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Empirical correlation between inelastic and elastic spectral displacement demands

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Inelastic spectral displacement demand is arguably one of the most effective, simplified means of relating earthquake intensity to building damage. However, seismic hazard assessment is typically conducted using empirical ground-motion prediction equations (GMPEs) that only provide indications of elastic spectral response quantities, which an engineer subsequently relates to inelastic demands using empirical relationships such as the equal-displacement rule. An alternative approach is to utilize relationships for the inelastic spectral displacement demand directly within the seismic hazard assessment process. Such empirical relationships are developed in this work, as a function of magnitude, distance, building period and yield strength coefficient, for four different hysteretic models that are representative of a wide range of possible structural typologies found in practice. The new relationships are likely to be particularly useful for performance-based seismic design and assessment.

INTRODUCTION

Earthquake shaking can generate a range of different types of damage in a building, to both structural and non-structural elements. Some types of damage, such as the out-of-plane failure of masonry walls or shear failure of reinforced concrete (RC) walls, is brittle in nature and is, arguably, best related to accelerations or internal force demands. However, damage to most types of structural elements and a large number of non-structural elements, is best gauged via deformation demands. This can be appreciated considering damage states within the idealized force-displacement response of a building, shown in Figure 1. It is apparent that from the onset of damage, to a state in which flexural plastic hinges within RC columns have deformed beyond repair and drift demands on partitions and glazing have exceeded their capacity, the displacement demand increases significantly whereas the forces (and therefore accelerations) remain relatively constant. This reinforces the point of Priestley et al. (2007), amongst others, that inelastic displacement demands should form the basis of design and assessment methods.

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Figure 1. Schematic illustration of possible damage states within the force-displacement response of a RC frame building.

Within current seismic design and assessment procedures, hazard is typically represented using elastic response spectra and inelastic displacement demands are established by using simplified relationships, such as the equal displacement rule (Veletsos and Newmark, 1960), in order to correlate the inelastic spectral response with the elastic demand at the estimated period of the structure. Numerous proposals have been made over the years (Iwan 1980, Vidic et al. 1994, Miranda and Ruiz-Garcia 2002, Grant et al. 2005, Pennucci et al. 2011) to relate elastic and inelastic spectral displacement demands, with modern expressions differentiating between the hysteretic properties of the system and spectral shape (Pennucci et al. 2011). However, such approaches will be of limited accuracy because the ratio of the inelastic to elastic spectral displacement displays, for example, magnitude and distance dependence, as pointed out by Miranda (2000), Ruiz-Garcia and Miranda (2003), Chopra and Chintanapakdee (2004). To address this, Tothong and Cornell (2006) presented an empirical ground-motion prediction model for inelastic (as opposed to elastic) spectral displacement, so as to directly provide site-specific information on inelastic spectral displacement demands that can be used for design or assessment. The model of Tothong and Cornell (2006) is a function of two earthquake parameters, moment magnitude (M_w) and closest distance to rupture (R_{rup}), and two bilinear oscillator parameters; an undamped elastic period (T) and a strength reduction factor (R).

While the model presented by Tothong and Cornell (2006) represents what the authors consider to be an important conceptual development, this paper develops improved models with the following features:

- Four different hysteretic models are considered, recognizing that the hysteretic force-displacement response properties of a structure can significantly affect inelastic demands (Grant et al. 2005, Priestley et al. 2007, Pennucci et al. 2011). The model of Tothong and Cornell (2006) is applicable only to systems with bi-linear hysteretic properties. In addition, this work adopts a tangent-stiffness proportional damping model for the nonlinear dynamic analyses that is considered (Priestley et al. 2007, Petrini et al. 2008) superior to an initial-stiffness proportional damping model.
- Similarly to the proposal of Tothong and Cornell (2006), the new empirical relationships are a function of M_w , R_{rup} , and T , but instead of using R , which requires knowledge of the elastic spectral displacement, the yield strength coefficient, C_y , is adopted, defined as the ratio of the lateral force at yield to the effective seismic weight of the building.
- Improved consideration is made of variance of the inelastic displacements through careful statistical treatment of the data.

By enabling the formation of GMPEs for inelastic displacement, this work will be beneficial for both the performance-based design and assessment of structures. For example, inelastic displacement demands would be useful for the Direct displacement-based seismic design approach of Priestley et al. (2007), which currently realizes inelastic displacement spectra through the use of empirical equivalent viscous damping relationships (see also Pennucci et al., 2011). Inelastic displacement demands would also be useful for seismic assessment methods such as the capacity spectrum method (Freeman, 1978; Fajfar, 1999) or the assessment version of Direct DBD (Welch et al. 2014), where again, in both cases empirical relationships are currently used to relate elastic and inelastic displacement demands at the structural analysis stage. Moreover, the findings are particularly useful for probabilistic seismic-demand analyses requiring estimates of inelastic displacements, both for individual structures, as pointed out by Tothong and Cornell (2006), as well as for portfolios of buildings within regional risk assessments.

NONLINEAR OSCILLATOR PROPERTIES

To establish new empirical relationships between the inelastic and elastic spectral displacement, a large set of nonlinear SDOF oscillators have been created, varying the following characteristics:

- Initial period of vibration: $T = 0.1s, 0.2s, 0.3s, 0.4s, 0.5s, 0.6s, 0.8s, 1.0s, 1.5s, 2.0s, 2.5s, 3.0s, \text{ and } 4.0s$.

- Yield strength coefficient (equal to the lateral yield strength divided by the seismic weight): $C_y = 2.5\%, 5\%, 7.5\%, 10\%, 12.5\%, 15\%, 20\%, 30\%, 40\%$ and 50% .
- Hysteretic model: Bi-linear, Takeda, SINA, and Flag-shaped (see Figure 2 and Table 1).

The equivalent viscous damping for each model was set as $\xi=5\%$ and was modeled with a tangent-stiffness proportional model for reasons provided by Priestley et al. (2007) and Petrini et al. (2008). As such, a total of 560 models have been created and each subject to 4812 ground motions, the details of which are reported in the next section.

The various periods of vibration were selected considering values that would be representative of typical building structures and also to ensure that both short and long period ranges of the response spectrum were examined. The yield strength coefficients vary from a low value of 2.5% , that would not be accepted in most modern code designs (such as the Eurocode 8, CEN 2004 which limits the minimum base shear coefficient to 5.0%) but is considered representative of some medium-rise pre-code buildings, up to a value of 50% which could be considered representative of well-designed short-period structures expected to possess limited ductility capacity. The motives for the selected hysteretic models and discussed shortly.

The period of vibration, T , and equivalent viscous damping coefficient, ξ , are the only parameters required to quantify the dynamic response of a single-degree of freedom (SDOF) structure subject to earthquake shaking, provided the response remains within the elastic range. Under intense earthquake shaking, however, the yield strength, F_y , of a structure will be exceeded and evaluation of the dynamic response also requires knowledge of the full force-displacement hysteretic response. The hysteretic response of a structure can be affected by many factors, including material typology, the yielding mechanism, construction detailing, and internal force levels. Figure 2 illustrates four different hysteretic models that are considered in this work. The hysteretic models have been selected to enable consideration of a wide range of structural typologies as explained subsequently.

Figure 2. Force-displacement hysteretic models considered in this work; (a) Bi-linear, (b) Takeda, (c) SINA, and (d) Flag-shaped models.

The first of the models depicted in Figure 2 is the bi-linear hysteretic model and is characterized by an initial stiffness, k_0 , yield force, F_y , and post-yield stiffness ratio, r , noting that the unloading stiffness of the bi-linear model is taken equal to the initial stiffness. The bi-linear model could be considered representative of the hysteretic behaviour of steel structures or structures base-isolated on friction pendulum devices or lead-rubber bearings. The second of the models shown is the Takeda model (Otani, 1981), and in addition to the parameters k_0 , F_y , and r , common to the bi-linear model, it requires definition of an unloading stiffness parameter, α , and a re-loading parameter, β (according to the Emori-Schnobrich, 1978 approach). The Takeda model is commonly used to represent well-detailed RC building structures and bridges (Priestley et al. 2007). The third model shown in Figure 2 is the SINA model (Saiidi and Sozen, 1979) that has been simplified in this work to possess a bi-linear (instead of trilinear) backbone force-displacement curve and symmetric positive and negative loading behavior. The SINA model is characterized by significant stiffness degradation with increasing deformation demand, exhibiting a “pinched” type of hysteretic loop. The model again requires specification of

parameters that are common to the bi-linear model, but with the amount of pinching controlled through the definition of a force F_{cc} (see Figure 2c). This third model can be considered representative of poorly detailed structures, such as older RC structures constructed with smooth reinforcing bars (Braga et al. 2009) or possessing members subject to high axial load ratios. The final hysteretic model considered in this work is the Flag-shaped model which loads up along a bi-linear force-displacement curve and then unloads with the same stiffness as loading until the lateral force reduces by an amount $\beta_f F_y$, see Figure 2(d), after which point the system re-centers. The flag-shaped model is commonly used to represent modern post-tensioned pre-cast concrete or timber structures with self-centering properties (e.g. Pampanin et al. 2001, Cebellos and Sullivan 2012) and can generally be considered representative of rocking systems that possess some energy dissipation.

With knowledge of the period, T , or angular frequency, ω_0 , of a structure and its yield strength coefficient, C_y , then for any nonlinear oscillator of mass, m , compatible values of yield strength, F_y , and initial stiffness, k_0 , are defined by Equations (1) and (2).

$$F_y = C_y mg \quad (1)$$

$$k_0 = 4\pi^2 \frac{m}{T^2} = \omega_0^2 m \quad (2)$$

where g is the acceleration due to gravity. These equations therefore provide two of the hysteretic parameters common to all the four hysteretic models described above. The other parameters which define the hysteretic properties of the models examined in this work are shown in Table 1. The post-yield force-displacement stiffness ratio of $r = 0.05$ shown in Table 1 is a value commonly adopted in practice, considering typical strain-hardening levels and secondary load paths. The alpha and beta values shown for the Takeda model in Table 1 are those recommended by Priestley et al. (2007) for RC wall and bridge structures. For modern, well-detailed RC frame structures Priestley et al. (2007) recommend slightly larger values, reflecting the additional energy dissipation that RC frame systems can provide. The β_f value of 0.3 for the flag-shaped hysteretic curve is considered to represent a rocking system with moderate levels of energy dissipation, noting that modern post-tensioned rocking systems can achieve greater β_f values (close to 1.0), whereas systems on rocking foundations may well possess lower β_f values.

Table 1. Parameters used to define the hysteretic properties of the nonlinear oscillators (after Sullivan et al. 2013).

Hysteretic Model	Parameters
Bi-linear	$r = 0.05$
Takeda (Thin)	$r = 0.05, \alpha = 0.5, \beta = 0.0$
SINA	$r = 0.05, F_{cc} = 0.3 F_y$
Flag-shaped	$r = 0.05, \beta_f = 0.3$

Models of the non-linear oscillators were created in Ruaumoko (Carr, 2009) and consist of a lumped mass connected to the ground via to a force-displacement spring element, the hysteretic properties of which have been described above. Nonlinear dynamic analyses were conducted using a Newmark integration scheme with an integration time-step of 0.01s. For each accelerogram (details of which are provided in the next section) the peak inelastic displacement for each oscillator was obtained and a ductility demand computed (according to the relationship shown in Figure 2). Elastic spectral demands for each oscillator were also computed, using the period of the oscillator and considering the 5% damped elastic spectral demand of each accelerogram. The next section of this paper describes the ground motions used for the analyses. Subsequently, the statistical processing of the data used to arrive at new empirical relationships for the inelastic displacement demand is explained and results presented.

GROUND MOTION RECORDS

The ground-motion records used to determine the inelastic displacements are a subset of the PEER NGA strong-motion database (Chiou et al., 2008). The particular subset is the same as that used in previous studies (such as Stafford and Bommer, 2009, and Stafford et al., 2009), and consists of 2406 accelerograms, each with two horizontal components, from 114 earthquakes. The process followed to define this subset is described in detail in Stafford and Bommer (2009), in which the particular events and numbers of records from each event are tabulated. The range of magnitudes covered by the data is 4.79-7.9 and rupture distances up to 100 km are considered. The site characteristics, as defined by the average shear-wave velocity, span the range 116-2017 m/s but are concentrated over velocities corresponding to NEHRP C and D classes.

While the total number of accelerogram components that are considered is 4812, which combined with the 10 values of the yield strength coefficient leads to 48120 analyses per hysteretic model and response period, the results from this full number are rarely used. For large values of the yield strength coefficient the response may remain elastic and the model formulation, described in the following section, dictates that these elastic responses need not be considered. However, even for very weak systems where many records will cause nonlinear

response, the overall dataset must still be reduced to account for the limited bandwidth of the recording instruments. To this end, the lowest usable frequency for each component (Chiou et al., 2008) is used to limit the numbers of records considered for each period. Therefore, the actual numbers of records that constrain the predictive relationships vary with both period and yield strength coefficient, as shown in Figure 3. From this figure it is clear that there is far greater empirical constraint at short periods than for long response periods.

Figure 3. Numbers of records used as a function of period and yield strength coefficient. The gray contours correspond to constant numbers of records.

MODEL FORMULATION

Two main approaches have been adopted in the literature for formulating predictive models for inelastic response. The first approach is to define ratios of inelastic to elastic displacements that are then combined with traditional ground-motion models for elastic ordinates to obtain inelastic predictions. Within this approach there are two subtly different options that one may employ. Most of the studies referred to in the introduction have developed models from ratios of computed inelastic and computed elastic response. The study of Tothong and Cornell (2006) is notably different from these in that their formulation works with the ratio of computed inelastic, but *predicted* elastic response. For the purposes of undertaking probabilistic hazard calculations directly for the inelastic response the latter approach of Tothong and Cornell (2006) offers important benefits and more readily enables coupling with ground-motion models for elastic spectral ordinates.

The second main approach is that adopted by Bozorgnia et al. (2010). This study considered elastic-perfectly-plastic oscillators and constant ductility spectra, but made direct predictions of their inelastic response quantity by using the same functional form as that used by Campbell and Bozorgnia (2008) for elastic spectral accelerations. This approach results in models that can be directly incorporated into existing software for hazard analysis. However, it also makes the implicit assumption that the functional form appropriate for elastic ordinates is also meaningful for inelastic quantities.

In the present study the approach adopted is most similar to that used by Tothong and Cornell (2006) in which the final predictive model is developed to be used with predictions from ground-motion models for elastic spectral ordinates. However, rather than deriving this model for inelastic-to-elastic displacement ratios directly in terms of the predicted elastic displacements (which has the undesirable effect of tying the predictive model to a particular elastic ground-motion model), a more general approach is adopted that enables the new model to be used interchangeably with any ground-motion model for elastic ordinates that has been developed for shallow active crustal regions.

GENERAL FRAMEWORK FOR INELASTIC DISPLACEMENT PREDICTION

A given inelastic spectral displacement, $S_{d,i}$, can be defined in terms of an elastic spectral displacement, $S_{d,e}$, as:

$$S_{d,i} = S_{d,e} \left(S_{d,i} / S_{d,e} \right) \quad (3)$$

As commonly assumed, the inelastic displacements are well represented by a lognormal distribution, leading to the transformed version of the above equation:

$$\ln S_{d,i} = \ln S_{d,e} + \ln \left(S_{d,i} / S_{d,e} \right) \quad (4)$$

For simplicity of notation, this equation is cast in terms of equivalent variables $y^i = \ln S_{d,i}$, $y^e = \ln S_{d,e}$, and $\Gamma = \ln \left(S_{d,i} / S_{d,e} \right)$. Therefore, the generic expression defining the inelastic response in terms of the elastic response is $y^i = y^e + \Gamma$.

Now, for a particular observation j from an earthquake i we would write:

$$y_{ij}^i = y_{ij}^e + \Gamma_{ij} \quad (5)$$

The two terms on the right-hand-side of this expression are known when undertaking a regression analysis (they are both computed from the accelerograms), but will not be known when making a prediction. It is therefore necessary to represent the terms on the right-hand-side

by quantities that may be predicted, plus any associated error in this prediction. The first term for the elastic displacements, y_{ij}^e , can be written in terms of a median prediction of these displacements as:

$$y_{ij}^e = \hat{y}_{ij}^e + \eta_i^e + \varepsilon_{ij}^e \quad (6)$$

where, the term $\hat{y}_{ij}^e$ is the median prediction (in logarithmic terms), and can be provided by a standard ground-motion model for elastic spectral ordinates, and the η_i^e and ε_{ij}^e are a random effect for event i and the residual respectively.

We must also develop a model for making predictions of the logarithmic inelastic displacement ratio that links elastic and inelastic displacements and we write:

$$\Gamma_{ij} = \hat{\Gamma}(y_{ij}^e, C_y) + \eta_i^\Gamma + \varepsilon_{ij}^\Gamma \quad (7)$$

as we will make the prediction of this logarithmic ratio a function of the *actual* elastic displacement and the strength C_y . Equation (7) states that the actual ratio can be represented by a model for the ratio and some variability denoted by $\eta_i^\Gamma + \varepsilon_{ij}^\Gamma$.

With these two components defined we can write our generic expression for the logarithmic inelastic displacement as in Equation (8).

$$y_{ij}^i = \hat{y}_{ij}^e + \hat{\Gamma}(y_{ij}^e, C_y) + \eta_i^e + \varepsilon_{ij}^e + \eta_i^\Gamma + \varepsilon_{ij}^\Gamma \quad (8)$$

However, the functional expression for the median logarithmic inelastic, $\hat{\Gamma}$, is still a function of the *true*, but unknown (for future predictions), elastic displacement. In order to deal with this issue we can make use of a first order Taylor series expansion of the logarithmic ratios about the median predicted logarithmic elastic displacement. That is:

$$\hat{\Gamma}(y_{ij}^e, C_y) \approx \hat{\Gamma}(\hat{y}_{ij}^e, C_y) + \left. \frac{\partial \hat{\Gamma}(y_{ij}^e, C_y)}{\partial y_{ij}^e} \right|_{\hat{y}_{ij}^e} (y_{ij}^e - \hat{y}_{ij}^e) = \hat{\Gamma}(\hat{y}_{ij}^e, C_y) + \left. \frac{\partial \hat{\Gamma}(y_{ij}^e, C_y)}{\partial y_{ij}^e} \right|_{\hat{y}_{ij}^e} (\eta_i^e + \varepsilon_{ij}^e) \quad (9)$$

and, for simplicity of notation, define:

$$\left. \frac{\partial \hat{\Gamma}(y_{ij}^e, C_y)}{\partial y_{ij}^e} \right|_{\hat{y}_{ij}^e} = \hat{\Gamma}'(\hat{y}_{ij}^e, C_y) \quad (10)$$

The final expression that is used within the regression framework for defining an observed logarithmic inelastic displacement, in terms of the sum of median predictions of logarithmic

elastic displacements and the logarithmic inelastic ratio as well as various random effects and residual errors, can then be given by:

$$y_{ij}^i = \hat{y}_{ij}^e + \hat{\Gamma}(\hat{y}_{ij}^e, C_y) + [1 + \hat{\Gamma}'(\hat{y}_{ij}^e, C_y)] \eta_i^e + [1 + \hat{\Gamma}'(\hat{y}_{ij}^e, C_y)] \varepsilon_{ij}^e + \eta_i^\Gamma + \varepsilon_{ij}^\Gamma \quad (11)$$

This equation is expressed entirely in terms of independent variables that are known, or can be computed, when making future predictions, with the primary independent variable being $\hat{y}_{ij}^e$. Although it is a rather elaborate expression, it should be borne in mind that this holds for the regression analysis. For making forward predictions the model must provide terms for the median inelastic displacement for a given scenario as well as the variance of these inelastic displacements. From the above equation, the expression for the median inelastic displacement is the far more compact:

$$\hat{y}_{ij}^i = \hat{y}_{ij}^e + \hat{\Gamma}(\hat{y}_{ij}^e, C_y) \quad (12)$$

where the median logarithmic inelastic displacement is simply the median logarithmic elastic displacement plus the median of the logarithmic ratios of inelastic-to-elastic displacement, predicted as a function of the median elastic displacement.

Equation (11) contains four random variables. The random effects from the elastic prediction and the inelastic ratios η_i^e and η_i^Γ , respectively, as well as the residual errors ε_{ij}^e and ε_{ij}^Γ . All of these variates are zero-mean normally distributed with variances equal to $\tau_{y^e, i}^2$, $\tau_{\Gamma, i}^2$, $\sigma_{y^e, ij}^2$, and $\sigma_{\Gamma, ij}^2$ respectively. Each of these pairs of variables are correlated and so the expression for the variance of the inelastic displacements, given this linearized framework is:

$$\begin{aligned} \sigma_{y^i, ij}^2 = & [1 + \hat{\Gamma}'(\hat{y}_{ij}^e, C_y)]^2 \tau_{y^e, i}^2 + [1 + \hat{\Gamma}'(\hat{y}_{ij}^e, C_y)]^2 \sigma_{y^e, ij}^2 + \tau_{\Gamma, i}^2 + \sigma_{\Gamma, ij}^2 \\ & + 2\rho_{y^e, \Gamma}^\eta [1 + \hat{\Gamma}'(\hat{y}_{ij}^e, C_y)] \tau_{y^e, i} \tau_{\Gamma, i} + 2\rho_{y^e, \Gamma}^\varepsilon [1 + \hat{\Gamma}'(\hat{y}_{ij}^e, C_y)] \sigma_{y^e, ij} \sigma_{\Gamma, ij} \end{aligned} \quad (13)$$

The correlation coefficients $\rho_{y^e, \Gamma}^\eta$ and $\rho_{y^e, \Gamma}^\varepsilon$ respectively relate to correlations between the random effects and the residual errors for the elastic displacements and inelastic ratios. In this expression the variance components have subscripts included to denote the fact that these may not be homoskedastic and may vary from event-to-event and from site-to-site. It is also important to note from this expression that irrespective of whether or not the variance components of the elastic ground-motion model, or of the inelastic ratio, are heteroskedastic or not, the variance $\sigma_{y^i}^2$ will be heteroskedastic (for both the inter-event and intra-event components) as a result of the dependence upon $\hat{\Gamma}'$.

MODEL FOR THE MEDIAN INELASTIC DISPLACEMENT RATIO

The main component of interest in the previous set of equations is $\hat{\Gamma}(y^e, C_y)$. In order to gain an appreciation for what form this median function should adopt an exploratory data analysis was initially conducted. A particular issue that arose in this case, especially for short period systems, is that the dataset contains so many observations that it is often difficult to identify trends in the data. For that reason, trends were primarily identified through the use of plots of binned response values, with both mean and median values shown. An example of such a plot is shown in Figure 4.

Figure 4. Demonstration of the degree of dispersion in the data and how the use of binned response plots enables clear trends to be observed.

Binned response plots were initially created for each of the hysteretic models for each of the considered response periods. The primary purpose of this exercise was to ascertain how the four hysteretic models behaved and whether a single common functional form could be used to represent the scaling in all four cases. Figure 5 shows an example of the scaling that is obtained for a period of 0.1 seconds, and from figures such as this it can be appreciated that while there are differences in the actual amplitudes of the inelastic ratios, the general shapes share many common features. From the outset it was seen that the primary predictor variable would be the elastic spectral displacement and by normalizing these displacements by the yield displacement for the oscillators the motions resulting nonlinear response could easily be identified. Therefore, plots like that shown in Figure 5 are presented in terms of a relative demand, which is defined as

the logarithm of the ratio of elastic to yield displacement. Note that as the yield displacement is directly a function of the yield strength coefficient ($S_{d,y} = gC_y / \omega_0^2$), the relative demand directly reflects the yield strength coefficient, as shown in Figure 5. Indeed, from Figure 5 it is possible to appreciate that the principle scaling of the inelastic displacement ratio depends upon the relative magnitudes of the elastic displacement and C_y for any given response period.

Figure 5. Generic scaling of the inelastic response ratios with respect to the relative demand for four different response periods.

In addition to the primary scaling with respect to the elastic displacement, the behavior with respect to other independent variables was investigated. As mentioned earlier, Tothong and Cornell (2006) identified a dependence upon magnitude that they attributed to the effects of spectral shape. Therefore, during the diagnostic phase of the regression analysis binned plots like those shown in Figure 6, with binning in terms of the secondary independent variables of magnitude, distance, site characteristics, source properties, yield strength coefficient, *etc.*, were created and inspected for all response periods. From these inspections it was found that magnitude had by far the strongest influence upon the inelastic spectral ratios of these secondary variables and that the other independent variables had a relatively weak impact.

Figure 6. Scaling of the inelastic displacement ratio at 0.1 seconds with respect to the relative demand for all four hysteretic models and for ranges of magnitude shown in the panel titles.

While the behavior shown in Figures 5 and 6 suggest that a common functional form might be applicable for all hysteretic models, when longer response periods are considered this is no longer the case. In particular, the bilinear model exhibits scaling for low levels of relative demand that differs from that of the other models. Specifically, while the other models tend to show a relatively abrupt transition from elastic to inelastic response, the bilinear model initially shows insensitivity to increasing relative demand beyond yield before then having a more gradual transition into increasing values of the inelastic ratio. This can be seen even for the short-period system shown in Figure 6 for the smaller magnitude ranges. For large levels of relative demand all four models tend to have a similar scaling of the inelastic ratios. As a result, a common functional form was adopted for the SINA, Takeda and Flag models, but a different form was selected to capture the smoother scaling just beyond yield that occurs for the bilinear model.

For the SINA, Takeda and Flag models the functional form for the inelastic ratios is given by:

$$\hat{\Gamma}(y_{ij}^e, C_y, M_w) = [\phi_1 + \phi_5 (M_w - 6)] \left[1 - \exp \left\{ -\exp [\phi_2 + \phi_4 (M_w - 6)] (r_x - \phi_3) \right\} \right] \quad (14)$$

where $r_x = \max(y_{ij}^e - y_{ij}^y, \phi_3)$, and the ϕ_i with $i \in \{1, \dots, 5\}$ representing regression coefficients. The coefficients for all hysteretic models were obtained through a regression analysis using the software R (R Core Team, 2015) and the nlme package in particular (Pinheiro et al., 2015).

For the Bilinear model, the functional form is slightly simpler, involving one less regression coefficient, and is defined by:

$$\hat{\Gamma}_{ij}(y_{ij}^e, C_y, M_w) = \frac{\phi_1 + \phi_4(M_w - 6)}{1 + \exp[(\phi_2 - r_x)/\phi_3]} \quad (15)$$

where now $r_x = y_{ij}^e - y_{ij}^y$. In both of these expressions, the logarithmic yield displacement, y_{ij}^y , can be found from the logarithm of the ratio of Equations (1) and (2). That is, the logarithmic yield displacement is defined using the following expression that includes the dependence upon the yield strength coefficient as well as the natural angular frequency of the oscillator ω_0 :

$$y_{ij}^y = \ln\left(\frac{F_y}{k_0}\right) = \ln\left(\frac{gC_y T^2}{4\pi^2}\right) = \ln\left(\frac{gC_y}{\omega_0^2}\right) \quad (16)$$

MODEL FOR THE VARIANCE OF THE INELASTIC DISPLACEMENT RATIO

For the median models presented in Equations (14)-(16) it should be apparent that when the elastic displacements are small, *i.e.*, below the value corresponding to ϕ_3 for the Flag, SINA and Takeda models, and below yield for the Bilinear model, the logarithmic inelastic displacement ratios will either equal zero, or will be very close to zero (for the Bilinear case). This fact places an immediate constraint upon variance for these inelastic ratios. Figure 4 shows that the variance of the inelastic ratios is very small just above yield, as should be expected, but then rapidly increases as the relative demand increases before later reaching some apparently stable value. It is important to account for this heteroskedasticity both when developing the model and making future predictions. In the case that this heteroskedasticity was ignored the large variance for high relative demands would act to significantly inflate the variance in cases when the relative demand only just implied inelastic response. From a model fitting perspective the implication would be that insufficient constraint would be imposed for cases just above yield, and this would influence the way the inelastic ratios transitioned into the elastic regime. From a forward prediction perspective, the implication of using an inflated variance just above yield would be to drastically overestimate the probabilities of exceeding limit-states associated with slight damage.

In order to account for this heteroskedastic variance in the regression, as well as for the predictive model itself, the variance structure used within the random effects regressions included residual variances that depended upon the relative demand according to a sigmoid function. That is, the intra-event standard deviation is defined as in Equation (17).

$$\sigma_{\Gamma}(r_x) = \sigma_{\Gamma} \frac{\omega_1}{1 + \exp[(\omega_2 - r_x)/\omega_3]} \quad (17)$$

In addition to this heteroskedasticity for the intra-event residuals, the regression analysis also took into account differences in the variances of both inter-event and intra-event residuals for observations associated with aftershocks and mainshocks. Finally, for the predictive model the inter-event standard deviation is set to scale in a manner that replicates the scaling of the median predictions. Therefore, for the Flag, SINA and Takeda models, the inter-event standard deviation is defined as:

$$\tau_{\Gamma}(r_x) = \tau_{\Gamma} \left[1 - \exp \left\{ -\exp \left[\phi_2 + \phi_4 (M_w - 6) \right] (r_x - \phi_3) \right\} \right] \quad (18)$$

while for the Bilinear model the inter-event standard deviation is defined as:

$$\tau_{\Gamma}(r_x) = \frac{\tau_{\Gamma}}{1 + \exp[(\phi_2 - r_x)/\phi_3]} \quad (19)$$

The total variance for the inelastic ratios is found from the sum of the squared variances associated with Equations (17) to (19).

MODEL COEFFICIENTS

The regression coefficients found for all response periods and for each of the four hysteretic models are presented in Tables 2-5.

Table 2. Coefficients for the Bilinear model. Additionally, note that $\psi_3 = 0.97886$ for all periods.

T	ϕ_1	ϕ_2	ϕ_3	ϕ_4	τ_{Γ}	σ_{Γ}	ω_1	ω_2	ω_3	$\rho_{y^e, \Gamma}^{\eta}$	$\rho_{y^e, \Gamma}^{\epsilon}$
0.1	1.7901	0.7823	0.2361	0.5043	0.7344	1.0484	0.7191	0.4340	0.1218	-0.3669	-0.1833
0.2	0.8669	1.1479	0.3044	0.4925	0.6088	1.0158	0.6716	0.9474	0.2793	-0.1703	-0.2428
0.3	0.5246	1.3955	0.3449	0.5031	0.5093	0.9516	0.6235	1.2987	0.3732	-0.1110	-0.2538
0.4	0.2450	1.4386	0.3466	0.4709	0.4981	0.9654	0.5825	1.4996	0.4221	-0.0878	-0.2534
0.5	0.0551	1.5316	0.3744	0.4909	0.4589	0.9365	0.5476	1.6115	0.4461	-0.0772	-0.2492
0.6	-0.0174	1.5669	0.3824	0.4976	0.3889	0.9327	0.5175	1.6701	0.4560	-0.0717	-0.2436
0.8	-0.1651	1.4546	0.3539	0.4336	0.3226	0.9318	0.4678	1.6999	0.4542	-0.0658	-0.2316
1.0	-0.2166	1.1774	0.3296	0.3383	0.2446	0.9433	0.4278	1.6729	0.4400	-0.0608	-0.2200
1.5	-0.1766	0.6486	0.1941	0.1333	0.0964	0.9525	0.3535	1.5351	0.3956	-0.0430	-0.1959
2.0	-0.2111	0.7234	0.2121	0.1067	0.1132	0.9485	0.3007	1.4004	0.3620	-0.0175	-0.1778
2.5	-0.2477	0.6138	0.1821	0.1339	0.0902	0.9015	0.2603	1.3007	0.3441	0.0133	-0.1640
3.0	-0.2438	0.6803	0.2002	0.0768	0.1038	0.8943	0.2279	1.2387	0.3404	0.0472	-0.1533
3.5	-0.2711	0.5973	0.1843	0.0655	0.1200	0.9371	0.2011	1.2106	0.3487	0.0828	-0.1449
4.0	-0.2402	0.4751	0.1571	0.0725	0.0888	0.9147	0.1784	1.2118	0.3670	0.1194	-0.1383

Table 3. Coefficients for the Flag model. Additionally, note that $\psi_3 = 0.98551$ for all periods.

T	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	τ_Γ	σ_Γ	ω_1	ω_2	ω_3	$\rho_{y^e, \Gamma}^\eta$	$\rho_{y^e, \Gamma}^\varepsilon$
0.1	2.8248	0.5308	0.2043	0.2172	0.2798	0.6359	1.0735	0.7267	0.2930	0.0506	-0.3757	-0.2130
0.2	1.6673	0.2405	0.2876	0.0627	0.4861	0.6072	1.0003	0.6411	0.4522	0.0869	-0.2436	-0.2334
0.3	1.2516	0.0785	0.2949	-0.0800	0.5369	0.5412	0.9680	0.5894	0.6594	0.1691	-0.1611	-0.2442
0.4	0.8991	0.1892	0.2950	-0.2047	0.5453	0.4631	0.9876	0.5582	0.8080	0.2269	-0.1050	-0.2490
0.5	0.6977	0.2969	0.2952	-0.3148	0.5376	0.3332	0.9750	0.5378	0.9100	0.2642	-0.0648	-0.2502
0.6	0.6162	0.3866	0.2962	-0.4135	0.5232	0.3146	0.9316	0.5231	0.9795	0.2875	-0.0350	-0.2491
0.8	0.4177	0.7191	0.2999	-0.5850	0.4875	0.2328	0.9166	0.5023	1.0567	0.3083	0.0047	-0.2427
1.0	0.3309	0.9270	0.3039	-0.7315	0.4507	0.2189	0.8886	0.4859	1.0852	0.3104	0.0283	-0.2332
1.5	0.1712	1.3685	0.3089	-1.0276	0.3693	0.1978	0.9173	0.4468	1.0621	0.2858	0.0512	-0.2033
2.0	0.1101	1.8103	0.3045	-1.2611	0.3054	0.1664	0.9493	0.4024	0.9897	0.2528	0.0492	-0.1703
2.5	0.0697	2.3414	0.2914	-1.4556	0.2555	0.1996	0.8900	0.3520	0.9064	0.2245	0.0357	-0.1369
3.0	0.0377	2.8366	0.2712	-1.6232	0.2161	0.1631	0.8646	0.2965	0.8245	0.2038	0.0162	-0.1039
3.5	-0.0387	3.4125	0.2455	-1.7710	0.1848	0.1366	0.9704	0.2372	0.7486	0.1909	-0.0068	-0.0717
4.0	-0.0445	3.6049	0.2153	-1.9036	0.1599	0.1174	0.9425	0.1749	0.6801	0.1850	-0.0317	-0.0405

Table 4. Coefficients for the SINA model. Additionally, note that $\psi_3 = 1.0296$ for all periods.

T	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	τ_Γ	σ_Γ	ω_1	ω_2	ω_3	$\rho_{y^e, \Gamma}^\eta$	$\rho_{y^e, \Gamma}^\varepsilon$
0.1	2.7211	0.3682	0.1917	0.5654	0.1836	0.7684	1.0300	0.7083	0.2542	0.0452	-0.3960	-0.2850
0.2	1.6321	-0.2341	0.1267	0.2422	0.4503	0.7260	1.0334	0.6019	0.3838	0.1115	-0.2263	-0.2760
0.3	1.1982	-0.4567	0.1380	0.0708	0.5055	0.6396	1.0028	0.5667	0.6449	0.2402	-0.1533	-0.2747
0.4	0.6992	-0.0023	0.1623	-0.0428	0.5102	0.4488	1.0388	0.5452	0.8318	0.3250	-0.1125	-0.2731
0.5	0.5061	-0.0004	0.1882	-0.1264	0.4983	0.3545	1.0005	0.5277	0.9565	0.3755	-0.0863	-0.2704
0.6	0.4103	0.0852	0.2128	-0.1917	0.4803	0.3114	0.9759	0.5119	1.0379	0.4033	-0.0677	-0.2670
0.8	0.1850	0.7545	0.2561	-0.2895	0.4407	0.2149	0.9852	0.4824	1.1208	0.4195	-0.0429	-0.2582
1.0	0.1149	0.7767	0.2917	-0.3607	0.4035	0.2035	0.9307	0.4548	1.1438	0.4086	-0.0264	-0.2479
1.5	-0.0239	0.6043	0.3560	-0.4800	0.3306	0.2131	0.9563	0.3920	1.0992	0.3463	0.0001	-0.2190
2.0	-0.0645	0.7859	0.3975	-0.5568	0.2817	0.1971	0.9574	0.3366	1.0169	0.2843	0.0184	-0.1886
2.5	-0.1149	0.9693	0.4055	-0.6117	0.2494	0.2189	0.8946	0.2877	0.9406	0.2394	0.0336	-0.1582
3.0	-0.1588	1.1998	0.4055	-0.6537	0.2287	0.2030	0.8940	0.2441	0.8813	0.2133	0.0472	-0.1283
3.5	-0.2303	1.4612	0.4055	-0.6871	0.2163	0.1958	1.0150	0.2048	0.8410	0.2046	0.0600	-0.0992
4.0	-0.2468	1.8637	0.4055	-0.7145	0.2100	0.1336	0.9491	0.1694	0.8188	0.2109	0.0721	-0.0709

Table 5. Coefficients for the Takeda model. Additionally, note that $\psi_3 = 1.00303$ for all periods.

T	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	τ_Γ	σ_Γ	ω_1	ω_2	ω_3	$\rho_{y^e, \Gamma}^\eta$	$\rho_{y^e, \Gamma}^\varepsilon$
0.1	2.6214	-0.0376	0.1374	0.6173	-0.0950	0.8143	1.0258	0.5759	0.2882	0.0688	-0.3959	-0.2483
0.2	1.2832	-0.0900	0.2355	0.1921	0.4317	0.6542	1.0219	0.5317	0.5088	0.1526	-0.1993	-0.2583
0.3	0.8516	-0.1271	0.2993	-0.0362	0.4995	0.5450	0.9877	0.4948	0.7716	0.2492	-0.1213	-0.2641
0.4	0.5141	0.1055	0.3454	-0.1890	0.4896	0.4507	0.9995	0.4692	0.9528	0.3107	-0.0814	-0.2661
0.5	0.3463	0.2385	0.3808	-0.3023	0.4631	0.3486	0.9744	0.4506	1.0729	0.3476	-0.0580	-0.2654
0.6	0.2740	0.3646	0.4092	-0.3916	0.4349	0.3130	0.9362	0.4364	1.1520	0.3685	-0.0430	-0.2631
0.8	0.1370	0.6798	0.4522	-0.5261	0.3857	0.2427	0.9185	0.4151	1.2350	0.3831	-0.0256	-0.2555
1.0	0.0619	0.7875	0.4835	-0.6253	0.3485	0.2290	0.8916	0.3986	1.2616	0.3790	-0.0160	-0.2454
1.5	-0.0802	1.2838	0.5339	-0.7938	0.2896	0.1905	0.9225	0.3641	1.2291	0.3434	-0.0038	-0.2159
2.0	-0.1158	1.4098	0.5634	-0.9042	0.2542	0.1737	0.9469	0.3310	1.1556	0.3059	0.0037	-0.1843
2.5	-0.1546	1.9369	0.5819	-0.9846	0.2271	0.1751	0.8786	0.2967	1.0824	0.2782	0.0105	-0.1526
3.0	-0.1679	2.3871	0.5937	-1.0469	0.2021	0.1650	0.8619	0.2608	1.0206	0.2618	0.0176	-0.1216
3.5	-0.2417	2.5545	0.6013	-1.0972	0.1764	0.1758	0.9663	0.2235	0.9730	0.2560	0.0251	-0.0914
4.0	-0.2190	3.5198	0.6059	-1.1390	0.1491	0.1025	0.9162	0.1851	0.9395	0.2594	0.0329	-0.0621

MODEL FOR THE VARIANCE OF THE INELASTIC DISPLACEMENTS

The expressions previously presented for the total variance of the inelastic displacements included the partial derivatives of the displacement-ratio models in accordance with standard expressions for error propagation. However, these expressions are most relevant in cases where the linearized function is well approximated by the first-order expansion over a region comparable to the variance of the variable about which the expansion was made. In our case, the variance of the elastic displacements is relatively large and the inelastic ratio models (particularly for the Flag, SINA and Takeda) cases are highly nonlinear, and not continuous for cases just beyond yield. In particular, the term $\hat{\Gamma}'$ changes from being zero to being very large either side of $r_x = \phi_3$ and so the propagated variance would have a significant jump at this point. Such considerations dictate that first-order error propagation formulations are not appropriate and provide the motivation for the adoption of an alternative approach based upon Monte Carlo simulation.

For a given value of the standard deviation for elastic displacements (which can be found from a ground-motion model) one can sample large numbers of possible elastic displacements that are consistent with a given value of the median, $\hat{y}_{ij}^e$, and then pass these simulated values through the model for the inelastic displacement ratio. Taking the standard deviation of these predicted ratios then gives a measure of the impact of approximating the true elastic displacement by the median prediction.

Following this approach, the behavior of these standard deviations from the propagated variance in the elastic displacements can be inspected and checked for homoskedasticity with respect to magnitude, period, *etc.* In the case of the Flag, SINA and Takeda models, the decreasing slope with increasing relative demand for the median inelastic ratio models results in a propagated distribution that has a slight positive skew that is most significant just above $r_x = \phi_3$. Elsewhere, however, the shape of the distribution is very well modeled by a normal distribution, albeit with . Therefore, rather than using the first-order error propagation framework to provide estimates of the total variance of the inelastic displacements separate models for this propagated error are developed.

The first-order error propagation expression of Equation (13) can be rewritten by grouping the terms that are associated with the base elastic and inelastic-ratio models and those terms associated with the error propagation, as shown in Equation (20).

$$\begin{aligned}
\sigma_{y^i,ij}^2 = & \underbrace{\tau_{y^e,i}^2 + \tau_{\Gamma,i}^2 + 2\rho_{y^e,\Gamma}^\eta \tau_{y^e,i} \tau_{\Gamma,i}}_{\text{inter-event terms from base models}} + \underbrace{\sigma_{y^e,ij}^2 + \sigma_{\Gamma,ij}^2 + 2\rho_{y^e,\Gamma}^\epsilon \sigma_{y^e,ij} \sigma_{\Gamma,ij}}_{\text{intra-event terms from base models}} \\
& + \underbrace{\hat{\Gamma}'(\hat{y}_{ij}^e, C_y)^2 \tau_{y^e,i}^2 + 2\rho_{y^e,\Gamma}^\eta \hat{\Gamma}'(\hat{y}_{ij}^e, C_y) \tau_{y^e,i} \tau_{\Gamma,i}}_{\text{propagated inter-event contributions}} \\
& + \underbrace{\hat{\Gamma}'(\hat{y}_{ij}^e, C_y)^2 \sigma_{y^e,ij}^2 + 2\rho_{y^e,\Gamma}^\epsilon \hat{\Gamma}'(\hat{y}_{ij}^e, C_y) \sigma_{y^e,ij} \sigma_{\Gamma,ij}}_{\text{propagated intra-event contributions}}
\end{aligned} \tag{20}$$

Note that in this expression, the propagated error terms are associated with the first-order contributions that are defined as:

$$\Delta\tau(\hat{y}_{ij}^e) = \hat{\Gamma}'(\hat{y}_{ij}^e, C_y) \tau_{y^e,i} \quad \text{and} \quad \Delta\sigma(\hat{y}_{ij}^e) = \hat{\Gamma}'(\hat{y}_{ij}^e, C_y) \sigma_{y^e,ij} \tag{21}$$

and the total propagated variance coming from the elastic displacements is defined by:

$$\Delta\sigma_{y^i,ij}^2(\hat{y}_{ij}^e) = \hat{\Gamma}'(\hat{y}_{ij}^e, C_y)^2 \left[\tau_{y^e,i}^2 + \sigma_{y^e,ij}^2 \right] = \hat{\Gamma}'(\hat{y}_{ij}^e, C_y)^2 \sigma_{T,y^e,ij}^2 \tag{22}$$

Therefore, given that we know this total propagated variance we can then partition this into inter- and intra-event contributions on a *pro rata*. That is, given $\Delta\sigma_{y^i,ij}^2(\hat{y}_{ij}^e)$ we can define the inter- and intra-event terms as:

$$\Delta\tau(\hat{y}_{ij}^e)^2 = \frac{\tau_{y^e,i}^2}{\tau_{y^e,i}^2 + \sigma_{y^e,ij}^2} \Delta\sigma_{y^i,ij}^2(\hat{y}_{ij}^e) \quad \text{and} \quad \Delta\sigma(\hat{y}_{ij}^e)^2 = \frac{\sigma_{y^e,ij}^2}{\tau_{y^e,i}^2 + \sigma_{y^e,ij}^2} \Delta\sigma_{y^i,ij}^2(\hat{y}_{ij}^e) \tag{23}$$

This means that the overall expression for the variance can be defined in terms of these propagated error contributions as:

$$\begin{aligned}
\sigma_{y^i,ij}^2 = & \underbrace{\tau_{y^e,i}^2 + \tau_{\Gamma,i}^2 + 2\rho_{y^e,\Gamma}^\eta \tau_{y^e,i} \tau_{\Gamma,i}}_{\text{inter-event terms from base models}} + \underbrace{\sigma_{y^e,ij}^2 + \sigma_{\Gamma,ij}^2 + 2\rho_{y^e,\Gamma}^\epsilon \sigma_{y^e,ij} \sigma_{\Gamma,ij}}_{\text{intra-event terms from base models}} \\
& + \underbrace{\Delta\sigma_{y^i,ij}^2(\hat{y}_{ij}^e)^2 + 2\rho_{y^e,\Gamma}^\eta \Delta\tau(\hat{y}_{ij}^e) \tau_{\Gamma,i} + 2\rho_{y^e,\Gamma}^\epsilon \Delta\sigma(\hat{y}_{ij}^e) \sigma_{\Gamma,ij}}_{\text{propagated error contributions}}
\end{aligned} \tag{24}$$

With this framework in mind we can then define the models for the total propagated error using three new parameters (that are derived from existing model terms).

$$\Delta\sigma_{y^i,ij}(\hat{y}_{ij}^e) = \frac{\psi_1}{\psi_3} \phi\left(\frac{r_x - \psi_2}{\psi_3}\right) \tag{25}$$

Here $\phi(z)$ is the probability density function for the standard normal distribution, and for the Flag, SINA and Takeda models ψ_1 and ψ_2 are defined in Equations (26) and (27), while ψ_3 is a constant for all periods and magnitudes and is given in Tables 3-5.

$$\psi_1 = |0.7\phi_5 (M_w - 6 + \phi_1/\phi_5)| \quad (26)$$

$$\psi_2 = \phi_3 + \frac{0.7}{\exp[\phi_2 + \phi_4 (M_w - 6)]} \quad (27)$$

Equations (26) and (27) do not apply for the Bilinear case. The first-order error propagation approach actually performs relatively well for the Bilinear model as a result of its more smoothly varying nature that leads to relatively low gradients. However, the variance can still be reduced by using a more precise definition of the propagated errors and any such reduction has important practical implications. The value of ψ_1 is defined in Equation (28), while $\psi_2 = \phi_2$ and ψ_3 is again constant and is given in Table 2.

$$\psi_1 = |0.8\phi_4 (M_w - 6 + \phi_1/\phi_4)| \quad (28)$$

BEHAVIOR OF MODEL FOR INELASTIC DISPLACEMENTS

In order to demonstrate the behavior of the complete model for the inelastic displacements it is necessary to combine the predictions of the inelastic ratios with elastic predictions from traditional ground motion models. To that end, the recently released NGA-West2 models of Campbell and Bozorgnia (2014) and Chiou and Youngs (2014), referred to as CB2014 and CY2014 hereafter, have been chosen to provide the elastic predictions for the following demonstration. It should be noted that the main reason that these models were selected was that they were developed from a database that shares many of the records used in the present study, and that they represent state-of-the-art efforts in semi-empirical ground-motion modeling. However, there is no particular reason why other models, including models developed from different regions could not be used with the newly presented expressions for the inelastic ratios. That said, by using these recent NGA-West2 models coupled with the new expressions for the inelastic ratios, the derived model for inelastic displacements implicitly accounts for attributes, such as those for finite-fault effects, nonlinear site response, sediment depth, and regionality. All of these features are passed into the model for inelastic displacements through the dependence of the inelastic ratios upon the predicted median elastic displacement.

Figure 7 provides an example of the shape of inelastic spectra for a given scenario and for each of the hysteretic models. It can be appreciated from this figure that at long periods the systems do not experience inelastic response for this particular yield strength. It can also be noted that differences in the inelastic predictions inevitably arise from differences in the underlying elastic predictions of the CB2014 and CY2014 models. While there is undoubtedly epistemic uncertainty associated with the expressions for the inelastic ratios, a significant

portion of the overall epistemic uncertainty in the inelastic displacement predictions will actually come directly from the uncertainty in the elastic predictions. This is an advantage of deriving the median inelastic ratio models in terms of the computed elastic displacements, but then using the Taylor series to recast the prediction equation in terms of the predicted elastic displacements. The advantage comes from being able to apply the models for the inelastic ratios to arbitrary elastic ground-motion models.

Figure 7. Predictions of inelastic and elastic displacements for the scenario shown in the panel headers, and for a vertical strike-slip rupture with average shear-wave velocity, $V_{s,30}$, of 350m/s.

Figure 8 expands upon Figure 7 by demonstrating the main influence of the expressions for the inelastic ratios. In particular, this figure highlights the effect of changing the strength of the oscillator as well as the intensity of the earthquake scenario. Each of these effects influences the relative demand that is primarily responsible for the scaling of the inelastic ratios, while the change of magnitude plays a secondary role in influencing the shape of the inelastic ratios. The consideration of multiple values of C_y in Figure 8 allows the points of deviation from the elastic spectra to be clearly seen, as well as the extent to which the elastic displacements are (generally) amplified through the nonlinear behavior of the systems. The departure of the inelastic displacements from their elastic counterparts, and the dependence of this departure upon both the strength of the system and the magnitude of the earthquake scenario have clear and important implications for design approaches based upon the assumption of equal displacements.

Figure 8. Demonstration of the influence of the yield strength coefficient on inelastic spectra for a moderate (left column) and strong (right column) scenario. All panels correspond to a strike-slip rupture and a site with $V_{S,30}$ of 350m/s. Elastic predictions are from CY2014.

Figure 9. Scaling of the inelastic ratios with respect to ductility demand for a number of combinations of response period and magnitude.

The behavior of the inelastic spectra at long response periods shown in Figure 8 is simply the result of the median elastic predictions providing very weak demands at these periods. In order to emphasis the point that significant departures from the elastic predictions do still occur at intermediate-to-long response periods, Figure 9 isolates the scaling of the inelastic displacement ratios for a number of combinations of period and earthquake magnitude. In Figure 9, these inelastic ratios are plotted against the displacement ductility demand that is a direct function of C_y . What can be appreciated from this figure, particularly in the panels related to $T=1$ seconds, is that the inelastic displacements will differ in a very significant manner from their elastic counterparts when low values of C_y exist (which is often the case for long-period structures).

While the scaling of the median inelastic displacements is clearly shown in Figure 8, for hazard and risk applications the standard deviation of the inelastic displacements is arguably as,

if not more, important. Figure 10 shows the behavior of the variance components against period for a strong earthquake scenario and relatively weak systems.

Figure 10. Variance components plotted as a function of period for all hysteretic models. The predictions are again for a vertical strike-slip rupture and for a site with $V_{s,30}$ of 350m/s.

It is clear from this figure that standard deviations are greater at short periods. However, the results of Figure 8 should be kept in mind in that for all systems the response at periods beyond about 2 seconds is essentially elastic. The variance components shown in this period range are therefore effectively just those of the elastic models of CB2014 and CY2014. Great care was taken when developing the model for the variance to ensure that the variance associated with the inelastic ratios did not double-count any of the variance associated with the elastic predictions. The developed model for the inelastic ratios therefore ensures that the properties of the elastic prediction equation (both in terms of median and standard deviation) are preserved when scenarios are considered that lead to elastic response (with the exception of a slight, and deliberate inflation of the variance just below yield, discussed in the following section). To demonstrate this latter point, in Figure 11 the total standard deviations for the logarithmic inelastic displacements are plotted against the ductility demand for the same scenarios that were considered in Figure 9. Figure 11 demonstrates that the standard deviations increase with increasing ductility demand in a manner roughly consistent with the trends in the inelastic ratios. The scaling is more complex in the transition region from elastic to inelastic response as here the effects of the error propagation are strongest. These standard deviations are also compared

against the equivalent standard deviations of the elastic displacements and are seen to be larger to an extent that depends upon the ductility.

Figure 11. Variation of the total standard deviation of logarithmic inelastic spectral displacements as a function of ductility demand for combinations of period and magnitude. The vertical dotted line is positioned at a ductility of unity.

Tables 2-5 include correlation coefficients between the random effects and residual errors for the elastic displacements and inelastic ratios that are used when computing the variance of the inelastic displacements in Equation (24). To compute these values it is necessary to consider normalized residuals from the inelastic ratio models *and* normalized residuals from the elastic models. For that reason the correlation coefficients can vary from one elastic prediction equation to another. However, in developing the model the sensitivity of these coefficients to the chosen elastic model was investigated. It was seen that very similar values of this correlation were obtained for both the CB2014 and CY2014 models. The values presented in Tables 2-5 are therefore based upon the elastic residuals obtained from both of these models and represent a generic average value. These correlations can be used when coupling the inelastic ratio models with other elastic models. Even though the correlations are small, and could arguably be ignored to simplify the expressions for the variance, their inclusion does help to ensure that no double-counting takes place and generally leads to lower variance for the inelastic displacements.

COMPARISON WITH TOTHONG & CORNELL (2006) FORMULATION

The adopted formulation of defining the inelastic ratios in terms of the median predicted elastic displacements is similar to the approach used by Tothong and Cornell (2006). However, there are also some very important differences that are worth highlighting. Tothong and Cornell (2006) developed the model for inelastic ratios directly using the median elastic predictions from a particular ground-motion model. This offers advantages in terms of the treatment of the variance as there is no need to propagate any variability through the inelastic ratios, but carries the disadvantage of complicating the model for the median. This is because predicted elastic median displacements below the yield displacement do not necessarily imply elastic behavior. Therefore, the Tothong and Cornell (2006) approach leads to a more complex model for the median inelastic ratios (especially just above and below yield), but a simpler model for the variance. This is exacerbated by them not decomposing the variability into inter- and intra-event components (which is required for spatial hazard and risk analyses).

The approach adopted herein allows for a simpler definition of the median inelastic ratios, but leads to a more complex variance structure. In this study, this increased complexity is also due to the consideration of heteroskedasticity with respect to the relative demand, the treatment of inter- and intra-event components, and the correlations between the inelastic ratios and the elastic displacements.

The main advantage of the present approach, however, is the fact that the model is not tied to any particular elastic prediction equation. The new models are therefore more regionally transportable and allow for epistemic uncertainty in inelastic displacements to be treated more cleanly.

A quantitative comparison of the predicted inelastic displacement ratios using the models of Tothong and Cornell (2006) and this study is shown in Figure 12. It is clear from this figure that there is relatively good agreement for small magnitude events, but that there are significant differences between the two models for larger events. However, it must be recognized that although both models make use of bilinear oscillators with the same post-yield stiffness ratio there are other differences in the modelling approach that complicate this comparison. For example, in the present study tangent stiffness proportional damping has been adopted whereas initial stiffness proportional damping was almost certainly adopted in Tothong and Cornell (2006). More importantly, the Tothong and Cornell (2006) model depends directly upon the now obsolete elastic spectral acceleration model of Abrahamson and Silva (1997). As mentioned above, this limits the transportability of the model and dictates that any biases in the elastic prediction equation must be compensated for in the model for the inelastic ratios.

Figure 12. Comparison of the inelastic displacement ratios computed using the model of Tothong and Cornell (2006) and the model of this study.

DISCUSSION & AREAS FOR FUTURE RESEARCH

Current design methods typically employ simple expressions, or concepts like 'equal displacement' or 'equal energy', to relate inelastic demands to their elastic counterparts. The elastic demands are defined from probabilistic hazard analysis, and reflect the earthquake characteristics that influence a given site, but the connection to inelastic response is usually independent of such characteristics. The models presented herein enable hazard analyses to be conducted directly in terms of inelastic displacements and result in spectral predictions like those shown in Figure 8 defining inelastic demands in terms of the response period and system strength. While the spectra shown in Figure 8 correspond to a single scenario, the use of the new models within hazard analysis would result in uniform hazard spectra, or other associated forms of conditional spectra, that are defined in terms of these fundamental structural characteristics. At the same time, the characteristics of the earthquakes that influence these demands are included throughout the process.

The results of this study have confirmed findings in previous studies that the earthquake magnitude and hysteretic model significantly affects the likely inelastic displacement demands. Current spectrum-based seismic design and assessment procedures tend to neglect these dependencies by adopting simplified relationships between elastic and inelastic displacements,

such as the equal displacement rule, often referred to as R - μ - T relationships. As such, by providing models that permit inelastic displacement demands to be computed during the hazard analysis stage, the results of this work could be used in place of elastic spectra and R - μ - T relationships, thereby offering the possibility for more accurate, yet simplified seismic design and assessment procedures.

Inelastic displacement expressions for four hysteretic models have been provided in this work and care has been made to ensure that these hysteretic models are representative of a wide range of different structural typologies. Nevertheless, it is also recognized that a large number of hysteretic models are available in the literature and therefore future research should aim to develop inelastic spectral displacement expressions for other such models. In addition, note that for assessment of existing structures P-delta effects will tend to modify the effective hysteretic response of a structure (such that the r -factor in Figure 2 can become negative) and amplify the ratio of inelastic to elastic spectral displacement demands. Expressions for inelastic displacement that account for P-delta effects should also therefore be accounted for in future studies.

CONCLUSIONS

Most empirical ground-motion prediction equations (GMPEs) used in seismic hazard assessment currently provide indications of elastic spectral response quantities. However, as numerous types of damage in a structure are a function of the inelastic deformation demand, it is typically left to the engineer, or building code prescriptions, to relate elastic with inelastic demands. An alternative approach, proposed by Tothong and Cornell (2006), is to implement relationships between elastic and inelastic spectral displacement demands within the seismic hazard assessment (SHA) process so as to directly provide site-specific information on inelastic spectral displacement demands that can be used for design or assessment. Recognizing the benefits of this approach, such relationships have been developed in this work for four different hysteretic models that are considered to represent a large range of possible structural typologies found in practice. The new relationships have been formulated as a function of building period and yield strength coefficient, thus making them independent of elastic spectral response quantities (the models use elastic displacement predictions, but these are defined directly in terms of seismological and site characteristics). Furthermore, care has been taken to correctly treat the statistical correlation between the elastic and inelastic spectral displacement values, providing improved consideration of variance compared to the previous proposal of Tothong and Cornell (2006). It is envisaged that the new relationships could be very beneficial for performance-based earthquake engineering, providing improved accuracy for spectrum-based

design and assessment methods by allowing hazard characteristics to be explicitly accounted for within demand estimates.

The careful treatment of the variance in the inelastic ratios, and their propagation to inelastic displacements, is a very important contribution of this article. One of the greatest advantages of making direct predictions of inelastic displacements is that it circumvents the need to break the process of estimating demand into steps related to hazard analysis, definition of hazard-consistent earthquake scenarios, accelerogram selection and modification, and nonlinear dynamic analysis. Despite advances in each of these areas in recent years, it is likely that some elements of variability in inelastic displacements are still being double-counted. The equations and framework presented in this article provide a robust reference against which the more fragmented approach referred to above can be compared. The predicted variance of the inelastic displacements is found to be greater than that for elastic displacements (and accelerations), but not by as much as is often assumed.

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