach. In this respect, to eliminate the need for co-rotational transformations for the additional zigzag displacement parameters, a 2D curvilinear shell system is employed for the direct definition of these parameters, such that a simple and fixed transformation of these additional parameters to their counterparts in the local element system holds throughout the analysis.

The proposed multi-layer shell modelling approach has been implemented for a 9-noded co-rotational shell element, which employs a bisector definition of the co-rotational system, and utilises the MITC9 strain mapping approach in the local system to address locking. Linear and geometrically nonlinear numerical examples are finally solved with the proposed multi-layer shell formulation, where excellent accuracy is generally achieved in comparison with elasticity solutions, and superior performance is typically demonstrated compared to existing models for laminated shells with alternating stiff/soft lay-ups.

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APPENDIX A: GLOBAL-TO-LOCAL DISPLACEMENT TRANSFORMATIONS

A bisector definition of the co-rotational system, previously proposed for quadrilateral shell elements with varying orders and types with the use of four corner nodal coordinates only [20], is adopted. The triad $(\mathbf{c}_x, \mathbf{c}_y, \mathbf{c}_z)$, which describes the current orientation of the local coordinate system relative to the global system, is simply obtained as follows [20]:

$$\mathbf{c}_x = \frac{\mathbf{c}_{13} - \mathbf{c}_{24}}{|\mathbf{c}_{13} - \mathbf{c}_{24}|}, \quad \mathbf{c}_y = \frac{\mathbf{c}_{13} + \mathbf{c}_{24}}{|\mathbf{c}_{13} + \mathbf{c}_{24}|}, \quad \mathbf{c}_z = \mathbf{c}_x \times \mathbf{c}_y \quad (\text{A1})$$

with:

$$\mathbf{c}_{ij} = \frac{\mathbf{v}_{ij}}{|\mathbf{v}_{ij}|}, \quad \mathbf{v}_{ij} = \mathbf{v}_{ij}^o + \mathbf{d}_j - \mathbf{d}_i \quad (\text{A2})$$

where $\mathbf{v}_{ij}^o$ is the vector connecting node i to node j in the initial element configuration, and $\mathbf{d}_i = \langle \mathbf{u}_{X,i}, \mathbf{u}_{Y,i}, \mathbf{u}_{Z,i} \rangle^T$ represents the global translational displacements of node i .

The transformation between global and local translational displacements can be expressed as [30]:

$$\mathbf{t}_i = \mathbf{R} \mathbf{d}_i + (\mathbf{R} - \mathbf{R}^o) \mathbf{v}_{io}^o \quad (i = 1 \rightarrow 9) \quad (\text{A3})$$

where $\mathbf{t}_i = \langle \mathbf{u}_{x0,i}, \mathbf{u}_{y0,i}, \mathbf{u}_{z0,i} \rangle^T$ represents the local translational displacements of node i ; $\mathbf{R}^o$ and $\mathbf{R}$ are the orientation matrices of the local co-rotational system in the initial and current configurations, respectively, defined as:

$$\mathbf{R}^o = [\mathbf{c}_x^o \quad \mathbf{c}_y^o \quad \mathbf{c}_z^o]^T, \quad \mathbf{R} = [\mathbf{c}_x \quad \mathbf{c}_y \quad \mathbf{c}_z]^T \quad (\text{A4})$$

and $\mathbf{v}_{io}^o$ is a vector describing the initial position of node i relative to a reference point o , which is selected at the internal node for the 9-noded shell element. It should be noted that choice of the reference point does not impact on accuracy, since a change in location of o merely results in the addition of an increment of rigid body translations to which the local element formulation should be insensitive.

The transformation between global and local rotations is [20,30]:

$$\bar{\mathbf{r}}_i = \bar{\mathbf{R}} \mathbf{n}_i, \quad \bar{\mathbf{R}} = [\mathbf{c}_x \quad \mathbf{c}_y]^T \quad (i = 1 \rightarrow 9) \quad (\text{A5})$$

where $\bar{\mathbf{r}}_i = \langle \theta_{x,i}, \theta_{y,i} \rangle^T$ represents the two rotations at node i in the local x-z and y-z planes, respectively; $\mathbf{n}_i$ is the normal vector at node i .

APPENDIX B: MURAKAMI'S ZIGZAG FUNCTION WITH TAYLOR EXPANSION

Murakami's zigzag function [11] is given as:

$$f(z) = (-1)^k \zeta_k, \quad \zeta_k = z_k / 2h_k \quad \zeta_k \in [-1, 1] \quad (B1)$$

The refinement of the classical theories with Murakami's zigzag function gives [13]:

$$f_{1M}(z) = c^0 + c^1 z + c^M f(z) \quad (B2)$$

The inclusion of Murakami's zigzag function to higher-order Taylor expansions gives [13]:

$$f_{NM}(z) = c^0 + c^1 z + c^2 z^2 + \dots + c^{N-1} z^{N-1} + c^N z^N + c^M f(z) \quad (B3)$$

Table 1: Number of displacement fields for the considered models.

Model	Number of displacement fields
FSDT-MITC9	5
L3-MITC9	7
L5-MITC9	9
L7-MITC9	11
L9-MITC9	13
L11-MITC9	15
EDZ2*-MITC9	9
EDZ3*-MITC9	11
EDZ4*-MITC9	13
EDZ5*-MITC9	15
EDZ1 [14]	9
EDZ4 [14]	18
EDZ7 [14]	27

Table 2: Non-dimensional deflection of 3-layer plate with varying SR and (a/h).

a/h	Model	SR			
		10		10^5	
		$\bar{u}_z$	Relative error	$\bar{u}_z$	Relative error
4	Elasticity [34]	3.01123	-	0.013159	-
	L3-MITC9	2.98319	0.93%	0.011907	9.51%
	FSDT	1.58218	47.46%	0.000180	98.63%
	EDZ1 [14]	2.34412	22.15%	0.000837	93.64%
	EDZ4 [14]	2.97886	1.07%	0.012629	4.03%
	EDZ7 [14]	2.99670	0.48%	0.013136	0.17%
100	Elasticity [34]	1.51021	-	0.002089	-
	L3-MITC9	1.51027	0.00%	0.002089	0.01%
	FSDT	1.10845	26.60%	0.000120	94.26%
	EDZ1 [14]	1.15866	23.28%	0.000163	92.18%
	EDZ4 [14]	1.51017	0.00%	0.001163	44.34%
	EDZ7 [14]	1.51019	0.00%	0.002021	3.30%

Table 3: Layer thicknesses and materials for 5-layer plate.

Layer index	Layer thickness	Layer material
1	$(2/14)h$	(B)
2	$(5/14)h$	(C)
3	$(1/14)h$	(A)
4	$(5/14)h$	(C)
5	$(1/14)h$	(A)

Table 4: Non-dimensional deflection of 5-layer plate with varying SR and (a/h).

a/h	Model	SR					
		10		10^3		10^5	
		$\bar{u}_z$	Relative error	$\bar{u}_z$	Relative error	$\bar{u}_z$	Relative error
10	Elasticity [34]	2.02302	-	0.22248	-	0.02572	-
	L5-MITC9	2.03898	0.79%	0.22096	0.68%	0.02548	0.94%
	EDZ2*-MITC9	2.02446	0.07%	0.06702	69.88%	0.00080	96.88%
	FSDT-MITC9	1.90176	5.99%	0.02153	90.32%	0.00022	99.16%
100	Elasticity [34]	1.81666	-	0.02252	-	0.00220	-
	L5-MITC9	1.81738	0.04%	0.02253	0.05%	0.00221	0.38%
	EDZ2*-MITC9	1.81724	0.03%	0.02086	7.39%	0.00021	90.45%
	FSDT-MITC9	1.81601	0.04%	0.02039	9.48%	0.00020	90.73%

Table 5: Layer thicknesses and materials for 7-layer plate.

Layer index	Layer thickness	Layer material
1	$(2/21)h$	(F)
2	$(5/21)h$	(C)
3	$(1/21)h$	(F)
4	$(5/21)h$	(C)
5	$(1/21)h$	(F)
6	$(5/21)h$	(C)
7	$(2/21)h$	(F)

Table 6: Non-dimensional deflection of 7-layer plate with varying SR and (a/h).

a/h	Model	SR					
		10		10^3		10^5	
		$\bar{u}_z$	Relative error	$\bar{u}_z$	Relative error	$\bar{u}_z$	Relative error
10	Elasticity [34]	1.8341	-	0.22225	-	0.03861	-
	L7-MITC9	1.8448	0.59%	0.22083	0.64%	0.03744	3.02%
	EDZ3*-MITC9	1.8452	0.61%	0.21799	1.91%	0.02760	28.51%
	FSDT-MITC9	1.7020	7.20%	0.01903	91.44%	0.00019	99.51%
100	Elasticity [34]	1.6245	-	0.02001	-	0.00220	-
	L7-MITC9	1.6253	0.05%	0.02002	0.06%	0.00221	0.58%
	EDZ3*-MITC9	1.6253	0.05%	0.02002	0.05%	0.00218	0.72%
	FSDT-MITC9	1.6238	0.04%	0.01789	10.57%	0.00018	91.84%

Table 7: Non-dimensional deflection of multi-layer plate with varying number of layers.

Number of layers	Model	Lay-up scheme (1)		Lay-up scheme (2)	
		$\bar{u}_z$	Relative error	$\bar{u}_z$	Relative error
5	Elasticity [34]	0.23578	-	0.04313	-
	L5-MITC9	0.23428	0.64%	0.04298	0.35%
	EDZ2*-MITC9	0.23428	0.64%	0.04298	0.35%
7	Elasticity [34]	0.24237	-	0.04884	-
	L7-MITC9	0.24090	0.61%	0.04867	0.35%
	EDZ3*-MITC9	0.23915	1.33%	0.04835	1.00%
9	Elasticity [34]	0.24746	-	0.05177	-
	L9-MITC9	0.24594	0.62%	0.05153	0.46%
	EDZ4*-MITC9	0.24191	2.24%	0.05075	1.98%
11	Elasticity [34]	0.25166	-	0.05341	-
	L11-MITC9	0.25014	0.61%	0.05315	0.48%
	EDZ5*-MITC9	0.24389	3.09%	0.05189	2.85%

Table 8: Lay-ups for 5-layer hemispherical shell.

Layer index	Layer material	Layer thickness		
		Scheme (1)	Scheme (2)	Scheme (3)
1	(F)	(1/5)h	(1/15)h	(3/25)h
2	(C)	(1/5)h	(2/15)h	(5/25)h
3	(F)	(1/5)h	(3/15)h	(9/25)h
4	(C)	(1/5)h	(4/15)h	(5/25)h
5	(F)	(1/5)h	(5/15)h	(3/25)h

Figure 1: Through thickness distribution of transverse shear stress/strain with various SRs (3 layers).

a. transverse shear stress

b transverse shear strain

Figure 2: Through thickness distribution of transverse shear stress/strain with various SRs (5 layers).

a. transverse shear stress

b. transverse shear strain

Figure 3: Through thickness distribution of transverse shear stress/strain with various SRs (7 layers).

Figure 4: Laminated plate and local coordinate system.

Figure 5: Zigzag modes for a 5-layer lamination with alternating stiff/soft lay-up.

Figure 6: 2D curvilinear shell coordinate system.

Figure 7: Relative orientation between the local element and shell systems.

Figure 8: An alternative definition of shell coordinate system using a reference vector.

Figure 9: Definition of the bisector co-rotational framework.

Figure 10: Positions of tying points for MITC9 element ($a = 1/\sqrt{3}$, $b = \sqrt{3/5}$, and $c = 1$).

Figure 11: Laminated plate under bidirectional sinusoidal loading.

Figure 12: Piecewise approximation of Taylor expansion.

a. $SR=10$, and $a/h=100$

b. $SR=10^5$, and $a/h=100$

Figure 13: Through-thickness distribution of non-dimensional in-plane stress $\bar{\sigma}_x$ for three-layer plate.

a. $SR=10$, and $a/h=4$

b. $SR=10^5$, and $a/h=4$

Figure 14: Through-thickness distribution of non-dimensional transverse shear stress $\bar{\sigma}_{xz}$ for three-layer plate.

Figure 15: Through-thickness distribution of non-dimensional transverse shear strain $\bar{\gamma}_{xz}$ for three-layer plate (SR=10, and $a/h=4$).

a. SR=10, and a/h=100

b. SR=10³, and a/h=100

Figure 16: Through-thickness distribution of non-dimensional planar stress $\bar{\sigma}_x$ for five-layered plate.

a. $SR=10$, and $a/h=10$

b. $SR=10^3$, and $a/h=10$

Figure 17: Through-thickness distribution of non-dimensional transverse shear strain $\bar{\gamma}_{xz}$ for five-layered plate.

a. $SR=10$, and $a/h=100$

b. $SR=10^3$, and $a/h=100$

Figure 18: Through-thickness distribution of non-dimensional planar stress $\bar{\sigma}_x$ for seven-layered plate.

a. $SR=10$, and $a/h=10$

b. $SR=10^3$, and $a/h=10$

Figure 19: Through-thickness distribution of non-dimensional transverse shear strain $\bar{\gamma}_{xz}$ for seven-layered plate.

Figure 20: Clamped circular laminated panel under uniform loading.

Figure 21: Mesh pattern for a 6×6 mesh of 9-noded laminated shell elements

Figure 22: Dimensionless load-deflection curves at point O of various models.

Figure 23: Pinched laminated hemispherical shell with a 18° hole.

Figure 24: Load-deflection curves at point A and B of various models with lay-up scheme (1).

Figure 25: Load-deflection curves at point A and B of various models with lay-up scheme (2).

Figure 26: Load-deflection curves at point A and B of various models with lay-up scheme (3).